CONGRUENCES FOR THE COEFFICIENTS OF WEAKLY HOLOMORPHIC MODULAR FORMS

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Abstract

Recent works have used the theory of modular forms to establish linear congruences for the partition function and for traces of singular moduli. In each case, the values of the given arithmetic function appear as the Fourier coefficients of a weakly holomorphic modular form. We show that similar congruences exist for the coefficients of any weakly holomorphic modular form on any congruence subgroup $\Gamma_0(N)$. In particular, we give congruences for a wide class of partition functions and for traces of CM values of arbitrary modular functions on certain congruence subgroups of prime level. Finally, we make a more general statement about simultaneous congruences for the coefficients of weakly holomorphic modular forms on $\Gamma_1(N)$.

For my parents

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Chapter 1

Introduction and motivation

A modular form is a meromorphic function on the complex upper half-plane which transforms with respect to $SL_2(\mathbb{Z})$ or one of its subgroups. Modular forms are central to much of modern number theory. Famously, Wiles proved Fermat's Last Theorem by showing that all rational elliptic curves are associated to modular forms. They have also been used to investigate properties of certain arithmetic functions. We start with examples of two functions whose behavior has been explored through the theory of modular forms. We then propose a general question which is the topic of the rest of this work.

1.1 The partition function $p(n)$

Our first example is an arithmetic function with a rich history dating back to Euler.

Definition. A *partition* of a positive integer n is a nonincreasing sequence of positive integers which sums to n. The number of partitions of n is denoted by $p(n)$.

Euler discovered the following generating function for $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots, \tag{1.1}$$

where we agree that $p(0) = 1$. This generating function is a powerful tool in the study of partitions.

Ramanujan made considerable contributions to partition theory. Arguably his most famous result in this area is his discovery of the following properties of $p(n)$, which hold for every n:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Ramanujan proved these remarkable congruences using (1.1) together with some results from q-series. His work inspired a wealth of research into other congruences of the partition function. Congruences modulo powers of 5, 7 and 11, which had been conjectured by Ramanujan, were proved by Watson [38] and Atkin [5]. Atkin [6], Atkin-O'Brien [8], Hjelle-Kløve [17], Kløve [20, 19], and Newman [28] found congruences modulo small powers of other primes less than 31.

Recently, Ono [30], Ahlgren [1] and Ahlgren-Ono [3] showed that in fact there are infinitely many families of congruences for $p(n)$. Their method exploits the fact that if $q := e^{2\pi i z}$, a slight modification of (1.1) yields

$$\sum_{n \equiv -1 \pmod{24}} p\left(\frac{n+1}{24}\right) q^n = \frac{1}{q \prod_{n=1}^{\infty}(1-q^{24n})},$$

which is a *weakly holomorphic modular form.* Their method in [3] explains every known linear congruence for $p(n)$.

For each prime $\ell \geq 5$, define $\delta_\ell := \frac{\ell^2-1}{24}$, and let S_ℓ be the set

$$S_\ell := \left\{\beta \in \{0,1,\ldots,\ell-1\} : \left(\frac{\beta+\delta_\ell}{\ell}\right) = 0 \text{ or } -\left(\frac{-6}{\ell}\right)\right\}.$$

Theorem 1.1. [3, Theorem 1] *If $\ell \geq 5$ is prime, m is a positive integer, and $\beta \in S_\ell$, then* *a positive proportion of the primes* $Q \equiv -1 \pmod{24\ell}$ *have the property that*

$$p\left(\frac{Q^3 n + 1}{24}\right) \equiv 0 \pmod{\ell^m}$$

for all $n \equiv 1 - 24\beta \pmod{24\ell}$ *with* $(Q, n) = 1$.

Theorem 1.1 leads to the following corollary.

Corollary 1.2. [3, Theorem 3] *If* $\ell \geq 5$ *is prime,* m *is a positive integer, and* $\beta \in S_\ell$*, then there are infinitely many non-nested arithmetic progressions* $\{An + B\} \subseteq \{\ell n + \beta\}$*, such that for every integer* n *we have*

$$p(An + B) \equiv 0 \pmod{\ell^m}.$$

Remark. Corollary 1.2 and the Chinese Remainder Theorem yield congruences for any modulus M coprime to 6.

1.2 Traces of singular moduli

Linear congruences have also been found for traces of singular moduli. Let D be a positive integer, and let $\mathcal{Q}_D$ be the set of positive definite integral binary quadratic forms

$$F(x, y) = ax^2 + bxy + cy^2$$

of discriminant $-D = b^2 - 4ac$. The group $\overline{\Gamma} := \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ with finitely many equivalence classes. For each $F \in \mathcal{Q}_D$, define α_F to be the unique root of $F(x, 1)$ in the complex upper half-plane $\mathbb{H}$. Then the *singular modulus* $j(\alpha_F)$ is an algebraic integer, where

$$j(z) := \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots,$$

is the usual elliptic modular function on $\mathrm{SL}_2(\mathbb{Z})$. Singular moduli are related to many important objects in number theory. They generate ring class fields over imaginary quadratic fields, and are closely connected to the theory of elliptic curves with complex multiplication.

Following Zagier [39], we set $J(z) := j(z) - 744$ and consider the sequence of modular functions defined as follows. Let $T_0(m)$ be the normalized weight zero Hecke operator of index m (see 2.25 for a definition). Set $J_0(z) := 1$, and for each positive integer m, define

$$J_m(z) := J(z)|T_0(m).$$

Then the *mth Hecke trace* of the singular moduli of discriminant $-D$ is

$$t_m(D) := \sum_{F \in \mathcal{Q}_D/\overline{\Gamma}} \frac{J_m(\alpha_F)}{w_F}, \tag{1.2}$$

where w_F is the size of the stabilizer of F under the action of $\overline{\Gamma}$. Ahlgren and Ono [4] showed that infinitely many congruences exist for the Hecke traces.

Theorem 1.3. [4, Theorem 1] *Suppose that p is an odd prime and that s and m are positive integers with $p \nmid m$. Then a positive proportion of the primes ℓ have the property that*

$$t_m(\ell^3 n) \equiv 0 \pmod{p^s}$$

for every positive integer n coprime to ℓ such that p is inert or ramified in $\mathbb{Q}(\sqrt{-n\ell})$.

Remark. As with $p(n)$, the generating function for $t_m(D)$ is a weakly holomorphic modular form.

1.3 A general question

The method used to prove each of these results relies on the ability to realize both $\{p(n)\}$ and $\{t_m(D)\}$ as the coefficients of certain half-integral weight weakly holomorphic modular forms.

Other results obtained in a similar way include congruences for the number of partitions of n into distinct parts due to Lovejoy [25] and Ahlgren-Lovejoy [2], Swisher's congruences for the Andrews-Stanley partition function [37], and Mahlburg's congruences for the crank function [26]. It is natural, then, to ask how common such phenomena are. We answer this question by proving a general result for weakly holomorphic modular forms. We show that an infinite number of linear congruences exist for the coefficients of every weakly holomorphic modular form of any weight, on any congruence subgroup $\Gamma_0(N)$, and with any character χ. This includes every form which can be written as a quotient of eta-functions. In particular, we can find linear congruences for a wide class of partition functions.

Work of Deligne and Serre involving Galois representations provides a method for showing the existence of congruences for the coefficients of integral weight cusp forms. This coupled with Shimura's correspondence leads to congruences for the coefficients of half-integral weight cusp forms. The challenge, then, is to relate half-integral weight weakly holomorphic modular forms to half-integral weight cusp forms. There are two approaches in the literature for building cusp forms from weakly holomorphic forms, one which uses the U-operator and the other which uses a quadratic twist. We will prove general statements about the congruences that may be achieved through each approach.

In Chapter 2, we establish the definitions, facts and theorems that we will require. Chapter 3 details the results of the U-operator approach. In Chapter 4, we present the results of the quadratic twist approach, and give a comparison of the congruences predicted by each method. Finally, in Chapter 5 we extend our main results to modular forms on $\Gamma_1(N)$, and show that simultaneous congruences exist for any finite set of weakly holomorphic modular forms.

Chapter 2

Preliminaries and background material

In this chapter we collect the definitions and results which we will require from the theory of modular forms, as well as facts about modular Galois representations and quadratic Gauss sums. Unless otherwise noted, these facts can be found in more detail in [21], [31], or [35].

2.1 Integral weight modular forms

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The *modular group* $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on $\hat{\mathbb{C}}$ by linear fractional transformations. That is, for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we define

$$\gamma z := \frac{az+b}{cz+d} \text{ and } \gamma\infty := \frac{a}{c}.$$

The complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is preserved by the action of Γ, since

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

For each $N \in \mathbb{N}$, the *principal congruence subgroup* of level N is the subgroup of Γ defined by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}.$$

Any group Γ' with $\Gamma(N) \leqslant \Gamma' \leqslant \Gamma$ is called a *congruence subgroup of level N*. We will consider in particular the congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \,(\mathrm{mod}\, N) \right\},$$

and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \,(\mathrm{mod}\, N), c \equiv 0 \,(\mathrm{mod}\, N) \right\}.$$

Members of the set $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ are called *cusps.* We will always write elements of $\mathbb{Q}$ in lowest terms (i.e. $\frac{a}{c}$ with $(a,c) = 1$). There are finitely many equivalence classes of cusps under the action of any congruence subgroup Γ'. The cusps are so called because a fundamental domain for $\Gamma' \backslash \mathbb{H}$ may be identified at its edges to form a Riemann surface. Then each equivalence class of cusps corresponds to a point which must be added to compactify the Riemann surface.

Example 2.1. For each $\frac{a}{c} \in \mathbb{Q}$, there exist integers b and d such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}$; hence Γ has only one equivalence class of cusps. We say in this case that Γ has a single cusp at ∞.

Example 2.2. A complete set of representatives for the cusps of $\Gamma_0(N)$ is given in [27] as

$$\left\{ \frac{a_c}{c} \in \mathbb{Q} : c|N,\ 1 \le a_c \le N,\ (a_c, N) = 1, a_c \equiv a_{c'} \,(\mathrm{mod}\,(c, N/c)) \iff a_c = a_{c'} \right\}.$$

Suppose that $f(z)$ is a meromorphic function on $\mathbb{H}$, and let $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$. The *slash operator* $|_k$ is defined by

$$f(z)|_k\gamma := (\det\gamma)^{\frac{k}{2}}(cz+d)^{-k} f(\gamma z). \tag{2.1}$$

Since $\left(\frac{d}{dz}(\gamma z)\right)^{\frac{k}{2}} = (\det\gamma)^{\frac{k}{2}}(cz+d)^{-k}$, the chain rule implies that for $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{Q})$, we have

$$f(z)|_k\gamma_1\gamma_2 = (f(z)|_k\gamma_1)|_k\gamma_2.$$

If $f(z)$ is meromorphic and $f(z+1) = f(z)$ for all $z \in \mathbb{H}$, then the transformation $z \mapsto q := e^{2\pi i z}$ takes $\mathbb{H}$ to the punctured open unit disk. Defining $g(q) := f(z)$ implies that $g(q)$ is meromorphic for $0 < |q| < 1$, so $g(q)$ has a Fourier expansion $g(q) = \sum_{n=-\infty}^{\infty} a(n)q^n$. We call $f(z) = \sum_{n=-\infty}^{\infty} a(n)q^n$ the *Fourier expansion of f at ∞.*

Throughout this chapter, let $k \in \mathbb{Z}$ and $N \in \mathbb{N}$, and let Γ' be a congruence subgroup of level N.

Definition. A function f is a *weakly holomorphic modular form* of weight k for Γ' if

(a) f is holomorphic on $\mathbb{H}$,

(b) $f(z)|_k\gamma = f$ for all $\gamma \in \Gamma'$, and

(c) $\gamma \in \Gamma$ implies that $f(z)|_k\gamma$ has a Fourier expansion of the form

$$f(z)|_k\gamma = \sum_{n \geq n_\gamma} a_\gamma(n) q_N^n$$

where $q_N := e^{\frac{2\pi i z}{N}}$ and $a_\gamma(n_\gamma) \neq 0$.

Remark. The fact that $f(z)|_k\gamma$ has a Fourier expansion in powers of q_N is explained in [21, page 125]. Part (c) of the definition requires that f is *meromorphic* at each cusp. It is shown in [21, Ch. III §3, Prop. 16] that this is a finite set of conditions, since it is sufficient to check one element from each equivalence class of cusps.

Definition. If $n_\gamma \geq 0$ (resp. $n_\gamma > 0$) for a particular γ then f *is holomorphic* (resp. *vanishes*) at the cusp $s = \gamma\infty$. A weakly holomorphic modular form f that is holomorphic at each cusp is a *holomorphic modular form.* If f vanishes at every cusp, then it is a *cusp form.*

We write $\mathcal{M}_k(\Gamma')$ to denote the $\mathbb{C}$-vector space of weakly holomorphic modular forms of weight k for Γ'. The subspaces of holomorphic modular forms and cusp forms are denoted by $M_k(\Gamma')$ and $S_k(\Gamma')$, respectively.

Example 2.3. If $k \geq 4$ is even, the Eisenstein series of weight k is defined by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\Gamma),$$

where B_k is the kth Bernoulli number, and $\sigma_t(n) = \sum_{d|n} d^t$. For example,

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \in M_4(\Gamma),$$

and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \in M_6(\Gamma).$$

Example 2.4.

$$\Delta(z) := \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + 252q^3 - \cdots \in S_{12}(\Gamma). \tag{2.2}$$

Suppose that χ is a Dirichlet character modulo N. Define

$$\mathcal{M}_k(\Gamma_0(N), \chi) := \left\{ f \in \mathcal{M}_k(\Gamma_1(N)) : f|_k \gamma = \chi(d) f \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

The spaces $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ are defined similarly.

Example 2.5.

$$\Theta^2(z) := \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^2 \in M_1\left(\Gamma_0(4), \left(\frac{-4}{\bullet}\right)\right).$$

Proposition 2.1. [21, Ch. III, Prop. 28]

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi \bmod N} \mathcal{M}_k(\Gamma_0(N), \chi).$$

The same statement holds with $\mathcal{M}_k$ replaced by M_k or S_k.

Let K be a number field with ring of integers $\mathcal{O}_K$, and let $f(z) = \sum a(n) q^n$ be a formal power series with coefficients in $\mathcal{O}_K$. For any ideal $\mathfrak{m}$ of $\mathcal{O}_K$, we define the *order of f modulo* $\mathfrak{m}$ to be

$$\text{ord}_{\mathfrak{m}}(f) := \min\{n : a(n) \notin \mathfrak{m}\}.$$

If $a(n) \in \mathfrak{m}$ for all n, then we write $\text{ord}_{\mathfrak{m}}(f) = +\infty$.

Proposition 2.2. (Sturm, [36, Theorem 1]) *Let $f(z) = \sum a(n)q^n \in M_k(\Gamma_0(N), \chi) \cap \mathcal{O}_K[\![q]\!]$ for some number field K, and suppose that $\mathfrak{m}$ is an ideal of $\mathcal{O}_K$. If*

$$\text{ord}_{\mathfrak{m}}(f) > \frac{k}{12}[\Gamma : \Gamma_0(N)]$$

then $\text{ord}_{\mathfrak{m}}(f) = +\infty$.

Remark. Suppose that $f(z) = \sum a(n)q^n$ and $g(z) = \sum b(n)q^n$ lie in $M_k(\Gamma_0(N), \chi) \cap \mathcal{O}_K[\![q]\!]$. We write

$$f(z) \equiv g(z) \pmod{\mathfrak{m}}$$

if $a(n) - b(n) \in \mathfrak{m}$ for all n. If we write $f(z) \equiv g(z) \pmod{M}$ for some $M \in Z$, we mean that $a(n) - b(n) \in M\mathcal{O}_K$ for all n. Proposition 2.2 asserts that these conditions only need to be verified for $n \leq \frac{k}{12}[\Gamma : \Gamma_0(N)]$.

2.2 Half-integral weight modular forms

Modular forms may also have half-integral weight. In this case, taking a direct analog of the slash operator by replacing k with $\frac{k}{2}$ in (2.1) is ambiguous because of the two possible choices for the square root of $cz + d$. Greater care is therefore needed to define these forms. Many of the fundamental properties of half-integral weight modular forms were discovered by Shimura [35].

Define the set

$$G := \left\{ (\alpha, \phi(z)) : \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}) \text{ and } \phi^2(z) = \frac{\pm(cz+d)}{\sqrt{\det \alpha}} \right\}.$$

Then G is a group under the operation

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)), \tag{2.3}$$

with identity $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$ and inverse $(\alpha, \phi(z))^{-1} = \left(\alpha^{-1}, \frac{1}{\phi(\alpha^{-1}z)} \right)$ for each $(\alpha, \phi(z)) \in G$. Restriction to matrices in Γ yields the subgroup

$$G' := \{(\alpha, \phi(z)) \in G : \alpha \in \Gamma\}.$$

For each meromorphic function f on $\mathbb{H}$ and each integer k, the *slash operator* $|_{\frac{k}{2}}$ is defined for $\xi = (\alpha, \phi(z)) \in G$ as

$$f(z)|_{\frac{k}{2}}\xi := \phi(z)^{-k} f(\alpha z). \tag{2.4}$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathbb{H}$, define

$$j(\gamma, z) := \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d}, \tag{2.5}$$

where

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases} \tag{2.6}$$

and $\left(\frac{c}{d}\right)$ is defined as follows. If d is an odd prime, then $\left(\frac{c}{d}\right)$ is the usual Legendre symbol. If d is any odd positive integer, then $\left(\frac{c}{d}\right)$ is defined multiplicatively. If d is a negative odd integer, then

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{|d|}\right) & \text{if } c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if } c < 0. \end{cases}$$

Finally, we let $\left(\frac{0}{\pm 1}\right) = 1$. Next, set

$$\tilde{\gamma} := (\gamma, j(\gamma, z)).$$

For any congruence subgroup $\Gamma' \leqslant \Gamma_0(4)$, let $\widetilde{\Gamma'} := \{\tilde{\gamma} : \gamma \in \Gamma'\}$.

Definition. A function f on $\mathbb{H}$ is called a *weakly holomorphic modular form* of weight $\frac{k}{2}$ for $\widetilde{\Gamma'}$ if

(a) f is holomorphic on $\mathbb{H}$,

(b) $f|_{\frac{k}{2}}\tilde{\gamma} = f$ for all $\tilde{\gamma} \in \widetilde{\Gamma'}$, and

(c) f is meromorphic at the cusps.

Again, for part (c) it is sufficient to check one element from each equivalence class of cusps. We say f is a *holomorphic modular form* if it is holomorphic at the cusps, and a *cusp form* if it vanishes at the cusps. The spaces of weight $\frac{k}{2}$ weakly holomorphic, holomorphic and cusp forms for $\widetilde{\Gamma'}$ are denoted by $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma'})$, $M_{\frac{k}{2}}(\widetilde{\Gamma'})$ and $S_{\frac{k}{2}}(\widetilde{\Gamma'})$, respectively.

Example 2.6.

$$\Theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots \in M_{\frac{1}{2}}(\widetilde{\Gamma_0(4)}). \tag{2.7}$$

We now describe the expansions of $f(z) \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma'})$ at the cusps, which are derived in [21, pp. 181–182]. Let $\psi = (\alpha, \phi(z)) \in G'$, let $s = \alpha\infty$, and set $\Gamma'_s := \{\gamma \in \Gamma' : \gamma s = s\}$.

Then $\alpha^{-1}\Gamma'_s\alpha$ has one of the forms

$$\left\{\pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^j\right\},\quad \left\{\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^j\right\},\quad \text{or}\quad \left\{-\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^j\right\} \tag{2.8}$$

for some positive integers h and j. The number h is called the *width* of s as a cusp of Γ'. If Γ' is of level N, then $h \mid N$. Let $t \in \{\pm 1, \pm i\}$ be the unique element such that

$$\pm\psi\left(\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}, t\right)\psi^{-1} \in \widetilde{\Gamma_0(4)}.$$

The numbers h and t depend only on the Γ'-equivalence class of s. Fix an integer k, and let $r \in \{0,1,2,3\}$ be the number that satisfies

$$t^k = e^{\frac{2\pi i r}{4}}. \tag{2.9}$$

If $f(z) \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma}')$, then the Fourier expansion of $f(z)|_{\frac{k}{2}}\psi$ has the form

$$f(z)|_{\frac{k}{2}}\psi = \sum_{n \geq n_\psi} a_\psi(n) q_h^{n+\frac{r}{4}}. \tag{2.10}$$

In the case $\Gamma' = \Gamma_0(N)$, we can explicitly calculate h. Writing $\alpha = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and calculating

$$\alpha\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\alpha^{-1} = \begin{pmatrix}1-cah & a^2h\\ -c^2h & 1+cah\end{pmatrix}, \tag{2.11}$$

we see that for the product in (2.11) to be an element of $\Gamma_0(N)_s$, we must have $-c^2h \equiv 0 \pmod{N}$. Since $\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}$ is a generator of $\alpha^{-1}\Gamma_0(N)_s\alpha$, then h must be the smallest positive number to satisfy that congruence. Therefore,

$$h = \frac{N}{(c^2, N)}.$$

Suppose that $4 \mid N$, and let χ be a Dirichlet character modulo N. Then

$$\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) := \left\{ f \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)}) : f|_{\frac{k}{2}}\tilde{\gamma} = \chi(d) f \text{ for all } \tilde{\gamma} \in \widetilde{\Gamma_0(N)} \text{ with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}.$$

The spaces $M_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ and $S_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ are defined similarly.

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma_0(N)$ for all N, this implies that $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) = 0$ if χ is an odd character (that is, $\chi(-1) = -1$). In light of this, we have the following analog of Proposition 2.1.

Proposition 2.3.

$$\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)}) = \bigoplus_{\chi \text{ even}} \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi).$$

The same statement holds with $\mathcal{M}_{\frac{k}{2}}$ replaced by $M_{\frac{k}{2}}$ or $S_{\frac{k}{2}}$.

The next result says that if a holomorphic modular form $f(z) \in M_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ has algebraic coefficients, then the coefficients have bounded denominators.

Proposition 2.4. (Serre, Stark, [34, Lemma 8]) *If $f(z) = \sum a(n)q^n \in M_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \cap \mathcal{O}_K[\![q]\!]$ for some number field K, then there is a nonzero $D \in \mathbb{Z}$ such that $D \cdot a(n) \in \mathcal{O}_K$ for all n.*

We have the following correspondence between spaces of integral and half-integral weight modular forms for $\Gamma_0(N)$ with character χ.

Proposition 2.5. *Let $\chi_{-4}(n) := \left(\frac{-4}{n}\right)$. Then if $\frac{k}{2} \in \mathbb{Z}$, we have*

$$\mathcal{M}_{\frac{k}{2}}(\Gamma_0(N), \chi) = \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi\chi_{-4}^{k/2}).$$

Proof. Let $\frac{k}{2} \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. For any meromorphic function $f(z)$ on $\mathbb{H}$, we have

$$f(z)|_{\frac{k}{2}}\tilde{\gamma} = j(\gamma, z)^{-k} f(z)$$

$$= \left(\left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d}\right)^{-k} f(z)$$
$$= \left(\frac{-1}{d}\right)^{k/2}(cz+d)^{-k/2}f(z)$$
$$= \chi_{-4}(d)^{k/2} f|_{\frac{k}{2}}\gamma.$$

Proposition 2.5 now follows from this fact. □

2.3 Eta-quotients

Dedekind's eta-function, $\eta(z)$, is defined by the infinite product

$$\eta(z) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{2.12}$$

Recalling the modular form $\Delta(z)$ from (2.2), we have

$$\eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \Delta(z).$$

The eta-function generates a wide class of weakly holomorphic modular forms.

Definition. An *eta-quotient* is any function $f(z)$ of the form

$$f(z) = \prod_{\delta | N}\eta^{r_\delta}(\delta z) \tag{2.13}$$

where each r_δ is an integer. If each $r_\delta \geq 0$, then $f(z)$ is called an *eta-product.*

Proposition 2.6. [16] *If* $f(z) = \prod_{\delta|N}\eta^{r_\delta}(\delta z)$ *is an eta-quotient with* $k := \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$ *such that*

(a) $\sum_{\delta|N} r_\delta \delta \equiv 0 \pmod{24}$, *and*

(b) $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,

then $f(z) \in \mathcal{M}_k(\Gamma_0(N), \chi)$ *with* χ *defined by* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$ *where* $s := \prod_{\delta | N} \delta^{r_\delta}$.

The generating functions for many partition functions are given by eta-quotients.

Example 2.7. Let $D(n)$ be the number of partitions of n into distinct parts. Then

$$\sum_{n=0}^{\infty} D(n) q^n = \prod_{n=1}^{\infty} (1 + q^n).$$

Rewriting this generating function slightly, and using Theorem 2.6, we have

$$\sum_{n \equiv 1 \pmod{24}} D\left(\frac{n-1}{24}\right) q^n = \frac{\eta(48z)}{\eta(24z)} \in \mathcal{M}_0\left(\Gamma_0(1152), \left(\frac{2}{\bullet}\right)\right).$$

We have the following corollary of Proposition 2.6 for half-integral weight eta-quotients.

Corollary 2.7. *If* $4 \mid N$ *and* $f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z)$ *is an eta-quotient with* $k := \sum_{\delta | N} r_\delta$ *odd such that*

(a) $\sum_{\delta | N} r_\delta \delta \equiv 0 \pmod{24}$, *and*

(b) $\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,

then $f(z) \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ *with* χ *defined by* $\chi(d) := \left(\frac{s}{d}\right)$ *where* $s := 2 \prod_{\delta | N} \delta^{r_\delta}$.

Proof. The function $\Theta(z)$ in (2.7) can be expressed as the eta quotient

$$\Theta(z) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)} \in M_{\frac{1}{2}}(\widetilde{\Gamma_0(4)}).$$

Let $g(z) = f(z) \cdot \Theta(z)$. Then $g(z)$ is an eta-quotient satisfying (a) and (b) above and has integer weight $\frac{k+1}{2}$, so by Proposition 2.6, $g(z) \in \mathcal{M}_{\frac{k+1}{2}}(\Gamma_0(N), \chi)$, where $\chi(d) := \left(\frac{(-1)^{(k+1)/2} s}{d}\right)$ for $s = 2 \prod_{\delta | N} \delta^{r_\delta}$. By Proposition 2.5,

$$g(z) \in \mathcal{M}_{\frac{k+1}{2}}(\widetilde{\Gamma_0(N)}, \chi \chi_{-4}^{(k+1)/2}) = \mathcal{M}_{\frac{k+1}{2}}\left(\widetilde{\Gamma_0(N)}, \left(\frac{s}{\bullet}\right)\right).$$

Finally,

$$f(z) = \frac{g(z)}{\Theta(z)} \in \mathcal{M}_{\frac{k}{2}}\left(\widetilde{\Gamma_0(N)}, \left(\tfrac{s}{\bullet}\right)\right).$$

□

Example 2.8. Let k be a positive integer, and let $p_k(n)$ be the number of k-colored partitions of n, that is, the number of partitions of n where each part is assigned one of k colors. The generating function for $p_k(n)$ is given by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k}.$$

Using (2.12), Proposition 2.6 and Corollary 2.7, we can write

$$\sum_{n \equiv -k \pmod{24}} p_k\left(\frac{n+k}{24}\right) q^n = \frac{1}{\eta^k(24z)} \in \begin{cases} \mathcal{M}_{-\frac{k}{2}}(\Gamma_0(576), \chi_{-4}^{k/2}) & \text{if } k \text{ is even,} \\ \mathcal{M}_{-\frac{k}{2}}(\widetilde{\Gamma_0(576)}, \left(\tfrac{12}{\bullet}\right)^k) & \text{if } k \text{ is odd.} \end{cases}$$

Example 2.9. An *overpartition* of n is a partition in which the first occurrence of a number may be overlined. The number of overpartitions of n is denoted by $\overline{p}(n)$. See Corteel and Lovejoy [11] for more about overpartitions. The generating function for $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = \frac{\eta(2z)}{\eta^2(z)} \in \mathcal{M}_{-\frac{1}{2}}(\widetilde{\Gamma_0(16)}). \tag{2.14}$$

Now we show that if $f(z)$ is an eta-quotient of level N, its Fourier expansion at a cusp $\frac{a}{c}$ can be written in terms of the local variable q_{h_0}, where $h_0 = \frac{N}{(c^2,N)}$ is the width of $\frac{a}{c}$ as a cusp of $\Gamma_0(N)$.

Proposition 2.8. *Suppose that* $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ *is an eta-quotient of integral or half-integral weight satisfying*

(a) $\sum_{\delta|N} r_\delta \delta \equiv 0 \pmod{24}$, *and*

(b) $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$.

Let $k := \sum_{\delta|N} r_\delta$. *Then for any* $\psi \in G'$ *with* $\psi\infty = \frac{a}{c}$, *we can write*

$$f(z)|_{\frac{k}{2}}\psi = \sum_{n \geq n_\psi} a_\psi(n) q_{h_0}^{n+\frac{r}{4}}, \tag{2.15}$$

where $h_0 = \frac{N}{(c^2,N)}$ *and* $r \in \{0, 1, 2, 3\}$.

Proof. By Propositions 2.5 and 2.6 and Corollary 2.7, we have $f(z) \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ for some real character χ. Let $\psi = (\alpha, \phi(z)) \in G'$ with $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and choose $t \in \{\pm 1, \pm i\}$ so that

$$\pm\psi \left(\begin{pmatrix} 1 & h_0 \\ 0 & 1 \end{pmatrix}, t \right) \psi^{-1} \in \widetilde{\Gamma_0(4)}.$$

Let $r \in \{0, 1, 2, 3\}$ be the number that satisfies

$$t^k \chi(1 + ach_0) = e^{\frac{2\pi i r}{4}}. \tag{2.16}$$

This step is possible since χ is real. Now by an argument similar to the one in [21, pp. 181–182], we have (2.15). □

Remark. As before, the choice of r depends only on k and the $\Gamma_0(N)$-equivalence class of the cusp $\frac{a}{c}$.

The next result gives the order of vanishing of an eta-quotient at any cusp.

Proposition 2.9. [24, 27] *Let* $a, c, N \in \mathbb{N}$ *with* $(a, c) = 1$, *and let* $h = \frac{N}{(c^2,N)}$. *If* $f(z)$ *is an eta-quotient, then the order of vanishing of* $f(z)$ *at the cusp* $\frac{a}{c}$ *with respect to the local variable* q_h *is*

$$\mathrm{ord}_{\frac{a}{c}}(f) = \frac{N}{24(c^2, N)} \sum_{\delta|N} \frac{(c, \delta)^2}{\delta} r_\delta.$$

2.4 Linear operators

Here we present four linear operators which are crucial to our study of the coefficients of weakly holomorphic modular forms. Each can be defined on spaces of integral or half-integral weight modular forms, but we will state the first three only for half-integral weight modular forms, as this is all we will need.

Let $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. Suppose that v and t are integers with $t \geq 1$. Set

$$\sigma_{v,t} := \left(\begin{pmatrix} 1 & v \\ 0 & t \end{pmatrix}, t^{1/4} \right) \in G. \tag{2.17}$$

The *U-operator* $U_t : \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \to \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0([N,t])}, \chi\chi_t^k)$ is defined by

$$f(z)|U_t = t^{\frac{k}{4}-1} \sum_{v=0}^{t-1} f(z)|_{\frac{k}{2}} \sigma_{v,t} = t^{-1} \sum_{v=0}^{t-1} f\left(\frac{z+v}{t}\right) = \sum a(tn)q^n. \tag{2.18}$$

The *V-operator* $V_t : \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \to \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(Nt)}, \chi\chi_t^k)$ is defined by

$$f(z)|V_t = t^{-\frac{k}{4}} f(z)|_{\frac{k}{2}} \left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, t^{-1/4} \right) = f(tz) = \sum a(n)q^{tn}.$$

These facts are stated in [35, Propositions 1.3 and 1.4] for holomorphic modular forms of weight $\frac{k}{2}$ with $k \geq 1$. However, one can verify with the same arguments given in [35] that these definitions extend to weakly holomorphic modular forms of weight $\frac{k}{2}$ for any integer k. Set

$$\tau_{v,t} := \left(\begin{pmatrix} 1 & -v/t \\ 0 & 1 \end{pmatrix}, 1 \right). \tag{2.19}$$

Applying U_t followed by V_t, we have

$$f(z)|U_t|V_t = t^{-1} \sum_{v=0}^{t-1} f(z)|_{\frac{k}{2}} \tau_{v,t} = \sum a(tn)q^{tn}. \tag{2.20}$$

If ψ is a Dirichlet character modulo m, then the *twist* of f by ψ is given by

$$f \otimes \psi := \sum \psi(n) a(n) q^n \in \mathcal{M}_{\frac{k}{2}}(\Gamma_0(\widetilde{Nm^2}), \chi\psi^2) \tag{2.21}$$

[31, See §2.2]. Note that if χ_t^{triv} is the trivial character modulo t, then

$$f(z)|U_t|V_t = f(z) - f(z) \otimes \chi_t^{\text{triv}}. \tag{2.22}$$

Let $k \in \mathbb{Z}$. For each prime p, the integral weight *Hecke operator* $T_{k,N,\chi}(p)$ preserves the space $\mathcal{M}_k(\Gamma_0(N), \chi)$. When $p \nmid N$, the effect of $T_{k,N,\chi}(p)$ on the Fourier expansion of $f(z) = \sum a(n) q^n \in \mathcal{M}_k(\Gamma_0(N), \chi)$ is given by

$$f(z)|T_{k,N,\chi}(p) = \sum \left[a(pn) + \chi(p) p^{k-1} a\left(\frac{n}{p}\right)\right] q^n, \tag{2.23}$$

where $a(\frac{n}{p}) = 0$ if $p \nmid n$.

The $T_{k,N,\chi}(p)$ generate Hecke operators $T_{k,N,\chi}(n)$ for each $n \in \mathbb{N}$ via the following rules of composition:

(1) For all $\nu \geq 1$, $T_{k,N,\chi}(p^{\nu+1}) = T_{k,N,\chi}(p^\nu) T_{k,N,\chi}(p) - \chi(p) p^{k-1} T_{k,N,\chi}(p^{\nu-1})$, and

(2) if $(m, n) = 1$ then $T_{k,N,\chi}(mn) = T_{k,N,\chi}(m) T_{k,N,\chi}(n)$.

A *newform* is a cusp form that is an eigenform for each $T_{k,N,\chi}(n)$ and has $a(1) = 1$. The following facts about newforms are due to Atkin-Lehner [7] and Li [23]. If $f(z) = \sum a(n) q^n \in S_k(\Gamma_0(N), \chi)$ is a newform, then $f(z)|T_{k,N,\chi}(n) = a(n) f(z)$ for each n. Also, there exists a number field K such that $a(n) \in \mathcal{O}_K$ for all n.

We write $S_k^{\text{new}}(\Gamma_0(N), \chi)$ to denote the subspace of $S_k(\Gamma_0(N), \chi)$ consisting of cusp forms that do not originate in a lower level. This new subspace has a basis of newforms. We can write

$$S_k(\Gamma_0(N), \chi) = \bigoplus_{M|N} \bigoplus_{dM|N} S_k^{\text{new}}(\Gamma_0(M), \chi)|V_d.$$

It follows that each $f(z) \in S_k(\Gamma_0(N), \chi)$ can be written as

$$f(z) = \sum \alpha(i, \delta) f_i(\delta z) \tag{2.24}$$

where the $f_i(z)$ are newforms in $S_k(\Gamma_0(N), \chi)$.

In the half-integral weight case, for each prime p, the Hecke operator $T_{\frac{k}{2},N,\chi}(p^2)$ preserves the space $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. For $p \nmid N$, the effect of $T_{\frac{k}{2},N,\chi}(p^2)$ on the Fourier expansion of $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ is

$$\begin{aligned} &f(z)|T_{\frac{k}{2},N,\chi}(p^2) \\ &\qquad = \sum \left[a(p^2 n) + \chi(p) \left(\frac{(-1)^{\frac{k-1}{2}} n}{p} \right) p^{\frac{k-3}{2}} a(n) + \chi(p^2) p^{k-2} a\left(\frac{n}{p^2}\right) \right] q^n. \end{aligned} \tag{2.25}$$

Using Proposition 2.3, this definition may be extended linearly to define Hecke operators $T_{\frac{k}{2},N}(p^2)$ for the space $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)})$. Then if $p \equiv -1 \pmod N$ is prime and $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)})$, we have

$$f(z)|T_{\frac{k}{2},N}(p^2) = \sum \left[a(p^2 n) + \left(\frac{(-1)^{\frac{k-1}{2}} n}{p} \right) p^{\frac{k-3}{2}} a(n) + p^{k-2} a\left(\frac{n}{p^2}\right) \right] q^n. \tag{2.26}$$

2.5 Galois representations

Here we give the facts we will use from the theory of Galois representations. This material is described in more depth in [14, Ch. 9].

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. The set of all automorphisms of $\overline{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$, denoted by $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. Each $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ restricts to an element $\sigma|_F \in \mathrm{Gal}(F/\mathbb{Q})$ for every finite Galois extension F of $\mathbb{Q}$. In fact, $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ can be described as the inverse limit of the finite Galois groups

$$\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) = \varprojlim_{F} \mathrm{Gal}\left(F/\mathbb{Q}\right).$$

By giving $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ the Krull topology, we get a suitable analog of the fundamental theorem of Galois theory. In this case, there is a one-to-one correspondence between subextensions $\overline{\mathbb{Q}} \supset K \supset \mathbb{Q}$ and closed subgroups of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. The open subgroups of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ correspond to the finite subextensions of $\overline{\mathbb{Q}}/\mathbb{Q}$.

Let $\overline{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Let $p \in \mathbb{Z}$ be prime and fix a maximal ideal $\mathfrak{p} \subset \overline{\mathbb{Z}}$ over p. The *decomposition group* of $\mathfrak{p}$ is

$$D_{\mathfrak{p}} := \{\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) : \mathfrak{p}^{\sigma} = \mathfrak{p}\}.$$

A reduction map $\overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$ can be defined by taking the kernel to be $\mathfrak{p}$. Then $(x+\mathfrak{p})^{\sigma} = x^{\sigma}+\mathfrak{p}$ for each $\sigma \in D_{\mathfrak{p}}$, so σ acts on $\overline{\mathbb{Z}}/\mathfrak{p}$ and hence also on $\overline{\mathbb{F}}_p$. This gives a surjective map $D_{\mathfrak{p}} \to \mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right)$. The group $\mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right)$ is topologically generated by the *Frobenius automorphism* $\phi_p : a \mapsto a^p$. Any preimage in $D_{\mathfrak{p}}$ of ϕ_p is called an *absolute Frobenius element*, and is denoted $\mathrm{Frob}_{\mathfrak{p}}$. Hence $\mathrm{Frob}_{\mathfrak{p}}$ is only defined up to the kernel of the map $D_{\mathfrak{p}} \to \mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right)$, which is given by

$$I_{\mathfrak{p}} := \{\sigma \in D_{\mathfrak{p}} : x^{\sigma} \equiv x \pmod{\mathfrak{p}} \text{ for all } x \in \overline{\mathbb{Z}}\},$$

and called the *inertia group* of $\mathfrak{p}$.

Definition. Let K be an algebraic number field with ring of integers $\mathcal{O}_K$, and let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$. A *Galois representation* is a continuous homomorphism

$$\rho : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K\right).$$

Since $\mathrm{Frob}_{\mathfrak{p}}$ is defined only up to $I_{\mathfrak{p}}$, we must have $I_{\mathfrak{p}} \subset \ker \rho$ in order for $\rho(\mathrm{Frob}_{\mathfrak{p}})$ to be well-defined. This leads to the following definition.

Definition. A Galois representation is said to be *unramified* at a prime $p \in \mathbb{Z}$ if $I_{\mathfrak{p}} \subset \ker \rho$ for every maximal ideal $\mathfrak{p} \subset \overline{\mathbb{Z}}$ lying over p.

The following result due to Deligne and Serre shows that the coefficients of integral weight newforms are determined by the images of Frobenius elements under certain Galois representations.

Theorem 2.10. [12, 13] *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ be a newform, and let K be a number field containing the Fourier coefficients $a(n)$ and the values of χ. If $\mathfrak{m}$ is an ideal of the ring of integers $\mathcal{O}_K$ with norm M, then there is a Galois representation*

$$\rho : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K\right)$$

such that

(a) ρ is unramified at all primes $p \nmid MN$, and

(b) for every prime $p \nmid MN$, we have

$$\mathrm{Tr}(\rho(\mathrm{Frob}_p)) \equiv a(p) \pmod{\mathfrak{m}}.$$

Remark. The absolute Frobenius elements are all conjugate in $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. Although the image $\rho(\mathrm{Frob}_{\mathfrak{p}})$ depends on the choice of $\mathfrak{p}$ above p, the trace of $\rho(\mathrm{Frob}_{\mathfrak{p}})$ depends only on the conjugacy class of $\rho(\mathrm{Frob}_{\mathfrak{p}})$, so it is unambiguous to write Frob_p in part (b) to denote any absolute Frobenius element for p.

The next result, due to Serre [33, Exercise 6.4], is central to the proofs of Theorems 1.1 and 1.3, as well as to the main theorems of this thesis.

Theorem 2.11. *Let $k \in \mathbb{Z}$ and $N \in \mathbb{N}$, and let χ be a Dirichlet character modulo N. Suppose that F is a number field with ring of integers $\mathcal{O}_F$, and let $\mathfrak{m}$ be an ideal of $\mathcal{O}_F$ with norm M. Then a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have the property that*

$$f(z)|T_{k,N,\chi}(p) \equiv 0 \pmod{\mathfrak{m}}$$

for each $f(z) \in S_k(\Gamma_0(N), \chi) \cap \mathcal{O}_F[\![q]\!]$.

Proof. Let $\{f_1(z), \ldots, f_d(z)\}$ be all of the newforms in $S_k(\Gamma_0(N), \chi)$, and write their Fourier expansions as $f_i(z) = \sum_{n=1}^{\infty} a_i(n)q^n$ for $1 \leq i \leq d$. Suppose that K is a finite extension of F containing all the coefficients of the f_i and all values of χ. Denote by $S_k(\Gamma_0(N), \chi)_{\mathcal{O}_F/\mathfrak{m}}$ the set of reductions modulo $\mathfrak{m}$ of the cusp forms in $S_k(\Gamma_0(N), \chi)$ whose coefficients lie in $\mathcal{O}_F$. For each $g(z) \in S_k(\Gamma_0(N), \chi)_{\mathcal{O}_F/\mathfrak{m}}$, choose an $h_g(z) \in S_k(\Gamma_0(N), \chi) \cap \mathcal{O}_F[\![q]\!]$ so that

$$h_g(z) \equiv g(z) \pmod{\mathfrak{m}}.$$

Then by (2.24), each $h_g(z)$ has the form

$$h_g(z) = \sum \alpha(i, \delta) f_i(\delta z)$$

where the $\alpha(i, \delta)$ are algebraic. By Proposition 2.2, $S_k(\Gamma_0(N), \chi)_{\mathcal{O}_F/\mathfrak{m}}$ is a finite set, so only finitely many $\alpha(i, \delta)$ appear as g ranges over $S_k(\Gamma_0(N), \chi)_{\mathcal{O}_F/\mathfrak{m}}$. Hence we can extend K so that it contains all of the $\alpha(i, \delta)$. We can also choose a nonzero integer C so that $C \cdot \alpha(i, \delta) \in \mathcal{O}_K$ for each $\alpha(i, \delta)$. Set $\mathfrak{m}' := C\mathfrak{m}$ and $M' := CM$. Since the $f_i(z)$ are newforms, if $p \nmid N$ then

$$f_i(\delta z)|T_{k,N,\chi}(p) = a_i(p) f_i(\delta z)$$

for each i and δ, so that

$$h_g(z)|T_{k,N,\chi}(p) = \sum_{i=1}^{d}\sum_{\delta|N}\alpha(i,\delta)a_i(p)f_i(\delta_i z).$$

Since each $f(z) \in S_k(\Gamma_0(N),\chi) \cap \mathcal{O}_F[\![q]\!]$ is congruent modulo $\mathfrak{m}$ to one of the $h_g(z)$, it suffices to show that a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have $a_i(p) \equiv 0 \pmod{\mathfrak{m}'}$ for each i.

For each i, Theorem 2.10 implies the existence of a Galois representation

$$\rho_i : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\mathcal{O}_K/\mathfrak{m}'\mathcal{O}_K\right),$$

unramified outside $M'N$, such that $\mathrm{Tr}(\rho(\mathrm{Frob}_p)) \equiv a_i(p) \pmod{\mathfrak{m}'}$ for each $p \nmid M'N$. Let μ be a primitive (MN)th root of unity. A character $\epsilon : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to (\mathbb{Z}/MN\mathbb{Z})^\times$ may be defined in the following way. For each $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, the restriction $\sigma|_{\mathbb{Q}(\mu)}$ takes $\mu \mapsto \mu^a$ for some $(a, MN) = 1$, so we define $\epsilon(\sigma) := a$. We now take the sum

$$\rho := \epsilon \oplus \bigoplus_{i=1}^{d}\rho_i : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to (\mathbb{Z}/MN\mathbb{Z})^\times \oplus \bigoplus_{i=1}^{d}\mathrm{GL}_2\left(\mathcal{O}_K/\mathfrak{m}'\mathcal{O}_K\right).$$

Let $H \lhd \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ be the kernel of ρ, and let E be the fixed field of H. Since H is normal and closed in $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, the extension E/Q is Galois and $\mathrm{Gal}\left(E/\mathbb{Q}\right) \cong \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)/H$ is isomorphic to the image of ρ, which is finite. Hence E is a number field and ρ factors through $\mathrm{Gal}\left(E/\mathbb{Q}\right)$. The restriction $\rho|_E$ must also be unramified outside of $M'N$.

Let $c \in \mathrm{Gal}\left(E/\mathbb{Q}\right)$ denote complex conjugation. Then $\rho_i(c)$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for each i, so $\mathrm{Tr}(\rho_i(c)) = 0$. Now the Chebotarev density theorem implies that there is a positive proportion of primes $p \nmid M'N$ such that Frob_p is conjugate to c in $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. For each such p, we have $a_i(p) \equiv \mathrm{Tr}(\rho_i(\mathrm{Frob}_p)) \equiv \mathrm{Tr}(\rho_i(c)) \equiv 0 \pmod{\mathfrak{m}'}$ for each i. Now $\mathrm{Frob}_p|_{\mathbb{Q}(\mu)} : \mu \mapsto \mu^p$, so $\epsilon(\mathrm{Frob}_p) = p$. But $c(\mu) = \mu^{-1}$, so $\epsilon(c) = -1$. Since Frob_p and c are conjugate in $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, we must have $p \equiv -1 \pmod{MN}$. This completes the proof of Theorem 2.11. □

2.6 Shimura's correspondence

In [35], Shimura describes the following relationship between cusp forms of integral and half-integral weight.

Definition. If χ is a Dirichlet character, then the *Dirichlet L-function* for χ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for real $s > 1$.

Theorem 2.12. (Shimura, [35]) *Let* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)},\chi)$ *with* $\lambda > 1$ *an integer. Let* $t > 0$ *be a squarefree integer, and define the Shimura lift* $S_t f(z) := \sum_{n=1}^{\infty} A_t(n)q^n$, *where the* $A_t(n)$ *are determined by*

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s-\lambda+1, \chi\chi_{(-1)^{\lambda}t}) \sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s}.$$

Then

(a) $S_t f(z) \in S_{2\lambda}(\Gamma_0(N/2),\chi^2)$, *and*

(b) the Shimura lift commutes with Hecke operators, that is, if $p \nmid N$ *is prime, then*

$$S_t(f(z)|T_{\lambda+\frac{1}{2},N,\chi}(p^2)) = (S_t f(z))|T_{2\lambda,N/2,\chi^2}(p).$$

Remark. Shimura showed only that the level of $S_t f(z)$ divided N. Niwa [29] proved that it was in fact $N/2$.

Remark. It follows from [35, Cor. 1.8] that (b) holds in the case that $f(z)$ is an eigenform for all of the $T_{\lambda+\frac{1}{2},N,\chi}(p^2)$ where $p \nmid N$. Then since $S_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)},\chi)$ has a basis of such eigenforms, the property extends additively to any $f(z) \in S_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)},\chi)$.

Theorem 2.12 makes possible the following analog of Theorem 2.11 for half-integral weight modular forms. The argument given here is due to Ono [30].

Theorem 2.13. *Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\frac{k}{2}}(\widetilde{\Gamma_0(N)},\chi) \cap \mathcal{O}_K[\![q]\!]$ for some number field K, and that $\mathfrak{m} \subset \mathcal{O}_K$ is an ideal with norm M. Furthermore, suppose that $k > 3$. Then a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have the property that*

$$f(z)|T_{\frac{k}{2},N,\chi}(p^2) \equiv 0 \pmod{\mathfrak{m}}.$$

Proof. For each squarefree $t > 0$, we have the Shimura lift $S_t f(z) \in S_{k-1}(\widetilde{\Gamma_0(N/2)},\chi^2) \cap \mathcal{O}_K[\![q]\!]$. By Theorem 2.11, a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have the property that

$$(S_t f(z))|T_{k-1,N/2,\chi^2}(p) \equiv 0 \pmod{\mathfrak{m}}$$

for all t. By Theorem 2.12(b), we have

$$S_t(f(z)|T_{\frac{k}{2},N,\chi}(p^2)) \equiv 0 \pmod{\mathfrak{m}}$$

for all t. Write $S_t(f(z)|T_{\frac{k}{2},N,\chi}(p^2)) = \sum_{n=1}^{\infty} A_t(n)q^n$, and $f(z)|T_{\frac{k}{2},N,\chi}(p^2) = \sum_{n=1}^{\infty} b(n)q^n$. Then for $n \geq 1$, the definition of the Shimura lift implies that

$$A_t(n) = \sum_{\substack{cd=n \\ c,d\geq 1}} \chi\chi_{(-1)^{\lambda}t}(c)c^{\lambda-1}b(td^2). \tag{2.27}$$

To see that $b(tn^2) \equiv 0 \pmod{\mathfrak{m}}$ we induct on n. First, $b(t) \equiv A_t(1) \equiv 0 \pmod{\mathfrak{m}}$. Now suppose that $b(tn^2) \equiv 0 \pmod{\mathfrak{m}}$ for all $n < n_0$. Then by (2.27), $A_t(n_0) \equiv b(tn_0^2) \pmod{\mathfrak{m}}$.

Therefore $b(tn_0^2) \equiv 0 \pmod{\mathfrak{m}}$ for all t, so we have

$$f(z)|T_{\frac{k}{2},N,\chi}(p^2) \equiv 0 \pmod{\mathfrak{m}}.$$

This concludes the proof of Theorem 2.13. $\square$

2.7 Gauss sums

Quadratic Gauss sums are related to twists of modular forms by quadratic characters, and will appear in the proof of Theorem 4.1 below. Here we derive the main fact we will use. Throughout this section, fix an odd prime p.

Definition. For any integer a, we call $g_a := \sum_{t=0}^{p-1} \left(\frac{t}{p}\right) e^{\frac{2\pi i a t}{p}}$ a *quadratic Gauss sum.* We write $g := g_1$.

Proposition 2.14. [18, Proposition 6.3.1] *For each integer a, we have* $g_a = \left(\frac{a}{p}\right) g \left(\frac{a}{p}\right)$

Proposition 2.15. [18, Proposition 6.3.2] $g^2 = \left(\frac{-1}{p}\right) p$.

Proposition 2.16. *Let $k \in \mathbb{Z}$ and $N \in \mathbb{N}$ with $4 \mid N$, and let $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. Then*

$$f \otimes \left(\frac{\bullet}{p}\right) = \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f(z) \bigg|_{\frac{k}{2}} \left(\begin{pmatrix} 1 & -v/p \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Proof. Using Propositions 2.14 and 2.15, we have

$$\begin{aligned} \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f(z) \bigg|_{\frac{k}{2}} \left(\begin{pmatrix} 1 & -v/p \\ 0 & 1 \end{pmatrix}, 1 \right) &= \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f\left(z - \frac{v}{p}\right) \\ &= \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) \sum a(n) e^{2\pi i n \left(z - \frac{v}{p}\right)} \\ &= \frac{g}{p} \sum a(n) q^n \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) e^{-2\pi i n v/p} \end{aligned}$$

$$
\begin{aligned}
&= \frac{g}{p} \sum \left(\frac{-n}{p} \right) g a(n) q^n \\
&= \sum \left(\frac{n}{p} \right) a(n) q^n \\
&= f(z) \otimes \left(\frac{\bullet}{p} \right).
\end{aligned}
$$

□

Chapter 3

Congruences using the U-operator approach

Our first method for establishing congruences for the coefficients of weakly holomorphic modular forms uses the U-operator defined in (2.18). Our main result shows that the phenomena in Theorems 1.1 and 1.3 are quite general.

Theorem 3.1. *Suppose that p is an odd prime, and that k and m are integers with k odd. Let N be a positive integer with $4 \mid N$ and $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let K be a number field with ring of integers $\mathcal{O}_K$, and suppose that $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \cap \mathcal{O}_K((q))$. If m is sufficiently large, then for each positive integer j, a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have the property that*

$$a(Q^3p^m n) \equiv 0 \pmod{p^j}$$

for all n coprime to Qp.

For completeness, we record the analogous result for integer weight modular forms.

Theorem 3.2. *Suppose that p is an odd prime, and that k and m are integers. Let N be a positive integer with $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let K be a number field with ring of integers $\mathcal{O}_K$, and suppose that $f(z) = \sum a(n)q^n \in \mathcal{M}_k(\Gamma_0(N), \chi) \cap \mathcal{O}_K((q))$. If m is sufficiently large, then for each positive integer j,*

(i) $a(p^m n) \equiv 0 \pmod{p^j}$ for almost all n coprime to p, and

(ii) a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ *have the property that*

$$a(Qp^m n) \equiv 0 \pmod{p^j}$$

for all n *coprime to* Qp.

Remark. In Theorem 3.2 (i), we mean "almost all" in the sense of density:

$$\#\{n \leq x : a(p^m n) \equiv 0 \pmod{p^j}\} \sim x \text{ as } x \to \infty.$$

In light of part (i) of Theorem 3.2, the conclusion of part (ii) is less surprising, so the most interesting result is really Theorem 3.1.

Remark. In each theorem, the integer m is determined by the order of vanishing of f at the cusps $\frac{a}{c}$ with $p^2 \mid c$. If there is a pole at a particular cusp, the corresponding order of vanishing is negative.

Remark. Note that Theorems 3.1 and 3.2 together with the Chinese Remainder Theorem imply linear congruences for all odd moduli M coprime to N.

3.1 Proof of Theorem 3.1

The following result is critical for proving Theorem 3.1.

Theorem 3.3. *Suppose that* p *is an odd prime, that* k, m *and* N *are integers with* $(N,p) = 1$, *and that* χ *is a Dirichlet character modulo* N. *Let* K *be a number field with ring of integers* $\mathcal{O}_K$. *Let* $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \cap \mathcal{O}_K((q))$. *If* m *is sufficiently large, then for every positive integer* j, *there is an integer* $\beta \geq j-1$ *and a cusp form*

$$g_{p,j}(z) \in S_{\frac{k}{2}+\frac{p^\beta(p^2-1)}{2}}(\widetilde{\Gamma_0(Np^2)}, \chi\chi_p^{km}) \cap \mathcal{O}_K[\![q]\!]$$

with the property that

$$g_{p,j}(z) \equiv \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} a(p^m n)q^n \pmod{p^j}.$$

To prove Theorem 3.3, we show that we can pick an integer m large enough so that $f(z)|U_{p^m} = \sum a(p^m n)q^n$ is holomorphic at every cusp $\frac{a}{c}$ with $p^2 \mid c$. We then define the modular form $f_m(z) := \sum_{p\nmid n} a(p^m n)q^n$, which vanishes at each of these cusps. Finally for each $j \geq 1$, we form $g_{p,j}(z)$ by multiplying $f_m(z)$ by an eta-quotient which vanishes at all of the cusps with $p^2 \nmid c$, and is congruent to 1 modulo p^j. The product $g_{p,j}(z)$ is then a cusp form congruent to $f_m(z)$ modulo p^j.

First we need to know the explicit form of the Fourier expansion of $f(z)|U_{p^m}$ at a cusp $\frac{a}{c}$ with $p^2 \mid c$.

Proposition 3.4. *Suppose that p is an odd prime, k and N are integers with $(N,p) = 1$, and χ is a Dirichlet character modulo N. Let $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. Suppose $\mu \in \{\pm 1, \pm i\}$, and that $\xi := \left(\begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix}, \mu\sqrt{cp^2 z + d}\right) \in G'$, with $ac > 0$. Then there exists an integer n_0, a sequence $\{a_0(n)\}_{n \geq n_0}$, a positive integer $h|N$, and an $r \in \{0,1,2,3\}$ such that for each $m \geq 1$, we have*

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = \sum_{\substack{n \geq n_0 \\ 4n+r \equiv 0 \pmod{p^m}}} a_0(n) q_{hp^m}^{n+\frac{r}{4}}.$$

Proof. Fix $m \geq 1$, and let $\sigma_{v,t}$ be defined as in (2.17). By (2.18),

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = (p^m)^{\frac{k}{4}-1} \sum_{v=0}^{p^m-1} f(z)|_{\frac{k}{2}}\sigma_{v,p^m}\xi. \tag{3.1}$$

For each v appearing in (3.1), we choose an integer $s_v \equiv 0 \pmod 4$ so that

$$s_v N \equiv (a + vcp^2)^{-1}(b + vd) \pmod{p^m}, \tag{3.2}$$

and set

$$w_v := s_v N. \tag{3.3}$$

We can ensure that $4 \mid s_v$ since any integer s_v satisfying the congruence can be replaced by $(1 - p^{2m})s_v$. We require a lemma to show that the w_v are distinct modulo p^m.

Lemma 3.5. *Let $w_v = s_v N$, with s_v defined as in* (3.2)*. The integers w_v run through the residue classes mod p^m as v does.*

Proof. Suppose $w_v \equiv w_u \pmod{p^m}$. Then using (3.2) we have

$$(a + ucp^2)(b + vd) \equiv (a + vcp^2)(b + ud) \pmod{p^m}. \tag{3.4}$$

Expanding and simplifying (3.4) yields

$$avd + bucp^2 \equiv aud + bvcp^2 \pmod{p^m}. \tag{3.5}$$

If $u \neq v$ then write

$$u = v + v_1 p^e \tag{3.6}$$

with $v_1 \neq 0$, $p \nmid v_1$ and $e \geq 0$. Substituting expression (3.6) in (3.5) and simplifying, we have

$$v_1 bcp^{e+2} \equiv adv_1 p^e \pmod{p^m}.$$

Since $p \nmid adv_1$, this gives a contradiction unless $e \geq m$. The lemma then follows from (3.6). □

Now we return to the proof of Proposition 3.4. Recall the definitions of w_v (3.3) and $\sigma_{v,t}$

(2.17). For each v, define

$$\gamma_v := \begin{pmatrix} a + vcp^2 & \frac{b+vd-aw_v-w_v vcp^2}{p^m} \\ cp^{m+2} & d - w_v cp^2 \end{pmatrix}, \tag{3.7}$$

and

$$\alpha_v := (\gamma_v, \mu\sqrt{cp^2(p^m z - w_v) + d}). \tag{3.8}$$

It is easy to verify that the determinant of γ_v is 1, and that the choice of w_v makes the upper right entry of γ_v an integer. Therefore $\gamma_v \in \Gamma$. A computation using (2.3) shows that

$$\sigma_{v,p^m}\xi = \alpha_v \sigma_{w_v,p^m}, \tag{3.9}$$

so from (3.1) and (3.9), we have

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = (p^m)^{\frac{k}{4}-1} \sum_{v=0}^{p^m-1} f(z)|_{\frac{k}{2}}\alpha_v \sigma_{w_v,p^m}. \tag{3.10}$$

For each v, some computations using (2.3) and (3.7) show that

$$\begin{aligned} \alpha_v\alpha_0^{-1} &= \left(\gamma_v, \mu\sqrt{cp^2(p^m z - w_v) + d}\right)\left(\gamma_0, \mu\sqrt{cp^2(p^m z - w_0) + d}\right)^{-1} \\ &= \left(\gamma_v, \mu\sqrt{cp^2(p^m z - w_v) + d}\right)\left(\gamma_0^{-1}, \frac{1}{\mu\sqrt{cp^2(p^m(\gamma_0^{-1}z) - w_0) + d}}\right) \\ &= \left(\gamma_v\gamma_0^{-1}, \frac{\sqrt{cp^2(p^m(\gamma_0^{-1}z) - w_v) + d}}{\sqrt{cp^2(p^m(\gamma_0^{-1}z) - w_0) + d}}\right) \\ &= \left(\gamma_v\gamma_0^{-1}, \sqrt{(w_v - w_0)c^2p^{m+4}z + 1 + (w_0 - w_v)acp^2}\right), \end{aligned} \tag{3.11}$$

where

$$\gamma_v\gamma_0^{-1} = \begin{pmatrix} 1 + (w_v - w_0)(acp^2 + vc^2p^4) & \frac{v}{p^m} + (w_0 - w_v)(\frac{a^2+avcp^2}{p^m}) \\ (w_v - w_0)c^2p^{m+4} & 1 + (w_0 - w_v)acp^2 \end{pmatrix}. \tag{3.12}$$

The next lemma shows that $\alpha_v\alpha_0^{-1} \in \widetilde{\Gamma_1(N)}$.

Lemma 3.6. *Let α_v and α_0 be defined as in* (3.8). *Then*

$$\alpha_v\alpha_0^{-1} = (\gamma_v\gamma_0^{-1}, j(\gamma_v\gamma_0^{-1}, z)) \in \widetilde{\Gamma_1(N)}.$$

Proof. By (3.3) we have $N|(w_v - w_0)$, so $\gamma_v\gamma_0^{-1} \in \Gamma_1(N)$. By (3.11) it remains to show that

$$j(\gamma_v\gamma_0^{-1}, z) = \sqrt{(w_v - w_0)c^2p^{m+4}z + 1 + (w_0 - w_v)acp^2}.$$

By (2.5) and (3.12),

$$j(\gamma_v\gamma_0^{-1}, z) = \left(\frac{(w_v - w_0)c^2p^{m+4}}{1 + (w_0 - w_v)acp^2}\right)\varepsilon_{1+(w_0-w_v)acp^2}^{-1}\sqrt{(w_v - w_0)c^2p^{m+4}z + 1 + (w_0 - w_v)acp^2}.$$

Each $w_v \equiv 0 \pmod 4$, so $\varepsilon_{1+(w_0-w_v)acp^2} = 1$ by (2.6). To evaluate the Jacobi symbol requires two cases: either $w_v > w_0$ or $w_v < w_0$. Note that the case $w_v = w_0$ is trivial. We will treat the case where $w_v > w_0$.

Define $\varepsilon := \{0, 1\}$ by $\varepsilon \equiv m \pmod 2$. Since $ac > 0$, we have $1 + (w_0 - w_v)acp^2 < 0$, so

$$\left(\frac{(w_v - w_0)c^2p^{m+4}}{1 + (w_0 - w_v)acp^2}\right) = \left(\frac{(w_v - w_0)p^\varepsilon}{(w_v - w_0)acp^2 - 1}\right).$$

Set $w_v - w_0 = 2^e r$, where r is odd. Then using properties of Jacobi symbols, and the fact that $(w_v - w_0)acp^2 \equiv 0 \,(\mathrm{mod}\, 8)$, we find that

$$\begin{aligned}\left(\frac{(w_v - w_0)p^\varepsilon}{(w_v - w_0)acp^2 - 1}\right) &= \left(\frac{2}{(w_v - w_0)acp^2 - 1}\right)^e \left(\frac{rp^\varepsilon}{(w_v - w_0)acp^2 - 1}\right)\\ &= \left(\frac{(w_v - w_0)acp^2 - 1}{rp^\varepsilon}\right)(-1)^{\left(\frac{rp^\varepsilon - 1}{2}\right)}\\ &= \left(\frac{-1}{rp^\varepsilon}\right)(-1)^{\left(\frac{rp^\varepsilon - 1}{2}\right)}\\ &= 1.\end{aligned}$$

The other case is handled similarly. This proves Lemma 3.6. □

Returning to the proof of Proposition 3.4, we see that (3.10) and Lemma 3.6 yield

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = (p^m)^{\frac{k}{4}-1}\sum_{v=0}^{p^m-1} f(z)|_{\frac{k}{2}}\alpha_0\sigma_{w_v,p^m}. \tag{3.13}$$

By (2.10), $f(z)|_{\frac{k}{2}}\alpha_0$ has the form

$$f(z)|_{\frac{k}{2}}\alpha_0 = \sum_{n\geq n_0} a_0(n)q_h^{n+\frac{r}{4}} = \sum_{n\geq n_0} a_0(n)q_{4h}^{4n+r}, \tag{3.14}$$

where h and r are defined as in (2.8) and (2.9). Now (2.4), (3.14) and (2.17) yield

$$\begin{aligned}\sum_{v=0}^{p^m-1} f(z)|_{\frac{k}{2}}\alpha_0\sigma_{w_v,p^m} &= \sum_{v=0}^{p^m-1} p^{-km/4}\sum_{n\geq n_0} a_0(n)\exp\left(\left(\frac{2\pi i}{4h}\right)\left(\frac{z+w_v}{p^m}\right)(4n+r)\right) \\ &= p^{-km/4}\sum_{n\geq n_0} a_0(n)\exp\left(\frac{2\pi iz}{4hp^m}(4n+r)\right)\sum_{v=0}^{p^m-1}\exp\left(\frac{2\pi iw_v}{4hp^m}(4n+r)\right).\end{aligned} \tag{3.15}$$

For each v, we have $4h \mid 4N \mid w_v$ and $(4h, p^m) = 1$, so by Lemma 3.5, the numbers $\frac{w_v}{4h}$ run through the residue classes modulo p^m as v does. Therefore,

$$\sum_{v=0}^{p^m-1}\exp\left(\frac{2\pi iw_v}{4hp^m}(4n+r)\right) = \sum_{v=0}^{p^m-1}\exp\left(\frac{2\pi iv}{p^m}(4n+r)\right) = \begin{cases} p^m & \text{if } 4n+r\equiv 0 \pmod{p^m}, \\ 0 & \text{otherwise.}\end{cases} \tag{3.16}$$

Putting (3.15) and (3.16) together, we obtain

$$\sum_{v=0}^{p^m-1} f(z)|_{\frac{k}{2}}\alpha_0\sigma_{w_v,p^m} = p^{m(1-\frac{k}{4})}\sum_{\substack{n\geq n_0 \\ 4n+r\equiv 0\,(\mathrm{mod}\, p^m)}} a_0(n)\exp\left(\frac{2\pi iz}{4hp^m}(4n+r)\right). \tag{3.17}$$

Finally, (3.13) and (3.17) imply that

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = \sum_{\substack{n\geq n_0 \\ 4n+r\equiv 0 \pmod{p^m}}} a_0(n) q_{hp^m}{}^{n+\frac{r}{4}}.$$

This concludes the proof of Proposition 3.4. □

Proposition 3.7. *Suppose that p is an odd prime, k and N are integers with $(N,p)=1$, and χ is a Dirichlet character modulo N. Let $f(z)=\sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)},\chi)$. For each nonnegative integer m, define*

$$f_m(z) := f(z)|U_{p^m} - f(z)|U_{p^{m+1}}|V_p = \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} a(p^m n) q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(Np^2)}, \chi\chi_p^{km}).$$

Then for m sufficiently large, f_m vanishes at each cusp $\frac{a}{cp^2}$.

Proof. Without loss of generality, we may assume that $ac>0$. By Proposition 3.4, for each m and each

$$\xi := \left(\begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix}, \mu\sqrt{cp^2z+d}\right) \in G' \tag{3.18}$$

with $ac>0$ and $\mu\in\{\pm 1, \pm i\}$, we have

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = \sum_{\substack{n\geq n_0 \\ 4n+r\equiv 0 \pmod{p^m}}} a_0(n) q_{hp^m}{}^{n+\frac{r}{4}}. \tag{3.19}$$

Let γ_0 be defined as in (3.7). The integers n_0 and r are determined by the equivalence class of the cusp $\gamma_0\infty$ under the action of $\Gamma_1(N)$, so there are finitely many such distinct pairs (n_0, r) as we run through all cusps of the form $\frac{a}{cp^2}$. If m is sufficiently large, then

$$-p^m < 4n_0 + r$$

for all such pairs. Fix such an m, and suppose ξ has the form (3.18). In the corresponding

Fourier expansion (3.19), if $a_0(n) \neq 0$ and $4n+r \equiv 0 \pmod{p^m}$, then $4n+r \geq 4n_0+r > -p^m$, so $4n+r \geq 0$, from which $n \geq 0$. Therefore,

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = \sum_{\substack{n\geq 0 \\ 4n+r\equiv 0 \,(\mathrm{mod}\, p^m)}} a_0(n) q_{hp^m}^{n+\frac{r}{4}}, \tag{3.20}$$

so $f(z)|U_{p^m}$ is holomorphic at the cusp $\frac{a}{cp^2}$. Now

$$f_m(z)|_{\frac{k}{2}}\xi = (f(z)|U_{p^m})|_{\frac{k}{2}}\xi - (f(z)|U_{p^m})|U_p|V_p|_{\frac{k}{2}}\xi. \tag{3.21}$$

By (2.20), the second term in (3.21) is

$$(f(z)|U_{p^m})|U_p|V_p|_{\frac{k}{2}}\xi = p^{-1}\sum_{v=0}^{p-1}(f(z)|U_{p^m})|_{\frac{k}{2}}\tau_{v,p}\xi. \tag{3.22}$$

For each v, we choose an integer s_v so that

$$4Ns_v \equiv a^{-1}vd \,(\mathrm{mod}\, p), \tag{3.23}$$

and set

$$w_v := 4Ns_v. \tag{3.24}$$

Define

$$\delta_v := \begin{pmatrix} 1 - aw_vcp + vw_vc^2p^2 & \frac{a^2w_v - avd}{p} - acvw_v + bvcp \\ -w_vc^2p^3 & 1 + aw_vcp \end{pmatrix}, \tag{3.25}$$

and

$$\beta_v := \left(\delta_v, \sqrt{-w_vc^2p^3z + 1 + aw_vcp}\right). \tag{3.26}$$

A computation shows that

$$\tau_{v,p}\xi = \beta_v\xi\tau_{w_v,p}. \tag{3.27}$$

The next lemma shows that $\beta_v \in \widetilde{\Gamma_1(Np)}$.

Lemma 3.8. *Let β_v be defined as in* (3.26). *Then $\beta_v \in \widetilde{\Gamma_1(Np)}$.*

Proof. By the definitions of w_v in (3.24) and δ_v in (3.25), it is clear that $\delta_v \in \Gamma_1(Np)$. It remains to show that

$$j(\delta_v, z) = \sqrt{-w_v c^2 p^3 z + 1 + w_v acp}.$$

By definition,

$$j(\delta_v, z) = \left(\frac{-w_v c^2 p^3}{1 + w_v acp}\right) \varepsilon_{1+w_v acp}^{-1} \sqrt{-w_v c^2 p^3 z + 1 + w_v acp}.$$

Since $4 \mid w_v$, we have $\varepsilon_{1+w_v acp} = 1$. Now

$$\left(\frac{-w_v c^2 p^3}{1 + w_v acp}\right) = \left(\frac{-w_v p}{1 + w_v acp}\right).$$

Write $w_v = 2^e r$, where r is odd. Then using properties of Jacobi symbols,

$$\left(\frac{-w_v p}{1 + w_v acp}\right) = \left(\frac{-1}{1 + w_v acp}\right)\left(\frac{2}{1 + w_v acp}\right)^e \left(\frac{rp}{1 + w_v acp}\right) = 1,$$

since $w_v acp \equiv 0 \pmod 8$. This proves Lemma 3.8. □

Then by (3.27) and Lemma 3.8, for each v appearing in (3.22), we have

$$(f(z)|U_{p^m})|_{\frac{k}{2}} \tau_{v,p}\xi = (f(z)|U_{p^m})|_{\frac{k}{2}} \beta_v \xi \tau_{w_v,p} = (f(z)|U_{p^m})|_{\frac{k}{2}} \xi \tau_{w_v,p}. \tag{3.28}$$

Now we rewrite (3.22) using (3.28), (3.20), (2.4) and (2.19) to get

$$\begin{aligned}(f(z)|U_{p^m})|U_p|V_p|_{\frac{k}{2}}\xi &= p^{-1}\sum_{v=0}^{p-1}(f(z)|U_{p^m})|_{\frac{k}{2}}\xi\tau_{w_v,p} \\ &= p^{-1}\sum_{v=0}^{p-1}\left(\sum_{\substack{n\geq 0 \\ 4n+r\equiv 0 \pmod{p^m}}} a_0(n) q_{hp^m}^{n+\frac{r}{4}}\right)\Bigg|_{\frac{k}{2}} \tau_{w_v,p}\end{aligned}$$

$$= p^{-1}\sum_{v=0}^{p-1}\sum_{\substack{n\geq 0\\ 4n+r\equiv 0\,(\mathrm{mod}\,p^m)}} a_0(n)\exp\left(\frac{2\pi i(z-\frac{w_v}{p})}{hp^m}\left(n+\frac{r}{4}\right)\right)$$

$$= p^{-1}\sum_{\substack{n\geq 0\\ 4n+r\equiv 0\,(\mathrm{mod}\,p^m)}} a_0(n)q_{hp^m}^{n+\frac{r}{4}}\sum_{v=0}^{p-1}\exp\left(\frac{-2\pi i w_v}{4hp^{m+1}}(4n+r)\right)$$

$$= p^{-1}\sum_{\substack{n\geq 0\\ 4n+r\equiv 0\,(\mathrm{mod}\,p^m)}} a_0(n)q_{hp^m}^{n+\frac{r}{4}}\sum_{v=0}^{p-1}\exp\left(\frac{-2\pi i w_v}{4hp}\left(\frac{4n+r}{p^m}\right)\right). \quad (3.29)$$

Recall the definition of w_v (3.24). Note that since a, d, h and 4 are all coprime to p, the numbers $-w_v/4h$ run through the residue classes modulo p as v does, so

$$\sum_{v=0}^{p-1}\exp\left(\frac{-2\pi i w_v}{4hp}\left(\frac{4n+r}{p^m}\right)\right) = \sum_{v=0}^{p-1}\exp\left(\frac{2\pi i v}{p}\left(\frac{4n+r}{p^m}\right)\right) = \begin{cases} p & \text{if } p \mid \frac{4n+r}{p^m}, \\ 0 & \text{otherwise.}\end{cases} \quad (3.30)$$

Putting together (3.29) and (3.30), we have

$$(f(z)|U_{p^m})|U_p|V_p|_{\frac{k}{2}}\xi = \sum_{\substack{n\geq 0\\ 4n+r\equiv 0\,(\mathrm{mod}\,p^{m+1})}} a_0(n)q_{hp^m}^{n+\frac{r}{4}}. \quad (3.31)$$

Now using (3.21), (3.20) and (3.31), we have

$$f_m(z)|_{\frac{k}{2}}\xi = \sum_{\substack{n\geq 0\\ 4n+r\equiv 0\,(\mathrm{mod}\,p^m)}} a_0(n)q_{hp^m}^{n+\frac{r}{4}} - \sum_{\substack{n\geq 0\\ 4n+r\equiv 0\,(\mathrm{mod}\,p^{m+1})}} a_0(n)q_{hp^m}^{n+\frac{r}{4}}. \quad (3.32)$$

If $r \neq 0$ then neither series in (3.32) has a constant term, and if $r = 0$, then the constant term in each expansion is $a_0(0)$, so they cancel. Therefore $f_m(z)$ vanishes at the cusp $\frac{a}{cp^2}$. This concludes the proof of Proposition 3.7. □

Now for each odd prime p, we define the eta-quotient

$$F_p(z) := \begin{cases} \frac{\eta^{p^2}(z)}{\eta(p^2 z)} \in M_{\frac{p^2-1}{2}}(\Gamma_0(p^2)) & \text{if } p \geq 5, \\ \frac{\eta^{27}(z)}{\eta^3(9z)} \in M_{12}(\Gamma_0(9)) & \text{if } p = 3. \end{cases} \tag{3.33}$$

Using Theorem 2.9, we see that $F_p(z)$ vanishes at every cusp $\frac{a}{c}$ with $p^2 \nmid c$. By the definition of $\eta(z)$ (2.12), it is clear that $F_p(z) \equiv 1 \pmod{p}$, and an easy induction argument shows that $F_p(z)^{p^{s-1}} \equiv 1 \pmod{p^s}$ for any integer $s \geq 1$.

Proof of Theorem 3.3. Let f be as in the hypotheses of Theorem 3.3. Let m be chosen to satisfy Proposition 3.7 for f, and fix j. If $\beta \geq j-1$ is sufficiently large, then

$$g_{p,j}(z) := f_m(z) \cdot F_p(z)^{p^\beta} \equiv f_m(z) \pmod{p^j}$$

vanishes at all cusps $\frac{a}{c}$ for which $p^2 \nmid c$. By Proposition 3.7, $g_{p,j}(z)$ vanishes at the cusps $\frac{a}{c}$ for which $p^2 | c$, so we have

$$g_{p,j}(z) \in S_{\frac{k}{2}+\frac{p^\beta(p^2-1)}{2}}(\widetilde{\Gamma_0(Np^2)}, \chi\chi_p^{km}) \cap \mathcal{O}_K[\![q]\!]. \tag{3.34}$$

By (2.18) and (2.20), we have

$$g_{p,j}(z) \equiv f_m(z) \equiv \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} a(p^m n) q^n \pmod{p^j}. \tag{3.35}$$

Combining (3.34) and (3.35) proves Theorem 3.3. □

Proof of Theorem 3.1. Let f and p be as in the hypotheses of Theorem 3.1. Fix an integer

$j \geq 1$, and let

$$g_{p,j}(z) \equiv \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} a(p^m n)q^n \pmod{p^j}$$

be the cusp form guaranteed by Theorem 3.3, with β chosen so that $\kappa := k + p^\beta(p^2-1) > 3$. By Proposition 2.13, a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have

$$g_{p,j}(z)|T_{\frac{\kappa}{2},Np^2,\chi\chi_p^{km}}(Q^2) \equiv 0 \pmod{p^j}. \tag{3.36}$$

If we write $g_{p,j}(z) = \sum_{n=1}^{\infty} b(n)q^n$, then (2.25) and (3.36) yield

$$\begin{aligned} g_{p,j}(z)&|T_{\frac{\kappa}{2},Np^2,\chi\chi_p^{km}}(Q^2) \\ &= \sum_{n=1}^{\infty}\left(b(Q^2n) + \chi\chi_p^{km}(Q)\left(\frac{(-1)^{\frac{\kappa-1}{2}}n}{Q}\right)Q^{\frac{\kappa-3}{2}}b(n)\right. \\ &\qquad \left. + \chi\chi_p^{km}(Q^2)Q^{\kappa-2}b\left(\frac{n}{Q^2}\right)\right)q^n \equiv 0 \pmod{p^j}. \end{aligned} \tag{3.37}$$

Replacing n by Qn in (3.37), we have

$$\begin{aligned} \sum_{n=1}^{\infty}\left(b(Q^3n) + \chi\chi_p^{km}(Q)\left(\frac{(-1)^{\frac{\kappa-1}{2}}Qn}{Q}\right)Q^{\frac{\kappa-3}{2}}b(Qn) + \chi\chi_p^{km}(Q^2)Q^{\kappa-2}b\left(\frac{Qn}{Q^2}\right)\right)q^{Qn} \\ \equiv 0 \pmod{p^j}. \end{aligned} \tag{3.38}$$

If $(Q,n) = 1$, then the coefficient of q^{Qn} in (3.38) is just $b(Q^3n)$. So

$$a(p^mQ^3n) \equiv b(Q^3n) \equiv 0 \pmod{p^j}$$

for all n coprime to Qp. This completes the proof of Theorem 3.1. □

3.2 Proof of Theorem 3.2

We now prove the integral weight analog to Theorem 3.1. For part (a) of Theorem 3.2, we use the following result of Serre.

Proposition 3.9. (Serre, [32, Corollaire du Théorème 1]) *Let*

$$f(z) = \sum_{n=0}^{\infty} c_n q_M^n, \quad M \geq 1,$$

be a modular form of integral weight $k \geq 1$ on a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and suppose that the coefficients c_n lie in the ring of integers of an algebraic number field K. Then for any integer $m \geq 1$,

$$c_n \equiv 0 \pmod{m}$$

for almost all n.

Proof of Theorem 3.2. Let f and p be as in the hypotheses of Theorem 3.2, and fix $j \geq 1$. First, if $4 \nmid N$, we may consider $f(z)$ as a modular form on $\Gamma_0(4N)$, so we will assume that $4 \mid N$. Then by Theorem 2.5 and Theorem 3.3, there exists a cusp form $g_{p,j}$ of positive integral weight with algebraic integer coefficients such that

$$g_{p,j}(z) \equiv \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} a(p^m n) q^n \pmod{p^j}. \tag{3.39}$$

The first assertion of Theorem 3.2 now follows from (3.39) and Proposition 3.9.

For the second assertion, it follows from (3.39) and Theorem 2.11 that if $g_{p,j}(z)$ has weight κ and character ψ, then a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have the property that

$$g_{p,j}(z) | T_{\kappa, Np^2, \psi}(Q) \equiv 0 \pmod{p^j}. \tag{3.40}$$

If we write $g_{p,j}(z) = \sum_{n=1}^{\infty} b(n)q^n$, then (2.23) and (3.40) imply that

$$g_{p,j}(z)|T_{\kappa,Np^2,\psi}(Q) = \sum_{n=1}^{\infty}\left(b(Qn) + \psi(Q)Q^{\kappa-1}b\left(\frac{n}{Q}\right)\right)q^n \equiv 0 \pmod{p^j}. \tag{3.41}$$

If $(Q,n) = 1$, then the coefficient of q^n in (3.41) is just $b(Qn)$. Therefore

$$a(Qp^m n) \equiv b(Qn) \equiv 0 \pmod{p^j}$$

for all n coprime to Qp. This concludes the proof of Theorem 3.2. □

3.3 Congruences for eta-quotients

Next we show that infinitely many congruences exist for the coefficients of any eta-quotient. Let $f(z) = \sum_{\delta|N} \eta^{r_\delta}(\delta z)$ be an eta-quotient as in (2.13). We require in addition that

$$\sum_{\delta|N} r_\delta \delta \equiv 0 \pmod{24}. \tag{3.42}$$

This is to ensure that $f(z)$ has a Fourier expansion of the form $\sum a(n)q^n$. Any eta-quotient can be made to satisfy condition (3.42) by replacing each δ with 24δ.

Theorem 3.10. *Suppose p is an odd prime and N is an integer with $4 \mid N$ and $(N,p) = 1$. Let $f(z) = \sum_{\delta|N} \eta^{r_\delta}(\delta z)$ satisfy (3.42), and suppose that $f(z)$ has the Fourier expansion $f(z) = \sum a(n)q^n$. Set $k := \sum_{\delta|N} r_\delta$, and let m be a sufficiently large integer.*

(a) If k is odd, then for each positive integer j, a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have

$$a(Q^3 p^m n) \equiv 0 \pmod{p^j}$$

for all n coprime to Qp.

(b) If k is even, then for each positive integer j,

(i) $a(p^m n) \equiv 0 \pmod{p^j}$ *for almost all* n *coprime to* p*, and*

(ii) a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ *have*

$$a(Qp^m n) \equiv 0 \pmod{p^j}$$

for all n *coprime to* Qp*.*

Proof. Let f be as in the hypotheses of Theorem 3.10. We claim that we can replace N by a power of N, if necessary, so that $f(z)$ satisfies

$$N\sum_{\delta|N}\frac{r_\delta}{\delta} \equiv 0 \pmod{24}. \tag{3.43}$$

To see this, we first note that $N\sum_{\delta|N}\frac{r_\delta}{\delta}$ is an integer, and $4 \mid N$ by hypothesis. If $3 \mid N$, then $24 \mid N^2$, and hence

$$N^3\sum_{\delta|N}\frac{r_\delta}{\delta} \equiv 0 \pmod{24}.$$

If $3 \nmid N$, then $\delta^2 \equiv 1 \pmod 3$ for each $\delta \mid N$, so

$$N\sum_{\delta|N}\frac{r_\delta}{\delta} \equiv N\sum_{\delta|N} r_\delta\delta \equiv 0 \pmod{3}, \tag{3.44}$$

where the second equivalence follows from (3.42). Since $4 \mid N$,

$$N^3\sum_{\delta|N}\frac{r_\delta}{\delta} \equiv 0 \pmod{8}. \tag{3.45}$$

Then (3.44) and (3.45) imply that

$$N^3\sum_{\delta|N}\frac{r_\delta}{\delta} \equiv 0 \pmod{24}.$$

Now by Corollary 2.7, f is a weakly holomorphic modular form of weight $\frac{k}{2}$ and some character χ with level N. Since $\eta(z)$ has integer coefficients, so does $f(z)$. Therefore, we may apply either Theorem 3.1 or Theorem 3.2 to f, depending on the parity of k, to complete the proof of Theorem 3.10. □

Remark. By Proposition 2.8, an eta-quotient $f(z)$ of level N satisfying (3.42) and (3.43) has Fourier expansions of the form

$$f(z)|_{\frac{k}{2}}\psi = \sum_{n \geq n_\psi} a_\psi(n) q_{h_0}^{n+\frac{r}{4}} \tag{3.46}$$

for each $\psi \in G'$ with $\psi\infty = \frac{a}{c}$, where $h_0 = \frac{N}{(c^2,N)}$. If we replace (3.14) with an expansion of the form in (3.46), the conclusion of Proposition 3.4 becomes

$$(f(z)|U_{p^m})|_{\frac{k}{2}}\xi = \sum_{\substack{n \geq n_0 \\ 4n+r \equiv 0 \pmod{p^m}}} a_0(n) q_{h_0 p^m}^{n+\frac{r}{4}}. \tag{3.47}$$

Now, if we replace (3.19) with (3.47), then Proposition 3.7 still holds, since the only property we required of h was that $h \mid N$, and this is also true for h_0. Thus, m is sufficiently large for $f(z)$ if it satisfies

$$-p^m < 4n_0 + r \tag{3.48}$$

for all distinct pairs (n_0, r) which come from Fourier expansions of the form (3.47).

Theorem 3.10 yields congruences for many partition functions. We mention only two examples here for brevity.

Example 3.1. Recall from Example 2.8 that the generating function for $p_k(n)$, the number of k-colored partitions of n, is

$$f(z) = \sum_{n \equiv -k \pmod{24}} p_k\left(\frac{n+k}{24}\right) q^n = \frac{1}{\eta^k(24z)},$$

a weakly holomorphic modular form of level 576. By Proposition 2.9, the order of vanishing of $f(z)$ at a cusp $s = \frac{a}{c}$ is

$$\mathrm{ord}_s(f(z)) = \frac{576}{24(c^2, 576)}\left(-k \cdot \frac{(c, 24)^2}{24}\right) = -k \cdot \frac{(c, 24)^2}{(c^2, 24^2)} = -k.$$

Then the Fourier expansion of $f(z)$ at s has the form

$$C_s \cdot q_{h_s}^{-k} + O(1), \tag{3.49}$$

for some $C_s \in \mathbb{C}$ and $h_s = \frac{576}{(c^2, 576)}$.

Theorem 3.10 applies to $f(z)$ with any prime $\ell \nmid 576$, so let $\ell \geq 5$. Then by (3.49) and (3.48), m must satisfy $-\ell^m < -4k$. For such m and for each positive integer j, we have congruences of the form

$$p_k\left(\frac{Q^3\ell^m n + k}{24}\right) \equiv 0 \pmod{\ell^j}$$

when k is odd, and of the form

$$p_k\left(\frac{Q\ell^m n + k}{24}\right) \equiv 0 \pmod{\ell^j}$$

when k is even. Note that when $k = 1$, we recover congruences for $p(n)$ of the form guaranteed by Ahlgren [1] and Ono [30].

Example 3.2. Consider the generating function

$$f(z) = \sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = \frac{\eta(2z)}{\eta^2(z)} \in \mathcal{M}_{-\frac{1}{2}}(\widetilde{\Gamma_0(16)})$$

for overpartitions given in Example 2.9. By Proposition 2.9, we have

$$\mathrm{ord}_s(f(z)) = \frac{16}{24(c^2, 16)}\left(\frac{(c, 2)^2}{2} - 2\right) = \begin{cases} 0 & \text{if } 2 \mid c, \\ -1 & \text{if } 2 \nmid c. \end{cases} \tag{3.50}$$

Then by (3.48), for each prime $p \neq 2$, we need m large enough so that $-p^m < -4$. Hence for $p = 3$, we need $m \geq 2$, and for $p \geq 5$ we only need $m \geq 1$. Then for such m, Theorem 3.10 yields congruences of the form

$$\overline{p}(Q^3 \ell^m n) \equiv 0 \pmod{\ell^j}.$$

When the modulus is 5, we find an explicit infinite family of such congruences.

Proposition 3.11. *Let $Q \equiv -1 \pmod 5$ be prime. Then*

$$\overline{p}(5Q^3 n) \equiv 0 \pmod 5$$

for all n coprime to Q.

Proof. Consider the theta functions

$$\Theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots \in M_{\frac{1}{2}}(\widetilde{\Gamma_0(4)}),$$

and

$$\Theta_1(z) := \frac{\eta^2(z)}{\eta(2z)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots \in M_{\frac{1}{2}}(\widetilde{\Gamma_0(16)}).$$

Let χ_2^{triv} be the trivial character modulo 2. Using (2.21), it is easy to verify that these theta functions satisfy

$$\Theta_1(z)^3 = \Theta(z)^3 - 2\Theta(z)^3 \otimes \chi_2^{\text{triv}} \in M_{\frac{3}{2}}(\widetilde{\Gamma_0(16)}). \tag{3.51}$$

Recall the definition of $\Delta(z)$ in (2.2), and the fact that the generating function for overpartitions is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{\eta(2z)}{\eta^2(z)} = \frac{1}{\Theta_1(z)}.$$

Set

$$f(z) := \frac{\frac{\Delta^2(z)}{\Delta(2z)}\Big| T_{12,2}(5)}{\frac{\eta^{10}(z)}{\eta^5(2z)}}. \tag{3.52}$$

Since the numerator of $f(z)$ is of level 2, and the 24th power of the denominator is also of level 2, it suffices to check holomorphicity at 0 and ∞, the two cusps of $\Gamma_0(2)$. It is clear by (2.12) and (2.23) that f is holomorphic at infinity. A computation using (2.23) shows that

$$\frac{\Delta^2(z)}{\Delta(2z)}\Big| T_{12,2}(5) = 48828126\frac{\Delta^2(z)}{\Delta(2z)} + 2342387712\Delta(z) + 4630511616\Delta(2z). \tag{3.53}$$

Using Theorem 2.9 with (3.52) and (3.53), we calculate that the expansion of $f(z)$ at 0 has the form

$$c \cdot q^{\frac{3}{16}} + \cdots$$

for some constant $c \in \mathbb{C}$. Therefore

$$f(z) \in M_{\frac{19}{2}}(\widetilde{\Gamma_0(16)}).$$

Since $T_{12,2}(5)$ is the same as the operator U_5 modulo 5, it can be verified using (2.12) and (2.18) that

$$\begin{aligned} f(z) &\equiv \frac{\frac{\Delta^2(z)}{\Delta(2z)}\Big| U_5}{\frac{\eta^{10}(z)}{\eta^5(2z)}} \pmod 5 \\ &\equiv \frac{\left(\frac{\eta^{50}(z)}{\eta^{25}(2z)} \cdot \sum \overline{p}(n)q^n\right)\Big| U_5}{\frac{\eta^{10}(z)}{\eta^5(2z)}} \pmod 5 \\ &\equiv \sum \overline{p}(5n)q^n \pmod 5. \end{aligned} \tag{3.54}$$

Recall the Eisenstein series

$$E_4(z) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \in M_4(\Gamma), \tag{3.55}$$

which clearly satisfies

$$E_4(z) \equiv 1 \pmod 5. \tag{3.56}$$

Using (3.51), (3.55) and (3.56), we have

$$\Theta_1(z)^3 \cdot E_4^2(z) \in M_{\frac{19}{2}}(\widetilde{\Gamma_0(16)})$$

with

$$\Theta_1(z)^3 \equiv \Theta_1(z)^3 \cdot E_4^2(z) \pmod 5. \tag{3.57}$$

By Theorem 2.2, two holomorphic modular forms in $M_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ are congruent modulo m if their coefficients are congruent modulo m for each index $n \leq \frac{k}{24}[\Gamma : \Gamma_0(N)]$. (To see this, apply Theorem 2.2 to the squares of the two forms.) Using (3.57), the congruence

$$f(z) \equiv \Theta_1(z)^3 \pmod 5 \tag{3.58}$$

can be easily verified with a computer algebra system, well beyond the Sturm bound of 19. Then (3.54) and (3.58) imply that

$$\Theta_1(z)^3 \equiv \sum \overline{p}(5n)q^n \pmod 5. \tag{3.59}$$

It is well known that $\Theta^3(z)$, considered as a form of level 16, is a normalized eigenform for $T_{\frac{3}{2},16}(Q^2)$, satisfying

$$\Theta(z)^3 | T_{\frac{3}{2},16}(Q^2) = (Q+1)\Theta^3(z) \tag{3.60}$$

for each odd prime Q.

The Hecke operator $T_{\frac{3}{2},16}(Q^2)$ commutes with the quadratic twist χ_2^{triv}, so using (3.51) and (3.60) we have

$$\Theta_1(z)^3 | T_{\frac{3}{2},16}(Q^2) \equiv (Q+1)\Theta_1(z)^3 \equiv 0 \pmod 5$$

when $Q \equiv -1 \pmod 5$. Then using (3.59) and an argument as in (3.38), we have

$$\overline{p}(5Q^3n) \equiv 0 \pmod 5,$$

when n is coprime to Q. This concludes the proof of Proposition 3.11. □

3.4 Traces of CM values of modular functions

For our final application of Theorem 3.1, we obtain linear congruences for the trace of an arbitrary weakly holomorphic modular function (a modular form of weight zero) on $\Gamma_0^*(p) = \langle \Gamma_0(p), W_p \rangle$, where p is prime (or $p = 1$) and $W_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ is the Fricke involution. Following Bruinier and Funke [10], let D be a positive integer, and let $\mathcal{Q}_{D,p}$ be the subset of quadratic forms $F(x,y) = ax^2 + bxy + cy^2$ in $\mathcal{Q}_D$ with $a \equiv 0 \pmod p$. Then $\Gamma_0^*(p)$ acts on $\mathcal{Q}_{D,p}$ with finitely many equivalence classes. For any weakly holomorphic modular function f on $\Gamma_0^*(p)$, define the modular trace function

$$t_f^*(D) := \sum_{F \in \mathcal{Q}_{D,p}/\Gamma_0^*(p)} \frac{1}{|\Gamma_0^*(p)_F|} f(\alpha_F), \tag{3.61}$$

where $\Gamma_0^*(p)_F$ is the stabilizer of F in $\Gamma_0^*(p)$. Note that if $p = 1$ and $f = J_m$, the trace function $t_m(n)$ from (1.2) is recovered. Using Theorem 3.1 and recent work of Bruinier and Funke [10], we have the following corollary.

Theorem 3.12. *Suppose that p is prime (or $p = 1$), k is an integer and $\zeta_p := e^{\frac{2\pi i}{p}}$. Let f be a weakly holomorphic modular function on $\Gamma_0^*(p)$ with Fourier expansion $f(z) = \sum a(n)q^n \in \mathbb{Q}(\zeta_p)((q))$, and suppose $a(0) = 0$. Let ℓ be an odd prime with $\ell \neq p$.*

(a) There exists an integer M such that $Mt_f^*(D)$ is an algebraic integer for each $D > 0$.

(b) Let M be as in (a). If m is a sufficiently large integer, then for each positive integer j,

a positive proportion of the primes $Q \equiv -1 \pmod{4p^2\ell^j}$ *have the property that*

$$Mt_f^*(Q^3\ell^m D) \equiv 0 \pmod{\ell^j}$$

for each D *coprime to* $Q\ell$.

Remark. There is no hypothesis controlling the principal part of the function $f(z)$, so a result as general as Theorem 3.1 is required in order to get congruences for $t_f^*(D)$.

In [10], Bruinier and Funke realize the traces of arbitrary modular functions on any congruence subgroup as the Fourier coefficients of certain weakly holomorphic modular forms. These forms are obtained by integrating the modular functions against a theta series associated to a certain lattice and a certain Schwartz function. The authors give the following theorem as a concrete example of their more general result.

Define $\sigma_1(n) := \sum_{t|n} t$ for $n \in \mathbb{Z}_{>0}$, set $\sigma_1(0) := -\frac{1}{24}$, and let $\sigma_1(x) := 0$ for $x \notin \mathbb{Z}_{\geq 0}$.

Theorem 3.13. (Bruinier-Funke [10], Theorem 1.1) *Let* $f \in \mathcal{M}_0(\Gamma_0^*(p))$ *have the Fourier expansion* $f(z) = \sum a(n)q^n$ *with* $a(0) = 0$. *Then*

$$G(z, f) := \sum_{D>0} t_f^*(D)q^D + \sum_{n\geq 0}(\sigma_1(n) + p\sigma_1(n/p))a(-n) - \sum_{m>0}\sum_{n>0} ma(-mn)q^{-m^2}$$

is a weakly holomorphic modular form of weight $3/2$ *for the group* $\widetilde{\Gamma_0(4p)}$.

Proof of Theorem 3.12. For simplicity, let $G(z) := G(z, f)$ be the modular form guaranteed by Theorem 3.13. The following lemma shows that $G(z)$ has algebraic coefficients.

Lemma 3.14. *Let* f *be a weakly holomorphic modular function for* $\Gamma_0^*(p)$ *whose Fourier coefficients with respect to* q_p *lie in* $\mathbb{Q}(\zeta_p)$. *Then for each discriminant* $D > 0$, $t_f^*(D)$ *is algebraic.*

Proof. We can view f as a modular function for the principal congruence subgroup $\Gamma(p)$, which consists of matrices in Γ congruent to the identity modulo p. Let k_p be the field of modular functions for Γ_p whose Fourier expansions with respect to q_p have coefficients in $\mathbb{Q}(\zeta_p)$. Fix a discriminant $D > 0$ and a quadratic form $F \in \mathcal{Q}_{D,p}$, and let α_F be the associated root in $\mathbb{H}$. Set $K = \mathbb{Q}(\alpha_F)$. By the theory of complex multiplication, the field $Kk_p(\alpha_F)$, generated over K by all values $f(\alpha_F)$ with $f \in k_p$ and f defined at α_F, is the ray class field over K with conductor p [22, Ch. 10 §1, Corollary to Theorem 2] . So in particular, $f(\alpha_F)$ is algebraic. Lemma 3.14 then follows from definition (3.61). □

By Lemma 3.14, we have $G(z) \in \overline{\mathbb{Q}}((q))$. The next lemma proves conclusion (a) of Theorem 3.12.

Lemma 3.15. *Let f be as in the hypotheses of Theorem 3.12, and let $G(z) := G(z, f)$ be the modular form guaranteed by Theorem 3.13. Then there exists a nonzero integer M and an algebraic number field L such that $MG(z) \in \mathcal{O}_L((q))$.*

Proof. Recall the function $\Delta(z)$ from (2.2). Since $G(z)$ is meromorphic at the cusps of $\Gamma_0(4p)$ and $\Delta(z)$ vanishes at each cusp, then for sufficiently large h, the function $\Delta^h(z)G(z)$ is a cusp form for $\widetilde{\Gamma_0(4p)}$. Since $\Delta(z)$ has integer coefficients,

$$\Delta^h(z)G(z) \in \overline{\mathbb{Q}}[\![q]\!].$$

Then by Proposition 2.4, there is an algebraic number field L and a nonzero integer M such that

$$M\Delta^h(z)G(z) \in \mathcal{O}_L[\![q]\!].$$

But $\frac{1}{\Delta^h(z)}$ also has integer coefficients, so we have

$$MG(z) \in \mathcal{O}_L((q)).$$

□

Now we prove Theorem 3.12(b). By Lemma 3.15, we can apply Theorem 3.1 to $MG(z)$.

Hence for each $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{4p\ell^j}$ have

$$Mt_f^*(Q^3\ell^m D) \equiv 0 \pmod{\ell^j}$$

for all D coprime to $Q\ell$. This concludes the proof of Theorem 3.12. □

Chapter 4

Congruences using the quadratic twist approach

We now present a second approach for producing congruences for the coefficients of a weakly holomorphic modular form $f(z)$. Instead of applying a U-operator to $f(z)$ to build cusp forms as in Theorem 3.1, we instead twist $f(z)$ by a quadratic character, as defined in (2.21). For the quadratic twist method to be successful, we require a set of conditions on the Fourier expansions of $f(z)$ at each cusp.

Let $\Gamma' \leqslant \Gamma_0(4)$ be a congruence subgroup of level N. Associate to each cusp $s = \frac{a}{c}$ the integers h_s and r_s (as defined in 2.8 and 2.9) and an element $\psi_s \in G'$ with $\psi_s\infty = s$. If $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma'})$, then for each s, write

$$f(z)|_{\frac{k}{2}}\psi_s = \sum_{n \geq n_s} a_{\psi_s}(n) q_{h_s}^{n+\frac{r_s}{4}}.$$

Definition. We say that f satisfies *condition C* for a prime p if there exists $\epsilon_p \in \{\pm 1\}$ such that for each ψ_s, the following is true: for all $n < 0$ with $p \nmid (4n + r_s)$ and $a_{\psi_s}(n) \neq 0$, we have

$$\left(\frac{4n + r_s}{p}\right) = \epsilon_p \left(\frac{h_s}{p}\right).$$

Remark. In the case $s = \infty$, condition C requires that $\left(\frac{n}{p}\right) = \epsilon_p$ for all $n < 0$ with $p \nmid n$ and $a(n) \neq 0$.

Remark. Condition C is well-defined. If a different element $\beta_s \in G'$ were chosen with $\beta_s\infty = s$, then $f|_{\frac{k}{2}}\beta_s$ and $f|_{\frac{k}{2}}\psi_s$ would be the same up to multiplication by a fourth root of unity. Further, it is sufficient to check the condition for one element from each equivalence class of cusps, because if $s' = \gamma s$ for $\gamma \in \Gamma'$, then $\widetilde{\gamma}\psi_s\infty = s'$, and $f|_{\frac{k}{2}}\widetilde{\gamma}\psi_s = f|_{\frac{k}{2}}\psi_s$.

We now state our main theorem for congruences using the quadratic twist method.

Theorem 4.1. *Suppose that p is an odd prime, and that k is an odd integer. Let N be a positive integer with $4 \mid N$ and $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let K be a number field with ring of integers $\mathcal{O}_K$, and suppose that $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \cap \mathcal{O}_K((q))$. Let f satisfy condition C for p. Then for each positive integer j, a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have the property that*

$$a(Q^3 n) \equiv 0 \pmod{p^j}$$

for all n coprime to Q such that $\left(\frac{-n}{p}\right) = -\epsilon_p$.

As before, we have an analog of Theorem 4.1 for integer weight modular forms. If $f(z) \in \mathcal{M}_k(\Gamma_0(N), \chi)$, we say that $f(z)$ satisfies condition C for p if it does so when viewed as a half-integral weight modular form via Proposition 2.5.

Theorem 4.2. *Suppose that p is an odd prime, and that k is an integer. Let N be a positive integer with $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let K be a number field with ring of integers $\mathcal{O}_K$, and suppose that $f(z) = \sum a(n)q^n \in \mathcal{M}_k(\Gamma_0(N), \chi) \cap \mathcal{O}_K((q))$. Let f satisfy condition C for p. Then for each positive integer j,*

(i) $a(n) \equiv 0 \pmod{p^j}$ for almost all n with $\left(\frac{-n}{p}\right) = -\epsilon_p$, and

(ii) a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have the property that

$$a(Qn) \equiv 0 \pmod{p^j}$$

for all n coprime to Q such that $\left(\frac{-n}{p}\right) = -\epsilon_p$.

The next result shows that every weakly holomorphic modular form of integral or half-integral weight on $\Gamma_0(N)$ with character χ satisfies condition C for a positive proportion of primes p.

Theorem 4.3. *Let k and N be integers with $N \geq 1$ and $4 \mid N$, and let χ be a Dirichlet character modulo N. Suppose that $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. Then a positive proportion of primes p have the property that f satisfies condition C for p.*

4.1 Proofs of Theorems 4.1, 4.2 and 4.3

The following result is critical for proving Theorem 4.1.

Theorem 4.4. *Suppose that p is an odd prime, and that k is an odd integer. Let N be a positive integer with $4 \mid N$ and $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let K be a number field with ring of integers $\mathcal{O}_K$, and suppose $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi) \cap \mathcal{O}_K((q))$. Let f satisfy condition C for p. Then for every positive integer j, there exists an integer $\beta \geq j-1$ and a cusp form*

$$g_{p,j}(z) \in S_{\frac{k}{2}+\frac{p^{\beta}(p^2-1)}{2}}(\widetilde{\Gamma_0(Np^2)}, \chi) \cap \mathcal{O}_K[[q]],$$

with the property that

$$g_{p,j}(z) \equiv \sum_{\left(\frac{n}{p}\right)=-\epsilon_p} a(n)q^n \pmod{p^j}.$$

To prove Theorem 4.4, we need the following technical result.

Proposition 4.5. *Suppose that p is an odd prime, and that k is an integer. Let N be a positive integer with $4 \mid N$ and $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. Suppose that $\mu \in \{\pm 1, \pm i\}$, and that $\xi :=$*

$\left(\begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix}, \mu\sqrt{cp^2z+d}\right) \in G'$ *with* $ac > 0$. *Let*

$$f(z)|_{\frac{k}{2}}\xi = \sum_{n \geq n_\xi} a_\xi(n) q_h^{n+\frac{r}{4}}.$$

Then

$$\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi = \left(\frac{h}{p}\right) \sum_{n \geq n_\xi} \left(\frac{4n+r}{p}\right) a_\xi(n) q_h^{n+\frac{r}{4}}.$$

Proof. Let g be the quadratic Gauss sum defined in §2.6. Then recalling Proposition 2.16 and (2.19), we have

$$f(z) \otimes \left(\frac{\bullet}{p}\right) = \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f(z)|_{\frac{k}{2}} \tau_{v,p},$$

so

$$\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi = \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f(z)|_{\frac{k}{2}} \tau_{v,p} \xi.$$

Recall the definitions of s_v (3.23), w_v (3.24) and β_v (3.26). Then by (3.27), we have

$$\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi = \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f(z)|_{\frac{k}{2}} \beta_v \xi \tau_{w_v,p}.$$

Since $\widetilde{\Gamma_1(Np)} < \widetilde{\Gamma_1(N)}$, Lemma 3.8 shows that $\beta_v \in \widetilde{\Gamma_1(N)}$, so

$$\begin{aligned}
\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi &= \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f(z)|_{\frac{k}{2}} \xi \tau_{w_v,p} \\
&= \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) \left(\sum_{n \geq n_\xi} a_\xi(n) q_h^{n+\frac{r}{4}}\right)\Bigg|_{\frac{k}{2}} \tau_{w_v,p} \\
&= \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) \sum_{n \geq n_\xi} a_\xi(n) \exp\left(\frac{2\pi i(z - w_v/p)(4n+r)}{4h}\right) \\
&= \frac{g}{p} \sum_{n \geq n_\xi} a_\xi(n) q_h^{n+\frac{r}{4}} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) \exp\left(\frac{-2\pi i w_v(4n+r)}{4hp}\right). \qquad (4.1)
\end{aligned}$$

Now recalling the definition of s_v (3.23), we see that since 4, a, d, and N are coprime to

p, the $s_v = w_v/4N$ run through the residue classes modulo p as v does. Also, $v \equiv w_v ad^{-1}$ (mod p). Then writing $N = hh'$, we have

$$\begin{aligned}
\sum_{v=0}^{p-1} \left(\frac{v}{p}\right) e^{\frac{-2\pi i w_v(4n+r)}{4hp}} &= \sum_{v=0}^{p-1} \left(\frac{w_v ad^{-1}}{p}\right) \exp\left(\frac{2\pi i w_v}{p}\left(\frac{-4n-r}{4h}\right)\right) \\
&= \left(\frac{ad^{-1}}{p}\right) \sum_{v=0}^{p-1} \left(\frac{4Ns_v}{p}\right) \exp\left(\frac{2\pi i(4Ns_v)}{p}\left(\frac{-4n-r}{4h}\right)\right) \\
&= \left(\frac{Nad^{-1}}{p}\right) \sum_{v=0}^{p-1} \left(\frac{s_v}{p}\right) \exp\left(\frac{2\pi i s_v}{p}(-h'(4n+r))\right) \\
&= \left(\frac{Nad^{-1}}{p}\right)\left(\frac{-h'(4n+r)}{p}\right) g. \qquad (4.2)
\end{aligned}$$

The last line follows from Proposition 2.14. Combining (4.1) and (4.2), and using Proposition 2.15, we have

$$\begin{aligned}
\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi &= \frac{g}{p} \sum_{n \geq n_\xi} a_\xi(n) q_h^{n+\frac{r}{4}} \left(\frac{Nad^{-1}}{p}\right)\left(\frac{-h'(4n+r)}{p}\right) g \\
&= \left(\frac{-1}{p}\right)\left(\frac{Nad^{-1}}{p}\right) \sum_{n \geq n_\xi} \left(\frac{-h'(4n+r)}{p}\right) a_\xi(n) q_h^{n+\frac{r}{4}} \\
&= \left(\frac{Nad^{-1}h'}{p}\right) \sum_{n \geq n_\xi} \left(\frac{4n+r}{p}\right) a_\xi(n) q_h^{n+\frac{r}{4}}. \qquad (4.3)
\end{aligned}$$

Now we use the facts that $ad \equiv 1 \pmod p$, that $h' \equiv h^{-1}N \pmod p$, and that if b is coprime to p, then $\left(\frac{b^{-1}}{p}\right) = \left(\frac{b}{p}\right)$, to see that

$$\left(\frac{Nad^{-1}h'}{p}\right) = \left(\frac{N^2ad^{-1}h^{-1}}{p}\right) = \left(\frac{adh}{p}\right) = \left(\frac{h}{p}\right). \qquad (4.4)$$

Therefore by (4.3) and (4.4), we have

$$\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi = \left(\frac{h}{p}\right) \sum_{n \geq n_\xi} \left(\frac{4n+r}{p}\right) a_\xi(n) q_h^{n+\frac{r}{4}}.$$

This concludes the proof of Proposition 4.5. □

Proposition 4.6. *Suppose that p is an odd prime, and k is an integer. Let N be a positive integer with $4 \mid N$ and $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ satisfy condition C for p with ϵ_p. Define*

$$f_p(z) := f(z) \otimes \chi_p^{triv} - \epsilon_p f(z) \otimes \left(\frac{\bullet}{p}\right) \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(Np^2)}, \chi).$$

Then f_p vanishes at each cusp $\frac{a}{cp^2}$.

Proof. Assume without loss of generality that $ac > 0$, and let

$$\xi := \left(\begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix}, \mu\sqrt{cp^2z+d}\right) \in G'$$

with $\mu \in \{\pm 1, \pm i\}$. By (2.20) and (2.22), we can write

$$f(z) \otimes \chi_p^{triv} = f(z) - \frac{1}{p}\sum_{v=0}^{p-1} f(z)|_{\frac{k}{2}}\tau_{v,p},$$

so that

$$f_p(z)|_{\frac{k}{2}}\xi = f(z)|_{\frac{k}{2}}\xi - \frac{1}{p}\sum_{v=0}^{p-1} f(z)|_{\frac{k}{2}}\tau_{v,p}\xi - \epsilon_p f(z) \otimes \left(\frac{\bullet}{p}\right)|_{\frac{k}{2}}\xi. \tag{4.5}$$

Recall the definitions of s_v (3.23), w_v (3.24) and β_v (3.26). Using (3.27) and Lemma 3.8, we have

$$\begin{aligned}
\frac{1}{p}\sum_{v=0}^{p-1} f(z)|_{\frac{k}{2}}\tau_{v,p}\xi &= \frac{1}{p}\sum_{v=0}^{p-1} f(z)|_{\frac{k}{2}}\xi\tau_{w_v,p} \\
&= \frac{1}{p}\sum_{v=0}^{p-1}\left(\sum_{n\geq n_\xi} a_\xi(n)q_h^{n+\frac{r}{4}}\right)|_{\frac{k}{2}}\tau_{w_v,p} \\
&= \frac{1}{p}\sum_{v=0}^{p-1}\sum_{n\geq n_\xi} a_\xi(n)\exp\left(\frac{2\pi i(z-w_v/p)}{4h}(4n+r)\right)
\end{aligned}$$

$$= \frac{1}{p}\sum_{n\geq n_\xi} a_\xi(n) q_h^{n+\frac{r}{4}} \sum_{v=0}^{p-1} \exp\left(\frac{-2\pi i s_v}{p}(h'(4n+r))\right)$$
$$= \sum_{h'(4n+r)\equiv 0 \pmod p} a_\xi(n) q_h^{n+\frac{r}{4}}$$
$$= \sum_{4n+r\equiv 0 \pmod p} a_\xi(n) q_h^{n+\frac{r}{4}}. \tag{4.6}$$

The last line follows since $p \nmid h'$.

Now by (4.5), (4.6) and Proposition 4.5, we have

$$f_p(z)|_{\frac{k}{2}}\xi = \sum_{4n+r\not\equiv 0 \pmod p} a_\xi(n) q_h^{n+\frac{r}{4}} - \epsilon_p \left(\frac{h}{p}\right) \sum_{n\geq n_\xi} \left(\frac{4n+r}{p}\right) a_\xi(n) q_h^{n+\frac{r}{4}}. \tag{4.7}$$

Therefore, $f_p(z)$ vanishes at ξ if for each $n<0$ with $p \nmid (4n+r)$ and $a_\xi(n) \neq 0$, we have

$$\left(\frac{4n+r}{p}\right) = \epsilon_p \left(\frac{h}{p}\right).$$

This requirement is met since f satisfies condition C for p. This concludes the proof of Proposition 4.6. □

Proof of Theorem 4.4. A cusp $\frac{a}{c}$ either has $p^2 \mid c$ or $p^2 \nmid c$. Recall the eta-quotient $F_p(z)$ defined in (3.33). By Proposition 4.6, if $\beta \geq j-1$ is sufficiently large, then

$$f_p(z) \cdot F_p(z)^{p^\beta} \equiv f_p(z) \pmod{p^j}$$

vanishes at every cusp. Set

$$g_{p,j}(z) := \frac{1}{2} f_p(z) \cdot F_p(z)^{p^\beta} \in S_{\frac{k}{2}+\frac{p^\beta(p^2-1)}{2}}(\widetilde{\Gamma_0(Np^2)}, \chi) \cap \mathcal{O}_K[[q]]. \tag{4.8}$$

Then since

$$f_p(z) = \sum_{p\nmid n} a(n)q^n - \epsilon_p \sum \left(\frac{n}{p}\right) a(n)q^n = 2 \sum_{\left(\frac{n}{p}\right)=-\epsilon_p} a(n)q^n,$$

we have

$$g_{p,j}(z) \equiv \sum_{\left(\frac{n}{p}\right)=-\epsilon_p} a(n)q^n \pmod{p^j}. \tag{4.9}$$

Combining (4.8) and (4.9) proves Theorem 4.4. □

Proof of Theorem 4.1. Let f and p be as in the hypotheses of Theorem 4.1, fix an integer $j \geq 1$, and let

$$g_{p,j}(z) \equiv \sum_{\left(\frac{n}{p}\right)=-\epsilon_p} a(n)q^n \pmod{p^j}$$

be the cusp form guaranteed by Theorem 4.4. Set $\kappa := k + p^\beta(p^2-1)$. By Proposition 2.13, a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have the property that

$$g_{p,j}(z)|T_{\frac{\kappa}{2},Np^2,\chi}(Q^2) \equiv 0 \pmod{p^j}.$$

If we write $g_{p,j}(z) = \sum_{n=1}^{\infty} b(n)q^n$, then

$$g_{p,j}(z)|T_{\frac{\kappa}{2},Np^2,\chi}(Q^2)$$
$$= \sum_{n=1}^{\infty} \left(b(Q^2n) + \chi(Q)\left(\frac{(-1)^{\frac{\kappa-1}{2}}n}{Q}\right) Q^{\frac{\kappa-3}{2}} b(n) + \chi(Q^2)Q^{\kappa-2} b\left(\frac{n}{Q^2}\right)\right) q^n \equiv 0 \pmod{p^j}. \tag{4.10}$$

Replacing n by Qn in (4.10), we have

$$\sum_{n=1}^{\infty} \left(b(Q^3n) + \chi(Q)\left(\frac{(-1)^{\frac{\kappa-1}{2}}Qn}{Q}\right) Q^{\frac{\kappa-3}{2}} b(Qn) + \chi(Q^2)Q^{\kappa-2} b\left(\frac{Qn}{Q^2}\right)\right) q^{Qn}$$
$$\equiv 0 \pmod{p^j}. \tag{4.11}$$

If $(Q,n)=1$, then the coefficient of q^{Qn} in (4.11) is just $b(Q^3n)$. So

$$a(Q^3n) \equiv b(Q^3n) \equiv 0 \pmod{p^j}$$

for all n coprime to Q with $\left(\frac{Q^3n}{p}\right) = -\epsilon_p$. Since $Q \equiv -1 \pmod{p}$, this completes the proof of Theorem 4.1. □

Proof of Theorem 4.2. Theorem 4.2 follows from Theorem 4.4, Proposition 3.9 and Theorem 2.11 in the same manner as the proof of Theorem 3.2. □

Proof of Theorem 4.3. Let S be a complete set of representatives for the equivalence classes of cusps under the action of $\Gamma_1(N)$. For each $s \in S$, let h_s, r_s and ψ_s be as in the definition of condition C. Let $M_s := \{m \in \mathbb{Z}_{<0} : 4m + r_s \not\equiv 0 \pmod{p} \text{ and } a_{\psi_s}(m) \neq 0\}$. By the remark following the definition of condition C, the set M_s does not depend on our choice of representative s. If p is prime and a is an integer, then $p \equiv 1 \pmod{4a}$ implies that $\left(\frac{a}{p}\right) = 1$. Set $M := \prod_{s\in S}\prod_{m\in M_s}(4m + r_s)$, and $M' := \prod_{p|MN} p$. Then for any prime $p \equiv 1 \pmod{4M'}$, we have

$$\left(\frac{h_s}{p}\right) = 1$$

for each $s \in S$, and

$$\left(\frac{4m + r_s}{p}\right) = 1$$

for each $m \in M_s$ and $s \in S$. Then f satisfies condition C for each $p \equiv 1 \pmod{4M'}$. By Dirichlet's theorem on primes in arithmetic progression, the density of such primes is $\frac{1}{\phi(4M')}$. This is a lower bound for the density of primes p for which f satisfies condition C. This completes the proof of Theorem 4.3. □

4.2 Congruences for eta-quotients

We now give the congruences for eta-quotients that are obtained using the quadratic twist method. We state an alternate form of condition C for eta-quotients, which we will call condition C_{eq}.

Suppose that $f(z) \in \mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ is an eta-quotient. Associate to each cusp $s = \frac{a}{c}$ the integers $h_s = \frac{N}{(c^2,N)}$ and r_s (defined as in 2.16) and an element $\psi_s \in G'$ with $\psi_s\infty = s$. Then by Proposition 2.8, we can write

$$f(z)|_{\frac{k}{2}}\psi_s = \sum_{n \geq n_s} a_{\psi_s}(n) q_{h_s}^{n+\frac{r_s}{4}}.$$

Definition. We say that f satisfies *condition C_{eq}* for a prime p if there exists $\epsilon_p \in \{\pm 1\}$ such that for each ψ_s, the following is true: for all $n < 0$ with $p \nmid (4n + r_s)$ and $a_{\psi_s}(n) \neq 0$, we have

$$\left(\frac{4n + r_s}{p}\right) = \epsilon_p \left(\frac{h_s}{p}\right).$$

Remark. Just as before, condition C_{eq} is well-defined.

Theorem 4.7. *Suppose p is an odd prime and N is an integer with $(N,p) = 1$. Let $f(z) = \sum_{\delta|N} \eta^{r_\delta}(\delta z)$ satisfy (3.42), and suppose that $f(z)$ has Fourier expansion $f(z) = \sum a(n)q^n$. Let $f(z)$ satisfy condition C_{eq} for p, and set $k := \sum_{\delta|N} r_\delta$.*

(a) If k is odd, then for each positive integer j, a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ have

$$a(Q^3 n) \equiv 0 \pmod{p^j}$$

for all $(n, Q) = 1$ with $\left(\frac{-n}{p}\right) = -\epsilon_p$.

(b) If k is even, then for each positive integer j,

(i) $a(n) \equiv 0 \pmod{p^j}$ for almost all n with $\left(\frac{-n}{p}\right) = -\epsilon_p$, and

(ii) a positive proportion of the primes $Q \equiv -1 \pmod{Np^j}$ *have*

$$a(Qn) \equiv 0 \pmod{p^j}$$

for all $(n, Q) = 1$ *with* $\left(\frac{-n}{p}\right) = -\epsilon_p$.

Proof. In the statement of Proposition 4.5, ξ represents the cusp $\frac{a}{cp^2}$. By Proposition 2.8, we can replace

$$f(z)|_{\frac{k}{2}}\xi = \sum_{n \geq n_\xi} a_\xi(n) q_h^{n+\frac{r}{4}}$$

with

$$f(z)|_{\frac{k}{2}}\xi = \sum_{n \geq n_\xi} a_\xi(n) q_{h_0}^{n+\frac{r}{4}}, \tag{4.12}$$

where $h_0 = \frac{N}{((cp^2)^2,N)} = \frac{N}{(c^2,N)}$, in the hypothesis of Proposition 4.5, as well as in (4.1). Then the conclusion of Proposition 4.5 becomes

$$\left(f(z) \otimes \left(\frac{\bullet}{p}\right)\right)\Big|_{\frac{k}{2}} \xi = \left(\frac{h_0}{p}\right) \sum_{n \geq n_\xi} \left(\frac{4n+r}{p}\right) a_\xi(n) q_{h_0}^{n+\frac{r}{4}}.$$

Now Proposition 4.6 holds with condition C replaced by condition C_{eq}, by substituting (4.12) in (4.6) and (4.7), and using our alternate version of Proposition 4.5. Theorem 4.7 now follows in the same manner as Theorems 4.1 and 4.2. □

4.3 A comparison of the two methods

We now consider the advantages and disadvantages of our two approaches for finding congruences. The U-operator method succeeds in establishing congruences for the coefficients of weakly holomorphic modular forms modulo any prime power p^j, where p does not divide the level. It makes no requirement on the Fourier expansions of the forms at the cusps, which is helpful in practice, since these can be difficult to compute. However, the congruences that arise are limited because they occur in only one residue class modulo some (possibly large) power of p.

The twist method yields congruences for coefficients with indices of the form Q^3n, which are not divisible by p. Hence these congruences are not limited to a particular residue class modulo p, but they exclude those coefficients whose indices are divisible by p. Also, this approach does not, in general, guarantee congruences modulo every prime power, as we see in the following example.

Example 4.1. Consider the eta-quotient

$$\frac{1}{\eta^{48}(z)} = q^{-2} + 48q^{-1} + 1244 + \cdots \in \mathcal{M}_{-24}(\Gamma).$$

Condition C_{eq} requires that $\left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)$, so we only get congruences modulo powers of p where $p \equiv 1, 7 \pmod 8$.

4.3.1 Congruences for $p(n)$

The partition function $p(n)$ provides a good test case for our two methods. Recall that the generating function

$$f(z) := \frac{1}{\eta(24z)} = \sum_{n \equiv -1 \pmod{24}} p\left(\frac{n+1}{24}\right) q^n$$

for $p(n)$ is a weakly holomorphic modular form for $\Gamma_0(576)$ with character. By (3.49), the order of vanishing of $f(z)$ at a cusp $s = \frac{a}{c}$ is -1. Recall from Example 3.1 that for each prime $\ell \geq 5$, m must satisfy $-\ell^m < -4$ in order for Theorem 3.10 to hold, so let $m \geq 1$. Then for each $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{576\ell^j}$ have the property that

$$p\left(\frac{Q^3\ell^m n + 1}{24}\right) \equiv 0 \pmod{\ell^j}$$

for all $(n, Q\ell) = 1$ with $Q^3\ell^m n \equiv -1 \pmod{24}$. These congruences are of the form guaranteed by Ono [30] and Ahlgren [1].

Turning to the twist approach, we see that for $f(z)$ to satisfy condition C_{eq} for a prime $\ell \geq 5$, there must be a choice of $\epsilon_\ell \in \{\pm 1\}$ so that

$$\left(\frac{-4}{\ell}\right) = \epsilon_\ell \left(\frac{h_s}{\ell}\right)$$

for each cusp s. Since $h_s = \frac{576}{(c^2,576)}$ and 576 is a square, each h_s must be a square, so each $\left(\frac{h_s}{\ell}\right) = 1$. Then for each prime $\ell \geq 5$, $f(z)$ satisfies condition C_{eq} for ℓ with $\epsilon_\ell = \left(\frac{-1}{\ell}\right)$. Therefore by Theorem 4.7, for each $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{576\ell^j}$ have the property that

$$p\left(\frac{Q^3 n + 1}{24}\right) \equiv 0 \pmod{\ell^j} \tag{4.13}$$

for all $(n, Q) = 1$ with $Q^3 n \equiv -1 \pmod{24}$ and $\left(\frac{-n}{\ell}\right) = -\epsilon_\ell$.

Recall from §1.1 the definition $\delta_\ell := \frac{\ell^2-1}{24}$, and the set

$$S_\ell := \left\{\beta \in \{0, 1, \ldots, \ell-1\} : \left(\frac{\beta + \delta_\ell}{\ell}\right) = 0 \text{ or } -\left(\frac{-6}{\ell}\right)\right\}.$$

Since each $Q \equiv -1 \pmod{24}$, then $Q^3 n \equiv -1 \pmod{24}$ implies that $n \equiv 1 \pmod{24}$, so we may write $n = 1 - 24s$. Then

$$\left(\frac{-n}{\ell}\right) = \left(\frac{24s-1}{\ell}\right) = \left(\frac{24s + \ell^2 - 1}{\ell}\right) = \left(\frac{24}{\ell}\right)\left(\frac{s + \delta_\ell}{\ell}\right) = \left(\frac{6}{\ell}\right)\left(\frac{\beta + \delta_\ell}{\ell}\right)$$

for $\beta \in \{0, 1, \ldots, \ell - 1\}$ with $s \equiv \beta \pmod{\ell}$. If $\left(\frac{-n}{\ell}\right) = -\left(\frac{-1}{\ell}\right)$, then $\left(\frac{\beta+\delta_\ell}{\ell}\right) = -\left(\frac{-6}{\ell}\right)$. By writing $n = 1 - 24(\beta + k\ell)$, we see that $n \equiv 1 - 24\beta \pmod{24\ell}$. Therefore, (4.13) duplicates Theorem 1.1 for all $\beta \in S_\ell$ except for the case $\beta \equiv -\delta_\ell \pmod{\ell}$.

4.3.2 Congruences for $t_1(D)$

As a second comparison, we consider the trace function $t_1(D)$ (1.2). Zagier [39] showed that values of $t_1(D)$ are generated by the weakly holomorphic modular form

$$g_1(z) = \frac{\Theta_1(z)E_4(z)}{\eta^6(4z)} = q^{-1} - 2 - \sum_{0<D\equiv 0,3 \pmod 4} t_1(D)q^D \in \mathcal{M}_{\frac{3}{2}}(\widetilde{\Gamma_0(4)}).$$

There are three equivalence classes of cusps of $\Gamma_0(4)$, represented by 0, $\frac{1}{2}$ and $\frac{1}{4}$. To calculate the order of vanishing of $g_1(z)$ at each cusp, we note that since $E_4(z)$ is holomorphic and nonvanishing at every cusp, it is sufficient to calculate the order of

$$\frac{\Theta_1(z)}{\eta^6(4z)} = \frac{\eta^2(z)}{\eta(2z)\eta^6(4z)}.$$

The information we need is gathered in the following table.

Cusp	Fourier Expansion	h
$\frac{1}{4} \sim \infty$	$q^{-1} - 2 - \cdots$	1
0	holomorphic	4
$\frac{1}{2}$	$q^{-1/4} + \cdots$	1

Let p be an odd prime. To apply Theorem 3.10, we must choose m so that $-p^m < -4$, so we need $m \geq 2$ for $p = 3$ and $m \geq 1$ for $p \geq 5$. Then for each $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{4p^j}$ have the property that

$$t_1(Q^3p^mn) \equiv 0 \pmod{p^j}$$

for all $(n, Qp) = 1$.

Now for $g_1(z)$ to satisfy condition C_{eq} for an odd prime p, we must have

$$\left(\frac{-4}{p}\right) = \epsilon_p \left(\frac{1}{p}\right) \quad \text{and} \quad \left(\frac{-1}{p}\right) = \epsilon_p \left(\frac{1}{p}\right),$$

so let $\epsilon_p = \left(\frac{-1}{p}\right)$. Then by Theorem 4.7, for each $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{4p^j}$ have the property that

$$t_1(Q^3 n) \equiv 0 \pmod{p^j}$$

for all $(n, Q) = 1$ with $\left(\frac{-n}{p}\right) = -\left(\frac{-1}{p}\right)$. This replicates the results of Theorem 1.3 for the first Hecke trace $t_1(D)$ when p is inert in $\mathbb{Q}(\sqrt{-Qn})$.

4.3.3 Congruences for $D(n)$

Finally, we recall from Example 2.7 that

$$f(z) = \sum_{n \equiv 1 \pmod{24}} D\left(\frac{n-1}{24}\right) q^n = \frac{\eta(48z)}{\eta(24z)}$$

is an eta-quotient on $\Gamma_0(1152)$. Using Theorem 2.9, we calculate that

$$\mathrm{ord}_{\frac{a}{c}}(f) = \frac{(c,48)^2}{(c^2,1152)} - \frac{2(c,24)^2}{(c^2,1152)} = \begin{cases} -1 & \text{if } 16 \nmid c, \\ 1 & \text{if } 16 \mid c. \end{cases}$$

Since we need $p \nmid 1152$, let $p \geq 5$ be prime. To apply Theorem 3.10, we must choose m so that $-p^m < -4$, so we just need $m \geq 1$. Then for each $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{1152p^j}$ have the property that

$$D\left(\frac{Qp^m n - 1}{24}\right) \equiv 0 \pmod{p^j}$$

for all $(n, Qp) = 1$. In the case $j = 1$, these congruences are guaranteed by Lovejoy [25].

If $s = \frac{a}{c}$ with $16 \nmid c$, then

$$h_s = \frac{1152}{(c^2, 1152)} = \frac{2(24)^2}{(c^2, 24^2)} = 2d^2$$

for some $d > 0$. Then for $f(z)$ to satisfy condition C_{eq} for a prime $p \geq 5$, we must have

$$\left(\frac{-1}{p}\right) = \epsilon_p \left(\frac{2d^2}{p}\right) = \epsilon_p \left(\frac{2}{p}\right).$$

Hence by Theorem 4.7, a positive proportion of the primes $Q \equiv -1 \pmod{1152p^j}$ have the property that

$$D\left(\frac{Qn-1}{24}\right) \equiv 0 \pmod{p^j}$$

for all $(n, Q) = 1$ with $\left(\frac{-n}{p}\right) = -\left(\frac{-2}{p}\right)$. This replicates the results of Ahlgren and Lovejoy [2] except for the case in which $p \mid n$.

Chapter 5

Further generalizations

In this chapter we prove some more general statements about congruences for weakly holomorphic modular forms. We extend Theorems 3.1 and 4.1 to show that simultaneous congruences exist for any finite set of weakly holomorphic modular forms for $\Gamma_1(N)$, even when the forms have different levels and weights. We first give the result found using the U-operator method.

Theorem 5.1. *Fix an integer $r \geq 1$. Associate to each $1 \leq i \leq r$ the integers k_i and $N_i \geq 1$ with $4 \mid N_i$, and a prime $p_i \nmid N_i$. Let $f_i(z) = \sum a_i(n)q^n \in \mathcal{M}_{\frac{k_i}{2}}(\widetilde{\Gamma_1(N_i)})$ for each i, and suppose that all of the coefficients of the $f_i(z)$ are contained in the ring of integers $\mathcal{O}_K$ of some number field K. If $m_1, \ldots, m_r$ are sufficiently large integers, then for any choice of integers $j_1, \ldots, j_r \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{\operatorname{lcm}(N_1, \ldots, N_r, p_1^{j_1}, \ldots p_r^{j_r})}$ have the property that*

$$a_i(Q^3 p_i^{m_i} n) \equiv 0 \pmod{p_i^{j_i}}$$

for each $1 \leq i \leq r$ and each $(n, Qp_1 \cdots p_r) = 1$.

Next we give the result for simultaneous congruences using the twist method.

Theorem 5.2. *Fix an integer $r \geq 1$. Associate to each $1 \leq i \leq r$ the integers k_i and $N_i \geq 1$ with $4 \mid N_i$, and a prime $p_i \nmid N_i$. Let $f_i(z) = \sum a_i(n)q^n \in \mathcal{M}_{\frac{k_i}{2}}(\widetilde{\Gamma_1(N_i)})$ for each i, and suppose that all of the coefficients of the $f_i(z)$ are contained in the ring of integers $\mathcal{O}_K$ of some number field K. Let each $f_i(z)$ satisfy condition C for p_i. Then for each choice of integers $j_1, \ldots j_r \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{\operatorname{lcm}(N_1, \ldots, N_r, p_1^{j_1}, \ldots, p_r^{j_r})}$*

have the property that

$$a_i(Q^3n) \equiv 0 \pmod{p_i^{j_i}}$$

for each $1 \leq i \leq r$ *and each* $(n, Q) = 1$ *satisfying the conditions* $\left\{\left(\frac{-n}{p_i}\right) = -\epsilon_{p_i} : 1 \leq i \leq r\right\}$.

5.1 Proofs of Theorem 5.1 and Theorem 5.2

To prove Theorems 5.1 and 5.2, we will need the following generalizations of Theorems 2.11 and 2.13.

Theorem 5.3. *Fix an integer* $r \geq 1$. *Associate to each* $1 \leq i \leq r$, *the integers* k_i *and* $N_i \geq 1$, *and the Dirichlet character* χ_i *modulo* N_i. *Suppose that* F *is a number field with ring of integers* $\mathcal{O}_F$, *and let* $\mathfrak{m}_i$ *be an ideal of* $\mathcal{O}_F$ *with norm* M_i *for each* i. *Then a positive proportion of the primes* $p \equiv -1 \pmod{\operatorname{lcm}(M_1, \ldots, M_r, N_1, \ldots, N_r)}$ *have the property that*

$$f(z)|T_{k_i,N_i,\chi_i}(p) \equiv 0 \pmod{\mathfrak{m}_i}$$

for every $1 \leq i \leq r$ *and each* $f(z) \in S_{k_i}(\Gamma_0(N_i), \chi_i) \cap \mathcal{O}_F[\![q]\!]$.

Proof. For each i, let $\{f_{i,1}(z), \ldots, f_{i,d(i)}(z)\}$ be all of the newforms in $S_{k_i}(\Gamma_0(N_i), \chi_i)$, and write their Fourier expansions as $f_{i,s}(z) = \sum_{n=1}^{\infty} a_{i,s}(n)q^n$ for $1 \leq s \leq d(i)$. Suppose that K is a finite extension of F containing all the coefficients of the $f_{i,s}(z)$ and all values of χ_i for each i. For each $g(z) \in S_{k_i}(\Gamma_0(N_i), \chi_i)_{\mathcal{O}_F/\mathfrak{m}_i}$, choose an $h_g(z) \in S_{k_i}(\Gamma_0(N_i), \chi_i) \cap \mathcal{O}_F[\![q]\!]$ so that

$$h_g(z) \equiv g(z) \pmod{\mathfrak{m}_i}.$$

Then by (2.24), each $h_g(z)$ has the form

$$h_g(z) = \sum \alpha(i, s, \delta) f_{i,s}(\delta z)$$

where the $\alpha(i, s, \delta)$ are algebraic. Since there are a finite number of g for each i, we can extend K so that it contains all the $\alpha(i,s,\delta)$, and choose a nonzero integer C so that $C \cdot \alpha(i,s,\delta) \in \mathcal{O}_K$ for each $\alpha(i,s,\delta)$. Set $\mathfrak{m}_i' := C\mathfrak{m}_i$ and $M_i' := CM_i$. If $p \nmid N_i$, then

$$h_g(z)|T_{k_i,N_i,\chi_i}(p) = \sum_{s=1}^{d(i)} \sum_{\delta | N_i} \alpha(i,s,\delta) a_{i,s}(p) f_{i,s}(\delta z)$$

for all g and i. Since each $f(z) \in S_{k_i}(\Gamma_0(N_i), \chi_i) \cap \mathcal{O}_F[\![q]\!]$ is congruent modulo $\mathfrak{m}_i$ to one of the $h_g(z)$, it is sufficient to show that a positive proportion of the primes $p \equiv -1 \pmod{\operatorname{lcm}(M_1, \ldots, M_r, N_1, \ldots, N_r)}$ have $a_{i,s}(p) \equiv 0 \pmod{\mathfrak{m}_i'}$ for each i and s.

Now let

$$\rho := \epsilon \oplus \bigoplus_{i,s} \rho_{i,s}$$

where $\epsilon : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to (\mathbb{Z}/\operatorname{lcm}(M_1, \ldots, M_r, N_1, \ldots, N_r)\mathbb{Z})^\times$ is defined as in the proof of Theorem 2.11, and each $\rho_{i,s} : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathcal{O}_K/\mathfrak{m}_i'\mathcal{O}_K)$ satisfies $\operatorname{Tr}(\rho_{i,s}(\operatorname{Frob}_p)) \equiv a_{i,s}(p) \pmod{\mathfrak{m}_i'}$ for each prime $p \nmid M_i'N_i$. Theorem 5.3 now follows as in the proof of Theorem 2.11. $\square$

Theorem 5.4. *Fix an $r \geq 1$ and let K be a number field. Associate to each $1 \leq i \leq r$ the integers k_i and $N_i \geq 1$ with $4 \mid N_i$. For each i, suppose that $f_1(z), f_2(z), \ldots, f_r(z)$ are cusp forms with each $f_i(z) = \sum a_i(n)q^n \in S_{\frac{k_i}{2}}(\widetilde{\Gamma_1(N_i)}) \cap \mathcal{O}_K[\![q]\!]$. For each i, let $\mathfrak{m}_i \subset \mathcal{O}_K$ be an ideal with norm M_i, and suppose that each $k_i > 3$. Then a positive proportion of the primes $p \equiv -1 \pmod{\operatorname{lcm}(M_1, \ldots, M_r, N_1, \ldots, N_r)}$ have the property that*

$$f_i(z)|T_{\frac{k_i}{2},N_i}(p^2) \equiv 0 \pmod{\mathfrak{m}_i}$$

for each $1 \leq i \leq r$.

Proof. By Proposition 2.3, we can write

$$f_i(z) = \sum_{\chi_i \bmod N_i} \alpha_{\chi_i} f_{\chi_i}(z) \tag{5.1}$$

for each i, where the α_{χ_i} are algebraic and each $f_{\chi_i}(z) \in S_{\frac{k_i}{2}}(\widetilde{\Gamma_0(N_i)}, \chi_i)$ has algebraic integer coefficients. There is a nonzero integer C such that each $C \cdot \alpha_{\chi_i}$ is an algebraic integer for all i and all χ_i. Let $\mathfrak{m}_i' := C\mathfrak{m}_i$ and $M_i' := CM_i$. Then it is sufficient to prove that a positive proportion of the primes $p \equiv -1 \pmod{\operatorname{lcm}(M_1, \ldots, M_r, N_1, \ldots, N_r)}$ have the property that

$$f_{\chi_i}(z) | T_{\frac{k_i}{2}, N_i, \chi_i}(p^2) \equiv 0 \pmod{\mathfrak{m}_i'}$$

for each i and each χ_i.

By Theorem 2.12(a), for each χ_i and each squarefree $t > 0$ we have a Shimura lift $S_t f_{\chi_i}(z) \in S_{k_i - 1}(\Gamma_0(N_i/2), \chi_i^2)$. By Theorem 5.3, a positive proportion of the primes $p \equiv -1 \pmod{\operatorname{lcm}(M_1, \ldots, M_r, N_1, \ldots, N_r)}$ have the property that

$$S_t f_{\chi_i}(z) | T_{k_i - 1, N_i/2, \chi_i^2}(p) \equiv 0 \pmod{\mathfrak{m}_i'}$$

for all i, χ_i and t. The result now follows as in the proof of Theorem 2.13. □

Proof of Theorem 5.1. Throughout the proof of Theorem 3.3, no properties of the weakly holomorphic modular form $f(z)$ were used other than those common to all elements of $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)})$. Further, the U and V operators may also be applied to elements of $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)})$. Therefore, for each i, we may choose an integer m_i large enough so that Theorem 3.3 guarantees cusp forms

$$g_{i,j}(z) \in S_{\frac{k_i}{2} + \frac{p_i^{\beta_i}(p_i^2 - 1)}{2}}(\widetilde{\Gamma_1(N_i p_i^2)}) \cap \mathcal{O}_K[\![q]\!],$$

with the property that

$$g_{i,j}(z) \equiv \sum_{\substack{n=1 \\ p_i \nmid n}}^{\infty} a_i(p_i^{m_i} n) q^n \pmod{p_i^j},$$

for each $j \geq 1$. Then by Theorem 5.4, for each r-tuple $(j_1, \ldots, j_r)$ with each $j_i \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{\operatorname{lcm}(N_1, \ldots, N_r, p_1^{j_1}, \ldots, p_r^{j_r})}$ have the property that

$$g_{i,j_i}(z) | T_{\frac{\kappa_i}{2}, N_i p_i^2}(Q^2) \equiv 0 \pmod{p_i^{j_i}},$$

where $\kappa_i := k_i + p_i^{\beta_i}(p_i^2 - 1)$, for each $1 \leq i \leq r$. Using (2.26), it follows as in the proof of Theorem 3.1 that

$$a_i(Q^3 p_i^{m_i} n) \equiv 0 \pmod{p_i^{j_i}}$$

for each $1 \leq i \leq r$ and each $(n, Qp_1 \cdots p_r) = 1$. This completes the proof of Theorem 5.1. □

Proof of Theorem 5.2. Throughout the proof of Theorem 4.4, no properties of the weakly holomorphic modular form $f(z)$ were used other than those common to all elements of $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)})$. Further, the twist operator may also be applied to elements of $\mathcal{M}_{\frac{k}{2}}(\widetilde{\Gamma_1(N)})$. Hence for each i, Theorem 4.4 guarantees cusp forms

$$g_{i,j}(z) \in S_{\frac{k_i}{2} + \frac{p_i^{\beta_i}(p_i^2-1)}{2}}(\widetilde{\Gamma_1(N_i p_i^2)}) \cap \mathcal{O}_K[\![q]\!],$$

with the property that

$$g_{i,j}(z) \equiv \sum_{\left(\frac{n}{p_i}\right) = -\epsilon_{p_i}} a_i(n) q^n \pmod{p_i^j},$$

for each $j \geq 1$. Then by Theorem 5.4, for each r-tuple $(j_1, \ldots, j_r)$ with $j_i \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{N_1 \cdots N_r p_1^{j_1} \cdots p_r^{j_r}}$ have the property that

$$g_{i,j_i}(z) | T_{\frac{\kappa_i}{2}, N_i p_i^2}(Q^2) \equiv 0 \pmod{p_i^{j_i}},$$

where $\kappa_i := k_i + p_i^{\beta_i}(p_i^2 - 1)$, for each $1 \leq i \leq r$. Using (2.26), it follows as in the proof of

Theorem 4.1 that

$$a_i(Q^3 n) \equiv 0 \pmod{p_i^{j_i}}$$

for each $1 \leq i \leq r$ and each $(n, Q) = 1$ satisfying the conditions $\left\{ \left(\frac{-n}{p_i}\right) = -\epsilon_{p_i} : 1 \leq i \leq r \right\}$.

□

Remark. Analogs of Theorems 5.1 and 5.2 may be stated for integer weight weakly holomorphic modular forms in a similar manner, but we will omit their statements.

5.2 Consequences of Theorems 5.1 and 5.2

Example 5.1. Dyson [15] defined the *rank* of a partition to be its largest part minus the number of its parts. Let $N(r, t; n)$ be the number of partitions of n whose rank is $r \pmod{t}$. For fixed $0 \leq r < t$, Bringmann and Ono [9] relate the function $N(r, t; n)$ to the coefficients of a weakly holomorphic modular form of weight 1/2 for some $\Gamma_1(N)$. Applying Theorem 3.3 to this form, they show that if $t \geq 1$ is odd and $\ell \nmid 6t$, then for all $j \geq 1$, there are infinitely many non-nested arithmetic progressions $An + B$ such that

$$N(r, t; An + B) \equiv 0 \pmod{\ell^j}.$$

These congruences may be viewed as a combinatorial decomposition of the congruences

$$p(An + B) \equiv 0 \pmod{\ell^j}$$

predicted by Corollary 1.2. Bringmann and Ono use Theorem 3.3 after making the argument that it extends to forms on $\Gamma_1(N)$ once they are written as linear combinations of forms on $\Gamma_0(N)$ with character, as in (5.1). We point out that the same result now follows directly from Theorem 5.1.

Theorems 5.1 and 5.2 lead to results like the following one for overpartitions (2.14) and

Hecke traces (1.2).

Proposition 5.5. *Let ℓ be an odd prime, and fix an integer m with $m \geq 2$ if $\ell = 3$ and $m \geq 1$ if $\ell \geq 5$.*

(a) For each integer $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{16\ell^j}$ have the property that

$$\overline{p}(Q^3\ell^m n) \equiv t_1(Q^3\ell^m n) \equiv 0 \pmod{\ell^j}$$

for all $(n, Q\ell) = 1$.

(b) For each integer $j \geq 1$, a positive proportion of the primes $Q \equiv -1 \pmod{16\ell^j}$ have the property that

$$\overline{p}(Q^3 n) \equiv t_1(Q^3 n) \equiv 0 \pmod{\ell^j}$$

for all n coprime to Q such that $\left(\frac{-n}{\ell}\right) = \left(\frac{-1}{\ell}\right)$.

Proof. Part (a) follows from Example 3.2, §4.3.2, and Theorem 5.1. For part (b), we see from (3.50) that for $\frac{\eta(2z)}{\eta^2(z)}$ to satisfy condition C_{eq} for an odd prime ℓ, we must have

$$\left(\frac{-4}{\ell}\right) = \epsilon_\ell \left(\frac{h_s}{\ell}\right)$$

for each cusp $s = \frac{a}{c}$ with $2 \nmid c$. Since $h_s = \frac{16}{(c^2,16)}$ is a square for each s, these conditions are satisfied by taking $\epsilon_\ell = \left(\frac{-1}{\ell}\right)$. By §4.3.2, $g_1(z)$ satisfies condition C_{eq} for all odd primes ℓ with the same choice of ϵ_ℓ. Therefore, Proposition 5.5 follows when Theorem 5.2 is applied to $\frac{\eta(2z)}{\eta^2(z)}$ and $g_1(z)$. □

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Stephanie Treneer was born in Seattle, Washington on May 21, 1978. She received a B.A. in Mathematics with honors from Whitman College in 2000. At the University of Illinois, she was awarded the Bateman Fellowship in Number Theory in 2005. Upon leaving the University of Illinois, her first postgraduate position will be as a John Wesley Young Instructor at Dartmouth College.