EXTENSIONS OF THE SELBERG-DELANGE METHOD

BY

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DISSERTATION

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Abstract

This dissertation involves two topics in analytic number theory. The first topic focuses on extensions of the Selberg-Delange Method, which are discussed in Chapters 2 and 3. The other topic, which is discussed in Chapter 4, is a new identity for Multiple Zeta Values.

The Selberg-Delange method is a method that is widely use to determined the asymptotic behavior of the sum of arithmetic functions whose corresponding Dirichlet's series can be written in the term of the Riemann zeta function, $\zeta(s)$. In Chapter 2, we first provide a history and recent developments of the Selberg-Delange method. Then, we provide a generalized version of the Selberg-Delange method which can be applied to a larger class of arithmetic functions. We devote Chapter 3 to the proofs of the results stated in Chapter 2.

In 1961, Matsuoka evaluated $\zeta(2)$ by means of evaluating the integral $\displaystyle\int_0^{\pi/2} x^2 \cos^{2n}(x)dx$. The last chapter of this dissertation generalize the idea of Matsuoka and obtains a new identity for Multiple Zeta Values.

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"Render therefore to all their dues: ... honour to whom honour."

– Romans 13:7, *King James Version*

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Chapter 1

Introduction

In this dissertation, we consider the study of several topics from the field of analytic number theory. Number theory is a field devoted to the study of the integers, especially prime numbers, and objects made out of integers, such as rational numbers and algebraic integers. The term analytic refers to the use of analytic tools, such as tools in real analysis, complex analysis, and, in recent years, harmonic analysis. One may wonder how tools in the "continuous" world help the study of objects in the "discrete" world. The first connection between these two worlds dates back to the $18th$ century in the time of the great mathematician Leonhard Euler. In 1737, Euler communicated a paper entitled "Variae observationes circa series infinitas" [16]. In the paper, he observed that there is a connection between an infinite product which runs over the set of prime numbers and an infinite sum that runs over the set of all natural numbers, which we quote.

Theorem 1.0.1 (Theorem 8)**.** *The expression formed from the sequence of prime numbers*

$$\frac{2^n \cdot 3^n \cdot 5^n \cdot 7^n \cdot 11^n \cdots}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\cdots}$$

has the same value as the sum of the series

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\cdots. \tag{1.0.1}$$

In this theorem, Euler was only concerned when n is a natural number. Approximately a century later, the German mathematician, Bernhard Riemann, extended the study of this connection by replacing n in (1.0.1) with a complex value s. The sum (1.0.1) is now know as the famous *Riemann Zeta Function*,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (\Re(s)>1).$$

Riemann was able to establish several analytic properties of $\zeta(s)$. Most importantly, he showed that $\zeta(s)$ can be extended as a meromorphic function to the entire complex plane with a simple pole at $s = 1$ by using the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

This analytic continuation enabled French mathematician, Jacques Hadamard, and Belgian mathematician, Chales Jean de la Vallée Poussin, to independently complete the proof of the Prime Number Theorem, which can be stated as follows.

Theorem 1.0.2 (Prime Number Theorem)**.** *Let $\pi(x)$ denote the number of prime numbers not exceeding x. Then the function $\pi(x)$ is asymptotic to $\frac{x}{\log x}$ as x tends to infinity.*

In other words the function $\pi(x)$ behaves similarly to the function $\frac{x}{\log x}$ when x is sufficiently large. Many mathematicians considers the proof of the Prime Number Theorem as the birth of the analytic number theory.

Even after the Prime Number Theorem was proved, many mathematicians continued to study analytic properties of $\zeta(s)$. The study brought light and elegance to many problems in the field, in particular the problems on the asymptotic behavior of arithmetic functions and the sum of arithmetic functions. One of the tools used in the study of the asymptotic behavior of arithmetic functions is a method called the Selberg-Delange method, which is the first topic we will study in this dissertation.

Before the discovery of the Selberg-Delange method, the main tool in the study of the asymptotic behavior of the summatory function of arithmetic functions was Perron's Formula, which can also be viewed as a special case of an inverse Mellin's transform.

Theorem 1.0.3 (Perron's Formula)**.** *Let $f(n)$ be an arithmetic function, and let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be the corresponding Dirichlet series. Assume that the Dirichlet series $F(s)$ is absolutely convergent* *for* $\Re(s) > \sigma_a$. *Then*

$$A(x) := \sum_{n<x} f(n) + \frac{1}{2} f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds$$

for $c > \sigma_a$ *and* $x > 0$.

By using Perron's Formula and analytic properties of $\zeta(s)$, especially the zero-free region of $\zeta(s)$, we are able to estimate the summatory function of arithmetic functions which can be simply expressed in term of $\zeta(s)$. For example, by using Perron's Formula on the functions $\frac{1}{\zeta(s)}$, and $\zeta(s)^2$, we can obtain estimations

$$M(x) := \sum_{n \leq x} \mu(n) = O\left(xe^{-c\sqrt{\log x}}\right), \tag{1.0.2}$$

$$D(x) := \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}), \tag{1.0.3}$$

where

$$\mu(n) = \begin{cases} 1, \text{ if } n = 1, \\ (-1)^k, \text{ if } n = p_1 p_2 \cdots p_k, \\ 0, \text{ otherwise,} \end{cases}$$

where $p_1, ..., p_k$ are distinct prime numbers. The function $d(n)$ is the number of divisors of n, γ is Euler's constant, and the notation $g(x) = O(f(x))$ means that there exists a positive constant M such that $|g(x)| \leq M\,|f(x)|$ for all x sufficiently large.

One limitation to the above method is that all the singularities of the corresponding Dirichlet series must be at most poles. For example, the method fails to give an asymptotic of the summatory function for the corresponding Dirichlet Series $F(s) = \sqrt{\zeta(s))}$.

The Selberg-Delange Method, essentially developed by Atle Selberg [52] and Hubert Delange [9] [11], is an extension of the method that is used to prove (1.0.2) and (1.0.3). The most important part of the Selberg-Delange method is that the method enables one to work with a Dirichlet series whose singularities are not poles. In particular, if the corresponding Dirichlet series admits a representation of the type

$$F(s) = H(s)\zeta(s)^z$$

for $\Re(s) > 1$, for a certain complex number z, and for an analytic function $H(s)$ that satisfies a certain rate of growth.

In recent years, several mathematicians such as Naimi and Smida [40], Lau and Wu [36], and Ben Saïd and Nicolas [2], extended the methods in several directions and in various settings. In the work of Lau and Wu, they adapted the method to give an estimate of summatory function sof the form

$$\sum_{g(n)<x} f(n),$$

where $g(n)$ is a positive real-valued multiplicative function under certain conditions, and the corresponding Dirichlet series admits a representation of the type

$$F_g(s) = H_g(s)\zeta(\theta s)^{\frac{\kappa}{\alpha^s}}$$

for $s \geq \frac{1}{\theta}$ and some fixed parameters κ and α.

In Chapter 2 and Chapter 3 of this dissertation, we extend the work of Naimi and Smida, Lau and Wu, and Ben Saïd and Nicolas to a class of summatory functions that have a corresponding Dirichlet series representation of the type

$$F_g(s) = H_g(s) \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)}$$

for $\Re(s) > \frac{1}{\theta}$, a Dirichlet character χ, a set of prime numbers $\mathcal{P}$ under certain conditions, and analytic functions $H_g(s)$ and $h_p(s)$ under certain conditions of growth rate.

In Chapter 2, we first provide a more detailed history and the most recent developments of the Selberg-Delange method. After that, we discuss in detail how we will generalize the method further. Lastly, we close Chapter 2 with statements of the two generalizations of the method. Chapter 3 is devoted entirely to the proof of these two main theorems. Due to complexity of the two theorems, we refrain from stating the full statements of these theorems at the present time.

In the same spirit as Chapter 2 and Chapter 3, the second topic of this dissertation is the asymptotic behavior of arithmetic functions in residue classes. A general question in this area is as follows: Given a positive integer N and integral-valued arithmetic function $f(n)$, how often does $f(n) \equiv a \pmod N$ for some integer a? Many mathematicians focus on a more specific question in the area, namely: What are the necessary and sufficient conditions such that $f(n)$ will fall into every residue class modulo N equally often, or in more technical terms, $f(n)$ is uniformly distributed modulo N? In 1961, Uchiyama gave such a criterion.

Theorem 1.0.4 (Uchiyama)**.** *The sequence of the integral-valued arithmetic function $\{f(n)\}$ is uniformly distributed modulo N if and only if for $r = 1, 2, ..., N-1$,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \exp(2\pi i f(n) r/N) = 0.$$

However, Uchimaya's criterion is somewhat difficult to apply in practice. In 1969, Delange gave a simpler criterions when $f(n)$ is an integral-valued additive function, such as $\omega(n)$, the number of distinct prime divisors of n [10]. This can be stated as follows.

Theorem 1.0.5 (Delange 1969)**.** *Let f be an integral-valued additive function, and let N be an integer greater than 1. The sequence $\{f(n)\}$ is uniformly distributed modulo N if and only if it satisfies one of the following conditions:*

(1) $\sum_{d \nmid f(p)} \frac{1}{p}$ *diverges for every divisor $d > 1$ of N.*

(2) $\frac{2f(2^r)}{d}$ *is an odd integer for every divisor $d > 1$ of N and every $r \geq 1$.*

At the end of Chapter 4, we extended the result of Delange to integral-valued additive functions with argument in arithmetic progression.

Theorem 1.0.6. *For every $m, q, a \in \mathbb{N}$ such that $(m, \varphi(q)) = 1$, and $0 < k < m$, let $f(n)$ be an* *integral-valued additive function such that $kf(p)$ is not a multiple of m for all primes p. Then*

$$\#\{n \leq x : n \equiv a \,(\mathrm{mod}\, q), f(n) \equiv k \,(\mathrm{mod}\, m)\} = \frac{x}{qm} + o(x).$$

Equivalently if $kf(p)$ is not a multiple of m for all prime p, then the function $f(qn+a)$ is uniformly distributed modulo m.

Many arithmetic functions may not be uniformly distributed in all residue classes, but uniformly distributed in the residue classes that are relatively prime to N. This phenomenon is known as weakly uniform distribution modulo N. Many important number theoretical functions are weakly uniformly distributed modulo N, for certain values of N. For example, J.P-Serre [53] gave necessary and sufficient conditions on N such that the sequence of Ramanujan's τ-function, $\tau(n)$ is weakly uniformly distributed modulo N. Another example is the following theorem due to Delange.

Theorem 1.0.7. *The sequence $\{d(n)\}$, where $d(n)$ is the number of divisors of n, is weakly uniformly distributed modulo N if and only if the least prime not dividing N is a primitive root modulo N.*

Another important multiplicative function is the Euler-totient function $\varphi(n)$, the number of positive integers less than n that are relatively prime to n, which we will be the main focus of our study in Chapter 4.

Theorem 1.0.8 (Narkiewicz)**.** *The sequence $\{\varphi(n)\}_n$ is weakly uniformly distributed modulo N if and only if N is relatively prime to 6.*

However, the method employed by Delange for $d(n)$ and Narkiewicz for $\varphi(n)$ does not give an asymptotic of the number n such that $d(n)$ (respectively $\varphi(n)$) $\equiv a \,(\mathrm{mod}\, N)$. Moreover, how are the values of $\varphi(n)$ distributed if N is not relatively prime to 6?

In Chapter 4, we mainly study the above questions. First, we consider the case when N is a power of 2. Since $\varphi(n)$ is always even except when $n = 1, 2$, we can disregard all the odd residue classes. We conducted a numerical experiment on how $\varphi(n)$ is distributed modulo 2^k. We find that there is a strong correlation between the number of $\varphi(n)$ congruent to a modulo 2^k and the highest power of 2 dividing a. This data led us to the proof of the following theorem.

Theorem 1.0.9. *Let k and r be positive integers and let $b \equiv 2^r a \pmod{2^k}$, where a is odd. Then*

$$\#\{n : n < x, \varphi(n) \equiv b \pmod{2^k}\} \sim \frac{3}{2^k} \frac{x(\log\log x)^{r-1}}{(r-1)! \log x}$$

as x tends to infinity.

The third and final topic in this dissertation are identities for multiple zeta values. The multiple zeta function is a generalization of $\zeta(s)$, which can be defined as follows.

Definition 1.0.10. *Let $s_1, s_2, \ldots, s_k$ be complex values such that $\Re(s_1) + \Re(s_2) + \cdots + \Re(s_m) > m$ for all $m \leq k$. We define the multiple zeta function by*

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{i=1}^{k} \frac{1}{n^{s_i}}. \tag{1.0.4}$$

Similar to $\zeta(s)$, multiple zeta functions have analytic continuations to $\mathbb{C}^k$ with possible simple poles at $s_k = 1$ and $s_j + \cdots + s_k = k - j + 2 - l$ for positive integers l and $1 \leq j < k$ [67]. If $s_1, s_2, \ldots, s_k$ are all positive integers greater than 1, then (1.0.4) are called multiple zeta values. The study of relation between multiple zeta values dates back to the the time of Euler [18, pp. $217 - 267$].

Theorem 1.0.11. *If $a, b > 1$, then*

$$\zeta(a, b) + \zeta(b, a) = \zeta(a)\zeta(b) - \zeta(a + b).$$

In particular,

$$\zeta(a, a) = \frac{1}{2}\left(\zeta^2(a) - \zeta(2a)\right).$$

Multiple zeta values also satisfy many other interesting relations. One example is the relation $\zeta(2, 1) = \zeta(3)$, which can be generalized in the following theorem.

Theorem 1.0.12 (Sum Theorem)**.** *Let n and k be natural numbers such that $n > k$. Then, for $n_1, ..., n_k \in \mathbb{N}$,*

$$\zeta(n) = \sum_{\substack{n_1 + \cdots + n_k = n \\ n_1 > 1}} \zeta(n_1, n_2, \ldots, n_k).$$

This theorem was proved for the case $k = 2$ by Euler, for $k = 3$ by Hoffman and Moen [27], and for the general case by Granville [22]. In Chapter 5, we derive a new identity for multiple zeta values using a similar idea to that used to evaluate $\zeta(2)$ by Matsuoka [39]. In particular, we prove the following theorem.

Theorem 1.0.13. *For any positive integer m, we have*

$$c_{m,0} + \sum_{l=1}^{m} \left(c_{m,l} \sum_{i=1}^{l} \sum_{r_1+\cdots+r_i=l} \zeta(2r_1, \ldots, 2r_i) \right) = 0.$$

where

$$c_{m,l} = (-1)^l \frac{\pi^{2(m-l)}}{2^{2m}} \frac{(2m)!}{(2(m-l)+1)!}.$$

Chapter 2

Selberg-Delange Method

2.1 Origin of the Method

At the turn of the 19*th* century, when Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved the Prime Number Theorem, Edmund Landau, a German mathematician, published an influential book "Handbuch der Lehre von der Verteilung der Primzahlen" [35], also know as "Landau's Primzaheln," for short. In "Primzahlen," Landau discussed the techniques that Hadamard and Vallée-Poussin used in their applications to the Prime Number Theorem. One of the applications concerns the behavior of the cardinality of the set of the natural numbers which have exactly v distinct prime factors, which can be stated as follows [35, p. 211].

Theorem 2.1.1. *(Landau) Let $\rho_k(x)$ be the number of integers $\leq x$ that are divisible by exactly k distinct primes, each occurring in any multiplicity. Then*

$$\rho_k(x) \sim \frac{1}{(k-1)!}\frac{x(\log\log x)^{k-1}}{\log x}.$$

In this theorem, Landau established the asymptotic of $\rho_k(x)$ for a fixed value of k. But, what will happen if the value of k is growing as a function of x? A similar question also appeared in Ramanujan's Lost Notebook, which we quote. [45, p. 337].

Entry 1. *$\varphi(x)$ is the number of numbers (not exceeding x) whose number of prime divisors doesn't exceed k.*

$$\varphi(x) \sim \frac{x}{\log x}\left(1 + \frac{\log\log x}{1!} + \frac{(\log\log x)^2}{1!} + \cdots + \frac{(\log\log x)^{[k]}}{[k]!}\right).$$

This is true when k is infinite. Is this true when k is a function of x?

This question was first answered by L.G. Sathe in the series of papers [48], [49], [50], [51]. Sathe's proof was rather complicated and very involved. Later in the same year, Atle Selberg [52] gave a much simpler proof of the theorem.

Theorem 2.1.2. *If $k \sim c \log\log x$ and $c < 2$, then*

$$\rho_k(x) \sim f(c) \prod_p \left(1 + \frac{c}{(p+c)(p-1)}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!},$$

where

$$f(c) = \frac{1}{\Gamma(c+1)} \left(\prod_p \left(1 - \frac{1}{p}\right) e^{1/p}\right)^c \prod_p \left(\left(1 + \frac{k}{p}\right) e^{-k/p}\right).$$

Even though Selberg's goal was to give an alternative approach to estimating $\rho_k(x)$, his method also applies to more general arithmetic functions, such as the sum of divisors function $\sigma(n)$. The main idea of Selberg was to create a Dirichlet series associated with the relevant arithmetical function and study the behavior of the series around the pole $s = 1$. This idea was later extended by Hubert Delange, [9] [11]. This method is now know as the Selberg-Delange Method, which is the main focus of our study.

2.2 The Development of the Method

Throughout this chapter, we will adopt the following notation. Let $s = \sigma + it$ be a complex number with real part σ and imaginary part t. Let $f(n)$ be a complex-valued function, not necessary multiplicative, which we want to study, and $f^+(n)$ be a positive real-valued function. Finally denote a Dirichlet series corresponding to $f(n)$ as

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

The core of Selberg's method relies on the following theorem, which we paraphrase in order to fit our future definitions.

Theorem 2.2.1. *Let*

$$F(s) = H(s, z)\zeta(s)^z,$$

where

$$H(s,z)=\sum_{n=1}^{\infty}\frac{b_z(n)}{n^s}\quad(\sigma>1),$$

and let

$$\sum_{n=1}^{\infty}|b_z(n)|\,\frac{(\log 2n)^{B+\delta}}{n}\tag{2.2.1}$$

be uniformly bounded for $|z|\le B$*. Then*

$$A(x):=\sum_{n\le x}f(n)=\frac{H(1,z)}{\Gamma(z)}\frac{x}{(\log x)^{1-z}}+O\left(\frac{x}{(\log x)^{2-z}}\right)\quad(x\to\infty),$$

uniformly for $|z|\le B$*, as* x *tend to infinity.*

In addition, Selberg proved the following theorem.

Theorem 2.2.2. *Under the assumptions of Theorem 2.2.1, let*

$$a_z(n)=\sum_{n=1}^{\infty}c_k(n)z^k,\qquad |z|\le A,$$

be an arithmetic function depending on a parameter z*. Moreover, if the second derivative of* $\frac{H(1,z)}{\Gamma(1+z)}$ *is uniformly bounded for* $|z|\le A$*, we have*

$$C_k(x):=\sum_{n\le x}c_k(n)=\frac{H\left(1,\frac{k-1}{\log\log x}\right)}{\Gamma\left(1+\frac{k-1}{\log\log x}\right)}\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}+O\left(\frac{x}{\log x}\frac{k^2(\log\log x)^{k-3}}{(k)!}\right),$$

uniformly for $k<(2-\delta)\log\log x$*.*

By letting $f(n)=z^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n, and applying Theorem 2.2.1 and Theorem 2.2.2, Selberg obtained Theorem 2.1.2.

Early applications of Selberg's theorems were used mostly for classes of functions $f(n)=z^{\alpha(n)}$, where $\alpha(n)$ is an additive function. In 1971 Delange [11] extended the result to the class of functions $f(n)=\chi(n)z^{\alpha(n)}$, where $\chi(n)$ is a Dirichlet character.

Before discussing the next development, we denote

$$\mathcal{D}_c:=\left\{s:\sigma\ge 1-\frac{c}{\log(3+|t|)}\right\},$$

the regionon which we will focus. Due to complexity and various parameters, Gerald Tenenbaum [57] has formulated the following terminology in order to apply Theorem 2.2.1 more effectively.

Definition 2.2.3. *Let $z \in \mathbb{C}, c_0 > 0, 0 \leq \delta < 1,$ and $M > 0$. We say that a Dirichlet series $F(s)$ has property $\mathcal{P}(z; c_0, \delta, M)$ if the following conditions hold.*

(1) $F(s)$ admits a representation of the type

$$F(s) = H(s, z)(\zeta(s))^z \tag{2.2.2}$$

for $\sigma \geq 1$.

(2) The function $H(s, z)$ in equation (2.2.2) is a complex-valued analytic function on the region $\mathcal{D}_{c_0}$, and satisfiies the inequality

$$|H(s, z)| \leq M(3 + |t|)^{\delta}. \tag{2.2.3}$$

Definition 2.2.4. *If the Dirichlet series $F(s)$ has property $\mathcal{P}(z; c_0, \delta, M)$, then we say $F(s)$ has property $\mathcal{P}^+(z, w; c_0, \delta, M)$ if there exists a positive real-valued function $f^+(n)$, such that $|f(n)| \leq f^+(n)$ for all $n \in \mathbb{N}$ and $F^+(s) = \sum_{n=1}^{\infty} \frac{f^+(n)}{n^s}$ has property $\mathcal{P}(w; c_0, \delta, M)$.*

Tenenbaum replaced the assumption on the convergence condition of the function (2.2.1) and it's derivatives by the analytic continuation of $H(s, z)$. The new theorem can be stated as follows.

Theorem 2.2.5. *Let $F(s)$ be a Dirichlet series that has property $\mathcal{P}^+(z, w; c_0, \delta, M)$. For $x \geq 3, N \geq 0, A > 0,$ and $|z|, |w| \leq A$, we have*

$$A(x) := \sum_{n \leq x} f(n) = \frac{x}{(\log x)^{1-z}} \left(\sum_{k=0}^{N} \frac{\lambda_k(z)}{(\log x)^k} + O_{c_0,\delta,A}\left(MR_N(x)\right) \right),$$

with

$$\lambda_k := \frac{1}{\Gamma(z-k)} \sum_{l+j=k} \frac{1}{l!j!} \gamma_j(z) \frac{d^l}{ds^l}\left(H(1; z)\right),$$

where

$$\gamma_j(z) = \frac{d^j}{ds^j}(s^{-1}((s-1)\zeta(s))^z)$$

and

$$R_N(x) = e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N + 1}{\log x}\right)^{N+1}$$

for some positive constants c_1 and c_2.

In 1996, M. Naimi and H. Smida [40] were able to replace the constant z in equation (2.2.2) by an analytic complex-valued function satisfying a certain rate of growth. We can state their theorem as follows.

Theorem 2.2.6. *Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be a Dirichlet series that has property $\mathcal{P}(h(s); c_0, \delta, M)$, and assume that for some $0 < \alpha < 1$*

$$|h(s)| \leq M \log(3 + |t|)^{\alpha} \quad (t \in \mathbb{R})$$

and $1 - \frac{c}{\log(3+|t|)} \leq \sigma \leq 2$.

Then there exists a polynomial $P_k(x)$ with degree at most k such that, uniformly for $N \geq 1$ and $x \geq 3$,

$$\sum_{n \leq x} f(n) = \frac{x}{(\log x)^{1-h(1)}} \left(\sum_{k=0}^{N} \frac{P_k(\log\log x)}{\log^k x} + O\left(R_N\right) \right)$$

with

$$R_N = M e^{-c_1\sqrt{\log x}} + M(c_2 N + 1)^{2N} \left(\frac{\log\log x}{\log x}\right)^{N+1}$$

for some constants c_1 and c_2 which depend only on c, α, δ and M.

Several years later Yuk-kam Lua and Jie Wu [36] gave another variation of Theorem 2.2.1. Their main purpose was to obtain the asymptotic behavior of the general sum

$$\sum_{g(n) < x} f(n)$$

where both $f(n)$ and $g(n)$ are multiplicative functions. Their main theorem can be stated as follows.

Theorem 2.2.7. *Let $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$. Suppose that $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{R}^+$ are two multiplicative functions such that for all primes p:*

(1) $|f(p)-\kappa| < \frac{c_1}{p^{\eta}}$, *for* $\eta > 0, C_1 \geq 0$ *and* $|\kappa| < \frac{1}{\eta}$,

(2) $g(p) = \alpha p^{\theta}$ *or* $g(p) = \alpha p^{\theta} + \alpha' p^{\theta'} + t(p)$, *where* $\left|t^{(l)}(u)\right| \leq (C_2 l + 1)^l u^{\theta''-l}$ *for* $\theta > 0$, $C_1 \geq 0, \alpha > 0, \alpha' \neq 0$ *and* $\theta > \theta' > \theta''$,

(3) $\displaystyle\sum_{\nu=2}^{\infty} \frac{|f(p^{\nu})|}{g(p^{\nu})^{1/\tilde{\theta}}} \leq \frac{C_3}{p^{\psi}}$, *where* $C_3 > 0, \psi > 1$, *and* $\tilde{\theta} > \theta$.

Then for any positive integer N, *we have*

$$A_g(x) = \sum_{g(n)<x} f(n) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\kappa/\alpha^{1/\theta}}} \left(\sum_{k=1}^{N} \frac{P_k(\log\log x)}{(\log x)^k} + O\left(R_{N,\lambda}(x)\right) \right),$$

where $P_k(x) = \displaystyle\sum_{l=1}^{k} \lambda_{k,l} x^l$ *and the coefficients* $\lambda_{k,l}$ *are given by*

$$\lambda_{k,l} := \frac{\theta^{-\kappa/\alpha^{\frac{1}{\theta}}}}{l!} \sum_{m=l}^{k} \sum_{n=l}^{m} \sum_{i=0}^{m-l} \lambda^*_{m,n,i},$$

where

$$\lambda^*_{m,n,i} := \frac{(-\log\alpha)^m (\kappa/\alpha^{1/\theta})^n (-\log\theta)^{n-l-i} a_{k,l} b_{n,m-n}}{(n-l-i)! i! \Gamma_i(\kappa/\alpha^{1/\theta} - k)}$$

where a_n *is the* n*-th coefficient of the Laurent series expansion of the function*

$$s^{-1} F_g(s) (\theta s - 1)^{\kappa/\alpha^s}$$

at $s = \frac{1}{\theta}$, *and where*

$$b_{m,n} := \sum_{n_1+n_2+\cdots+n_m=n} \frac{1}{(n_1+1)! \cdots (n_m+1)!}.$$

The error term $R_{N,\lambda}(x)$ *is given by*

$$R_{N,\lambda}(x) := \left((c_1 N + 1) \frac{\lambda \log\log x + c_2}{\log x} \right)^{N+1} + e^{-c_3 \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}},$$

with $0 < \lambda < 1$ *and for some constants* c_1, c_2, *and* c_3.

In the same year, Fethi Ben Saïd and Jean-Louis Nicolas [2] introduced the use of a Dirichlet character to obtain an asymptotic of certain arithmetic functions, with a restriction on the primes dividing n are in certain sets of arithmetic progressions. Their theorem can be stated as follows.

Theorem 2.2.8. *Let b be a positive integer, ξ be a Dirichlet character modulo b and $J \subset (\mathbb{Z}/k\mathbb{Z})^*$. Let $g(n)$ be a multiplicative function such that $g(n) > 0$ and $g(n) \to \infty$. Let $a_{J,\xi}(n)$ ($b_{J,\xi}(n)$, respectively) be a complex-valued multiplicative function (respectively, real-valued). Suppose that for all $n \geq 1$, $|a_{J,\xi}(n)| \leq b_{J,\xi}(n)$ and the series*

$$F_{g,J,\xi}(s) = \sum_{n=1}^{\infty} \frac{a_{J,\xi}(n)}{g(n)^s}$$

and

$$F^{+}_{g,J,\xi}(s) = \sum_{n=1}^{\infty} \frac{b_{J,\xi}(n)}{g(n)^s}$$

are analytic on the half plane $\sigma > 1$. Also suppose that there exist three real constants $B > 0$, $0 < c < \frac{1}{2}, 0 \leq \delta < 1$ and functions $(f_j(s))_{j \in J}$ and $f^+(s)$, analytic on the domain $\mathcal{D}_c$ such that

$$\max\{|f_j(s)|, f^+(s)\} \leq B(\log(3 + |t|)^{\delta}$$

for $j \in J$ and $s \in \mathcal{D}_c$. Also suppose that in a half-plane $\sigma > 1$ the series $F_{g,J,\xi}(s)$ admits a representation of the type

$$F_{g,J,\xi}(s) = H_{g,J,\xi}(s) \prod_{j \in J} \prod_{p \equiv j \pmod k} \left(1 - \frac{\xi(p)}{p^s}\right)^{-f_j(s)},$$

where $H_{g,J,\xi}(s)$ is analytic on $\mathcal{D}_c$, and satisfies the inequality

$$|H_{g,J,\xi}(s)| \leq B(3 + |t|)^{\delta}.$$

Similarly, in a half plane $\sigma > 1$ the series $F^{+}_{g,J,\xi}(s)$ admits a representation of the type

$$F^{+}_{g,J,\xi}(s) = H^{+}_{g,J,\xi}(s)\zeta(s)^{f^{+}(s)},$$

where $H^{+}_{g,J,\xi}(s)$ is analytic on $\mathcal{D}_c$ and satisfies the inequality

$$\left|H^{+}_{g,J,\xi}(s)\right| \leq B(3+|t|)^{\delta}.$$

Let

$$f(s) = \frac{1}{\varphi(k)} \sum_{j\in J} f_j(s).$$

Then for a non-principal character ξ,

$$A_{g,J,\xi}(x) := \sum_{g(n)<x} a_{J,\xi}(n) = O\left(x \frac{\log\log x}{(\log x)^2}\right),$$

and for a principal character ξ_0

$$A_{g,J,\xi_0}(x) = \frac{x}{(\log x)^{1-f(1)}} \left(\frac{H_{g,J,\xi_0}(1) C_{J,k}}{\Gamma(f(1))} + O\left(\frac{\log\log x}{\log x}\right)\right),$$

where the constant $C_{J,k}$ is defined by

$$C_{J,k} = \prod_{j\in J} \left(\prod_{p\equiv j \pmod k} \left(1-\frac{1}{p}\right)^{-f_j(1)} \prod_{p} \left(1-\frac{1}{p}\right)^{-f_j(1)/\varphi(k)} \prod_{p|b, p\equiv j \pmod k} \left(1-\frac{1}{p}\right)^{f_j(1)} \right).$$

The proof of Theorem 2.2.8 relies on another result of Naimi and Smida which can be stated as follows.

Theorem 2.2.9 (Theorem A, [2]). *Let $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ have property $\mathcal{P}^{+}(h, h^{+}, c, \delta, M)$, where $h(s)$ and $h^{+}(s)$ are analytic on $\mathcal{D}_c$ and satisfy the inequality*

$$\max\{|h(s)|, \left|h^{+}(s)\right|\} \leq \log(3+|t|)^{\delta}.$$

Also, let $A_g(x) = \sum_{n\geq 1, g(n)\leq x} f(n)$. Then

$$A_g(x) = x(\log x)^{h(1)-1} \left(\frac{H_h(1)}{\Gamma(h(1))} + O\left(\frac{\log\log x}{\log x}\right)\right).$$

We are unable to locate the paper of Naimi and Smida, so we will give a proof of a stronger version of this theorem in the next chapter.

2.3 Modifications

In this section, we will combine the ideas of Lau and Wu [36], Naimi and Smida [40], and Ben Saïd and Nicolas [2] with some of our modifications to obtain a more general version of Selberg's Theorem. We will use Definition 2.2.3, Definition 2.2.4, and Theorem 2.2.5 of Tenenbaum as the base of our modifications. As usual, we let $s = \sigma + it$.

2.3.1 Modification on the Dirichlet Series

The first modification we make is on the summatory function $A(x) = \sum_{n<x} f(n)$. We define an analogue,

$$A_g(x) := \sum_{\substack{n \geq 1 \\ g(n) < x}} f(n)$$

for some function $g(n)$. Following Selberg's idea, we need to create an associated Dirichlet series for $A_g(x)$, namely,

$$F_g(s) := \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}.$$

It is evident that one must put conditions on $F_g(s)$ and $g(n)$. In Selberg's proof, there is a part where he applies Perron's formula to the corresponding Dirchlet series. Thus, we need to find conditions such that we are able to apply analogues of Perron's formula. For this reason we need $F_g(s)$ and $g(n)$ to satisfy the following conditions.

(1) The Dirichlet series $F_g(s)$ have a finite abscissa of convergence.

(2) The function $g(n)$ is a real-valued multiplicative function such that $g : \mathbb{N} \to [1, \infty)$, and

$$g(n) \to \infty \tag{2.3.1}$$

as n tends to infinity.

(3) The limit superior of the ratio

$$\limsup_{n\to\infty} \frac{\log n}{\log g(n)}$$

exists and is non-negative.

In the paper of Ben Saïd and Nicolas [2], they only assume condition (2). We believe that this condition is not sufficient to carry out the proof without assuming that the Dirichlet series is absolutely convergent at some real number $\sigma > 0$.

2.3.2 Modification on the Representation of $F_g(s)$

The next modification we make is to the equation (2.2.2). We divide these modifications into two stages. Later on, we will state a result corresponding to each stage. For the current discussion, we assume that all functions have an analytic continuation and do not vanish on some region $\mathcal{D}$. We will discuss this region in more detail in the next subsection.

First Modification

In the first stage, similar to Lau and Wu [36], we introduce a parameter $\theta > 0$. We replace $\zeta(s)$ in equation (2.2.2) with $\zeta(\theta s)$. Next, similar to Naimi and Smida [40], we replace a complex constant z by an analytic function $h(s)$ such that

$$|h(s)| \leq M(\log(3+|t|))^{\alpha}$$

for some positive constant M and $0 \leq \alpha < 1$ in the region $\mathcal{D}$. With these modifications, the analogue of equation (2.2.2) can be written as

$$F_g(s) = H_g(s,h;\theta)\zeta(\theta s)^{h(s)} \tag{2.3.2}$$

where $H_g(s,h;\theta)$ has an analytic continuation to region $\mathcal{D}$ and satisfies the inequality

$$H_g(s,h;\theta) \leq M(3+|t|)^{\delta}$$

for some $0 \leq \delta < 1$ and $M > 0$ in the region $\mathcal{D}$.

Second Modification

In the second modification, we will modify the equation (2.3.2). We will first adopt the idea of Ben Saïd and Nicolas [2] and generalize it further. In the paper of Ben Saïd and Nicolas, they replaced $\zeta(s)^z$ in the equation (2.2.2) by

$$\prod_{j\in J}\prod_{p\equiv j\,(\mathrm{mod}\,k)}\left(1-\frac{\xi(p)}{p^s}^{-h_j(s)}\right), \tag{2.3.3}$$

where $J\subseteq(\mathbb{Z}/k\mathbb{Z})^{\times}$, χ is a Dirichlet character modulo q, and $h_j(s)$ are analytic functions satisfying a certain rate of growth. In the light of Ben Saïd and Nicolas, we wish to generalize (2.3.3) to a product on a certain set of primes $\mathcal{P}$, more precisely

$$\prod_{p\in\mathcal{P}}\left(1-\frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)}.$$

But some sets of prime numbers or some choices of $h_p(s)$ do not possess certain analytic properties that we need. Thus, we need to put conditions on $\mathcal{P}$ and $h_p(s)$. For the conditions on $\mathcal{P}$, first, let χ be a Dirichlet character modulo q and let $\tilde{q}$ be a multiple of q. Let $\pi(a,\tilde{q},x)$ denote the number of primes less than x and congruent to a modulo $\tilde{q}$. Next, let $\lambda:(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}\to[0,1]$. For all $\eta>\frac{1}{\theta}-\frac{c}{\theta\log 3}$ and for all $a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}$,

$$\#\left\{p: p<x, p\in\mathcal{P}, p\equiv a\,(\mathrm{mod}\,\tilde{q})\right\}=\lambda(a)\pi(a,\tilde{q},x)+O(x^{\eta}). \tag{2.3.4}$$

We will see later that the error term x^{η} in (2.3.4) is the best possible. Next, for the condition $h_p(s)$, we first introduce a function $h(s)$ analytic in the region $\mathcal{D}_{c,\theta}$. Also for all $s\in\mathcal{D}_{c,\theta}$

$$\max\{|h_p(s)|,|h(s)|\}\le M(\log(3+|t|))^{\alpha}$$

for $0\le\alpha<1$ and uniformly for all $p\in\mathcal{P}$. Next, we define a region

$$K_T:=K_{c,\theta,T}=\left\{s: s\in\bar{\mathcal{D}}_{c,\theta}, \sigma\le\frac{2}{\theta}, |t|\le T\right\}.$$

For all $\eta > \frac{1}{\theta} - \frac{c}{\theta \log 3}$,

$$\sum_{\substack{p \in \mathcal{P} \\ p < x}} \|h_p - h\|_{\infty,K} \leq x^{\eta},$$

where

$$\|f\|_{\infty,K} = \sup_{s \in K_T} |f(s)| \,.$$

In other words, on average the function $h_p(s)$ should behave similarly to $\lambda(p)\chi(p)h(s)$ on compact set K_T for all T.

2.3.3 Modification on the Analytic Region

The next modification is on the analytic region. The first necessary condition is that $\zeta(\theta s)$ needs to have an analytic continuation to the region with an exception of a simple pole at $s = \frac{1}{\theta}$. We also need $\zeta(\theta s)$ to not vanish in the region. Moreover, in order to compromise with equation (2.3.4), we also need $L(\theta s, \xi)$, where ξ is a Dirichlet character modulo $\tilde{q}$, to have an analytic continuation to the region with an exception of a simple pole at $s = \frac{1}{\theta}$, and not vanish in the region. The analyticity of $\zeta(\theta s)$ and $L(\theta s, \xi)$ can be showed by their functional equations which were proved by Riemann [46] and Hurwitz [29, pp 72-88], respectively. All that remains is to consider the zero-free regions of $\zeta(\theta s)$ and $L(\theta s, \xi)$. The first result along this line was first proved by de la Vellée Poussin [7]. In 1899, de la Vellée Poussin showed that there exists a constant $c_1 > 0$ such that the Riemann Zeta function $\zeta(s)$ does not vanish in the region

$$\sigma > 1 - \frac{c_1}{\log |t|}$$

for sufficiently large t. This was improved by several people. First, Littlewood [37] showed that there exists a constant c_2 such that $\zeta(s)$ dose not vanish in the region

$$\sigma > 1 - \frac{c_2 \log \log |t|}{\log |t|}.$$

Later Chudakov [4] extended the region to

$$\sigma > 1 - \frac{c_3}{(\log |t|)^{\frac{3}{4}+\epsilon}}$$

for some constant $c_3 > 0$. The most recent result was given by Korobov [31] and Vinogradov [61], who independently showed that there exists a constant $c_4 > 0$ such that $\zeta(s)$ does not vanish in the region

$$\sigma > 1 - \frac{c_4}{(\log |t|)^{\frac{2}{3}} (\log \log |t|)^{\frac{1}{3}}}$$

for sufficiently large t. These results also hold for $L(s, \chi)$ for a fixed χ. For our purpose, we will only use the classical zero-free region of de la Vallée Poussin. For $\theta > 0$, we define a region $\mathcal{D}_{c,\theta}$ to be the classical zero-free region on of $\zeta(\theta s)$, more precisely,

$$\mathcal{D}_{c,\theta} := \left\{ s : \sigma > \frac{1}{\theta} - \frac{c}{\theta \log(\max\{3, \theta |t|\})} \right\}. \tag{2.3.5}$$

One may ask, how large is the constant c_1 in the zero-free region of de la Vallée Poussin? de la Vallée Poussin showed that one can take $c_1 = \frac{1}{34.82}$. This result was recently improved by Kaidiri [30] to $c_1 = \frac{1}{5.69663}$. For our purposes, we will assume that $c_1 < \frac{1}{2}$.

One of the consequences of working in the larger analytic and zero-free region is the improvement on the error terms in the asymptotic formula for $A_g(x)$. We define $Err(x)$ to be an increasing function such that

$$|\pi(x) - Li(x)| \leq M x Err(x)^{-1}.$$

The function $Err(x)$ depends on the zero-free region of $\zeta(s)$; in our case

$$Err(x) = e^{c_5 \sqrt{\log x}}$$

for some constant $c_5 > 0$.

Remark: If one uses the region of Korobov and Vinogradov, then we can effect an improvement,

$$Err(x) = \exp\left(\frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right).$$

Moreover, under the assumption of the Riemann Hypothesis, von Koch [62] showed that

$$Err(x) = \frac{\sqrt{x}}{\log x}.$$

2.4 Statement of the Results

First, we will introduce a notation. For $z_0 \in \mathbb{C}$ and any positive integer k, we define

$$\frac{1}{\Gamma_k(z_0)} := \frac{d^k}{dz^k}\left(\frac{1}{\Gamma(z)}\right)\bigg|_{z=z_0} = \frac{k!}{2\pi i}\oint_\gamma \frac{1}{\Gamma(z)(z-z_0)^{k+1}}dz. \tag{2.4.1}$$

We are now introduce definitions analogue to Definition 2.2.3 and Defintion 2.2.4.

Definition 2.4.1. *Let* $0 < c < \frac{1}{2}, \theta > 0, M \geq 0, 0 \leq \delta < 1, 0 \leq \alpha < 1,$ *and* $\kappa \geq 0$*, and let* $h(s)$ *be an analytic function in* $\mathcal{D}_{c,\theta}$ *where*

$$\mathcal{D}_{c,\theta} = \left\{ s : \sigma > \frac{1}{\theta} - \frac{c}{\theta \log(\max\{3, |\theta t|\})} \right\}.$$

We say that a Dirichlet series $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *with a finite abscissa of convergence has property* $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$ *if the following conditions hold.*

(1) $g(n)$ *is a real-valued multiplicative function such that* $g : \mathbb{N} \to [1, \infty)$*,* $g(n)$ *tends to infinity as* n *tends to infinity, and*

$$\limsup_{n\to\infty} \frac{\log n}{\log g(n)} = \kappa$$

for some constant $\kappa \geq 0$*.*

(2) $F_g(s)$ *admits a representation of the type*

$$F_g(s) = H_g(s, h; \theta)\zeta(\theta s)^{h(s)}$$

for $\sigma > \frac{1}{\theta}$*.*

(3) The function $H_g(s, h; \theta)$ *has analytic continuation to the region* $\mathcal{D}_{c,\theta}$ *and satisfies the inequal-*

ity

$$|H_g(s,h;\theta)| \le M(3+|t|)^{\delta}$$

for all $s \in \mathcal{D}_{c,\theta}$.

(4) *The function* $h(s)$ *is analytic in the region* $\mathcal{D}_{c,\theta}$ *and satisfies the inequality*

$$|h(s)| \le M \log(3+|t|)^{\alpha}$$

for all $s \in \mathcal{D}_{c,\theta}$.

Definition 2.4.2. *We say that a function* $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *has property* $\mathcal{B}^+(h, h^+; c, \theta, M, \delta, \alpha, \kappa)$ *if* $F_g(s)$ *have property* $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$ *and there exists a positive real-valued function* $f^+(n)$, *such that* $|f(n)| \le f^+(n)$ *for all* $n \in \mathbb{N}$ *and* $F_g^+(s) = \displaystyle\sum_{n=1}^{\infty} \frac{f^+(n)}{g(n)^s}$ *has property* $\mathcal{B}(h^+; c, \theta, M, \delta, \alpha, \kappa)$.

Now, we are ready to state a stronger version of Theorem 2.2.9.

Theorem 2.4.3. *Let* $0 < c < \frac{1}{2}, \theta > 0, M \ge 0, 0 \le \delta < 1, 0 \le \alpha < 1,$ *and* $\kappa \ge 0$, *and let* $h(s)$ *and* $h^+(s)$ *be analytic functions in* $\mathcal{D}_{c,\theta}$

where

$$\mathcal{D}_{c,\theta} = \left\{ s : \sigma > \frac{1}{\theta} - \frac{c}{\theta \log(\max\{3, |\theta t|\})} \right\}.$$

Let a Dirichlet series $F_g(s) = \displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *have property* $\mathcal{B}^+(h, h^+; c, \theta, M, \delta, \alpha, \kappa)$. *Also, let*

$$A(s) := s^{-1} H_g(s,h;\theta)((\theta s - 1)\zeta(\theta s))^{h(s)}.$$

Then, uniformly for $N \ge 1$ *and* $x \ge 3$, *we have*

$$A_g(x) := \sum_{g(n)<x} f(n) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} \left(\sum_{m=0}^{N} \frac{P_m(\log\log x)}{(\log x)^m} + O\left(R_N\right) \right),$$

where

$$P_m(x) := \sum_{j=0}^{m} \sum_{n=j}^{m} \frac{e_{m,n}}{\theta^{h\left(\frac{1}{\theta}\right)}} \sum_{i=0}^{n-j} \binom{n}{i} \binom{n-i}{j} (\log\theta)^{n-i-j} \frac{(-1)^{j+i}}{\Gamma_i\left(h\left(\frac{1}{\theta}\right) - m\right)} x^j,$$

$$e_{m,n} := \frac{(-1)^n}{n!} \sum_{k=n}^{m} \frac{A^{(m-k)}\left(\frac{1}{\theta}\right) a_{k,n}}{(m-k)!},$$

$$a_{k,n} := \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_i \geq 1}} \prod_{i=1}^{n} \frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!},$$

and

$$R_N := \left(\frac{c_1 N + 1}{\log x}\left(c_2 + \log\log x\right)\right)^{N+1} + M Err(x)^{-c_3}$$

where

$$Err(x) = e^{c_4 \sqrt{\log x}}$$

for some positive constants c_1, c_2, c_3 *and* c_4 *depend on* $c, M, \delta, \theta,$ *and* α.

In particular, for $N = 1$,

$$A_g(x) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} \left(\frac{A\left(\frac{1}{\theta}\right)}{\theta^{h\left(\frac{1}{\theta}\right)} \Gamma\left(h\left(\frac{1}{\theta}\right)\right)} + O\left(\frac{\log\log x}{\log x}\right)\right). \tag{2.4.2}$$

By letting $\theta = 1$ in (2.4.2), we obtain Theorem 2.2.9.

Now, for the next theorem, we will introduce another definitions analogue to Definition 2.2.3 and Defintion 2.2.4.

Definition 2.4.4. *Let* $0 < c < \frac{1}{2}, \theta > 0, M \geq 0, 0 \leq \delta < 1, 0 \leq \alpha < 1,$ *and* $\kappa \geq 0$. *Let* $\mathcal{P}$ *be a set of prime numbers,* χ *be a Dirichlet character modulo* q, $\tilde{q}$ *be a positive integer divisible by* q *and define a function* $\lambda : (\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times} \to [0,1]$. *And lastly, let* $h(s)$ *and for all primes* $p \in \mathcal{P}$ *let* $h_p(s)$ *a complex-valued functions analytic in the region* $\mathcal{D}_{c,\theta}$ *where*

$$\mathcal{D}_{c,\theta} = \left\{ s : \sigma > \frac{1}{\theta} - \frac{c}{\theta \log(\max\{3, |\theta t|\})} \right\}.$$

We say that a Dirichlet series $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *has property* $\mathcal{A}(h_p, h; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$ *if the following conditions hold.*

(1) The function $g(n)$ *is a multiplicative function such that* $g : \mathbb{N} \to [1, \infty)$, $g(n)$ *tends to infinity as* n *tends to infinity, and*

$$\limsup_{n\to\infty} \frac{\log n}{\log g(n)} = \kappa$$

for some constant $\kappa \geq 0$*.*

(2) $F_g(s)$ *admits a representation of the type*

$$F_g(s) = H_g(s) \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)}$$

for $\sigma > \frac{1}{\theta}$*.*

(3) For all $\eta > 1 - \frac{c}{\log 3}$ *and for each* $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}$*,*

$$\mathcal{P}_{a,\tilde{q}}(x) := \#\left\{p : p < x, p \in \mathcal{P}, p \equiv a \,(\operatorname{mod} \tilde{q})\right\} = \lambda_{\tilde{q}}(a)\pi(a, \tilde{q}, x) + O(x^{\eta})$$

where

$$\pi(a, \tilde{q}, x) = \#\left\{p : p < x, p \equiv a \,(\operatorname{mod} \tilde{q})\right\}.$$

(4) The function $H_g(s)$ *has an analytic continuation to the region* $\mathcal{D}_{c,\theta}$ *and satisfies the inequality*

$$|H_g(s)| \leq M(3 + |t|)^{\delta}$$

for all $s \in \mathcal{D}_{c,\theta}$*.*

(5) The function $h_p(s)$ *and* $h(s)$ *are analytic in the region* $\mathcal{D}_{c,\theta}$ *and satisfies the inequality*

$$\max\{|h(s)|, |h_p(s)|\} \leq M \log(3 + |t|)^{\alpha_1}$$

for all $s \in \mathcal{D}_{c,\theta}$*, for all* $p \in \mathcal{P}$*.*

(6) For all sufficiently large T*, and for all* $\eta > \frac{1}{\theta} - \frac{c}{\theta \log 3}$*,*

$$\sum_{\substack{p \in \mathcal{P} \\ p < x}} \|h_p - h\|_{\infty, K_T} \leq C x^{\eta},$$

where constant C *depends on* $\mathcal{P}$ *and* η *and*

$$K_T := K_{c,\theta,T} = \left\{s : s \in \bar{\mathcal{D}}_{c,\theta}, \sigma \leq \frac{2}{\theta}, |t| \leq T\right\}.$$

and

$$\|f\|_{\infty,K} = \sup_{s\in K_T} |f(s)|\,.$$

Definition 2.4.5. *We say a function* $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *has property* $\mathcal{A}^+(h_p, h, h^+; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$ *if* $F_g(s)$ *has property* $\mathcal{A}(h_p, h; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$ *and there exists a positive real-valued function* $f^+(n)$*, such that* $|f(n)| \le f^+(n)$ *for all* $n \in \mathbb{N}$*, with the following properties holds.*

(1) $F_g^+(s) = \sum_{n=1}^{\infty} \frac{f^+(n)}{g(n)^s}$ *admits a representation of the type*

$$F_g^+(s) = H_g^+(s)\zeta(\theta s)^{h^+(s)},$$

for $\sigma > \frac{1}{\theta}$.

(2) The function $H_g^+(s)$ *has an analytic continuation to the region* $\mathcal{D}_{c,\theta}$ *and satisfies the inequality*

$$\left|H_g^+(s)\right| \le M(3+|t|)^{\delta}$$

for all $s \in \mathcal{D}_{c,\theta}$.

(3) $h^+(s)$ *is analytic in the region* $\mathcal{D}_{c,\theta}$ *and satisfies the inequality*

$$\left|h^+(s)\right| \le M\log(3+|t|)^{\alpha}.$$

for all $s \in \mathcal{D}_{c,\theta}$.

We are now ready to state our main result.

Theorem 2.4.6. *Let* $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *have property* $\mathcal{A}^+(h_p, h, h^+; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$*. Define*

$$T(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)} \zeta(\theta s)^{-\tilde{\lambda} h(s)},$$

where

$$\tilde{\lambda} = \sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}} \lambda(a)\chi(a),$$

and

$$A(s) := s^{-1}H_g(s)\left((\theta s-1)\zeta(\theta s)\right)^{\tilde{\lambda}h(s)}.$$

Then, for $x \geq 3$ and $N \geq 0$,

$$A_g(x) := \sum_{g(n)<x} f(n) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\tilde{\lambda}h\left(\frac{1}{\theta}\right)}}\left(\sum_{m=0}^{N}\frac{P_m(\log\log x)}{(\log x)^m} + O\left(R_N\right)\right) \tag{2.4.3}$$

where

$$P_m(x) := \sum_{j=0}^{m}\sum_{n=j}^{m}\frac{e_{m,n}}{\theta^{\tilde{\lambda}h\left(\frac{1}{\theta}\right)}}\sum_{i=0}^{n-j}\binom{n}{i}\binom{n-i}{j}(\log\theta)^{n-i-j}\frac{(-1)^{j+i}}{\Gamma_i\left(\tilde{\lambda}h\left(\frac{1}{\theta}\right)-m\right)}x^j, \tag{2.4.4}$$

$$e_{m,n} = \frac{(-1)^n}{n!}\sum_{k=n}^{m}\sum_{l=0}^{m-k}\binom{m-k}{l}T^{(l)}\left(\frac{1}{\theta}\right)\frac{A^{(m-k-l)}\left(\frac{1}{\theta}\right)a_{k,n}}{(m-k)!},$$

$$a_{k,n} = \tilde{\lambda}^n \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_i\geq 1}}\prod_{i=1}^{n}\frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!},$$

and

$$R_N = \left(\frac{c_1N+1}{\log x}\left(c_2+\log\log x\right)\right)^{N+1} + M\,Err(x)^{-c_3}, \tag{2.4.5}$$

where

$$Err(x) = e^{c_4\sqrt{\log x}}$$

for some positive constants c_1, c_2, c_3 and c_4. For our convenience, we define $\frac{1}{\Gamma_i(m)} = 0$ for integers $m \leq 0$ and all i.

In most known applications, $h_p(s) = z$, where z is a complex number for all primes p. In this setting, by choosing $h(s) = z$ we can show that

$$a_{k,n} = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We can reduce from equation (2.4.3) to

$$A_g(x) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\tilde{\lambda}h\left(\frac{1}{\theta}\right)}} \left(\sum_{m=0}^{N} \frac{\beta_m(z)}{(\log x)^m} + O\left(R_N^*\right) \right) \tag{2.4.6}$$

where

$$\beta_m(z) := \frac{1}{\theta^z \Gamma\left(\tilde{\lambda}z - m\right)} \sum_{k=0}^{m} \sum_{l=0}^{m-k} \binom{m-k}{l} T^{(l)}\left(z\right) \frac{A^{(m-k-l)}(z)}{(m-k)!} \tag{2.4.7}$$

and

$$R_N^* = \left(\frac{c_1 N + 1}{\log x}\right)^{N+1} + MErr(x)^{-c_3}. \tag{2.4.8}$$

In some applications, such as [20], interested in the secondary term of the asymptotic expansion of $A_g(s)$ of specific function $f(s)$. By letting $N = 1$, (2.4.3) can be written as

$$\begin{aligned} A_g(x) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\tilde{\lambda}h\left(\frac{1}{\theta}\right)}} & \left(\frac{A\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right)}{\Gamma\left(\tilde{\lambda}h\left(\frac{1}{\theta}\right)\right)} + \frac{\tilde{\lambda}h'\left(\frac{1}{\theta}\right) A\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right)}{\log(x)\Gamma_1\left(\tilde{\lambda}h\left(\frac{1}{\theta}\right) - 1\right)} \right. \\ & \left. + \frac{A\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right)\left(\tilde{\lambda}h'\left(\frac{1}{\theta}\right)\log\frac{\log x}{\theta} + 1\right) + A'\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right) + A\left(\frac{1}{\theta}\right) T'\left(\frac{1}{\theta}\right)}{\log(x)\Gamma\left(\tilde{\lambda}h\left(\frac{1}{\theta}\right) - 1\right)} + O\left(R_1\right) \right). \end{aligned} \tag{2.4.9}$$

Moreover, if $h(s) = z$ is a complex constant, we can simplify (2.4.9) further to obtain

$$A_g(x) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\tilde{\lambda}z}} \left(\frac{A\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right)}{\Gamma\left(\tilde{\lambda}z\right)} + \frac{A\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right) + A'\left(\frac{1}{\theta}\right) T\left(\frac{1}{\theta}\right) + A\left(\frac{1}{\theta}\right) T'\left(\frac{1}{\theta}\right)}{\log(x)\Gamma\left(\tilde{\lambda}z - 1\right)} + O\left(R_1\right) \right). \tag{2.4.10}$$

Chapter 3

Modification of the Selberg-Delange Method

3.1 Preliminary Results for Theorem 2.4.3

In this section, we will give the proofs of several lemmas which are necessary to prove Theorem 2.4.3. The first lemma, which we are proving, is concerned about the existence of the abscissa of absolute convergence of the Dirichlet series $F_g(s)$.

Lemma 3.1.1. *Let $g(n)$ be a real-valued function such that $g : \mathbb{N} \to [1, \infty)$, $g(n)$ tends to infinity as n tends to infinity, and*

$$\limsup_{n\to\infty} \frac{\log n}{\log g(n)} = \kappa,$$

where κ is a non-negative real number. We also let $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ have a finite abscissa of convergence σ_c. Then the abscissa of absolute convergence, σ_a, exists and satisfies

$$\sigma_c \le \sigma_a \le \sigma_c + \kappa.$$

Proof. Fix $g(n)$ as in the theorem and assume that $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ has a finite abscissa of convergence σ_c. Let $\epsilon > 0$. By definition of σ_c, the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)^{\sigma_c+\epsilon}}$$

converges. It follows that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)^{\sigma_c+\epsilon}} = 0.$$

Hence, there exists an N_ϵ such that, for all $n \geq N_\epsilon$,

$$\left| \frac{f(n)}{g(n)^{\sigma_c+\epsilon}} \right| < 1. \tag{3.1.1}$$

Next, since

$$\limsup_{n\to\infty} \frac{\log n}{\log g(n)} = \kappa,$$

for some $\kappa \geq 0$, we see that for $\delta > 0$, there exists N_δ such that for all $n \geq N_\delta$,

$$\log n \leq (\kappa + \delta) \log g(n).$$

Exponentiating both sides, we find that

$$n \leq g(n)^{\kappa+\delta}, \tag{3.1.2}$$

for all $n \geq N_\delta$. Now let $N = \max\{N_\epsilon, N_\delta\}$. By (3.1.1) and (3.1.2), we see that, for $n > N$,

$$\left| \frac{f(n)}{g(n)^{\sigma_c+\epsilon+\kappa+\delta+\epsilon(\kappa+\delta)}} \right| = \frac{|f(n)|}{g(n)^{\sigma_c+\epsilon}} \cdot \frac{1}{g(n)^{(\kappa+\delta)(1+\epsilon)}} < \frac{1}{\left(g(n)^{(\kappa+\delta)}\right)^{(1+\epsilon)}} \leq \frac{1}{n^{1+\epsilon}}.$$

It follows that for $\sigma \geq \sigma_c + \epsilon + \kappa + \delta + \epsilon(\kappa + \delta)$,

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{g(n)^s} = \sum_{n=1}^{N} \frac{|f(n)|}{g(n)^s} + \sum_{n=N}^{\infty} \frac{|f(n)|}{g(n)^s} \leq \sum_{n=1}^{N} \frac{|f(n)|}{g(n)^s} + \sum_{n=N}^{\infty} \frac{1}{n^{1+\epsilon}}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$$

is absolutely convergence for $\sigma \geq \sigma_c + \epsilon + \kappa + \delta + \epsilon(\kappa + \delta)$. Since this is true for any $\epsilon > 0$ and $\delta > 0$, then

$$\sigma_a \leq \sigma_c + \kappa.$$

This completes the proof. $\square$

Another essential component of the proof is the behavior of $F_g(s)$ when $s \in D_{c,\theta}$.

Lemma 3.1.2. *Let* $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *have property* $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$ *and let* $B > \frac{1}{\theta}$. *Then*

$$|F_g(s)| \leq M_{c,M,\delta,\theta,\alpha,B}(3 + |t|)^{\frac{\delta+1}{2}}$$

for $s \in \mathcal{D}_{c,\theta}, \sigma < B$ *and* $\left|s - \frac{1}{\theta}\right| > \frac{c}{\theta \log 3}$.

Proof. Let $F_g(s)$ have property $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$. Then there exists $H_g(s, h; \theta)$ such that

$$F_g(s) = H_g(s, h; \theta)\zeta(\theta s)^{h(s)} \tag{3.1.3}$$

and

$$|H_g(s, h; \theta)| \leq M(3 + |t|)^{\delta}. \tag{3.1.4}$$

Next, we need to obtain an upper bound for $\left|\zeta(\theta s)^{h(s)}\right|$. First, recall bounds of the Riemann Zeta function [58, p. 49, Theorem 3.5]. Uniformly for $1 - \frac{c}{\log(3+|t|)} \leq \sigma \leq 2$, and $t > t_c$ for some $t_c > 0$

$$|\zeta(s)| \leq C_1 \log(3 + |t|),$$

where C_1 is a positive constant depending on c. Since $\zeta(s)$ converges for $\sigma \geq 2$, thus uniformly for $1 - \frac{c}{\log(3+|t|)} \leq \sigma \leq B$, and $t > t_{c,B}$

$$|\zeta(s)| \leq C_2 \log(3 + |t|),$$

where C_1 is a positive constant depending on c and B. It follows that

$$|\zeta(\theta s)| \leq C_3 \log(3 + |t|), \tag{3.1.5}$$

where C_1 is a positive constant depending on c, B, and θ, uniformly for $\frac{1}{\theta} - \frac{c}{\theta \log(3+|t|)} \leq \sigma \leq B$, and $t > t_{c,\theta,B}$. Next, by property (4) of $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$, we see that

$$|h(s)| \leq M \left(\log(3 + |t|)\right)^{\alpha}, \tag{3.1.6}$$

for $0 \leq \alpha < 1$. By (3.1.5) and (3.1.6), for $t > t_{c,\theta,B}$,

$$\begin{aligned}
\left|\zeta(\theta s)^{h(s)}\right| &= |\zeta(\theta s)|^{|\Re h(s)|} e^{-Arg(\zeta(\theta s))|\Im h(s)|} \\
&\leq \exp\left\{|h(s)| \left(\log(\zeta(\theta s)) + \pi\right)\right\} \\
&\leq \exp\left\{M \left(\log\left(3+|t|\right)\right)^{\alpha} \left(\log\log(3+|t|) + \log(C_3) + \pi\right)\right\} \\
&\leq \exp\left\{C_4 \left(\log\left(3+|t|\right)\right)^{\alpha} \log\log(3+|t|)\right\},
\end{aligned}$$

where C_1 is a positive constant depending on c, B, and θ. Since $\alpha < 1$, then there exists $t_{c,\theta,\alpha,B}$ such that for $t > t_{c,\theta,\alpha,B}$,

$$C_4 \left(\log\left(3+|t|\right)\right)^{\alpha} \log\log(3+|t|) \leq \frac{1-\delta}{2} \log(3+|t|).$$

Thus, for $t > t_{c,\theta,\alpha,B}$,

$$\left|\zeta(\theta s)^{h(s)}\right| \leq \exp\left\{\frac{1-\delta}{2} \log(3+|t|)\right\} \leq \left(\log(3+|t|)\right)^{\frac{1-\delta}{2}}. \tag{3.1.7}$$

By combining (3.1.3), (3.1.4), and (3.1.7), for $t > t_{c,\theta,\alpha,B}$, we obtain

$$\begin{aligned}
|F_g(s)| &= |H_g(s, h; \theta)| \left|\zeta(\theta s)^{h(s)}\right| \\
&\leq M \log(3+|t|)^{\delta} \left(\log(3+|t|)\right)^{\frac{1-\delta}{2}} \\
&\leq M \left(\log(3+|t|)\right)^{\frac{1+\delta}{2}}.
\end{aligned}$$

Hence, for $t > 0$,

$$|F_g(s)| \leq M_{c,M,\delta,\theta,\alpha,B}(3+|t|)^{\frac{\delta+1}{2}},$$

as desired. This completes the proof. □

One crucial part of the proof of Theorem 2.4.3 is to establish an analogue of the effective form of Perron's formula. We now introduce an analogue of a normalized summatory function,

$$A_g^{\star}(x) = \sum_{n \geq 1, g(n) < x} f(n) + \frac{1}{2} \sum_{n \geq 1, g(n) = x} f(n)$$

and the function

$$\alpha(x) = \begin{cases} 1, & x > 1, \\ \frac{1}{2} & x = 1, \\ 0, & 0 < x < 1. \end{cases}$$

Lemma 3.1.3. *For any positive* $c, T,$ *and* T', *we have*

$$\left| \alpha(x) - \frac{1}{2\pi i} \int_{c-iT'}^{c+iT} x^s \frac{ds}{s} \right| \le \frac{x^c}{2\pi |\log x|} \left(\frac{1}{T} + \frac{1}{T'} \right) (x \ne 1) \tag{3.1.8}$$

and

$$\left| \alpha(1) - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} x^s \frac{ds}{s} \right| \le \frac{c}{T+c} \quad (x = 1). \tag{3.1.9}$$

The proof of this lemma can be found in Tenenbaum [57, p. 131] .

Theorem 3.1.4 (Analogue of Perron's Formula)**.** *Let* $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ *has abscissa of convergence* σ_c *and abscissa of absolute convergence* σ_a *such that*

$$\sigma_c \le \sigma_a \le \sigma_c + \kappa$$

for some $\kappa \ge 0$*. Let* $\xi > \max\{0, \sigma_c\}$*. Assume that for* $\sigma \ge \xi$,

$$|F_g(s)| \le M t^{\delta}$$

for some $0 \le \delta < 1$ *and positive constant* M *depending on* ξ*. Denote the set* $g(\mathbb{N})$ *as the image of the function* $g(n)$*. Then*

$$A_g^{\star}(x) := \sum_{n \ge 1, g(n) < x} f(n) + \frac{1}{2} \sum_{n \ge 1, g(n) = x} f(n) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} F_g(s) x^s s^{-1} ds, \tag{3.1.10}$$

where the integral is conditionally convergent for $x \in \mathbb{R} \backslash g(\mathbb{N})$ *and convergent in the sense of Cauchy's Principle Value for* $x \in g(\mathbb{N})$*.*

Proof. Fix $F_g(s), \sigma_c, \sigma_a, \kappa, \delta,$ and M as in the statement of the theorem. First, we will prove the theorem for $\sigma > \sigma_a$. Suppose $\xi > \sigma_a$. Since $F_g(s)$ is absolutely and uniformly convergent for

$\sigma \geq \sigma_a + \epsilon$ for a fixed $\epsilon > 0$, we are able to interchange the sum and the integral of the right hand side of the equation (3.1.10). We obtain

$$\frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT} F_g(s)\frac{x^s}{s}ds = \frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT}\sum_{n=1}^{\infty}\frac{f(n)}{g(n)^s}\frac{x^s}{s}ds = \frac{1}{2\pi i}\sum_{n=1}^{\infty} f(n)\int_{\xi-iT'}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s\frac{ds}{s}.$$

Then by (3.1.8), for $x \in \mathbb{R}^+\backslash g(\mathbb{N})$, we see that

$$\begin{aligned}
\left|A_g^\star(x) - \frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT} F_g(s)\frac{x^s}{s}ds\right| &= \left|\sum_{n\geq 1, g(n)<x} f(n) - \sum_{n=1}^{\infty} f(n)\frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s\frac{ds}{s}\right| \\
&= \left|\sum_{n=1}^{\infty} f(n)\left(\alpha\left(\frac{x}{g(n)}\right) - \frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s\frac{ds}{s}\right)\right| \\
&\leq \sum_{n=1}^{\infty}|f(n)|\left|\left(\alpha\left(\frac{x}{g(n)}\right) - \frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s\frac{ds}{s}\right)\right| \\
&\leq \sum_{n=1}^{\infty}|f(n)|\frac{x^\xi}{2\pi g(n)^\xi\left|\log\frac{x}{g(n)}\right|}\left(\frac{1}{T}+\frac{1}{T'}\right) \\
&\leq \frac{x^\xi}{2\pi}\left(\frac{1}{T}+\frac{1}{T'}\right)\sum_{n=1}^{\infty}\frac{|f(n)|}{g(n)^\xi\left|\log\frac{x}{g(n)}\right|}. \qquad (3.1.11)
\end{aligned}$$

Now, since $x \in \mathbb{R}^+\backslash g(\mathbb{N})$, there exists a constant $C_{x,g} > 0$ such that

$$\left|\frac{1}{\log\frac{x}{g(n)}}\right| \leq C_{x,g}$$

for all $n \geq 1$. Therefore, from equation (3.1.11),

$$\left|A_g^\star(x) - \frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT} F_g(s)\frac{x^s}{s}ds\right| \leq \frac{x^\xi}{2\pi}C_{x,g}\left(\frac{1}{T}+\frac{1}{T'}\right)\sum_{n=1}^{\infty}\frac{|f(n)|}{g(n)^\xi}.$$

for $x \in \mathbb{R}^+\backslash g(\mathbb{N})$. Next, since $\xi > \sigma_a$, the series $\sum_{n=1}^{\infty}\frac{|f(n)|}{g(n)^\xi}$ converges. Thus

$$\left|A_g^\star(x) - \frac{1}{2\pi i}\int_{\xi-iT'}^{\xi+iT} F_g(s)\frac{x^s}{s}ds\right| \leq M_1 x^\xi\left(\frac{1}{T}+\frac{1}{T'}\right) \qquad (3.1.12)$$

for some constant $M_1 > 0$. By letting T and T' tend to infinity independently, we complete the proof the first assertion of the theorem for $\xi > \sigma_a$.

For the second assertion, it is sufficient to take $T = T'$. By proceeding in the same manner as in the proof of the first assertion, and using (3.1.9) as the upper bound when $g(n) = x$, we can show that

$$\begin{aligned}
\left| A_g^\star(x) - \frac{1}{2\pi i}\int_{\xi-iT}^{\xi+iT} F_g(s)\frac{x^s}{s}ds \right| &= \left| \sum_{\substack{n\geq 1\\ g(n)\leq x}} f(n) + \frac{1}{2}\sum_{\substack{n\geq 1\\ g(n)=x}} f(n) - \sum_{n=1}^{\infty} f(n)\frac{1}{2\pi i}\int_{\xi-iT}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s \frac{ds}{s} \right| \\
&= \left| \sum_{n=1}^{\infty} f(n)\left(\alpha\left(\frac{x}{g(n)}\right) - \frac{1}{2\pi i}\int_{\xi-iT}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s \frac{ds}{s}\right)\right| \\
&\leq \sum_{\substack{n\geq 1\\ g(n)\neq x}} |f(n)| \left| \left(\alpha\left(\frac{x}{g(n)}\right) - \frac{1}{2\pi i}\int_{\xi-iT}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s \frac{ds}{s}\right)\right| + \\
&\quad + \sum_{\substack{n\geq 1\\ g(n)=x}} |f(n)| \left| \left(\alpha\left(1\right) - \frac{1}{2\pi i}\int_{\xi-iT}^{\xi+iT}\left(\frac{x}{g(n)}\right)^s \frac{ds}{s}\right)\right| \\
&\leq \sum_{\substack{n\geq 1\\ g(n)\neq x}} |f(n)| \frac{x^\xi}{2\pi g(n)^\xi \left|\log\frac{x}{g(n)}\right|}\frac{2}{T} + \sum_{\substack{n\geq 1\\ g(n)=x}} |f(n)| \left(\frac{\xi}{\xi+T}\right).
\end{aligned}$$

Next, note that since the domain of $g(n)$ is the set of nature numbers and $g(n)$ tends to infinity as n tends to infinity, then for any x there are only finitely many values of n such that $g(n) = x$. Thus,

$$\sum_{\substack{n\geq 1\\ g(n)=x}} |f(n)|$$

is a finite sum. Therefore, by (3.1.12),

$$\left| A_g^\star(x) - \frac{1}{2\pi i}\int_{\xi-iT}^{\xi+iT} F_g(s)\frac{x^s}{s}ds \right| \leq M_1 x^\xi \frac{2}{T} + M_2\left(\frac{\xi}{\xi+T}\right)$$

for some positive constants M_1 and M_2'. Thus by letting T tend to infinity, we complete the proof of the theorem for $\xi > \sigma_a$.

Now, suppose that $\sigma_c < \xi \leq \sigma_a$. By Lemma 3.1.1, $\xi + \kappa > \sigma_a$. Consider a rectangular contour integral

$$\frac{1}{2\pi i}\int_{\mathcal{R}} F_g(s)\frac{x^s}{s}ds,$$

where $\mathcal{R}$ is the positively oriented rectangle with vertices $(\xi \pm iT)$ and $(\xi + \kappa \pm iT)$. Since $|F_g(s)| \ll t^\delta$ for $0 \le \delta < 1$. Thus the contribution of a horizontal segment is

$$\left| \frac{1}{2\pi i} \int_{\xi \pm iT}^{\xi+\kappa \pm iT} F_g(x) \frac{x^s}{s} ds \right| \ll (T)^\delta \frac{x^{\xi+\kappa}}{T} \ll_x \frac{1}{T^\epsilon}.$$

for some $\epsilon > 0$, as T tends to infinity. Thus by the residue theorem,

$$\frac{1}{2\pi i} \int_{\xi - iT}^{\xi+iT} F_g(x) \frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{\xi+\kappa - iT}^{\xi+\kappa+iT} F_g(x) \frac{x^s}{s} ds + O\left(\frac{1}{T^\epsilon}\right).$$

Letting T tends to infinity, we find that

$$\frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi+i\infty} F_g(x) \frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{\xi+\kappa - i\infty}^{\xi+\kappa+i\infty} F_g(x) \frac{x^s}{s} ds.$$

Hence

$$\left| A_g^\star(x) - \frac{1}{2\pi i} \int_{\xi - iT}^{\xi+iT} F_g(s) \frac{x^s}{s} ds \right| = \left| A_g^\star(x) - \frac{1}{2\pi i} \int_{\xi+\kappa - iT}^{\xi+\kappa+iT} F_g(s) \frac{x^s}{s} ds \right|.$$

Since $\xi + \kappa > \sigma_a$, by proceeding in the same way as in the proof for the case $\xi > \sigma_a$, the theorem follows for the case $\sigma_c < \xi \le \sigma_a$. $\square$

The equation (3.1.10) of the analogue of the Perron's Formula is insufficient to prove Theorem 2.4.3. We need a more effective version of Theorem 3.1.4, which can be stated as follows.

Theorem 3.1.5. *Let $F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}$ have abscissa of convergence σ_c and abscissa of absolute convergence σ_a such that*

$$\sigma_c \le \sigma_a \le \sigma_c + \kappa$$

for some $\kappa \ge 0$. Assume that for $\sigma \ge \xi$,

$$|F_g(s)| \le Mt^\delta$$

for some $0 \le \delta < 1$ and positive constant M depending on ξ. Define

$$A_g(x) := \sum_{\substack{n \ge 1 \\ g(n) < x}} f(n).$$

Then

$$\int_1^x A_g(t)dt = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)x^{s+1}\frac{ds}{s(s+1)}. \tag{3.1.13}$$

Proof. Let $w \geq 0$ and $x \in \mathbb{R}^+ \backslash g(\mathbb{N})$. First note that

$$F_g(s) = \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s} = \sum_{n=1}^{\infty} \frac{f(n)g(n)^w}{g(n)^{s+w}}.$$

By Theorem 3.1.4 for $s' = s + w$, we see that

$$\sum_{\substack{n\geq 1\\ g(n)<x}} f(n)g(n)^w = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)\frac{x^{s+w}}{(s+w)}ds. \tag{3.1.14}$$

Also note that

$$\sum_{\substack{n\geq 1\\ g(n)<x}} f(n)x^w = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)\frac{x^{s+w}}{s}ds. \tag{3.1.15}$$

Therefore by subtracting (3.1.14) from (3.1.15), we obtain

$$\begin{aligned}\sum_{\substack{n\geq 1\\ g(n)<x}} f(n)\left(x^w - g(n)^w\right) &= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)\frac{x^{s+w}}{s}ds - \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)\frac{x^{s+w}}{(s+w)}ds \\ &= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)\frac{x^{s+w}w}{s(s+w)}ds. \qquad (3.1.16)\end{aligned}$$

The equation (3.1.16) still holds if $x \in g(\mathbb{N})$. Thus letting $w = 1$, we find that

$$\int_1^x A_g(t)dt = \sum_{\substack{n\geq 1\\ g(n)<x}} f(n)\left(x - g(n)\right) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F_g(s)\frac{x^{s+1}}{s(s+1)}ds.$$

This completes the proof. □

Another important estimation we will need is an estimation of a truncated Hankel contour integral. The Hankel contour $\mathcal{H}(a, r)$ is a path formed by joining the circle of radius r and center at a, excluding the point $s = a - r$, and the segment $(-\infty, a - r]$ traced twice as shown in Figure 3.1.

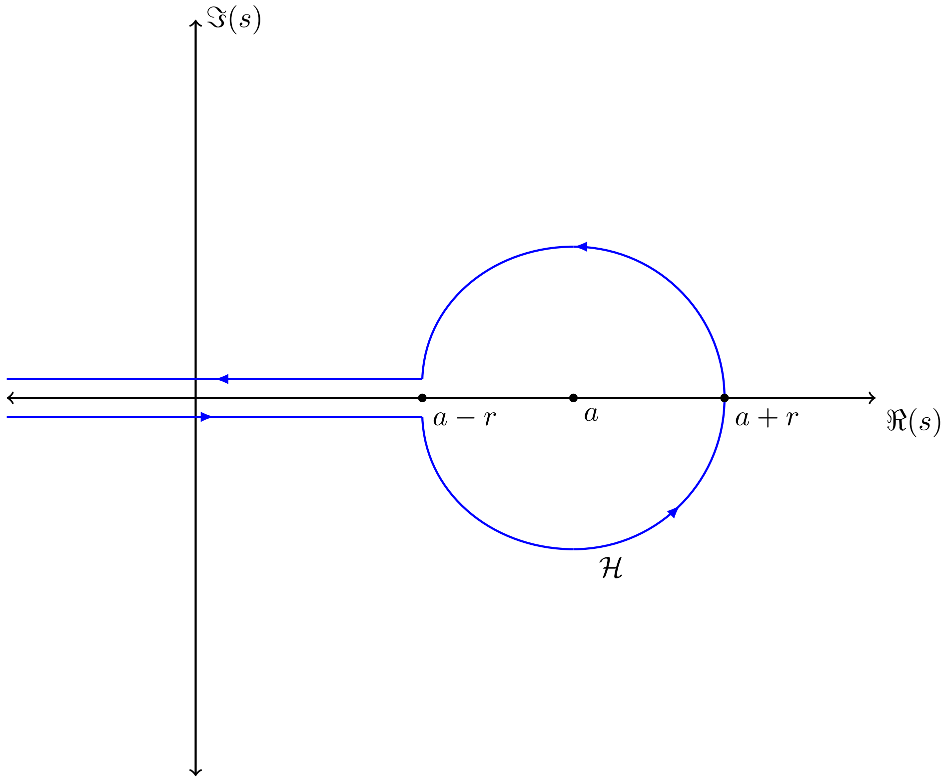

Figure 3.1: Hankel's contour center at a with radius r

One important fact about a Hankel contour integral is its connection to $\Gamma(z)$, which can be seen in the following theorem.

Theorem 3.1.6 (Hankel's Formula)**.** *For any complex number z and positive integer k, we have*

$$\frac{1}{\Gamma_k(z)} := \frac{d^k}{dz^k}\left(\frac{1}{\Gamma(z)}\right) = \frac{(-1)^k}{2\pi i}\int_{\mathcal{H}(0,r)} s^{-z}e^s(\log s)^k ds.$$

The proof of the theorem can be find in [40, pp. $11-12$]. Now, we define a truncated Hankel contour $\mathcal{H}(a,r,X)$ to be the part of the contour $\mathcal{H}(a,r)$ where $\sigma > -x$. By using Theorem 3.1.6, we obtain the following corollary.

Corollary 3.1.7. *Let $X > 1$, and let k and m be non-negative integers such that $k < m$. For an arbitrary complex number z, we have*

$$\frac{(-1)^k}{2\pi i}\int_{\mathcal{H}(0,r,X)} s^{m-z}e^s(\log s)^k ds = \frac{1}{\Gamma_k(z-m)} + E_{k,m,z}(X),$$

where

$$|E_{k,m,z}| \leq \frac{e^{\pi|\Im(z)|}}{\pi} \int_X^\infty \rho^{m-\Re(z)} e^{-\rho} \left|\log(\rho+\pi)\right|^k d\rho.$$

Proof. By Thoerem 3.1.6, we see that

$$\frac{(-1)^k}{2\pi i} \int_{\mathcal{H}(0,r,X)} s^{m-z} e^s (\log s)^k ds + \frac{(-1)^k}{2\pi i} \int_{\substack{s=\rho e^{\pm i\pi} \\ \rho \geq X}} s^{m-z} e^s (\log s)^k ds = \frac{1}{\Gamma_k(z-m)}.$$

Then for $\rho > 1$,

$$\begin{aligned} |E_{k,m,z}(X)| :&= \left| \frac{(-1)^k}{2\pi i} \int_{\substack{s=\rho e^{\pm i\pi} \\ \rho \geq X}} s^{m-z} e^s (\log s)^k ds \right| \\ &\leq \frac{1}{2\pi} \int_{\rho \geq |X|} \left|s^{m-z}\right| e^{|s|} \left|\log s\right|^k |ds| \\ &\leq \frac{1}{\pi} \int_X^\infty \rho^{m-\Re(z)} e^{\pi|\Im(z)|-\rho} \left|\log(\rho+\pi)\right|^k d\rho. \end{aligned}$$

This completes the proof. □

Lastly before we prove Theorem 2.4.3, we need to establish a series expansion of $F_g(s)$ near $s = \frac{1}{\theta}$. We will break up this process into several lemmas. But, first we will prove a lemma that we frequently use in our estimations.

Lemma 3.1.8. *Suppose a function $f(z)$ is analytic in an open disk γ of radius r and center at z_0 and $|f(z)| \leq m$ for all $z \in \gamma$. Then for each positive integer k,*

$$\left|f^{(k)}(z)\right| \leq \frac{k! m r}{(r - |z-z_0|)^{k+1}}$$

for all z in γ. In particular, if $f(z)$ is also continuous on the boundary of γ then

$$\left|f^{(k)}(z_0)\right| \leq k! r^{-k} \sup_{s \in \bar{\gamma}} |f(s)|.$$

Proof. The proof of the first assertion can be found in [43, *p.*167]. For the second assertion, fix $f(z)$ as in the statement of the theorem. Since $f(z)$ is continuous on the boundary of γ, then by the Maximum Modulus Principle, $|f(z)|$ attains its maximum on the boundary of γ. Thus, by applying

(3.1.8) at $z = z_0$,

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!r}{(r - |z_0 - z_0|)^{k+1}} \sup_{s\in\bar{\gamma}} |f(s)| = k!r^{-k} \sup_{s\in\bar{\gamma}} |f(s)| \,,$$

as desired. This completes the proof. $\square$

Lemma 3.1.9. *Let $0 \leq c < \frac{1}{2}, \theta > 0, M \geq 0, 0 \leq \delta < 1, 0 \leq \alpha < 1,$ and $\kappa \geq 0$. Let $h(s)$ be a complex-valued function and analytic in the region $\mathcal{D}_{c,\theta}$. Let $F_g(s)$ have property $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$. Define*

$$A(s) := s^{-1}H(s, h; \theta)((\theta s - 1)\zeta(\theta s))^{h(s)}.$$

Then for integers $N \geq 1$,

$$A(s) = \sum_{j=0}^{N} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j}(\theta s - 1)^j + O\left(M\left|\frac{\theta s - 1}{c}\right|^{N+1}\right),$$

where

$$\frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!} = \frac{1}{2\pi i}\int_{\left|s-\frac{1}{\theta}\right|=\frac{c}{2\theta}} \frac{A(s)}{(s-\frac{1}{\theta})^{j+1}}ds,$$

uniformly for $\left|s - \frac{1}{\theta}\right| < \frac{c}{2\theta}$ and $s \in \mathcal{D}_{c,\theta}$.

Proof. Fix $c, M, \delta, \alpha, \kappa, h(s), F_g(s)$ and $A(s)$ as in the statement of Lemma 3.1.9. The function $A(s)$ is analytic in the region $\left|s - \frac{1}{\theta}\right| < \frac{c}{2\theta}$. Thus

$$A(s) = \sum_{j=0}^{\infty} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!}\left(s - \frac{1}{\theta}\right)^j = \sum_{j=0}^{\infty} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j}(\theta s - 1)^j \,,$$

where

$$\frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!} = \frac{1}{2\pi i}\int_{\left|s-\frac{1}{\theta}\right|=\frac{c}{2\theta}} \frac{A(s)}{(s-\frac{1}{\theta})^{j+1}}ds.$$

Let γ be a disk of radius $\frac{c}{2\theta}$ and center $\frac{1}{\theta}$. Since $A(s)$ is analytic in the region $\left|s - \frac{1}{\theta}\right| < \frac{c}{\theta \log 3}$, then by Lemma 3.1.8,

$$\left|A^{(j)}\left(\frac{1}{\theta}\right)\right| \leq j!\left(\frac{c}{2\theta}\right)^{-j} \sup_{s\in\bar{\gamma}} |A(s)| \,. \tag{3.1.17}$$

Since $(\theta s - 1)\zeta(\theta s)$ has an analytic continuation to $\mathbb{C}$, thus there exists an analytic function $f_1(s)$, such that for $\sigma > \frac{1}{\theta}$,

$$(\theta s - 1)\zeta(\theta s) = e^{f_1(s)}.$$

Therefore, for $\left|s - \frac{1}{\theta}\right| < \frac{c}{2\theta}$,

$$|A(s)| = \left|s^{-1}\right| |H(s,h;\theta)| \left|((\theta s - 1)\zeta(\theta s))^{h(s)}\right| \leq 2(M(3 + |t|)e^{|h(s)||f_1(s)|} \ll M. \tag{3.1.18}$$

Thus, by equations (3.1.17) and (3.1.18), for $N \geq 1$,

$$\begin{aligned}
A(s) &= \sum_{j=0}^{N} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j} (\theta s - 1)^j + \sum_{j=N+1}^{\infty} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j} (\theta s - 1)^j \\
&= \sum_{j=0}^{N} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j} (\theta s - 1)^j + O\left(M\left|\frac{(N+1)!\theta^{N+1}}{c^{N+1}}\right|\left|\frac{(\theta s - 1)^{N+1}}{(N+1)!\theta^{N+1}}\right|\right). \\
&= \sum_{j=0}^{N} \frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j} (\theta s - 1)^j + O\left(M\left|\frac{(\theta s - 1)^{N+1}}{c^{N+1}}\right|\right).
\end{aligned}$$

This completes the proof. □

Lemma 3.1.10. *Let $h(s)$ be an analytic function on the region $\mathcal{D}_{c,\theta}$ satisfying the inequality*

$$|h(s)| \leq M(\log t)^{\alpha}$$

for some constant $M > 0$ and $0 \leq \alpha < 1$ in the region $\mathcal{D}_{c,\theta}$. For $N \geq 1$ and $s \in \mathcal{D}_{c,\theta} \backslash (-\infty, \frac{1}{\theta}]$, we have

$$(\theta s - 1)^{-h(s)} = (\theta s - 1)^{-h(\frac{1}{\theta})} \left(\sum_{k=0}^{N} \left(\sum_{n=0}^{k} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n \frac{a_{k,n}}{\theta^k}\right)(\theta s - 1)^k + O(R_N)\right), \tag{3.1.19}$$

where

$$\begin{aligned}
a_{0,n} &= 1, \\
a_{k,n} &= \sum_{\substack{k_1 + \cdots + k_n = k \\ k_i \geq 1}} \prod_{i=1}^{n} \frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!},
\end{aligned}$$

and

$$R_N = (2M)^{N+1} \left| \frac{\theta s - 1}{c} \right|^{N+1} |\log(\theta s + 1)|^{N+1}.$$

Proof. Fix $h(s)$ as in the statement of the lemma. For any positive integer N, we can write the left hand side of (3.1.19) as

$$\begin{aligned}
(\theta s - 1)^{-h(s)} &= (\theta s - 1)^{-h\left(\frac{1}{\theta}\right)} (\theta s - 1)^{-\left(h(s) - h\left(\frac{1}{\theta}\right)\right)} \\
&= (\theta s - 1)^{-h\left(\frac{1}{\theta}\right)} \exp\left(- \left(h(s) - h\left(\frac{1}{\theta}\right)\right) \log(\theta s - 1)\right) \\
&= (\theta s - 1)^{-h\left(\frac{1}{\theta}\right)} \left(\sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n!} \right) \\
&= (\theta s - 1)^{-h\left(\frac{1}{\theta}\right)} \left(\sum_{n=0}^{N} \frac{(-1)^n u^n}{n!} + \sum_{n=N+1}^{\infty} \frac{(-1)^n u^n}{n!} \right) \\
&= (\theta s - 1)^{-h\left(\frac{1}{\theta}\right)} \left(S_N + R_N^* \right),
\end{aligned}$$

where

$$S_N := \sum_{n=0}^{N} \frac{(-1)^n u^n}{n!}, \tag{3.1.20}$$

$$R_N^* := \sum_{n=N+1}^{\infty} \frac{(-1)^n u^n}{n!}, \tag{3.1.21}$$

and

$$u := \left(h(s) - h\left(\frac{1}{\theta}\right)\right) \log(\theta s - 1). \tag{3.1.22}$$

Since $h(s)$ is analytic on the disk $\left|s - \frac{1}{\theta}\right| < \frac{c}{2\theta}$, we can write

$$h(s) = \sum_{k=0}^{\infty} \frac{h^{(k)}\left(\frac{1}{\theta}\right)}{k!\theta^k} (\theta s - 1)^k.$$

Thus (3.1.22) can be written as

$$u = \log(\theta s - 1) \sum_{k=1}^{\infty} \frac{h^{(k)}\left(\frac{1}{\theta}\right)}{k!\theta^k} (\theta s - 1)^k. \tag{3.1.23}$$

Next, by raising (3.1.23) to the nth power, we see that

$$\begin{aligned}
u^n &= (\log(\theta s - 1))^n \left(\sum_{k=1}^{\infty} \frac{h^{(k)}\left(\frac{1}{\theta}\right)}{k!\theta^k} (\theta s - 1)^k \right)^n \\
&= (\log(\theta s - 1))^n \left(\sum_{k=n}^{\infty} \left(\sum_{\substack{k_1+\cdots+k_n=k \\ k_i \geq 1}} \prod_{i=1}^{n} \frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!} \right) \frac{1}{\theta^k} (\theta s - 1)^k \right) \\
&= (\log(\theta s - 1))^n \left(\sum_{k=n}^{\infty} a_{k,n} \frac{1}{\theta^k} (\theta s - 1)^k \right) \qquad (3.1.24)
\end{aligned}$$

with

$$a_{k,n} = \sum_{\substack{k_1+\cdots+k_n=k \\ k_i \geq 1}} \prod_{i=1}^{n} \frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!}. \qquad (3.1.25)$$

Thus, replacing u^n in (3.1.20) by (3.1.24), we find that

$$\begin{aligned}
S_N &= \sum_{n=0}^{N} \frac{(-1)^n u^n}{n!} \\
&= \sum_{n=0}^{N} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n \left(\sum_{k=n}^{\infty} a_{k,n} \frac{1}{\theta^k} (\theta s - 1)^k \right) \\
&= \sum_{n=0}^{N} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n \left(\sum_{k=n}^{N} a_{k,n} \frac{1}{\theta^k} (\theta s - 1)^k + \sum_{k=N+1}^{\infty} a_{k,n} \frac{1}{\theta^k} (\theta s - 1)^k \right) \\
&= \sum_{k=0}^{N} \left(\sum_{n=0}^{k} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n a_{k,n} \right) \frac{1}{\theta^k} (\theta s - 1)^k \\
&\quad + \sum_{k=N+1}^{\infty} \left(\sum_{n=0}^{N} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n a_{k,n} \right) \frac{1}{\theta^k} (\theta s - 1)^k \\
&= \sum_{k=0}^{N} \left(\sum_{n=0}^{k} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n a_{k,n} \right) \frac{1}{\theta^k} (\theta s - 1)^k + K_N
\end{aligned}$$

where

$$K_N := \sum_{k=N+1}^{\infty} \left(\sum_{n=0}^{N} \frac{(-1)^n}{n!} (\log(\theta s - 1))^n a_{k,n} \right) \frac{1}{\theta^k} (\theta s - 1)^k.$$

Next, by Lemma 3.1.8 and property (4) of $\mathcal{B}(h; c, \theta, M, \delta, \alpha, \kappa)$,

$$\left| \frac{h^{(k)}\left(\frac{1}{\theta}\right)}{k!\theta^k} \right| \leq \frac{1}{\theta^k} \sup_{\left|s - \frac{1}{\theta}\right| = \frac{c}{2\theta}} |h(s)| \frac{(2\theta)^k}{c^k} \leq c_1 M \left(\frac{1}{c} \right)^k. \qquad (3.1.26)$$

for some positive constant c_1. Thus, applying (3.1.26) to (3.1.25), we deduced that

$$\left|\frac{a_{k,n}}{\theta^k}\right| = \sum_{\substack{k_1+\cdots+k_n=k\\ k_i\geq 1}} \prod_{i=1}^{n} \left|\frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!\theta^{k_i}}\right| \leq c_2 \sum_{\substack{k_1+\cdots+k_n=k\\ k_i\geq 1}} M^n c^{-k} \leq c_2 \binom{k-1}{n-1} M^n c^{-k} \leq c_2 2^{k-1} M^n c^{-k} \tag{3.1.27}$$

for some positive constant c_2. Hence, for $\left|s-\frac{1}{\theta}\right| < \frac{c}{2\theta}$,

$$\begin{aligned}
|K_N| &= \left|\sum_{k=N+1}^{\infty}\left(\sum_{n=0}^{N}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n\frac{a_{k,n}}{\theta^k}\right)(\theta s-1)^k\right| \\
&\ll \left|\log(\theta s-1)\right|^N \sum_{k=N+1}^{\infty} 2^{k-1}\left|\frac{\theta s-1}{c}\right|^k \sum_{n=0}^{N}\left|\frac{(-1)^n M^n}{n!}\right| \\
&\ll 2^{N+1}M^{N+1}\left|\frac{\theta s-1}{c}\right|^{N+1} \left|\log(\theta s+1)\right|^{N+1}.
\end{aligned}$$

Therefore

$$S_N = \sum_{k=0}^{N}\left(\sum_{n=0}^{k}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n\frac{a_{k,n}}{\theta^k}\right)(\theta s-1)^k + O\left((2M)^{N+1}\left|\frac{\theta s-1}{c}\right|^{N+1}\left|\log(\theta s+1)\right|^{N+1}\right). \tag{3.1.28}$$

Lastly, we estimate R_N^*. For $\left|s-\frac{1}{\theta}\right| < \frac{c}{4\theta}$,

$$|R_N^*| = \left|\sum_{n=N+1}^{\infty}\frac{(-1)^n u^n}{n!}\right| \leq \frac{|u|^{N+1}}{(N+1)!}\left(\sum_{n=0}^{\infty}\frac{|u|^n}{n!}\right).$$

By (3.1.26), we see that

$$|u^n| \leq 2^n M^n \left|\frac{\theta s-1}{c}\right|^n \log(\theta s-1)^n.$$

Therefore

$$|R_N^*| \leq \frac{2^{N+1}M^{N+1}\left|\theta s-1\right|^{N+1}\left(\log(\theta s-1)\right)^{N+1}}{c^{N+1}(N+1)!}\left(\sum_{n=0}^{\infty}\frac{|2M|^n}{n!c^n}\left|(\theta s-1)\log(\theta s-1)\right|^n\right).$$

Since $|x\log x|$ attains a local maximum value of $\frac{1}{e}$ when $x=\frac{1}{e}$ for $0\leq x\leq 1$, we have

$$|\theta s-1|\left|\log\theta s-1\right| = |\theta s-1|\left(\log|\theta s-1|+\pi\right) \leq \frac{1}{e}+\frac{\pi c}{2}.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{|2M|^n}{n!c^n} |(\theta s-1)\log(\theta s-1)|^n \leq \sum_{n=0}^{\infty} \frac{(2M)^n}{n!c^n} \left|\frac{1}{e}+\frac{\pi c}{2}\right|^n = O(1).$$

Therefore

$$R_N^* \ll (2M)^{N+1} \left|\frac{\theta s-1}{c}\right|^{N+1} |\log(\theta s-1)|^{N+1}. \tag{3.1.29}$$

Combining the estimates (3.1.28) and (3.1.29) completes the proof. □

Lemma 3.1.11. *Let $0 \leq c < \frac{1}{2}, \theta > 0, M \geq 0, 0 \leq \delta < 1, 0 \leq \alpha < 1,$ and $\kappa \geq 0$. Let $h(s)$ be a complex-valued function and analytic in the region $\mathcal{D}_{c,\theta}$. Let $F_g(s)$ have property $\mathcal{B}(h; c, M, \theta, \delta, \alpha, \kappa)$. For $N \geq 1, \left|s-\frac{1}{\theta}\right| < \frac{c}{2\theta}$, and $s \in \mathcal{D}_{c,\theta} \backslash (-\infty, \frac{1}{\theta}]$ we have*

$$F_g(s) = s(\theta s-1)^{-h(\frac{1}{\theta})} \left(\sum_{m=0}^{N} Q_m\left(\log(\theta s-1)\right)(\theta s-1)^m + O(R_N)\right),$$

where

$$Q_m(x) = \sum_{n=0}^{m} \left(\sum_{k=n}^{m} \frac{A^{(m-k)}\left(\frac{1}{\theta}\right) a_{k,n}}{(m-k)!\theta^m}\right) \frac{(-1)^n x^n}{n!},$$
$$A(s) = s^{-1} H(s,h;\theta)((\theta s-1)\zeta(\theta s))^{h(s)}.$$

and

$$R_N = (2M)^{N+1} \left|\frac{\theta s-1}{c}\right|^{N+1} |\log(\theta s-1)|^{N+1}.$$

Proof. Fix $c, \theta, M, \delta, \alpha, \kappa, h(s)$ and $F_g(s)$ as in the statement of the theorem. By Lemma 3.1.10, and property (2) of $\mathcal{B}(h; c, M, \theta, \delta, \alpha, \kappa)$,

$$\begin{aligned}
F_g(s) &= H(s,h;\theta)\zeta(\theta s)^{h(s)} \\
&= sA(s)(\theta s-1)^{-h(s)} \\
&= sA(s)\left((\theta s-1)^{-h(\frac{1}{\theta})}\left(\sum_{k=0}^{N}\left(\sum_{n=0}^{k}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n \frac{a_{k,n}}{\theta^k}\right)(\theta s-1)^k + O(R_N)\right)\right) \\
&= s(\theta s-1)^{-h(\frac{1}{\theta})}\left(S_N + O(R_N')\right).
\end{aligned}$$

where

$$S_n := A(s)\left(\sum_{k=0}^{N}\left(\sum_{n=0}^{k}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n\frac{a_{k,n}}{\theta^k}\right)(\theta s-1)^k\right),$$

and

$$R_N' := A(s)(2M)^{N+1}\left|\frac{\theta s-1}{c}\right|^{N+1}\left|\log(\theta s+1)\right|^{N+1}.$$

Next, by Lemma 3.1.9 and rearranging the order of summation, we have

$$\begin{aligned}
S_N &= A(s)\left(\sum_{k=0}^{N}\left(\sum_{n=0}^{k}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n\frac{a_{k,n}}{\theta^k}\right)(\theta s-1)^k\right)\\
&= \left(\sum_{j=0}^{N}\frac{A^{(j)}\left(\frac{1}{\theta}\right)}{j!\theta^j}(\theta s-1)^j+O\left(\left|\frac{\theta s-1}{c}\right|^{N+1}\right)\right)\left(\sum_{k=0}^{N}\left(\sum_{n=0}^{k}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n\frac{a_{k,n}}{\theta^k}\right)(\theta s-1)^k\right)\\
&= \sum_{m=0}^{2N}\left(\sum_{k=0}^{m}\left(\frac{A^{(m-k)}\left(\frac{1}{\theta}\right)}{(m-k)!\theta^{m-k}}\left(\sum_{n=0}^{k}\frac{(-1)^n}{n!}\left(\log(\theta s-1)\right)^n\frac{a_{k,n}}{\theta^k}\right)\right)\right)(\theta s-1)^m+O\left(\left|\frac{\theta s-1}{c}\right|^{N+1}\right)\\
&= \sum_{m=0}^{2N}\left(\sum_{n=0}^{m}\left(\frac{(-1)^n}{n!}\sum_{k=n}^{m}\frac{A^{(m-k)}\left(\frac{1}{\theta}\right)a_{k,n}}{(m-k)!\theta^m}\right)\left(\log(\theta s-1)\right)^n\right)(\theta s-1)^m+O\left(\left|\frac{\theta s-1}{c}\right|^{N+1}\right)\\
&= \sum_{m=0}^{2N}Q_m(\log(\theta s-1))(\theta s-1)^m+O\left(\left|\frac{\theta s-1}{c}\right|^{N+1}\right),
\end{aligned}$$

where

$$Q_m(x) := \sum_{n=0}^{m}\left(\sum_{k=n}^{m}\frac{A^{(m-k)}\left(\frac{1}{\theta}\right)a_{k,n}}{(m-k)!\theta^m}\right)\frac{(-1)^n x^n}{n!}.$$

Next, by (3.1.27) and (3.1.17),

$$\begin{aligned}
\left|Q_m(\log(\theta s-1))\right| &\ll \sum_{n=0}^{m}\sum_{k=n}^{m}\left(Mc^{k-m}\right)\left(2^{k-1}M^nc^{-k}\right)\frac{\left|\log(\theta s-1))\right|^n}{n!}\\
&\ll c^{-m}2^m+\sum_{n=1}^{m}\sum_{k=n}^{m}2^{k-1}M^{n+1}c^{-m}\frac{\left|\log(\theta s-1))\right|^n}{n!}\\
&\ll c^{-m}M2^m+c^{-m}2^mM^{m+1}\left|\log(\theta s-1)\right|^m.
\end{aligned}$$

Thus

$$\sum_{m=N+1}^{2N}Q_m(\log(\theta s-1))(\theta s-1)^m \ll \sum_{m=N+1}^{2N}\left(c^{-m}M2^m+c^{-m}2^mM^{m+1}\left|\log(\theta s-1)\right|^m\right)\left|\theta s-1\right|^m$$

$$\ll M\left|\frac{2(\theta s-1)}{c}\right|^{N+1}+M^{N+1}\left|\frac{2(\theta s-1)}{c}\right|^{N+1}|\log(\theta s-1)|^{N+1}.$$

Hence

$$S_N=\sum_{m=0}^{N}Q_m(\log(\theta s-1))(\theta s-1)^m+O\left(M^{N+1}\left|\frac{2(\theta s-1)}{c}\right|^{N+1}|\log(\theta s-1)|^{N+1}\right). \quad (3.1.30)$$

Next, note that, for $\left|s-\frac{1}{\theta}\right|<\frac{c}{4\theta}$,

$$A(s)=1+O\left(\left|\frac{\theta s-1}{c}\right|\right).$$

Thus

$$R''_N\ll 2^{N+1}\left|\frac{\theta s-1}{c}\right|^{N+1}|\log(\theta s-1)|^{N+1}. \quad (3.1.31)$$

Therefore, by combining (3.1.30) and (3.1.31), we complete the proof. □

3.2 Proof of Theorem 2.4.3

The proof of the theorem is along the same lines as the general argument of Selberg. In our proof, we assume that c_i is a positive real constant for all i.

Proof. Let $\theta>0, M\geq 0, 0\leq\delta<1, 0\leq\alpha<1$, and $\kappa\geq 0$. Let c be a positive constant such that $\zeta(s)$ has no zeroes in the region

$$\sigma\geq 1-\frac{c}{\log(\max\{3,|t|\})}$$

for $|t|>0$. Let $h(s)$ and $h^+(s)$ be complex-valued functions analytic in the region $\mathcal{D}_{c,\theta}$. Let $F_g(s)=\sum_{n=1}^{\infty}\frac{f(n)}{g(n)^s}$ have property $\mathcal{B}^+(h,h^+;c,M,\theta,\delta,\alpha,\kappa)$. By Theorem 3.1.5,

$$\int_0^x A_g(t)dt=\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}F_g(s)x^{s+1}\frac{ds}{s(s+1)},$$

where $\eta = \frac{1}{\theta} + \frac{1}{\log x}$, and

$$A_g(x) = \sum_{\substack{g(n)<x \\ n\geq 1}} f(n).$$

Let $T > 1$ be a parameter which we will determine later. Since $\delta < 1$, by Lemma 3.1.2,

$$\frac{1}{2\pi i}\int_{\eta+iT}^{\eta+i\infty} F_g(s)x^{s+1}\frac{ds}{s(s+1)} \leq \int_T^\infty Mx^{1+\frac{1}{\theta}+\frac{1}{\log x}}(3+t)^{\frac{\delta+1}{2}}\frac{dt}{t^2} \ll Mx^{1+\frac{1}{\theta}}T^{\frac{\delta-1}{2}}.$$

Similarly, we obtain the same estimate for the integral along the half line $(\eta - i\infty, \eta - iT]$. Therefore

$$\int_0^x A_g(t)dt = \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} F_g(s)x^{s+1}\frac{ds}{s(s+1)} + O\left(Mx^{1+\frac{1}{\theta}}T^{\frac{\delta-1}{2}}\right).$$

Next, we claim that

$$\int_0^x A_g(t)dt = \frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{\theta}, \frac{c}{2\theta\log x}, \frac{c}{2\theta\log 3}-\frac{1}{\theta})} F_g(s)x^{s+1}\frac{ds}{s(s+1)} + O\left(Mx^{1+\frac{1}{\theta}}Err(x)^{-1}\right), \qquad (3.2.1)$$

where $Err(x) := e^{c_1\sqrt{\log x}}$. Consider the contour $\mathcal{H}$, which is formed by joining the following paths with positive orientation as in Figure 1.2:

L_1: a vertical segment $[\eta - iT, \eta + iT]$;

L_2: a horizontal segment $\left[\frac{1}{\theta} - \frac{c}{2\theta\log(\theta T)} + iT, \eta + iT\right]$;

L_3: a curve described by $\sigma_3(t) = \frac{1}{\theta} - \frac{c}{2\theta\log(\theta t)}$ for $\frac{3}{\theta} \leq t \leq T$;

L_4: a truncated Hankel's contour with radius $r = \frac{c}{2\log x}$, centered $\frac{1}{\theta}$, and $\sigma \geq \frac{1}{\theta} - \frac{c}{2\theta\log(3)}$;

L_5: a curve described by $\sigma_5(t) = \frac{1}{\theta} - \frac{c}{2\theta\log(|\theta t|)}$ for $-T \leq t \leq -\frac{3}{\theta}$;

L_6: a horizontal segment $\left[\frac{1}{\theta} - \frac{c}{2\theta\log(\theta T)} - iT, \eta - iT\right]$;

L_7: a vertical segment $\left[\frac{1}{\theta} - \frac{c}{2\theta\log(3)}, \frac{1}{\theta} - \frac{c}{2\theta\log(3)} + \frac{3}{\theta}i\right]$;

L_8: a vertical segment $\left[\frac{1}{\theta} - \frac{c}{2\theta\log(3)} - \frac{3}{\theta}i, \frac{1}{\theta} - \frac{c}{2\theta\log(3)}\right]$.

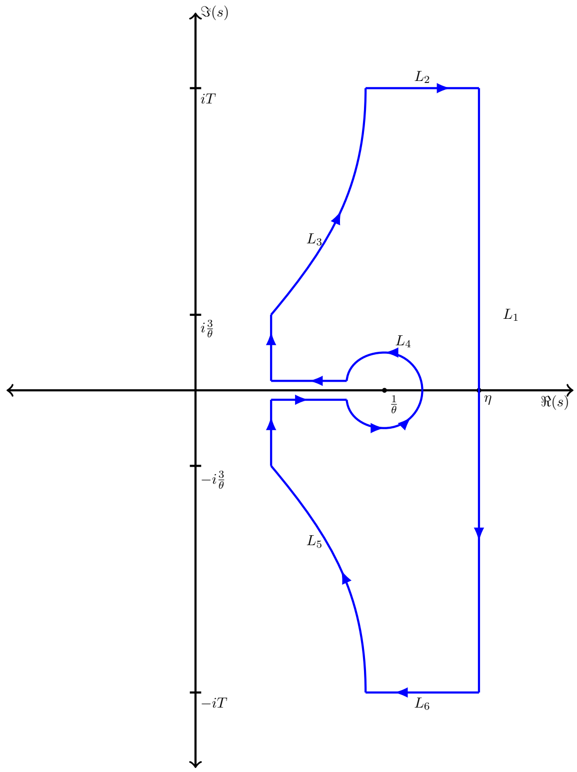

Figure 3.2: Contour $\mathcal{H}$

Since the region enclosed by $\mathcal{H}$ does not contain any pole of $F_g(s)$, then by the Residue Theorem, we deduce that

$$\frac{1}{2\pi i}\int_{\mathcal{H}} F_g(s)x^{s+1}\frac{ds}{s(s+1)} = 0.$$

We now estimate the integral over each piece of the contour. By Theorem 3.1.2, for $0 \leq \delta < 1$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{L_2} F_g(s)x^{s+1}\frac{ds}{s(s+1)}\right| &\ll M\int_{\frac{1}{\theta}-\frac{c}{2\theta\log(\theta T)}}^{\eta}(3+T)^{\frac{\delta+1}{2}}\frac{x^{1+\sigma}}{|\sigma+iT|\,|1+\sigma+iT|}d\sigma \\ &\ll Mx^{1+\frac{1}{\theta}}\frac{(3+T)^{\frac{\delta+1}{2}}}{T^2} \\ &\ll Mx^{1+\frac{1}{\theta}}T^{-\frac{3-\delta}{2}}.\end{aligned}$$

Next, we estimate the integral over the arc L_3. We find that

$$\left|\frac{1}{2\pi i}\int_{L_3} F_g(s)x^{s+1}\frac{ds}{s(s+1)}\right| \ll M\int_0^T(3+t)^{\frac{\delta+1}{2}}\frac{x^{1+\sigma_3(t)}}{|\sigma_3(t)+it|\,|1+\sigma_3(t)+it|}d\sigma$$

$$\ll Mx^{1+\frac{1}{\theta}-\frac{c}{2\theta\log(\theta T)}}\int_0^T \frac{(3+t)^{\frac{\delta+1}{2}}}{(t+1)^2}dt$$
$$\ll Mx^{1+\frac{1}{\theta}-\frac{c}{2\theta\log(\theta T)}}.$$

Lastly, we estimate the integral over the vertical segment L_7, and find that

$$\left|\frac{1}{2\pi i}\int_{L_7} F_g(s)x^{s+1}\frac{ds}{s(s+1)}\right| \ll M\int_0^{\frac{3}{\theta}}(3+t)^{\frac{\delta+1}{2}}\frac{x^{1+\frac{1}{\theta}-\frac{c}{2\theta\log 3}}}{\left|\frac{1}{\theta}-\frac{c}{2\theta\log 3}+it\right|\left|1+\frac{1}{\theta}-\frac{c}{2\theta\log 3}+it\right|}d\sigma$$
$$\ll Mx^{1+\frac{1}{\theta}-\frac{c}{2\theta\log(\theta T)}}.$$

The estimates on L_5, L_6 and L_8 are similar to these on L_3, L_2 and L_7. Hence

$$\int_{L_4} F_g(s)x^{s+1}\frac{ds}{s(s+1)} = -\int_{L_1} F_g(s)x^{s+1}\frac{ds}{s(s+1)} + O\left(Mx^{1+\frac{1}{\theta}-\frac{c}{2\theta\log(\theta T)}} + Mx^{1+\frac{1}{\theta}}T^{-\frac{3-\delta}{2}}\right).$$

By taking $T = Err(x)^{c_2}$, we obtain (3.2.1).

For convenience, we denote

$$\Phi(x) := \frac{1}{2\pi i}\int_{\mathcal{H}_1} F_g(s)x^{s+1}\frac{ds}{s(s+1)}$$

where

$$\mathcal{H}_1 = \mathcal{H}\left(\frac{1}{\theta}, \frac{c}{2\theta\log x}, \frac{c}{2\theta\log 3}-\frac{1}{\theta}\right).$$

Next, we will study the behavior of $\Phi(x)$. First note that $\Phi(x)$ is an infinitely differentiable function of x for real-valued $x > 0$. Thus, we have

$$\Phi'(x) = \frac{1}{2\pi i}\int_{\mathcal{H}_1} F_g(s)x^s\frac{ds}{s}, \quad \Phi''(x) = \frac{1}{2\pi i}\int_{\mathcal{H}_1} F_g(s)x^{s-1}ds.$$

First, we will show that

$$\Phi''(x) \ll Mx^{\frac{1}{\theta}-1}(\log x)^{\left|\Re(h(\frac{1}{\theta}))\right|}.$$

For $s \in \mathcal{D}_{c,\theta}\backslash(-\infty, \frac{1}{\theta}]$ and $\left|s-\frac{1}{\theta}\right| < \frac{c}{2\theta}$, by the Weierstrass Factorization Theorem [5, p. 170] there exists an entire function $\alpha(s)$ such that

$$F_g(s)(\theta s-1)^{h\left(\frac{1}{\theta}\right)} = H_g(s,h;\theta)e^{h(s)\alpha(s)}(\theta s-1)^{-\left(h(s)-h\left(\frac{1}{\theta}\right)\right)}.$$

Moreover, by Lemma 3.1.10,

$$(\theta s-1)^{-\left(h(s)-h\left(\frac{1}{\theta}\right)\right)} = O(1).$$

Therefore, by properties (3) and (4) of $\mathcal{B}(h;c,\theta,M,\delta,\alpha,\kappa)$, we see that for $s\in\mathcal{D}_{c,\theta}\backslash(-\infty,\frac{1}{\theta})$

$$F_g(s)(\theta s-1)^{h\left(\frac{1}{\theta}\right)} \ll M.$$

Thus

$$\begin{aligned}\left|\Phi''(x)\right| &\ll M\frac{1}{2\pi i}\int_{\mathcal{H}_1}\left|(\theta s-1)^{h\left(\frac{1}{\theta}\right)}\right|\left|x^{s-1}\right|ds\\ &\ll M\int_r^{\frac{c}{2\theta\log(3)}}\rho^{-\Re\left(h\left(\frac{1}{\theta}\right)\right)}e^{\left(\frac{1-\rho}{\theta}-1\right)\log x}d\rho + M\theta\int_{-\pi}^{\pi}r^{-\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|+1}x^{\left(\frac{r}{\theta}+\frac{1}{\theta}-1\right)}dt,\end{aligned}$$

where $r=\frac{c}{2\log x}$. Thus letting $\rho=\frac{u\theta}{\log x}$, we obtain the estimate

$$\begin{aligned}\left|\Phi''(x)\right| &\ll Mx^{\frac{1}{\theta}-1}(\log x)^{\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|-1}\int_{\frac{c}{2}}^{\frac{c}{2\log 3}\log x}u^{-\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|}e^{-u}du + M\theta x^{\frac{1}{\theta}-1}(\log x)^{\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|}\\ &\ll Mx^{\frac{1}{\theta}-1}(\log x)^{\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|}. \qquad (3.2.2)\end{aligned}$$

Now, we will give an estimation of $\Phi'(x)$. By Lemma 3.1.11,

$$\Phi'(x)=\frac{1}{2\pi i}\int_{\mathcal{H}_1}\frac{F_g(s)}{s}x^s ds = \Sigma_N + R_N,$$

where

$$\Sigma_1 := \frac{1}{2\pi i}\int_{\mathcal{H}_1}\sum_{m=0}^{N}\sum_{n=0}^{m}\left(\sum_{k=n}^{m}\frac{A^{(m-k)}\left(\frac{1}{\theta}\right)a_{k,n}}{(m-k)!\theta^m}\right)\frac{(-1)^n(\log(\theta s-1))^n}{n!}(\theta s-1)^{m-h\left(\frac{1}{\theta}\right)}x^s ds$$

and

$$R_N=\frac{1}{2\pi i}\int_{\mathcal{H}_1}2^{N+1}\left|\frac{\theta s-1}{c}\right|^{N+1}\left|\log(\theta s-1)\right|^{N+1}x^s ds.$$

First, we will focus on Σ_1. By rearranging the sum, we have

$$\Sigma_1 = \sum_{m=0}^{N}\sum_{n=0}^{m} \frac{e_{m,n}}{\theta^m}\frac{1}{2\pi i}\int_{\mathcal{H}_1} x^s \left(\log(\theta s-1)\right)^n (\theta s-1)^{m-h\left(\frac{1}{\theta}\right)}\,ds,$$

where

$$e_{m,n} = \frac{(-1)^n}{n!}\sum_{k=n}^{m}\frac{A^{(m-k)}\left(\frac{1}{\theta}\right)a_{k,n}}{(m-k)!}. \tag{3.2.3}$$

Next, letting $u = \left(s-\frac{1}{\theta}\right)\log x$, we have

$$\begin{aligned}
\Sigma_1 &= \sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}}{\theta^m}\frac{1}{2\pi i}\int_{\mathcal{H}_0} x^{u\log x+\frac{1}{\theta}}\left(\log\left(\frac{\theta u}{\log x}\right)\right)^n\left(\frac{\theta u}{\log x}\right)^{m-h\left(\frac{1}{\theta}\right)}\frac{du}{\log x}\\
&= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}}{(\log x)^m\theta^{h\left(\frac{1}{\theta}\right)}}\frac{1}{2\pi i}\int_{\mathcal{H}_0} e^u\left(\log(u)+\log\left(\frac{\theta}{\log x}\right)\right)^n u^{m-h\left(\frac{1}{\theta}\right)}du,
\end{aligned}$$

where

$$\mathcal{H}_0 = \mathcal{H}\left(0, \frac{c}{2\theta}, \frac{c}{2\theta\log 3}\log x\right).$$

Applying the binomial theorem and Corollary 3.1.7 gives us

$$\begin{aligned}
\Sigma_1 &= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}\theta^{-h\left(\frac{1}{\theta}\right)}}{(\log x)^m}\frac{1}{2\pi i}\int_{\mathcal{H}_0} e^u\sum_{i=0}^{n}\binom{n}{i}(\log u)^i\left(\log\left(\frac{\theta}{\log x}\right)\right)^{n-i}u^{m-h\left(\frac{1}{\theta}\right)}du\\
&= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}\theta^{-h\left(\frac{1}{\theta}\right)}}{(\log x)^m}\sum_{i=0}^{n}\binom{n}{i}\left(\log\left(\frac{\theta}{\log x}\right)\right)^{k-i}\frac{1}{2\pi i}\int_{\mathcal{H}_0} e^u(\log u)^i u^{m-h\left(\frac{1}{\theta}\right)}du\\
&= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}\theta^{-h\left(\frac{1}{\theta}\right)}}{(\log x)^m}\sum_{i=0}^{n}\binom{n}{i}\left(\log\left(\frac{\theta}{\log x}\right)\right)^{n-i}\times\\
&\quad\times\left(\frac{(-1)^i}{\Gamma_i\left(h\left(\frac{1}{\theta}\right)-m\right)}+E_{i,m,h\left(\frac{1}{\theta}\right)}(X)\right)\\
&= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}}{(\log x)^m\theta^{h\left(\frac{1}{\theta}\right)}}\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}\binom{n-i}{j}(\log\theta)^{n-i-j}(-1)^j(\log\log x)^j\times\\
&\quad\times\left(\frac{(-1)^i}{\Gamma_i\left(h\left(\frac{1}{\theta}\right)-m\right)}+E_{i,m,h\left(\frac{1}{\theta}\right)}(X)\right)
\end{aligned}$$

with $X := \frac{c}{2\theta}\log x$. Thus, the contribution of $\dfrac{(-1)^i}{\Gamma_i\left(h\left(\frac{1}{\theta}\right)-m\right)}$ to Σ_1 is

$$\frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\frac{P_m(\log\log x)}{(\log x)^m}, \tag{3.2.4}$$

where

$$\begin{aligned}
P_m(x) &= \sum_{n=0}^{m}\frac{e_{m,n}}{\theta^{h\left(\frac{1}{\theta}\right)}}\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}\binom{n-i}{j}(\log\theta)^{n-i-j}(-1)^j\frac{(-1)^i}{\Gamma_i\left(h\left(\frac{1}{\theta}\right)-m\right)}x^j\\
&= \sum_{n=0}^{m}\frac{e_{m,n}}{\theta^{h\left(\frac{1}{\theta}\right)}}\sum_{j=0}^{n}\sum_{i=0}^{n-j}\binom{n}{i}\binom{n-i}{j}(\log\theta)^{n-i-j}\frac{(-1)^{j+i}}{\Gamma_i\left(h\left(\frac{1}{\theta}\right)-m\right)}x^j\\
&= \sum_{j=0}^{m}\sum_{n=j}^{m}\frac{e_{m,n}}{\theta^{h\left(\frac{1}{\theta}\right)}}\sum_{i=0}^{n-j}\binom{n}{i}\binom{n-i}{j}(\log\theta)^{n-i-j}\frac{(-1)^{j+i}}{\Gamma_i\left(h\left(\frac{1}{\theta}\right)-m\right)}x^j.
\end{aligned}$$

Next, the contribution of $E_{i,m,h\left(\frac{1}{\theta}\right)}$ to Σ_1 is

$$\begin{aligned}
\Sigma_2 &:= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{e_{m,n}}{(\log x)^m\theta^{h\left(\frac{1}{\theta}\right)}}\sum_{i=0}^{n}\binom{n}{i}\log\left(\frac{\theta}{\log x}\right)E_{i,m,h\left(\frac{1}{\theta}\right)}\\
&\ll \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{|e_{m,n}|}{(\log x)^m\theta^{h\left(\frac{1}{\theta}\right)}}\left|\sum_{i=0}^{n}\binom{n}{i}\log\left(\frac{\theta}{\log x}\right)\int_X^\infty \rho^{m-\Re h\left(\frac{1}{\theta}\right)}e^{-\rho}\left|\log(\rho)+\pi\right|^i d\rho\right|\\
&\ll \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\sum_{n=0}^{m}\frac{|e_{m,n}|}{(\log x)^m\theta^{h\left(\frac{1}{\theta}\right)}}\int_X^\infty \rho^{m-\Re h\left(\frac{1}{\theta}\right)}e^{-\rho}\left|2\log(\rho)+2\log\log x\right|^n d\rho.
\end{aligned}$$

By combining (3.2.3), (3.1.17), and (3.1.27), we have

$$|e_{m,n}| \ll \frac{1}{n!}\sum_{k=n}^{m}M\frac{\theta^{m-k}}{c^{m-k}}\left(\frac{2^{k-1}M^n\theta^k}{c^k}\right) \ll \frac{M^{n+1}\theta^m}{n!c^m}.$$

Thus,

$$\begin{aligned}
\Sigma_2 &\ll \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\frac{1}{(\log x)^m\theta^{h\left(\frac{1}{\theta}\right)-m}}\sum_{n=0}^{m}\frac{M^{n+1}}{n!c^m}\int_X^\infty \rho^{m-\Re h\left(\frac{1}{\theta}\right)}e^{-\rho}\left|2\log(\rho)+2\log\log x\right|^n d\rho\\
&\ll M\frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\sum_{m=0}^{N}\frac{1}{(c\log x)^m\theta^{h\left(\frac{1}{\theta}\right)-m}}\int_X^\infty \rho^{m-\Re h\left(\frac{1}{\theta}\right)}e^{-\rho}e^{M|2\log(\rho)+2\log\log x|}d\rho
\end{aligned}$$

$$\ll M \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} \sum_{m=0}^{N} \frac{(\log x)^{2M}}{(c\log x)^m \theta^{h\left(\frac{1}{\theta}\right)-m}} \int_X^\infty \rho^{m-\Re h\left(\frac{1}{\theta}\right)+2M} e^{-\rho} d\rho.$$

Next, note that

$$\begin{aligned}
\int_X^\infty \rho^{m-\Re h\left(\frac{1}{\theta}\right)+2M} e^{-\rho} d\rho &\ll e^{-X/2} \int_X^\infty \rho^{m+\left|h\left(\frac{1}{\theta}\right)\right|+2M} e^{-\rho/2} d\rho \\
&\ll e^{-X/2} 2^m \int_{X/2}^\infty \rho^{m+\left|h\left(\frac{1}{\theta}\right)\right|+2M} e^{-\rho/2} d\rho \\
&\ll e^{-X/2} 2^m \Gamma\left(m + \left|h\left(\frac{1}{\theta}\right)\right| + 2M + 1\right) \\
&\ll e^{-X/2} 2^m (m+1)! \left(\left|h\left(\frac{1}{\theta}\right)\right| + 2M + 1\right)^m.
\end{aligned}$$

Since $X = \frac{c}{2\theta \log 3} \log x$, we find that

$$\begin{aligned}
\Sigma_2 &\ll M \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} \sum_{m=0}^{N} \frac{(\log x)^{2M}}{(c\log x)^m} e^{-X/2} (m+1)! \left(2\left|h\left(\frac{1}{\theta}\right)\right| + 4M + 2\right)^m \\
&\ll M \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} e^{-X/4} \sum_{m=0}^{N} \left(\frac{2\left|h\left(\frac{1}{\theta}\right)\right| + 4M + 2}{c\log x}\right)^m (m+1)! \\
&\ll M \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} e^{-X/4} \frac{(N+1)!}{(\log x)^N} \sum_{m=0}^{N} c_4(X)^{N-m} \frac{(m+1)!}{(N+1)!} \\
&\ll M \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} e^{-X/8} \left(\frac{1}{\log x}\right)^{N+1} (N+1)! \\
&\ll M \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} \left(\frac{c_3 N + 1}{\log x}\right)^{N+1}. \qquad (3.2.5)
\end{aligned}$$

By combining (3.2.4) and (3.2.5), we have

$$\Sigma_1 = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}} \left(\sum_{m=0}^{N} \frac{P_m(\log\log x)}{(\log x)^m} + O\left(\left(\frac{c_3 N + 1}{\log x}\right)^{N+1}\right)\right). \qquad (3.2.6)$$

Now, we will turn our attention to R_N. First note that $|\log(\theta s - 1)| \le \log\log x + c_4$ for $s \in \mathcal{H}_1$. By letting $u = \left(\frac{1}{\theta} - s\right)\log x$, we have

$$|R_N| = \frac{1}{2\pi i} \int_{\mathcal{H}_1} 2^{N+1} |\theta s - 1|^{-h\left(\frac{1}{\theta}\right)} \left|\frac{\theta s - 1}{c}\right|^{N+1} |\log(\theta s - 1)|^{N+1} |x^s| \, ds.$$

$$
\begin{aligned}
&\ll \int_{\frac{c}{2\theta}}^{\frac{c}{2\theta \log 3}\log x} 2^{N+1}\left|\frac{\theta u}{\log x}\right|^{N+1-h\left(\frac{1}{\theta}\right)} \left|\log\log x + c_4\right|^{N+1} x^{\frac{1}{\theta}} e^{-u}\frac{du}{\log x} + \\
&+ \qquad \int_{-\pi}^{\pi} 2^{N+1}\left|\frac{\theta c}{2\log x}\right|^{N+1-h\left(\frac{1}{\theta}\right)} \left|\log\log x + c_4\right|^{N+1} x^{\frac{1}{\theta}} e^{-\frac{c}{4\theta}}\frac{dt}{\log x} \\
&\ll \frac{x^{\frac{1}{\theta}}(\log\log x + c_4)^{N+1}}{(\log x)^{N+2-h\left(\frac{1}{\theta}\right)}}\left(\int_{\frac{c}{2\theta}}^{\frac{c}{2\theta\log 3}\log x} u^{N+1-h\left(\frac{1}{\theta}\right)} e^{-u} du + c_5\right) \\
&\ll \frac{x^{\frac{1}{\theta}}(\log\log x + c_4)^{N+1}}{(\log x)^{N+2-h\left(\frac{1}{\theta}\right)}}\Gamma\left(N+2-h\left(\frac{1}{\theta}\right)\right) \\
&\ll \frac{x^{\frac{1}{\theta}}(\log\log x + c_4)^{N+1}}{(\log x)^{N+2-h\left(\frac{1}{\theta}\right)}}\left(c_6 N + 1\right)^{N+1}. \qquad (3.2.7)
\end{aligned}
$$

Therefore by combining the estimates (3.2.6) and (3.2.7), we obtain

$$
\Phi'(x) = \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-h\left(\frac{1}{\theta}\right)}}\left(\sum_{m=0}^{N}\frac{P_m(\log\log x)}{(\log x)^m} + O\left(\left(\frac{c_5 N + 1}{\log x}\left(c_4 + \log\log x\right)\right)^{N+1}\right)\right).
$$

In order to complete the proof, we need to show that $\Phi'(x)$ is a good approximation for $A_g(x)$. Let ϵ be a positive real number such that $\epsilon < \frac{x}{2}$. By (3.2.1), we have

$$
\int_x^{x+\epsilon} A_g(t)dt = \Phi(x+\epsilon) - \Phi(x) + O\left(Mx^{1+\frac{1}{\theta}} Err(x)^{-c_1}\right).
$$

Since $\Phi(x)$ is twice differentiable for $x \geq 3$, by Taylor's formula we have

$$
\Phi(x+\epsilon) - \Phi(x) = \epsilon\Phi'(x) + \epsilon^2\int_0^1 (1-t)\Phi''(x+\epsilon t)dt.
$$

Appealing to (3.2.2) gives us

$$
\Phi(x+\epsilon) - \Phi(x) = \epsilon\Phi'(x) + O\left(\epsilon^2 M x^{\frac{1}{\theta}-1}(\log x)^{\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|}\right).
$$

Hence

$$
\int_x^{x+\epsilon} A_g(t)dt = \epsilon\Phi'(x) + O\left(\epsilon^2 M x^{\frac{1}{\theta}-1}(\log x)^{\left|\Re\left(h\left(\frac{1}{\theta}\right)\right)\right|}\right) + O\left(Mx^{1+\frac{1}{\theta}} Err(x)^{-c_1}\right). \qquad (3.2.8)
$$

Next, note that

$$\int_x^{x+\epsilon} A_g(t)dt = \epsilon A_g(x) + \int_x^{x+\epsilon} (A_g(t) - A_g(x))\, dt. \tag{3.2.9}$$

Next, by combining (3.2.8) and (3.2.9), we obtain

$$\epsilon A_g(x) - \epsilon\Phi'(x) = \int_x^{x+\epsilon} (A_g(t) - A_g(x))\, dt + O\left(\epsilon^2 M x^{\frac{1}{\theta}-1} (\log x)^{\left|\Re(h(\frac{1}{\theta}))\right|}\right) + O\left(M x^{1+\frac{1}{\theta}} Err(x)^{-c_1}\right).$$

By dividing both sides by ϵ, we obtain the estimate,

$$A_g(x) - \Phi'(x) \ll \frac{1}{\epsilon}\int_x^{x+\epsilon} |A_g(t) - A_g(x)|\, dt + O\left(\epsilon M x^{\frac{1}{\theta}-1} (\log x)^{\left|\Re(h(\frac{1}{\theta}))\right|}\right) + O\left(\frac{1}{\epsilon} M x^{1+\frac{1}{\theta}} Err(x)^{-c_1}\right). \tag{3.2.10}$$

Since $F_g(s)$ has property $\mathcal{B}^+(h, h^+; c, M, \theta, \delta, \alpha)$, there exists a positive real-valued function $f^+(n)$ such that $|f(n)| \leq f^+(n)$ for all n and $F_g^+(s) = \sum_{n=1}^{\infty} \frac{f^+(n)}{g(n)^s}$ has property $\mathcal{B}(h^+; c, M, \theta, \delta, \alpha, \kappa)$. Thus, there exists $\Psi(x)$ such that

$$\int_0^x A_g^+(t)dt = \Psi(x) + O\left(M x^{1+\frac{1}{\theta}} Err(x)^{-c_1}\right)$$

and

$$\Psi''(x) \ll M x^{\frac{1}{\theta}-1} (\log x)^{\left|\Re(h(\frac{1}{\theta}))\right|},$$

where

$$A_g^+(x) := \sum_{g(n)\leq x} f^+(n).$$

Next, note that

$$|A_g(t) - A_g(x)| \leq A_g^+(t) - A_g^+(x).$$

Therefore

$$\begin{aligned}
\int_x^{x+\epsilon} |A_g(t) - A_g(x)|\, dt &\leq \int_x^{x+\epsilon} A_g^+(t) - A_g^+(x)dt \\
&\leq \int_x^{x+\epsilon} A_g^+(t)dt - \int_{x-\epsilon}^{x} A_g^+(t)dt \\
&\leq \Psi(x+\epsilon) + \Psi(x-\epsilon) - 2\Psi(x) + O\left(M x^{1+\frac{1}{\theta}} Err(x)^{-c_6}\right)
\end{aligned}$$

$$\ll \epsilon^2 \sup_{x-\epsilon<y<x+\epsilon} \left|\Psi''(x)\right| + Mx^{1+\frac{1}{\theta}} Err(x)^{-c_6}$$

$$\ll \epsilon^2 Mx^{\frac{1}{\theta}-1}(\log x)^{\left|\Re(h^+(\frac{1}{\theta}))\right|} + Mx^{1+\frac{1}{\theta}} Err(x)^{-c_7}. \tag{3.2.11}$$

By combining (3.2.10) and (3.2.11), we have

$$A_g(x) - \Phi'(x) \ll \epsilon Mx^{\frac{1}{\theta}-1}(\log x)^{\left|\Re(h(\frac{1}{\theta}))\right|} + \frac{1}{\epsilon} Mx^{1+\frac{1}{\theta}} Err(x)^{-c_8}.$$

Finally, let $\epsilon = xErr(x)^{-c_9}$. We have

$$A_g(x) - \Phi'(x) \ll Mx^{\frac{1}{\theta}} Err(x)^{-c_{10}}.$$

This completes the proof of Theorem 2.4.3. □

3.3 Preliminary Results for Theorem 2.4.6

3.3.1 Analytic Continuation of $U(s)$

One of the essential parts of the proof of Theorem 2.4.6 is the analytic continuation of

$$U(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\tilde{\lambda}}$$

to the region $\mathcal{D}_{c,\theta}$, where $\mathcal{D}_{c,\theta}$ is defined in 2.3.5.

Theorem 3.3.1. *Let χ be a Dirichlet character modulo q and let $\tilde{q}$ be a positive integer such that $q \mid \tilde{q}$. Let $\theta > 0$ and let c be a positive constant such that $L(\theta s, \xi)$ does not vanish in the region $\mathcal{D}_{c,\theta}$ for all Dirichlet characters ξ modulo $\tilde{q}$. Let $\lambda : (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times \to [0,1]$. Let $\mathcal{P}$ be a set of primes such that for any $\eta > 1 - \frac{c}{\log 3}$*

$$\left|\#\left\{p : p < x, p \in \mathcal{P}, p \equiv a \,(\mathrm{mod}\, \tilde{q})\right\} - \lambda(a)\pi(a,q,x)\right| < x^\eta \tag{3.3.1}$$

for all $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$ and all sufficiently large x. Then the function

$$U(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\tilde{\lambda}},$$

where

$$\tilde{\lambda} = \frac{1}{\varphi(\tilde{q})} \sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a)\lambda(a),$$

has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$.

Before we embark on the proof of Theorem 3.3.1, a comment is in order. One may wonder whether the statement of the theorem still holds with a weaker assumption on the bound on the right side of inequality (3.3.1). As we will see later, if one does not change anything else in the statement of the theorem, except for the bound on the right side of (3.3.1), then having a bound of the form $O(x^\eta)$ for some $0 \le \eta < 1$ is necessary.

Proof. Fix $c, \theta, \chi, q, \tilde{q}$ and a function λ as in the statement of the theorem. Next, fix a set of primes $\mathcal{P}$ such that for any $\eta > 1 - \frac{c}{\log 3}$,

$$|\mathcal{P}_{a,\tilde{q}}(x) - \lambda(a)\pi(a,q,x)| < x^\eta$$

for each $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$ and all x sufficiently large, where

$$\mathcal{P}_{a,\tilde{q}}(x) := \{p : p < x, p \in \mathcal{P}, p \equiv a \,(\mathrm{mod}\, \tilde{q})\}.$$

We need to show that the function defined for $\sigma > \frac{1}{\theta}$ by

$$U(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\tilde{\lambda}}$$

has an analytic continuation to the region $\mathcal{D}_{c,\theta}$, and does not vanish in this region. In order to prove this, it is sufficient to show that for any real number $\frac{1}{\theta} - \frac{c}{\theta \log 3} < \sigma_1 < \frac{1}{\theta}$, the function $U(s)$

has an analytic continuation and does not vanish in the region

$$R_{c,\theta,\sigma_1} := \{s \in \mathcal{D}_{c,\theta} : \sigma > \sigma_1\}.$$

Fix σ_1. Next fix an η such that

$$1 - \frac{c}{\log 3} < \eta < \theta\sigma_1.$$

With η fixed, we now proceed to first study the function

$$G(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} G_a(s)^{-\lambda(a)\chi(a)},$$

where

$$G_a(s) = \prod_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1}. \tag{3.3.2}$$

For $\sigma > \frac{1}{\theta}$, we see that

$$\begin{aligned}
\log G(s) &= -\sum_{p \in \mathcal{P}} \log\left(1 - \frac{\chi(p)}{p^{\theta s}}\right) - \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \lambda(a)\chi(a) \log\left(G_a(s)\right) \\
&= -\sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \sum_{\substack{p \in \mathcal{P} \\ p \equiv a \,(\mathrm{mod}\, \tilde{q})}} \log\left(1 - \frac{\chi(p)}{p^{\theta s}}\right) + \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \lambda(a)\chi(a) \sum_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \log\left(1 - \frac{1}{p^{\theta s}}\right) \\
&= \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a) \sum_{\substack{p \in \mathcal{P} \\ p \equiv a \,(\mathrm{mod}\, \tilde{q})}} \sum_{k=1}^{\infty} \frac{\chi(a^{k-1})}{kp^{k\theta s}} - \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \lambda(a)\chi(a) \sum_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \sum_{k=1}^{\infty} \frac{1}{kp^{k\theta s}} \\
&= K_1(s) + K_2(s),
\end{aligned} \tag{3.3.3}$$

where

$$K_1(s) = \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a) \left(\sum_{\substack{p \in \mathcal{P} \\ p \equiv a \,(\mathrm{mod}\, \tilde{q})}} \frac{1}{p^{\theta s}} - \lambda(a) \sum_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \frac{1}{p^{\theta s}} \right)$$

and

$$K_2(s) = \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a) \left(\sum_{\substack{p \in \mathcal{P} \\ p \equiv a \,(\mathrm{mod}\, \tilde{q})}} \sum_{k=2}^{\infty} \frac{\chi(a^{k-1})}{kp^{k\theta s}} - \lambda(a) \sum_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \sum_{k=2}^{\infty} \frac{1}{kp^{k\theta s}} \right).$$

The multiple sums which define $K_2(s)$ are absolutely convergent for $\sigma > \frac{1}{2\theta}$. Therefore $K_2(s)$ has an analytic continuation to the half-plane $\sigma > \frac{1}{2\theta}$. Now, consider

$$K_{1,a}(s,x) = \sum_{\substack{p\in\mathcal{P} \\ p\equiv a \,(\mathrm{mod}\,\tilde{q}) \\ p<x}} \frac{1}{p^{\theta s}} - \lambda(a) \sum_{\substack{p\equiv a \,(\mathrm{mod}\,\tilde{q}) \\ p<x}} \frac{1}{p^{\theta s}}. \tag{3.3.4}$$

By applying Abel's summation,

$$\begin{aligned} K_{1,a}(s,x) &= \frac{\mathcal{P}_{a,\tilde{q}}(x) - \lambda(a)\pi(a,q,x)}{x^{\theta s}} + \theta s \int_1^x \frac{\mathcal{P}_{a,\tilde{q}}(u) - \lambda(a)\pi(a,q,u)}{u^{\theta s+1}} du \\ &= \frac{\mathcal{P}_{a,\tilde{q}}(x) - \lambda(a)\pi(a,q,x)}{x^{\theta s}} + \theta s \int_1^\infty \frac{\mathcal{P}_{a,\tilde{q}}(u) - \lambda(a)\pi(a,q,u)}{u^{\theta s+1}} du \\ &\quad - \theta s \int_x^\infty \frac{\mathcal{P}_{a,\tilde{q}}(u) - \lambda(a)\pi(a,q,u)}{u^{\theta s+1}} du \\ &= \theta s \int_1^\infty \frac{\mathcal{P}_{a,\tilde{q}}(u) - \lambda(a)\pi(a,q,u)}{u^{\theta s+1}} du + O\left(x^{\eta-\theta\sigma}\right) + O\left(\theta\,|s| \int_x^\infty u^{\eta-\theta\sigma-1} du\right), \end{aligned}$$

as x tends to infinity. All the integrals are absolutely convergent for $\sigma > \frac{\eta}{\theta}$. Thus

$$\sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a) \left(K_{1,a}(s,x) + \theta s \int_x^\infty \frac{\mathcal{P}_{a,\tilde{q}}(u) - \lambda(a)\pi(a,q,u)}{u^{\theta s+1}} du - \frac{\mathcal{P}_{a,\tilde{q}}(x) - \lambda(a)\pi(a,q,x)}{x^{\theta s}}\right) \tag{3.3.5}$$

provides an analytic continuation of $K_1(s)$ to the half-plane $\sigma > \frac{\eta}{\theta}$. Therefore

$$\log G(s) = K_1(s) + K_2(s)$$

has an analytic continuation to the half-plane $\sigma > \frac{\eta}{\theta}$. It follows that the function

$$G(s) = e^{K_1(s)+K_2(s)}$$

has an analytic continuation and does not vanish in the region R_{c,θ,σ_1}. Since this holds for any $\sigma_1 > \frac{1}{\theta} - \frac{c}{\theta\log 3}$, it follows that $G(s)$ has an analytic continuation and does not vanish in the open half plane $\sigma > \frac{1}{\theta} - \frac{c}{\theta\log 3}$.

Next, in order to obtain the analytic continuation and zero-free region for $U(s)$, it is sufficient to do this for the ratio $\frac{U(s)}{G(s)}$. First recall, for any $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$,

$$G_a(s) := \prod_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1}.$$

Thus, for $\sigma > \frac{1}{\theta}$ and a Dirichlet character ξ modulo $\tilde{q}$,

$$\begin{aligned}
\log(G_a(s)) &= -\sum_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \log\left(1 - \frac{1}{p^{\theta s}}\right) \\
&= -\frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \sum_p \xi(p) \log\left(1 - \frac{1}{p^{\theta s}}\right) \\
&= \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \sum_p \sum_{k=1}^{\infty} \frac{\xi(p)}{kp^{k\theta s}} \\
&= \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \left(\sum_p \sum_{k=1}^{\infty} \frac{\xi(p^k)}{kp^{k\theta s}} + \bar{\xi}(a) \sum_p \sum_{k=2}^{\infty} \frac{\xi(p) - \xi(p^k)}{kp^{k\theta s}} \right) \\
&= H_a(s, \theta, \xi) + H_a^*(s, \theta, \xi), \qquad (3.3.6)
\end{aligned}$$

where

$$H_a(s, \theta, \xi) = \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \sum_p \sum_{k=1}^{\infty} \frac{\xi(p^k)}{kp^{k\theta s}},$$

and

$$H_a^*(s, \theta, \xi) = \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \sum_p \sum_{k=2}^{\infty} \frac{\xi(p) - \xi(p^k)}{kp^{k\theta s}}.$$

The function $H_a^*(s, \theta, \xi)$ is absolutely convergent for $\sigma > \frac{1}{2\theta}$ and defines an analytic function on the half plane $\sigma > \frac{1}{2\theta}$. Now, note that for $\sigma > \frac{1}{\theta}$,

$$H_a(s, \theta, \xi) = \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \log \prod_p \left(1 - \frac{\xi(p)}{p^{\theta s}}\right)^{-1} = \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\, \tilde{q})} \bar{\xi}(a) \log L(\theta s, \xi). \qquad (3.3.7)$$

Therefore, by (3.3.6) and (3.3.7), we see that

$$G_a(s) = e^{H_a^*(s, \theta, \xi)} \prod_{\xi \,(\mathrm{mod}\, \tilde{q})} L(\theta s, \xi)^{\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}}.$$

Now, for $\xi = \xi_0$ and $\sigma > \frac{1}{\theta}$,

$$L(\theta s, \xi_0) = \prod_p \left(1 - \frac{\xi_0(p)}{p^{\theta s}}\right)^{-1} = \prod_{p|\tilde{q}} \left(1 - \frac{1}{p^{\theta s}}\right) \prod_p \left(1 - \frac{1}{p^{\theta s}}\right)^{-1} = \prod_{p|\tilde{q}} \left(1 - \frac{1}{p^{\theta s}}\right) \zeta(\theta s).$$

Thus for $\sigma > \frac{1}{\theta}$,

$$G_a(s) = e^{H_a^*(s,\theta,\xi)} \prod_{p|\tilde{q}} \left(1 - \frac{1}{p^{\theta s}}\right)^{\frac{1}{\varphi(\tilde{q})}} \zeta(\theta s)^{\frac{1}{\varphi(\tilde{q})}} \prod_{\substack{\xi \,(\mathrm{mod}\, \tilde{q}) \\ \xi \neq \xi_0}} L(\theta s, \xi)^{\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}} = M_a(s) \zeta(\theta s)^{\frac{1}{\varphi(\tilde{q})}},$$

where

$$M_a(s) = e^{H_a^*(s,\theta,\xi)} \prod_{p|\tilde{q}} \left(1 - \frac{1}{p^{\theta s}}\right)^{\frac{1}{\varphi(\tilde{q})}} \prod_{\substack{\xi \,(\mathrm{mod}\, \tilde{q}) \\ \xi \neq \xi_0}} L(\theta s, \xi)^{\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}}.$$

Thus

$$\prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} G_a(s)^{-\lambda(a)\chi(a)} = \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \left(M_a(s) \zeta(\theta s)^{\frac{1}{\varphi(\tilde{q})}}\right)^{-\lambda(a)\chi(a)} = \zeta(\theta s)^{-\lambda} \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} M_a(s)^{-\lambda(a)\chi(a)}, \tag{3.3.8}$$

where

$$\tilde{\lambda} = \frac{1}{\varphi(\tilde{q})} \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a)\lambda(a).$$

Since the Dirichlet L-functions $L(\theta s, \xi)$ for non-principal characters ξ do not vanish and have analytic continuations to the region $\mathcal{D}_{c,\theta}$, it follows that the product

$$\prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} M_a(s)^{-\lambda(a)\chi(a)}$$

has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$. Now, by (3.3.8),

$$\begin{aligned} U(s) &= \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda} \\ &= \left(\prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} G_a(s)^{-\lambda(a)\chi(a)}\right) \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} M_a(s)^{\lambda(a)\chi(a)} \end{aligned}$$

$$= G(s) \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} M_a(s)^{\lambda(a)\chi(a)} \tag{3.3.9}$$

Therefore $U(s)$ has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$ as desired. $\square$

Next, we prove a converse of a simplified version of Theorem 3.3.1, which does not involve a Dirichlet Character χ. The general question we address is the following. For which increasing functions $\beta : [1, \infty) \to [1, \infty)$ does the following claim hold, and for which does functions the claim does not hold?

Claim. *Let $c, \theta > 0$. Let $0 \leq \lambda \leq 1$ and let $\mathcal{P}$ be a set of primes such that the function defined for $\sigma > \frac{1}{\theta}$ by*

$$F_{\mathcal{P}}(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda}$$

has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$. Then for any sequence of primes $\tilde{\mathcal{P}}$ satisfying

$$\#\left(\left(\left(\mathcal{P} \backslash \tilde{\mathcal{P}}\right) \cup \left(\tilde{\mathcal{P}} \backslash \mathcal{P}\right)\right) \cap [1, x]\right) \leq \beta(x)$$

for all $x \geq 1$, the function defined for $\sigma > \frac{1}{\theta}$ by

$$F_{\tilde{\mathcal{P}}}(s) = \prod_{p \in \tilde{\mathcal{P}}} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda}$$

has an analytic continuation to $\mathcal{D}_{c,\theta}$ and does not vanish in the region.

From the proof of Theorem 3.3.1 above, it follows that the claim holds for any function β which satisfies the following condition: For any $\eta > 1 - \frac{c}{\log 3}$, we have

$$\beta(x) \leq x^{\eta} \tag{3.3.10}$$

for all x sufficiently large. We now prove a converse, namely that in order for the claim to hold it is necessary for β to satisfy inequalities of the form (3.3.10). That is to say, if β has a larger order of magnitude than the one allowed by (3.3.10), then there exists a set of primes $\tilde{\mathcal{P}}$ satisfying (3.3.1) for which the corresponding function $F_{\tilde{\mathcal{P}}}$ either vanishes in the region $\mathcal{D}_{c,\theta}$ or does not have an analytic continuation to the region. To be precise, we will prove the following theorem.

Theorem 3.3.2. *Let $c, \theta > 0$. Let $\lambda \geq 0$ and let $\mathcal{P}$ be a set of primes such that the function defined for $\sigma > \frac{1}{\theta}$ by*

$$F_{\mathcal{P}}(s) := \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda}$$

has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$. Let $\tilde{\mathcal{P}}$ be a set of primes containing $\mathcal{P}$ such that the function defined for $\sigma > \frac{1}{\theta}$ by

$$F_{\tilde{\mathcal{P}}}(s) := \prod_{p \in \tilde{\mathcal{P}}} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda}$$

has an analytic continuation to $\mathcal{D}_{c,\theta}$. Then, for any $\eta > 1 - \frac{c}{\log 3}$, we have

$$\#\left(\left(\tilde{\mathcal{P}} \backslash \mathcal{P}\right) \cap [1, x]\right) \leq x^{\eta}$$

for all x sufficiently large.

This not only shows that a bound of the form $\beta(x) = O\left(x^{\eta}\right)$, with $\eta < 1$, is necessary in order for the above associated claim to hold, but moreover, the constant $1 - \frac{c}{\log 3}$ is the best possible.

Proof. Fix $c, \theta, \lambda, \mathcal{P}$ and $\tilde{\mathcal{P}}$, as in the statement of the theorem. Thus the functions $F_{\mathcal{P}}(s)$ and $F_{\tilde{\mathcal{P}}}(s)$ have analytic continuations to $\mathcal{D}_{c,\theta}$, and $F_{\mathcal{P}}(s)$ does not vanish in this region. Define, for $\sigma > \frac{1}{\theta}$,

$$F(s) := \frac{F_{\tilde{\mathcal{P}}}(s)}{F_{\mathcal{P}}(s)},$$

so for $\sigma > \frac{1}{\theta}$,

$$F(s) = \prod_{p \in \tilde{\mathcal{P}} \backslash \mathcal{P}} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1},$$

and by our assumption, $F(s)$ has an analytic continuation to the region $\mathcal{D}_{c,\theta}$. Next, fix an η such that $1 - \frac{c}{\log 3} < \eta < 1$. We need to show that

$$\mathcal{P}^*(x) = \#\left(\left(\tilde{\mathcal{P}} \backslash \mathcal{P}\right) \cap [1, x]\right) \leq M x^{\eta}$$

for some constant $M > 0$ and for x sufficiently large.

In order to do this, we first show that $F(s)$ has an analytic continuation to the region $\sigma > \sigma_1$

for any $\frac{1}{\theta} - \frac{c}{\theta \log 3} < \sigma_1 < \frac{\eta}{\theta}$. Thus let $r_{c,\theta}$ be a real number such that $\frac{2-\eta}{\theta} < r_{c,\theta} < \frac{1}{\theta} + \frac{c}{\theta \log 3}$. Next, let γ be a circle centered at $\frac{2}{\theta}$ with radius $r_{c,\theta}$. Then, by Lemma 3.1.8,

$$\left| \frac{F^{(m)}\left(\frac{2}{\theta}\right)}{m!} \right| \leq \sup_{z \in \gamma} |F(z)| \, r_{c,\theta}^{-m} = c_1 r_{c,\theta}^{-m} \tag{3.3.11}$$

for some positive constant c_1. Next, for $s > \frac{1}{\theta}$, $F(s)$ can be expressed as a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\theta s}},$$

where $a_n = 1$ if $n = 1$ or if all prime factors of n are in $\tilde{\mathcal{P}} \backslash \mathcal{P}$; otherwise $a_n = 0$. It is crucial in our proof that the coefficients a_n are non-negative. Moreover, $F(s)$ is absolutely convergent for $s > \frac{1}{\theta}$. Hence

$$\frac{F^{(m)}(s)}{m!} = \frac{1}{m!} \sum_{n=1}^{\infty} \frac{a_n (-\theta)^m (\log n)^m}{n^{\theta s}} \tag{3.3.12}$$

for $s > \frac{1}{\theta}$. Now, let $s_0 = \frac{2}{\theta} + it$ for some $t \in \mathbb{R}$. By (3.3.11) and (3.3.12),

$$\left| \frac{F^{(m)}(s_0)}{m!} \right| \leq \frac{1}{m!} \sum_{n=1}^{\infty} \left| \frac{a_n (-\theta)^m (\log n)^m}{n^{\theta s}} \right| = \frac{1}{m!} \sum_{n=1}^{\infty} \frac{a_n \theta^m (\log n)^m}{n^2} = \left| \frac{F^{(m)}\left(\frac{2}{\theta}\right)}{m!} \right| \leq c_1 r_{c,\theta}^{-m}. \tag{3.3.13}$$

The function $F(s)$ has a Laurent series expansion as $s = s_0$,

$$F(s) = \sum_{n=1}^{\infty} \frac{F^{(n)}(s_0)}{n!} (s - s_0)^n,$$

with radius of convergence R_t. Thus

$$\frac{1}{R_t} = \limsup_{n \to \infty} \left| \frac{F^{(n)}(s_0)}{n!} \right|^{\frac{1}{n}} \leq \limsup_{n \to \infty} \left| c_2 r_{c,\theta}^{-n} \right|^{\frac{1}{n}} = \frac{1}{r_{c,\theta}}.$$

Hence $R_t \geq r_{c,\theta}$. Since these are true for any $t \in \mathbb{R}$, we have shown that $F(s)$ has an analytic continuation to the half-plane $\sigma > \frac{2}{\theta} - r_{c,\theta} =: \sigma_1$. Thus $F(s)$ has an analytic continuation for $\sigma \geq \frac{\eta}{\theta} > \sigma_1$.

Our next step is to apply Perron's formula to $F(s)$ and obtain a good estimation for

$$A(x) := \sum_{n<x} a_n,$$

where $a_n = 1$ if $n = 1$ or if all prime factors of n are in $\tilde{\mathcal{P}} \backslash \mathcal{P}$; otherwise $a_n = 0$.

First, we want to show that

$$\int_{\frac{2}{\theta}-i\infty}^{\frac{2}{\theta}+i\infty} F(s)\frac{x^{\theta s}}{s}ds = \int_{\frac{\eta}{\theta}-i\infty}^{\frac{\eta}{\theta}+i\infty} F(s)\frac{x^{\theta s}}{s}ds. \tag{3.3.14}$$

So, consider a positively oriented rectangular contour, Γ, with vertices $(\frac{\eta}{\theta} \pm iT)$ and $(\frac{2}{\theta} \pm iT)$ for some $T > 1$. Since $F(s)$ is analytic in the region $\sigma \geq \eta$, by the residue theorem,

$$\frac{1}{2\pi i}\int_{\Gamma} F(s)\frac{x^s}{s}ds = 0.$$

Thus, (3.3.14) follows if we can show that the integral over the two horizontal segments approachs 0 as T tends to infinity. First we will obtain an estimate for $F(s)$ for $\sigma \geq \frac{\eta}{\theta}$. By (3.3.13), for $\frac{\eta}{\theta} \leq \sigma_0 \leq \frac{2}{\theta}$,

$$|F(\sigma_0 + iT)| \leq \sum_{n=1}^{\infty}\left|\frac{F^{(n)}\left(\frac{2}{\theta}+iT\right)}{n!}\right|\left|\sigma_0 + iT - \left(\frac{2}{\theta}+iT\right)\right|^n \leq \sum_{n=1}^{\infty} c_1\left|\frac{\sigma_0 - \frac{2}{\theta}}{r_{c,\theta}}\right|^n.$$

Since $\left|\sigma_0 - \frac{2}{\theta}\right| < r_{c,\theta}$, then

$$|F(s)| \leq c_3 \tag{3.3.15}$$

for some constant $c_3 > 0$ and for all $\sigma \geq \frac{\eta}{\theta}$. It follows that

$$\left|\int_{\frac{2}{\theta}+iT}^{\frac{\eta}{\theta}+iT} F(s)\frac{x^{\theta s}}{s}ds\right| \leq \int_{\frac{\eta}{\theta}+iT}^{\frac{2}{\theta}+iT} |F(s)|\frac{\left|x^{\theta s}\right|}{|s|}ds \leq \int_{\frac{\eta}{\theta}}^{\frac{2}{\theta}} \frac{c_3 x^{\theta u}}{\sqrt{\left(\frac{\eta}{\theta}\right)^2 + T^2}}du \leq \frac{c_4 x^2}{T}$$

for some positive constant c_4. A similar estimate holds for the integral over the horizontal segment $[\frac{\eta}{\theta} + iT, \frac{2}{\theta} + iT]$. Thus by letting T tend to infinity, (3.3.14) follows.

Now, we turn our attention to the summatory function $A(x)$. Since a_n is non-negative for all n

and $a_p = 1$ for $p \in \tilde{\mathcal{P}} \backslash \mathcal{P}$, then

$$\mathcal{P}^*(x) \leq A(x)$$

Then by applying Perron's formula and combining (3.3.14) and (3.3.15), we obtain

$$\mathcal{P}^*(x) \leq A(x) = \frac{1}{2\pi i} \int_{\frac{2}{\theta}-i\infty}^{\frac{2}{\theta}+i\infty} \frac{F(s)}{\theta s} x^{\theta s} ds = \frac{1}{2\pi i} \int_{\frac{\eta}{\theta}-i\infty}^{\frac{\eta}{\theta}+i\infty} \frac{F(s)}{\theta s} x^{\theta s} ds \leq c_3 x^{\eta} \frac{1}{2\pi i} \int_{\frac{\eta}{\theta}-i\infty}^{\frac{\eta}{\theta}+i\infty} \frac{ds}{\theta s} \leq c_5 x^{\eta}$$

for some constant $c_5 > 0$ as desired. □

We are now ready to prove the converse to Theorem 3.3.1.

Theorem 3.3.3. *Let $c, \theta > 0$. Let $\lambda \geq 0$ and let χ be a Dirichlet character modulo q. Let $\mathcal{P}$ be a set of primes such that the function defined for $\sigma > \frac{1}{\theta}$ by*

$$F_{\mathcal{P}}(s) := \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda}$$

has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$ for some complex number λ. Let $\tilde{\mathcal{P}}$ be a set of primes containing $\mathcal{P}$ such that the function defined for $\sigma > \frac{1}{\theta}$ by

$$F_{\tilde{\mathcal{P}}}(s) := \prod_{p \in \tilde{\mathcal{P}}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\lambda}$$

has an analytic continuation to $\mathcal{D}_{c,\theta}$, and such that for all $p \in \tilde{\mathcal{P}} \backslash \mathcal{P}, \chi(p) = 1$. Then, for any $\eta > 1 - \frac{c}{\log 3}$, we have

$$\#\left(\left(\tilde{\mathcal{P}} \backslash \mathcal{P}\right) \cap [1, x]\right) \leq x^{\eta}$$

for all x sufficiently large.

Proof. Fix $c, \theta, \lambda, \mathcal{P}$ and $\tilde{\mathcal{P}}$, as in the statement of the theorem. Thus the functions $F_{\mathcal{P}}(s)$ and $F_{\tilde{\mathcal{P}}}(s)$ have analytic continuations to $\mathcal{D}_{c,\theta}$, and $F_{\mathcal{P}}(s)$ does not vanish in this region. Define, for $\sigma > \frac{1}{\theta}$,

$$F(s) := \frac{F_{\tilde{\mathcal{P}}}(s)}{F_{\mathcal{P}}(s)}$$

so for $\sigma > \frac{1}{\theta}$,

$$F(s) = \prod_{p \in \tilde{\mathcal{P}} \setminus \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} = \prod_{p \in \tilde{\mathcal{P}} \setminus \mathcal{P}} \left(1 - \frac{1}{p^{\theta s}}\right)^{-1}$$

and by our assumption, $F(s)$ has an analytic continuation to the region $\mathcal{D}_{c,\theta}$. The remainder of the proof proceeds in the same manner as the proof of Theorem 3.3.2. $\square$

3.3.2 The Distribution of Values of $U(s)$

Theorem 3.3.4. *Let χ be a Dirichlet character modulo q and let $\tilde{q}$ be a positive integer such that $q \mid \tilde{q}$. Let $\theta > 0$ and let c be a positive constant such that $L(\theta s, \xi)$ does not vanish in the region $\mathcal{D}_{c,\theta}$ for all Dirichlet characters ξ modulo $\tilde{q}$. Define $\lambda : (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times \to [0,1]$. Let $\mathcal{P}$ be a set of primes such that for any $\eta > 1 - \frac{c}{\log 3}$*

$$|\#\{p : p < x, p \in \mathcal{P}, p \equiv a \,(\mathrm{mod}\, \tilde{q})\} - \lambda(a)\pi(a,q,x)| < x^\eta \tag{3.3.16}$$

for all $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$ and all sufficiently large x. Define

$$U(s) := \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\tilde{\lambda}},$$

where

$$\tilde{\lambda} = \frac{1}{\varphi(\tilde{q})} \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \chi(a)\lambda(a).$$

Then, for $|t| > \frac{3}{\theta}$ and all $s \in \mathcal{D}_{c,\theta}$,

$$|U(s)| \le c_1(\log(3 + |t|))^A \tag{3.3.17}$$

for some positive constants c_1 and A.

Proof. In this proof, we will refer to several calculations we did in Theorem 3.3.1. By equation (3.3.9), we see that, for $\sigma > \frac{1}{\theta}$,

$$U(s) = G(s) \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} M_a(s)^{\lambda(a)\chi(a)},$$

where

$$G(s)=\prod_{p\in\mathcal{P}}\left(1-\frac{\chi(p)}{p^{\theta s}}\right)^{-1}\prod_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}}G_a(s)^{-\lambda(a)\chi(a)},$$

$$G_a(s)=\prod_{p\equiv a\,(\mathrm{mod}\,\tilde{q})}\left(1-\frac{1}{p^{\theta s}}\right)^{-1},$$

$$M_a(s)=e^{H_a^*(s,\theta,\xi)}\prod_{p|\tilde{q}}\left(1-\frac{1}{p^{\theta s}}\right)^{\frac{1}{\varphi(\tilde{q})}}\prod_{\substack{\xi\,(\mathrm{mod}\,\tilde{q})\\ \xi\neq\xi_0}}L(\theta s,\xi)^{\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}},$$

and

$$H_a^*(s,\theta,\xi)=\frac{1}{\varphi(\tilde{q})}\sum_{\xi\,(\mathrm{mod}\,\tilde{q})}\bar{\xi}(a)\sum_{p}\sum_{k=2}^{\infty}\frac{\xi(p)-\xi(p^k)}{kp^{k\theta s}}.$$

First, we will determine an upper bound for the function $|G(s)|$. By equation (3.3.3), for $\sigma>\frac{1}{\theta}$,

$$\log G(s)=K_1(s)+K_2(s),$$

where

$$K_1(s)=\sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}}\chi(a)\sum_{p\equiv a\,(\mathrm{mod}\,\tilde{q})}\frac{\mathcal{I}_{\mathcal{P}}(p)-\lambda(a)}{p^{\theta s}},$$

and

$$K_2(s)=\sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}}\chi(a)\sum_{p\equiv a\,(\mathrm{mod}\,\tilde{q})}\sum_{k=2}^{\infty}\frac{\chi(a^{k-1})\mathcal{I}_{\mathcal{P}}(p)-\lambda(a)}{kp^{k\theta s}},$$

where $\mathcal{I}_{\mathcal{P}}$ is the characteristic function on the set $\mathcal{P}$. For $K_1(s)$, suppose that $\sigma\geq\frac{2}{\theta}$. Since $0\leq\lambda(a)\leq 1$, then

$$|K_1(s)|\leq\sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}}|\chi(a)|\sum_{p\equiv a\,(\mathrm{mod}\,\tilde{q})}\frac{|\mathcal{I}_{\mathcal{P}}(p)-\lambda(a)|}{p^{\theta\sigma}}\sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}}\sum_{p\equiv a\,(\mathrm{mod}\,\tilde{q})}\frac{1}{p^2}\leq M_1$$

for some positive constant M_1. Now, suppose $\sigma<\frac{2}{\theta}$. By equation (3.3.5), for $x>1$,

$$K_1(s)=\sum_{a\in(\mathbb{Z}/\tilde{q}\mathbb{Z})^{\times}}\chi(a)\left(J_1(a,s,x)+J_2(a,s,x)+J_3(a,s,x)\right),\tag{3.3.18}$$

where

$$J_1(a,s,x) := \sum_{\substack{p\equiv a \pmod{\tilde{q}}\\ p<x}} \frac{\mathcal{I}_{\mathcal{P}}(p)-\lambda(a)}{p^{\theta s}},$$

$$J_2(a,s,x) := -\frac{\mathcal{P}_{a,\tilde{q}}(x)-\lambda(a)\pi(a,q,x)}{x^{\theta s}},$$

and

$$J_3(a,s,x) := \theta s\int_x^\infty \frac{\mathcal{P}_{a,\tilde{q}}(u)-\lambda(a)\pi(a,q,u)}{u^{\theta s+1}}du$$

is an analytic continuation of $K_1(s)$ to the region $\mathcal{D}_{c,\theta}$.

First note that, for $s\in\mathcal{D}_{c,\theta}$ and $|t|\geq\frac{3}{\theta}$, then $-\theta\sigma<-1+\frac{c}{\log|\theta t|}$. It follows that, for $s\in\mathcal{D}_{c,\theta}$ and $|t|\geq\frac{3}{\theta}$,

$$|J_1(a,s,x)| \leq \sum_{\substack{p\equiv a \pmod{\tilde{q}}\\ p<x}} \frac{|\mathcal{I}_{\mathcal{P}}(p)-\lambda(a)|}{p^{\theta\sigma}} \leq \sum_{p<x}\frac{1}{p^{\theta\sigma}} \leq \sum_{p<x}\frac{e^{\frac{c\log p}{\log|\theta t|}}}{p}.$$

By setting $x=|\theta t|^{\frac{1}{1-\eta}}$, we deduce that

$$\left|J_1(a,s,|\theta t|^{\frac{1}{1-\eta}})\right| \leq \sum_{p<|\theta t|^{\frac{1}{1-\eta}}} \frac{e^{\frac{c\log|\theta t|}{(1-\eta)\log|\theta t|}}}{p} \ll \sum_{p<|\theta t|^{\frac{1}{1-\eta}}}\frac{1}{p} \ll \log\log\left(|\theta t|^{\frac{1}{1-\eta}}\right) \ll \log\log(3+|t|) \tag{3.3.19}$$

for $s\in\mathcal{D}_{c,\theta}$ and $|t|>\frac{3}{\theta}$. Next, by (3.3.16), we see that, for $\sigma>\max\{\frac{\eta}{\theta},\frac{1}{\theta}-\frac{c}{\theta\log 3}\}$, $s\in\mathcal{D}_{c,\theta}$ and $|t|>\frac{3}{\theta}$,

$$|J_2(a,s,x)| \leq \frac{|\mathcal{P}_{a,\tilde{q}}(x)-\lambda(a)\pi(a,q,x)|}{x^{\theta\sigma}} \ll x^{\eta-\theta\sigma} \ll x^{\eta-1+\frac{c}{\log|\theta t|}}.$$

Again, by letting $x=|\theta t|^{\frac{1}{1-\eta}}$, we deduce that

$$\left|J_2(a,s,|\theta t|^{\frac{1}{1-\eta}})\right| \ll |\theta t|^{\frac{1}{1-\eta}\left(\eta-1+\frac{c}{\log|\theta t|}\right)} \ll |\theta t|^{-1}\, e^{\frac{c\log|\theta t|}{(1-\eta)\log(|\theta t|)}} \ll \frac{1}{|t|} \tag{3.3.20}$$

for $s\in\mathcal{D}_{c,\theta}$ and $|t|>\frac{3}{\theta}$.

Lastly, we estimate $J_3(a,s,x)$. Since $\sigma < \frac{2}{\theta}$, for $|t| > \frac{3}{\theta}$,

$$|s| \leq \sqrt{\left(\frac{2}{\theta}\right)^2 + t^2} \leq 2\,|t|\,.$$

It follows that, for $\sigma > \max\{\frac{\eta}{\theta}, \frac{1}{\theta} - \frac{c}{\theta \log 3}\}$, $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$,

$$\begin{aligned}|J_3(a,s,x)| &\leq |\theta s| \int_x^\infty \frac{|\mathcal{P}_{a,\tilde{q}}(u) - \lambda(a)\pi(a,q,u)|}{u^{\theta\sigma+1}} du \\ &\ll |\theta s| \int_x^\infty u^{\eta-\theta\sigma-1} du \ll \frac{|\theta t|}{\theta\sigma - \eta} x^{\eta-\theta\sigma} \ll |\theta t|\, x^{\eta - 1 + \frac{c}{\log|\theta t|}}.\end{aligned}$$

Letting $x = |\theta t|^{\frac{1}{1-\eta}}$, we see that

$$\left| J_3(a,s,|\theta t|^{\frac{1}{1-\eta}}) \right| \ll |\theta t|\, |\theta t|^{\frac{1}{1-\eta}\left(\eta - 1 + \frac{c}{\log|\theta t|}\right)} \ll |\theta t|\, |\theta t|^{-1}\, |\theta t|^{\frac{c}{(1-\eta)\log|\theta t|}} \ll 1 \tag{3.3.21}$$

for $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$.

Therefore, by combining (3.3.19), (3.3.20), and (3.3.21), we deduce that

$$K_1(s) \leq C_1 \log\log(3 + |t|),$$

for some positive constant C_1 for all $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$. Next, we will show that $K_2(s) = O(1)$. For $\sigma \geq \sigma_1 > \frac{1}{2\theta}$,

$$\begin{aligned}|K_2(s)| &\leq \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \sum_{p \equiv a \,(\mathrm{mod}\, \tilde{q})} \sum_{k=2}^\infty \left| \frac{\chi(a^{k-1})\mathcal{I}_{\mathcal{P}}(p) - \lambda(a)}{kp^{k\theta s}} \right| \\ &\leq \sum_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \sum_p \sum_{k=2}^\infty \frac{2}{kp^{k\theta\sigma}} \\ &\leq \varphi(\tilde{q}) \sum_p \frac{1}{p^{2\theta\sigma}} \left(\frac{p^{\theta\sigma}}{p^{\theta\sigma} - 1} \right) \\ &\leq \varphi(\tilde{q}) \frac{\sqrt{2}}{\sqrt{2} - 1} \sum_p \frac{1}{p^{2\theta\sigma}} \\ &\leq \varphi(\tilde{q})(2 + \sqrt{2})\zeta(2\sigma\theta) \leq \varphi(\tilde{q})(2 + \sqrt{2})\zeta(2\sigma_1\theta).\end{aligned}$$

Thus, by choosing $\sigma_1 < \frac{1}{\theta} - \frac{c}{\theta \log 3}$, we deduce that

$$|K_2(s)| \leq C_2$$

for some positive constant C_2 and for all $s \in \mathcal{D}_{c,\theta}$. Therefore

$$|\log G(s)| \leq C_1 \log\log|t| + C_2.$$

Thus for $|t| > \frac{3}{\theta}$ and $s \in \mathcal{D}_{c,\theta}$,

$$|G(s)| = e^{|\log G(s)|} \leq e^{C_1 \log\log|t| + C_2} \leq C_3(\log|t|)^{C_1} \tag{3.3.22}$$

for some positive constant C_3.

Next, we will begin estimating $M_a(s)$. Recall that

$$M_a(s) = e^{H_a^*(s,\theta,\xi)} \prod_{p|\tilde{q}} \left(1 - \frac{1}{p^{\theta s}}\right)^{\frac{1}{\varphi(\tilde{q})}} \prod_{\substack{\xi \,(\mathrm{mod}\,\tilde{q}) \\ \xi \neq \xi_0}} L(\theta s, \xi)^{\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}},$$

where

$$H_a^*(s,\theta,\xi) = \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\,\tilde{q})} \bar{\xi}(a) \sum_p \sum_{k=2}^{\infty} \frac{\xi(p) - \xi(p^k)}{kp^{k\theta s}}.$$

First, we estimate $H_a^*(s,\theta,\xi)$. For $\sigma > \sigma_2 > \frac{1}{2\theta}$, we see that

$$\begin{aligned}
|H_a^*(s,\theta,\xi)| &= \left| \frac{1}{\varphi(\tilde{q})} \sum_{\xi \,(\mathrm{mod}\,\tilde{q})} \bar{\xi}(a) \sum_p \sum_{k=2}^{\infty} \frac{\xi(p) - \xi(p^k)}{kp^{k\theta s}} \right| \\
&\leq \sum_p \sum_{k=2}^{\infty} \left| \frac{\xi(p) - \xi(p^k)}{kp^{k\theta s}} \right| \\
&\leq \sum_p \sum_{k=2}^{\infty} \frac{2}{kp^{k\theta\sigma}} \\
&\leq \sum_p \frac{1}{p^{2\theta\sigma}} \left(\frac{p^{\theta\sigma}}{p^{\theta\sigma} - 1} \right) \\
&\leq \frac{\sqrt{2}}{\sqrt{2} - 1} \sum_p \frac{1}{p^{2\theta\sigma}}
\end{aligned}$$

$$\leq (2+\sqrt{2})\zeta(2\sigma\theta) \leq \zeta(2\sigma_2\theta).$$

Thus, by choosing $\sigma_2 < \frac{1}{\theta} - \frac{c}{\theta \log 3}$, we deduce that

$$|H_a^*(s,\theta,\xi)| \leq C_4$$

for some positive constant C_4 for all $s \in \mathcal{D}_{c,\theta}$. Next, by [2, Lemma 3], for a non-principal Dirichlet character ξ, $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$,

$$|L(\theta s, \xi)| \leq C_5 \log |\theta t|$$

for some positive constant C_5 depending on $\tilde{q}$. It follows that, for $s \in \mathcal{D}_{c,\theta}$ and any $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$,

$$|M_a(s)| = e^{|H_a^*(s,\theta,\xi)|} \prod_{p|\tilde{q}} \left|1 - \frac{1}{p^{\theta s}}\right|^{\frac{1}{\varphi(\tilde{q})}} \prod_{\substack{\xi \pmod{\tilde{q}} \\ \xi \neq \xi_0}} \left| L(\theta s, \xi)^{\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}} \right| \ll \prod_{\substack{\xi \pmod{\tilde{q}} \\ \xi \neq \xi_0}} |C_5 \log |t||^{\left|\frac{\bar{\xi}(a)}{\varphi(\tilde{q})}\right|}.$$

Thus, for every $a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times$, $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$,

$$|M_a(s)| \leq C_7 (\log |t|)^{C_6}$$

for some positive constants C_6 and C_7. Lastly, by combining (3.3.22) and (3.3.2), we see that, for $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$,

$$|U(s)| = |G(s)| \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} \left| M_a(s)^{\lambda(a)\chi(a)} \right| \ll \log |T| \prod_{a \in (\mathbb{Z}/\tilde{q}\mathbb{Z})^\times} C_7 (\log(|t|)^{C_6 |\lambda(a)\chi(a)|} \ll (\log |t|)^A$$

for some positive constant A. Hence, for $s \in \mathcal{D}_{c,\theta}$,

$$|U(s)| \leq M_F (\log(3 + |t|))^A$$

as desired. □

3.4 Proof of Theorem 2.4.6

Proof. Fix $F_g(s), T(s), A(s), c, \theta, M, \delta, \alpha, \kappa, \mathcal{P}, \chi, \tilde{q}, \lambda, h_p(s), h(s)$ and η as stated in the theorem. Let $F_g(s)$ have property $\mathcal{A}^+(h_p, h, h^+; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$. Define

$$\tilde{\lambda} := \frac{1}{\varphi(\tilde{q})} \sum_{a \in \mathbb{Z}/\tilde{q}\mathbb{Z}} \chi(a)\lambda(a).$$

We want to show that $F_g(s)$ has property $\mathcal{B}^+(\tilde{\lambda}h, h^+; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. First, we will show that $F_g(s)$ has property $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. First, condition (1) of property $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$ follows directly from condition (1) of $\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$. Next, we will show that $F_g(s)$ satisfies condition (2) of property $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. By property (2) of $\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$,

$$F_g(s) = H(s) \prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)},$$

where $H(s)$ is an analytic function on the region $\mathcal{D}_{c,\theta}$ satisfing the inequality

$$|H(s)| \leq M(\log(3 + |t|))^{\delta}$$

for all $s \in \mathcal{D}_{c,\theta}$. Thus, for $\sigma > \frac{1}{\theta}$,

$$F_g(s) = H(s) \left(\prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)} \zeta(\theta s)^{-\tilde{\lambda}h(s)} \right) \zeta(\theta s)^{\tilde{\lambda}h(s)} = \tilde{H}(s)\zeta(\theta s)^{\tilde{h}(s)}, \tag{3.4.1}$$

where

$$\tilde{H}(s) = H(s) \left(\prod_{p \in \mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)} \zeta(\theta s)^{-\tilde{\lambda}h(s)} \right) = H(s)T(s),$$

and

$$\tilde{h}(s) = \tilde{\lambda}h(s).$$

This proves condition (2) of $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$.

Next, we will show that $F_g(s)$ satisfies condition (4) of $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. By property (5) of $\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$, for all $s \in \mathcal{D}_{c,\theta}$,

$$\left|\tilde{h}(s)\right| = \left|\tilde{\lambda}h(s)\right| \leq \left|\tilde{\lambda}\right| |h(s)| \leq c_2(\log(3+|t|))^{\alpha}$$

for $0 \leq \alpha < 1$ for some positive constant c_1. It remains to show that $F_g(s)$ satisfies condition (3) of $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. More precisely, we need to show that

$$\tilde{H}(s) = H(s)T(s)$$

has an analytic continuation to the region $\mathcal{D}_{c,\theta}$ and satisfies the inequality

$$\left|\tilde{H}(s)\right| \leq c_2(3+|t|)^{\frac{\delta+1}{2}}$$

for some positive constant c_2. First, we will focus on the analytic continuation and upper bound for $T(s)$. Consider, for $\sigma > \frac{1}{\theta}$,

$$T(s) = \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-h_p(s)} \zeta(\theta s)^{-\tilde{\lambda}h(s)} = T_1(s)T_2(s),$$

where

$$T_1(s) := \left(\prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\tilde{\lambda}}\right)^{h(s)},$$

and

$$T_2(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{h(s)-h_p(s)}.$$

By Theorem 3.3.1, the function

$$U(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi(p)}{p^{\theta s}}\right)^{-1} \zeta(\theta s)^{-\tilde{\lambda}}$$

has an analytic continuation and does not vanish in the region $\mathcal{D}_{c,\theta}$. Thus, by property (5) of

$\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$, $h(s)$ is analytic in the region $\mathcal{D}_{c,\theta}$. Hence

$$T_1(s) = U(s)^{h(s)}$$

has an analytic continuation to the region $\mathcal{D}_{c,\theta}$. Next, we calculate an upper bound of $T_1(s)$. By Theorem 3.3.4, there exist positive constants c_3 and c_4 such that

$$|U(s)| \leq c_3(\log(3 + |t|))^{c_4}.$$

It follows that

$$\log |U(s)| \leq \log(c_3(\log(3 + |t|))^{c_4}) = \log(c_3) + c_4 \log(\log(3 + |t|)) \leq c_5 \log\log(3 + |t|)$$

for some positive constant c_5. Moreover, by property (5) of $\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$,

$$|h(s)| \leq M(\log(3 + |t|))^{\alpha}.$$

Therefore,

$$\begin{aligned}
|T_1(s)| \leq |U(s)|^{|h(s)|} e^{\pi|h(s)|} &\leq \exp\{|h(s)| \left(\log |U(s)| + \pi\right)\} \\
&\leq \exp\{M \log(3 + |t|)^{\alpha}(c_5 \log\log(3 + |t|) + \pi)\} \\
&\leq \exp\{c_6(\log(3 + |t|))^{\alpha'}\}
\end{aligned}$$

for some positive constant c_6 and $\alpha < \alpha' < 1$. Since $\alpha' < 1$, it follows that for all sufficiently large t,

$$c_6 \log(3 + |t|)^{\alpha'} \leq \frac{1 - \delta}{4} \log(3 + |t|).$$

Therefore

$$|T_1(s)| \leq c_7(3 + |t|)^{\frac{1-\delta}{4}} \tag{3.4.2}$$

for some positive constant c_7.

Next, we turn our attention to $T_2(s)$. First, taking the logarithm of $T_2(s)$, we see that, for $\sigma > \frac{1}{\theta}$,

$$\begin{aligned}\log T_2(s) &= \sum_{p\in\mathcal{P}}(h(s)-h_p(s))\log\left(1-\frac{\chi(p)}{p^{\theta s}}\right)\\ &= \sum_{p\in\mathcal{P}}(h_p(s)-h(s))\sum_{k=1}^{\infty}\frac{\chi(p^k)}{kp^{k\theta s}}\\ &= J_1(s)+J_2(s),\end{aligned}$$

where

$$J_1(s) := \sum_{p\in\mathcal{P}}(h_p(s)-h(s))\frac{\chi(p)}{p^{\theta s}},$$

and

$$J_2(s) := \sum_{p\in\mathcal{P}}(h_p(s)-h(s))\sum_{k=2}^{\infty}\frac{\chi(p^k)}{kp^{k\theta s}}.$$

The function $J_2(s)$ converges for $\sigma > \frac{1}{2\theta}$. Moreover, by property (5) of $\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$, for $\sigma \geq \frac{1}{\theta} - \frac{c}{\theta\log 3}$,

$$\begin{aligned}|J_2(s)| &\leq \sum_{p\in\mathcal{P}}|h_p(s)-h(s)|\sum_{k=2}^{\infty}\left|\frac{\chi(p^k)}{kp^{k\theta s}}\right|\\ &\leq \sum_{p\in\mathcal{P}}(|h_p(s)|+|h(s)|)\sum_{k=2}^{\infty}\frac{1}{kp^{k\theta\sigma}}\\ &\leq 2M(\log(3+|t|))^{\alpha}\sum_{p}\frac{1}{p^{2\theta\sigma}}\\ &\leq 2M(\log(3+|t|))^{\alpha}\frac{\sqrt{2}}{\sqrt{2}-1}\sum_{p}\frac{1}{p^{2\theta\sigma}}\\ &\leq 2M(\log(3+|t|))^{\alpha}(2+\sqrt{2})\zeta(2\sigma\theta)\\ &\leq c_9(\log(3+|t|))^{\alpha} \qquad (3.4.3)\end{aligned}$$

for some positive constant c_9. Next, we will estimate $J_1(s)$. First, we define a function

$$B(x) := \sum_{\substack{p\in\mathcal{P}\\ p<x}}(h_p(s)-h(s))\chi(p).$$

Also, for $T > 0$, we define

$$K_T := K_{c,\theta,T} = \left\{ s : s \in \bar{\mathcal{D}}_{c,\theta}, \sigma \leq \frac{2}{\theta}, |t| \leq T \right\}.$$

Then by property (6) of $\mathcal{A}(h_p, h, ; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$, for $T > 0$ and for all $\eta > 1 - \frac{c}{\log 3}$,

$$|B(x)| \leq \sum_{\substack{p \in \mathcal{P} \\ p < x}} |h_p(s) - h(s)| \leq \sum_{\substack{p \in \mathcal{P} \\ p < x}} \|h_p(s) - h(s)\|_{\infty, K_T} \leq c_8 x^{\eta}, \tag{3.4.4}$$

for some constant c_8 depending on $\mathcal{P}, c, \theta$, and η. Let

$$J_1(s, x) := \sum_{\substack{p \in \mathcal{P} \\ p < x}} (h_p(s) - h(s)) \frac{\chi(p)}{p^{\theta s}}.$$

Then, by applying Abel's Summation formula, we see that, for $\sigma > \frac{1}{\theta}$,

$$J_1(s, x) = \frac{B(x)}{x^{\theta s}} + \theta s \int_1^x \frac{B(u)}{u^{\theta s + 1}} du.$$

By (3.4.4), we see that for $x > 1$,

$$|J_1(s, x)| \leq \frac{|B(x)|}{x^{\theta \sigma}} + |\theta s| \int_1^x \frac{|B(u)|}{u^{\theta \sigma + 1}} du \leq c_8 x^{\eta - \theta \sigma} + c_8 |\theta s| \int_1^x u^{\eta - \theta \sigma - 1} du. \tag{3.4.5}$$

As x tends to infinity, the right hand side of (3.4.5) converges when $\sigma > \frac{\eta}{\theta}$. It follows that

$$J_1(s, x) - \frac{B(x)}{x^{\theta s}} + \theta s \int_x^{\infty} \frac{B(u)}{u^{\theta s + 1}} du \tag{3.4.6}$$

is an analytic continuation of $J(s)$ to the region $\mathcal{D}_{c,\theta}$. We now determine an upper bound for $J_1(s)$. For $\sigma \geq \frac{2}{\theta}$, we see that

$$|J_1(s)| \leq \sum_{p \in \mathcal{P}} |h_p(s) - h(s)| \frac{1}{p^2} \leq 2M(\log(3 + |t|))^{\alpha} \sum_{p \in \mathcal{P}} \leq \frac{1}{p^2} c_{10} (\log(3 + |t|))^{\alpha}$$

for some positive constant c_{10}. So assume that $\sigma < \frac{2}{\theta}$. Next note that, for $s \in \mathcal{D}_{c,\theta}$ and $|t| \geq \frac{3}{\theta}$, then $-\theta\sigma < -1 + \frac{c}{\log|\theta t|}$. It follows that, for $s \in \mathcal{D}_{c,\theta}$ and $|t| \geq \frac{3}{\theta}$,

$$|J_1(s,x)| \leq \sum_{\substack{p\in\mathcal{P}\\ p<x}} |h_p(s) - h(s)| \frac{1}{p^{\theta\sigma}} \leq 2M(\log(3+|t|))^{\alpha} \sum_{p<x} \frac{1}{p^{\theta\sigma}} \leq 2M(\log(3+|t|))^{\alpha} \sum_{p<x} \frac{e^{\frac{c\log p}{\log|\theta t|}}}{p}.$$

By setting $x = |\theta t|^{\frac{1}{1-\eta}}$, we deduce that, for $s \in \mathcal{D}_{c,\theta}$ and $|t| > \frac{3}{\theta}$,

$$\begin{aligned}
\left|J_1(s, |\theta t|^{\frac{1}{1-\eta}})\right| &\leq 2M(\log(3+|t|))^{\alpha} \sum_{p<|\theta t|^{\frac{1}{1-\eta}}} \frac{e^{\frac{c\log|\theta t|}{(1-\eta)\log|\theta t|}}}{p} \\
&\ll M(\log(3+|t|))^{\alpha} \sum_{p<|\theta t|^{\frac{1}{1-\eta}}} \frac{1}{p} \\
&\ll M(\log(3+|t|))^{\alpha} \log\log\left(|\theta t|^{\frac{1}{1-\eta}}\right) \\
&\ll M(\log(3+|t|))^{\alpha} \log\log(3+|t|) \\
&\ll M(\log(3+|t|))^{\alpha''} \qquad (3.4.7)
\end{aligned}$$

for some $\alpha < \alpha'' < 1$. Next, by (3.4.6), we see that, for $s \in \mathcal{D}_{c,\theta}, \sigma < \frac{2}{\theta}$ and $|t| \geq \frac{3}{\theta}$,

$$\begin{aligned}
\left|-\frac{B(x)}{x^{\theta s}} + \theta s\int_x^{\infty} \frac{B(u)}{u^{\theta s+1}}du\right| &\leq \left|\frac{B(x)}{x^{\theta s}}\right| + \left|\theta s\int_x^{\infty} \frac{B(u)}{u^{\theta s+1}}du\right| \\
&\ll x^{\eta-\theta\sigma} + |\theta s|\int_x^{\infty} u^{\eta-\theta\sigma-1}du \\
&\ll x^{\eta-\theta\sigma} + \frac{|\theta t|}{\theta\sigma - \eta} x^{\eta-\theta\sigma} \\
&\ll x^{\eta-1+\frac{c}{\log|\theta t|}} + |\theta t|\, x^{\eta-1+\frac{c}{\log|\theta t|}}.
\end{aligned}$$

Again, by letting $x = |\theta t|^{\frac{1}{1-\eta}}$, we deduce that

$$\left|-\frac{B(x)}{x^{\theta s}} + \theta s\int_x^{\infty} \frac{B(u)}{u^{\theta s+1}}du\right| \ll |\theta t|^{-1+\frac{c}{(1-\eta)\log|\theta t|}} + |\theta s|\,|\theta t|^{-1+\frac{c}{(1-\eta)\log|\theta t|}} \ll \frac{1}{|\theta t|} + 1 \ll 1. \qquad (3.4.8)$$

Therefore, by (3.4.7) and (3.4.8), we obtain, for $s \in \mathcal{D}_{c,\theta}$,

$$|J_1(s)| \leq c_{12}(\log(3+|t|))^{\alpha''} \qquad (3.4.9)$$

for some positive constant c_{12} and $\alpha < \alpha'' < 1$. Then, by (3.4.9), (3.4.3) and $0 \le \alpha < \alpha'' < 1$, we see that, for sufficiently large t,

$$|\log(T_2(s))| \le |J_1(s)| + |J_2(s)| \le c_{12}(\log(3+|t|))^{\alpha''} + c_9(\log(3+|t|))^{\alpha} < \frac{1-\delta}{4}\log(3+|t|).$$

It follows that

$$|T_2(s)| \ll \exp\{|\log(T_1(s))|\} \ll e^{\frac{1-\delta}{4}\log(3+|t|)} \ll (3+|t|)^{\frac{1-\delta}{4}}. \tag{3.4.10}$$

Thus, for $s \in \mathcal{D}_{c,\theta}$,

$$|T(s)| = |T_1(s)|\,|T_2(s)| \ll (3+|t|)^{\frac{1-\delta}{4}}(3+|t|)^{\frac{1-\delta}{4}} \ll (3+|t|)^{\frac{1-\delta}{2}}.$$

Now, since $|H(s)| \le M(3+|t|)^{\delta}$ for all $s \in \mathcal{D}_{c,\theta}$, we obtain

$$\left|\tilde{H}(s)\right| = |H(s)|\,|T(s)| \ll M(3+|t|)^{\delta}(3+|t|)^{\frac{1-\delta}{2}}(3+|t|)^{\frac{1+\delta}{2}}.$$

By choosing $M_F > M$ sufficiently large, we obtain

$$\left|\tilde{H}(s)\right| \le M_F(3+|t|)^{\frac{1+\delta}{2}}$$

and

$$\left|\tilde{h}(s)\right| \le M_F(\log(3+|t|))^{\alpha}$$

for all $s \in \mathcal{D}_{c,\theta}$. These complete the proof of condition (2) of $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. Therefore $F_g(s)$ has property $\mathcal{B}(\tilde{\lambda}h; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. Lastly, we need to show that $F_g(s)$ satisfies the remaining conditions of $\mathcal{B}^+(\tilde{\lambda}h, h^+; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. Conditions (1) and (3) of $\mathcal{B}(\tilde{\lambda}h, h^+; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$ follow directly from conditions (1) and (3) of $\mathcal{A}^+(h_p, h, h^+; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$. It remains to show condition (2) of $\mathcal{B}(\tilde{\lambda}h, h^+; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. Since $\delta < \frac{1+\delta}{2}$ and $M_F > M$, by condition (2) of $\mathcal{A}^+(h_p, h, h^+; \mathcal{P}, \chi, \lambda, \tilde{q}, c, \theta, M, \delta, \alpha, \kappa)$, we obtain

$$\left|H^+(s)\right| \le M(3+|t|)^{\delta} \le M_F(3+|t|)^{\frac{\delta+1}{2}}.$$

Thus, $F_g(s)$ has property $\mathcal{B}^+(\tilde{\lambda} h, h^+; c, \theta, M_F, \frac{\delta+1}{2}, \alpha, \kappa)$. Therefore, by applying Theorem 2.4.3, we obtain, for $x \geq 3$ and $N \geq 0$,

$$\begin{aligned} A_g(x) := \sum_{g(n)<x} f(n) &= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\tilde{h}(\frac{1}{\theta})}} \left(\sum_{m=0}^{N} \frac{P_m(\log\log x)}{(\log x)^m} + O\left(R_N\right) \right) \\ &= \frac{x^{\frac{1}{\theta}}}{(\log x)^{1-\tilde{\lambda} h(\frac{1}{\theta})}} \left(\sum_{m=0}^{N} \frac{P_m(\log\log x)}{(\log x)^m} + O\left(R_N\right) \right), \end{aligned}$$

where

$$\begin{aligned} P_m(x) &:= \sum_{j=0}^{m} \sum_{n=j}^{m} \sum_{i=0}^{n-j} \frac{e_{m,n}}{\theta^{\tilde{h}(\frac{1}{\theta})}} \binom{n}{i} \binom{n-i}{j} \frac{(-1)^{i+j} (\log\theta)^{n-i-j}}{\Gamma_i\left(\tilde{h}\left(\frac{1}{\theta}\right) - m\right)} x^j \\ &= \sum_{j=0}^{m} \sum_{n=j}^{m} \sum_{i=0}^{n-j} \frac{e_{m,n}}{\theta^{\tilde{\lambda} h(\frac{1}{\theta})}} \binom{n}{i} \binom{n-i}{j} \frac{(-1)^{i+j} (\log\theta)^{n-i-j}}{\Gamma_i\left(\tilde{\lambda} h\left(\frac{1}{\theta}\right) - m\right)} x^j, \end{aligned}$$

$$\begin{aligned} e_{m,n} &= \frac{(-1)^n}{n!} \sum_{k=n}^{m} \frac{\left(T\left(\frac{1}{\theta}\right) A\left(\frac{1}{\theta}\right)\right)^{(m-k)} a_{k,n}}{(m-k)!} \\ &= \frac{(-1)^n}{n!} \sum_{k=n}^{m} \sum_{l=0}^{m-k} \binom{m-k}{l} T^{(l)}\left(\frac{1}{\theta}\right) \frac{A^{(m-k-l)}\left(\frac{1}{\theta}\right) a_{k,n}}{(m-k)!}, \end{aligned}$$

$$a_{k,n} = \tilde{\lambda}^n \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_i \geq 1}} \prod_{i=1}^{n} \frac{h^{(k_i)}\left(\frac{1}{\theta}\right)}{k_i!},$$

and

$$R_N = \left(\frac{c_{13}N+1}{\log x} \left(c_{14} + \log\log x\right) \right)^{N+1} + M Err(x)^{-c_{15}}, \tag{3.4.11}$$

where

$$Err(x) = e^{c_{16}\sqrt{\log x}}$$

for some constants c_{13}, c_{14}, c_{15} and c_{16}. This completes the proof. $\square$

Chapter 4

Partial Multiple Zeta Values Identities

4.1 History of Special Values of the Riemann Zeta Function

The problem of evaluating particular values of the zeta function has a long history. It first appeared in 1644, when Pietro Mengoli proposed the problem of evaluating $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is now known as the Basel problem. The Basel problem was first solved by Euler who used the infinite product representation of $\sin x$ to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Moreover, Euler [17, p.185] showed

$$\sum_{n=1}^{\infty} \frac{1}{n^{26}} = \frac{2^{24}}{27!}(76977927\pi^{26}) = \frac{1315862\pi^{26}}{11094481976030578125},$$

which is the largest even integer he able to computed. However, Euler's method failed to calculate the values for positive odd integers, which is still an open problem. In the past century, many mathematicians have found different proofs to obtain the solution to the Basel problem. These proofs were found by E.L Stark [56], F. Holme [28], D.P. Giesy [21], I. Papdimitriou [44], T. Apostol [1], A. Yaglom and I. Yaglom [65], B. R.Choe [3], G.F. Simmon [54], R. Kortram [32], J. Hofbauer [25], J.D. Harper [24], and D. Ritelli [47]. However, we will be focusing on the proof given by Y. Matuoka [39].

Y. Matsuoka proved the Basel problem by using the integral $\int_0^{\pi/2} x^2 \cos^{2n}(x)dx$ and showing that

$$\frac{\prod_{k=1}^{n} 2k}{\prod_{k=1}^{n}(2k-1)} \int_0^{\pi/2} x^2 \cos^{2n}(x)dx = \frac{\pi}{4}\left(\frac{\pi^2}{6} - \sum_{i=1}^{n} \frac{1}{i^2}\right).$$

Thus as n tends to infinity, this yields the solution of the Basel problem. In 2010, Kenneth Stolarsky asked for a similar result for $\zeta(4)$. However, we find that the result can not be generalized to $\zeta(4)$ directly, but it is connected to a similar object which is called a multiple zeta function.

4.2 Multiple Zeta Functions

The first type of multiple zeta function appeared in the 1742 letter of Goldbach to Euler in an attempt to evaluate the double sum

$$S_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{k^p}, \quad (q \geq 2, p, q \in \mathbb{N})$$

in terms of special values of the Riemann Zeta function at positive integers. This double sum is now know as a double Euler sum.

Definition 4.2.1. *Let $s_1, s_2, \ldots, s_k$ be complex values such that $\Re(s_1) + \Re(s_2) + \cdots + \Re(s_i) > i$ for all $i \leq k$. We define the multiple zeta function*

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{i=1}^{k} \frac{1}{n^{s_i}}.$$

When $s_1, \ldots, s_k$ are integers, then $\zeta(\bar{\nu})$ is often called a multiple zeta value or k-fold Euler sum. One primary goal is to obtain an identity for multiple zeta values. We refer the reader to the work of D. Zagier [66], A. Granville [22], C. Markett [38], V. Drinfeld [14], H. Tsumura [59], and M. Hoffman [26]. One useful identity proved by Euler is the Euler reflection formula which can be stated as follows. [18, pp. 217 - 267]

Proposition 4.2.2. *If $a, b > 1$, then*

$$\zeta(a, b) + \zeta(b, a) = \zeta(a)\zeta(b) - \zeta(a + b).$$

In particular,

$$\zeta(a, a) = \frac{1}{2} \left(\zeta^2(a) - \zeta(2a)\right). \tag{4.2.1}$$

Now, we will define the partial zeta function.

Definition 4.2.3. *Let $\bar{\nu} = (s_1, \ldots, s_k)$ be a k-tuple of complex values such that $\Re(s_1) + \Re(s_2) + \cdots + \Re(s_i) > i$ for all $i \leq k$. For $N > k$, we define a partial zeta function of order N,*

$$\zeta_N(\bar{\nu}) = \sum_{N \geq n_1 > n_2 > \cdots > n_k > 0} \prod_{i=1}^{k} \frac{1}{n^{s_i}}.$$

4.3 Main Result

First, for convenience, we introduce the following definition.

Definition 4.3.1. *For positive integers m and n, we define*

$$J(m,n) = \frac{\int_0^{\pi/2} x^{2m} \cos^{2n} x dx}{\int_0^{\pi/2} \cos^{2n} x \, dx}.$$

Observe that $J(0,n) = 1$. Now, we are ready to state our theorem.

Theorem 4.3.2. *For any positive integers m and n, we have*

$$J(m,n) = \frac{\int_0^{\pi/2} x^{2m} \cos^{2n} x dx}{\int_0^{\pi/2} \cos^{2n} x \, dx} = c_{m,0} + \sum_{l=1}^{m} \left(c_{m,l} \sum_{i=1}^{l} \sum_{r_1 + \cdots + r_i = l} \zeta_n(2r_1, \ldots, 2r_i) \right)$$

where

$$c_{m,l} = (-1)^l \frac{\pi^{2(m-l)}}{2^{2m}} \frac{(2m)!}{(2(m-l)+1)!}.$$

Moreover, by letting n tend to infinity, we obtain the identity

$$c_{m,0} + \sum_{l=1}^{m} \left(c_{m,l} \sum_{i=1}^{l} \sum_{r_1 + \cdots + r_i = l} \zeta(2r_1, \ldots, 2r_i) \right) = 0.$$

Proof. First, we will use integration by parts to find a recurrence relation of $J(m,n)$. To that end,

$$\begin{aligned}
I(m,n) &:= \int_0^{\pi/2} x^{2m} \cos^{2n}(x) dx \\
&= \frac{x^{2m+1}}{2m+1} \cos^{2n}(x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\pi/2} \frac{x^{2m+1}}{2m+1} \left(2n \cos^{2n-1}(x) \sin(x) \right) dx \\
&= -\frac{2n}{2m+1} \int_0^{\pi/2} x^{2m+1} \cos^{2n-1}(x) \sin(x) dx
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2n}{2m+1}\left(\frac{x^{2m+2}}{2m+2}\cos^{2n-1}(x)\sin(x)\bigg|_0^{\frac{\pi}{2}}\right.\\
&\quad\left.-\int_0^{\pi/2}\frac{x^{2m+2}}{2m+2}\left((2n-1)\cos^{2n-2}(x)\sin^2(x)+\cos^{2n}(x)\right)dx\right)\\
&= \frac{2n}{2m+1}\left(\int_0^{\pi/2}\frac{x^{2m+2}}{2m+2}\left((2n-1)\cos^{2n-2}(x)(1-\cos^2 x)\right)+\cos^{2n}(x)\right)dx\\
&= \frac{2n}{(2m+1)(2m+2)}\left((2n-1)\int_0^{\pi/2}x^{2m+2}\cos^{2n-2}(x)dx-2n\int_0^{\pi/2}x^{2m+2}\cos^{2n}(x)dx\right)\\
&= \frac{2n(2n-1)}{(2m+1)(2m+2)}I(m+1,n-1)-\frac{4n^2}{(2m+1)(2m+2)}I(m+1,n) \qquad (4.3.1)
\end{aligned}
$$

Also consider the following reduction formula

$$
\int_0^{\pi/2}\cos^{2n}(x)dx=\frac{2n-1}{2n}\int_0^{\pi/2}\cos^{2n-2}(x)\,dx. \qquad (4.3.2)
$$

Thus, by using (4.3.1) and (4.3.2), we deduce that

$$
\begin{aligned}
J(m,n) &= \frac{I(m,n)}{\int_0^{\pi/2}\cos^{2n}(x)dx}\\
&= \frac{2n(2n-1)}{(2m+1)(2m+2)}\frac{I(m+1,n-1)}{\int_0^{\pi/2}\cos^{2n}(x)dx}-\frac{4n^2}{(2m+1)(2m+2)}\frac{I(m+1,n)}{\int_0^{\pi/2}\cos^{2n}(x)dx}\\
&= \frac{4n^2}{(2m+1)(2m+2)}J(m+1,n-1)-\frac{4n^2}{(2m+1)(2m+2)}J(m+1,n)\\
&= -\frac{4n^2}{(2m+1)(2m+2)}\left(J(m+1,n)-J(m+1,n-1)\right).
\end{aligned}
$$

It follows that

$$
J(m+1,n)-J(m+1,n-1)=-\frac{(2m+1)(2m+2)}{4n^2}J(m,n).
$$

Thus

$$
J(m+1,n)-J(m+1,0)=-(2m+1)(2m+2)\sum_{k=1}^{n}\frac{1}{4k^2}J(m,k).
$$

Hence

$$
J(m+1,n)=\frac{1}{2m+3}\left(\frac{\pi}{2}\right)^{2m+2}-(2m+1)(2m+2)\sum_{k=1}^{n}\frac{1}{4k^2}J(m,k). \qquad (4.3.3)
$$

Now, we have the recurrence relations. We will prove the formula by induction on m. Clearly,

$$J(1,n)=\frac{1}{3}\left(\frac{\pi}{2}\right)^2-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k^2}=c_{1,0}+c_{1,1}\zeta_n(2).$$

Now, assume that

$$J(m,n)=c_{m,0}+\sum_{l=1}^{m}\left(c_{m,l}\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i=l}\zeta_n(2r_1,\ldots,2r_i)\right).$$

Then by (4.3.3), we deduce

$$\begin{aligned}
J(m+1,n)&=\frac{1}{2m+3}\left(\frac{\pi}{2}\right)^{2m+2}-(2m+1)(2m+2)\sum_{k=1}^{n}\frac{1}{4k^2}J(m,k)\\
&=\frac{1}{2m+3}\left(\frac{\pi}{2}\right)^{2m+2}\\
&\quad-(2m+1)(2m+2)\sum_{k=1}^{n}\frac{1}{4k^2}\left(c_{m,0}+\sum_{l=1}^{m}\left(c_{m,l}\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i=l}\zeta_n(2r_1,\ldots,2r_i)\right)\right)\\
&=c_{m+1,0}-(2m+1)(2m+2)\sum_{k=1}^{n}\frac{1}{4k^2}\left(c_{m,0}+\sum_{l=1}^{m}c_{m,l}\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i=l}\zeta_n(2r_1,\ldots,2r_i)\right).
\end{aligned}\tag{4.3.4}$$

Now, we have

$$\begin{aligned}
-\frac{(2m+1)(2m+2)}{4}c_{m,l}&=-\frac{(2m+1)(2m+2)}{4}\left((-1)^l\frac{\pi^{2(m-l)}}{2^{2m}}\frac{(2m)!}{(2(m-l)+1)!}\right)\\
&=(-1)^{l+1}\frac{\pi^{2((m+1)-(l+1)}}{2^{2m+2})}\left(\frac{(2m+2)!}{(2((m+1)-(l+1))+1)!}\right)\\
&=c_{m+1,l+1}.
\end{aligned}\tag{4.3.5}$$

Moreover,

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{k^2}\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i=l}\zeta_k(2r_1,\ldots,2r_i)&=\sum_{k=1}^{n}\frac{1}{k^2}\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i=l}\sum_{k\geq k_1>\cdots>k_i\geq 1}\frac{1}{k_1^{2r_1}\cdots k_i^{2r_i}}\\
&=\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i=l}\sum_{k=1}^{n}\sum_{k\geq k_1>\cdots>k_i\geq 1}\frac{1}{k^2k_1^{2r_1}\cdots k_i^{2r_i}}
\end{aligned}$$

$$= \sum_{i=1}^{l} \sum_{r_1+\cdots+r_i=l} (\zeta_n(2, 2r_1, \cdots, 2r_i) + \zeta_n(2r_1+2, \cdots, 2r_i))$$

$$= \sum_{i=1}^{l+1} \sum_{r_1+\cdots+r_i=l+1} \zeta_n(2r_1, \cdots, 2r_i). \tag{4.3.6}$$

Lastly, by combining (4.3.4), (4.3.5), and (4.3.6), we see that

$$\begin{aligned}
J(m+1,n) &= c_{m+1,0} - (2m+1)(2m+2)\sum_{k=1}^{n} \frac{1}{4k^2}\left(c_{m,0} + \sum_{l=1}^{m}\left(c_{m,l}\sum_{i=1}^{l}\sum_{r_1+\cdots+r_i} \zeta_n(2r_1, \ldots, 2r_i)\right)\right) \\
&= c_{m+1,0} - c_{m+1,l+1}\zeta_n(2) + \sum_{l=1}^{m} c_{m+1,l+1} \sum_{k=1}^{n} \frac{1}{k^2} \sum_{i=1}^{l} \sum_{r_1+\cdots+r_i} \zeta_n(2r_1, \ldots, 2r_i) \\
&= c_{m+1,0} - c_{m+1,l+1}\zeta_n(2) + \sum_{l=1}^{m} c_{m+1,l+1} \sum_{i=1}^{l+1} \sum_{r_1+\cdots+r_i=l+1} \zeta_n(2r_1, \ldots, 2r_i) \\
&= c_{m+1,0} - c_{m+1,l+1}\zeta_n(2) + \sum_{l=2}^{m+1} c_{m+1,l} \sum_{i=1}^{l} \sum_{r_1+\cdots+r_i=l} \zeta_n(2r_1, \ldots, 2r_i) \\
&= c_{m+1,0} + \sum_{l=1}^{m+1} c_{m+1,l} \sum_{i=1}^{l} \sum_{r_1+\cdots+r_i=l} \zeta_n(2r_1, \cdots, 2r_i),
\end{aligned}$$

as desired. It remain to show that $J(m,n)$ tends to 0 as n tends to infinity. By definition, we find that

$$\begin{aligned}
0 \le J(m,n) &= \frac{\int_0^{\pi/2} x^{2m} \cos^{2n}(x)dx}{\int_0^{\pi/2} \cos^{2n}(x)dx} \\
&\le \left(\frac{\pi}{2}\right)^{2m} \frac{\int_0^{\pi/2} \sin^2(x)\cos^{2n}(x)dx}{\int_0^{\pi/2} \cos^{2n}(x)dx} \\
&= \left(\frac{\pi}{2}\right)^{2m} \left(\frac{\int_0^{\pi/2} \cos^{2n}(x)dx}{\int_0^{\pi/2} \cos^{2n}(x)dx} - \frac{\int_0^{\pi/2} \cos^{2n+2}(x)dx}{\int_0^{\pi/2} \cos^{2n}(x)dx}\right) \\
&= \left(\frac{\pi}{2}\right)^{2m} \left(1 - \frac{2n+1}{2n+2}\right)
\end{aligned}$$

Thus by the Squeeze Theorem, $\lim_{n\to\infty} J(m,n) = 0$. □

In order to answer K. Stolarsky 's question regarding the values of $\zeta(4)$, we give the following corollary.

Corollary 4.3.3. *We have*

$$\zeta(4) = \frac{\pi^2}{3}\zeta(2) - \zeta^2(2) - \frac{\pi^4}{60} = \frac{\pi^4}{90}.$$

Proof. In Theorem 4.3.2, by letting $m = 2$ we find that

$$J(2,n) = \frac{\pi^4}{5 \cdot 2^4} - \frac{4\pi^2}{2^4}\zeta_n(2) + \frac{4!}{2^4}\left(\zeta_n(2,2) + \zeta_n(4)\right).$$

By letting n tends to infinity, we obtain

$$0 = \frac{\pi^4}{80} - \frac{\pi^2}{4}\zeta(2) + \frac{3}{2}\left(\zeta(2,2) + \zeta(4)\right).$$

By the identity (4.2.1), we find that

$$\zeta(2,2) + \zeta(4) = \frac{1}{2}\zeta^2(2) + \frac{1}{2}\zeta(4).$$

Therefore,

$$0 = \frac{\pi^4}{80} - \frac{\pi^2}{4}\zeta(2) + \frac{3}{4}\zeta^2(2) + \frac{3}{4}\zeta(4).$$

Hence

$$\zeta(4) = -\frac{\pi^4}{60} + \frac{\pi^2}{3}\zeta(2) - \zeta^2(2) = \frac{\pi^4}{90}$$

as desired. $\square$

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