IDENTITIES INVOLVING THETA FUNCTIONS AND ANALOGUES OF THETA FUNCTIONS

BY

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DISSERTATION

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Abstract

My dissertation is mainly about various identities involving theta functions and analogues of theta functions.

In Chapter 2, we give a completely elementary proof of Ramanujan's circular summation formula of theta functions and its generalizations given by S. H. Chan and Z. -G. Liu, and J. M. Zhu, who used the theory of elliptic functions. In contrast to all other proofs, our proofs are elementary. An application of this summation formula is given.

In Chapter 3, we analyze various generalized two-dimensional lattice sums, one of which arose from the solution to a certain Poisson equation. We evaluate certain lattice sums in closed form using results from Ramanujan's theory of theta functions, continued fractions and class invariants. Many nice explicit examples are given.

In Chapter 4, we study one page in Ramanujan's lost notebook that is devoted to claims about a certain integral with two parameters. One claim gives an inversion formula for the integral that is similar to the transformation formula for theta functions. Other claims are remindful of Gauss sums. In this chapter, we prove all the claims made by Ramanujan about this integral.

To my husband, Xiaokang, and our unborn baby boy.

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Chapter 1

Introduction

We might define a q-series to be one with summands containing expressions of the type

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 0, \tag{1.0.1}$$

where we interprete $(a;q)_0 = 1$ and $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$ when $|q| < 1$. However, this definition is not completely satisfactory. The main reason is that, in the theory of q-series, we often let parameters in the summands tend to 0 or to ∞, and consequently, factors of the type $(a;q)_n$ may be no longer be contained in the summands. Then theta functions frequently arise in the identities satisfied by series with $(a;q)_n$ in their summands. Dangling participle Ramanujan's notation, we define the general theta function by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.0.2}$$

The four most important special cases are defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.0.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.0.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.0.5}$$

$$\chi(q) := (-q;q^2)_\infty, \tag{1.0.6}$$

where the first three product representations above are special cases of the Jacobi triple product identity

$$f(a,b) = (-a,ab)_\infty (-b,ab)_\infty (ab,ab)_\infty, \tag{1.0.7}$$

the most useful theorem in the theory of theta functions that was first discovered by Gauss. The last equality in (1.0.5) is called Euler's pentagonal number theorem, which has its combinatorial interpretation in terms of the partitions. Theta functions play a prominent role in the theory of elliptic functions, modular forms and basic hypergeometric series. To learn how q-series articulate with theta functions and number theory, in particular in the spirit of Ramanujan, refer to [9]. My dissertation is mainly about various identities involving theta functions and analogues of theta functions, and applications of theta functions.

1.1 An elementary proof of Ramanujan's circular summation formula and its generalizations

In the history, several mathematicians, such as L. Euler, C.F. Gauss, E. Heine, F.H. Jackson, L.J. Rogers and S. Ramanujan played leading role in the development of q-series. Ramanujan undoubtedly contributed more to q-series than anyone either before or after his time. One of his most famous theorems is the ${}_1\psi_1$ summation theorem [31, Chap. 16, Entry 17]. Another famous theorem is his circular summation formula stated without proof in [33, p. 54]. In Chapter 2, I devise a completely elementary and neat proof of the circular summation formula that perhaps reflects Ramanujan's thinking more than previous proofs.

Ramanujan's general theta function $f(a, b)$ is defined by (1.0.2). We also introduce Jacobi theta functions $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$

$$\theta_2(z|\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)iz}, \tag{1.1.1}$$

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \tag{1.1.2}$$

where $q = e^{\pi i \tau}$ and $\mathtt{Im}\ \tau > 0$. On page 54 of his Lost Notebook [33], Ramanujan recorded a statement which is now known as Ramanujan's circular summation.

Theorem 1.1.1. *For any positive integer $n \geq 2$, if*

$$U_r = a^{r(r+1)/(2n)} b^{r(r-1)/(2n)} \quad \textit{and} \quad V_r = a^{r(r-1)/(2n)} b^{r(r+1)/(2n)},$$

then

$$\sum_{r=0}^{n-1} U_r^n f^n \left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r} \right) = f(a, b) H_n(ab).$$

When $n \geq 3$,

$$H_n(x) = 1 + 2nx^{(n-1)/2} + \cdots .$$

Ramanujan's circular summation can be restated in terms of classical theta function $\theta_3(z|\tau)$ defined by (1.1.2).

Theorem 1.1.2. *For any positive integer* $n \geq 2$,

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \theta_3^n(z + k\pi\tau | n\tau) = F_n(\tau)\theta_3(z|\tau). \tag{1.1.3}$$

When $n \geq 3$,

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots .$$

The first proof of Theorem 1.1.1 was given by S. S. Rangachari in [38]. He also discussed Ramanujan's explicit expressions of H_n for $n = 2, 3, 4, 5$ and 7. Rangachari proved these claims by using Mumford's theory of theta functions and some results on weight polynomials in coding theory. Theorem 1.1.1 was later proved by Son [39] using a much more elementary but lengthier proof. We devise an completely elementary and neat proof of the circular summation formula that perhaps reflects Ramanujan's thinking more than previous proofs.

With the theory of elliptic functions, H. H. Chan, Z. -G. Liu and S. T. Ng proved a dual form of the Ramanujan circular summation formula in [22]. The authors of [17] proved an additive decomposition of θ_3. Zeng [47] then proved a general result which unifies the two results in [22] and [17]. Other studies on Ramanujan's circular summation can be found in [3], [24], [25], [34]. Motivated by [47], S. H. Chan and Z. -G. Liu proved the following very general theta function identity using the theory of elliptic funtions in [23].

Theorem 1.1.3. *Let* m *and* n *be any positive integers. Suppose* $y_1, y_2, \ldots, y_n$ *are* n *complex numbers such that* $y_1 + y_2 + \cdots + y_n = 0$. *Then we have*

$$\sum_{k=0}^{mn-1} \prod_{j=1}^{n} \theta_3\left(z + y_j + \frac{k\pi}{mn}\middle|\, \tau\right) = G_{m,n}(y_1, y_2, \ldots, y_n|\tau)\theta_3(mnz|m^2n\tau), \tag{1.1.4}$$

where

$$G_{m,n}(y_1, y_2, \ldots, y_n|\tau) = mn \sum_{\substack{r_1, r_2, \ldots, r_n = -\infty \\ r_1 + r_2 + \cdots + r_n = 0}}^{\infty} q^{r_1^2 + r_2^2 + \cdots + r_n^2} e^{2i(r_1y_1 + r_2y_2 + \cdots + r_ny_n)}. \tag{1.1.5}$$

Applying the imaginary transformation formula for θ_3 to Theorem 1.1.3, the authors of [23] deduced the following alternative version of Theorem 1.1.3.

Theorem 1.1.4. *Let m and n be any positive integers. Suppose that $y_1, y_2, \ldots, y_n$ are n complex numbers such that $y_1 + y_2 + \cdots + y_n = 0$. Then we have*

$$\sum_{k=0}^{mn-1} q^{k^2} e^{2kiz} \prod_{j=1}^{n} \theta_3(mz + (y_j + km)\pi\tau | m^2 n\tau) = F_{m,n}(y_1, y_2, \ldots, y_n|\tau)\theta_3(z|\tau), \tag{1.1.6}$$

where

$$\begin{aligned} &F_{m,n}(y_1, y_2, \ldots, y_n|\tau) \\ =&\frac{(-i\tau)^{(1-n)/2}}{(m^2 n)^{n/2}} q^{-\frac{y_1^2+y_2^2+\cdots+y_n^2}{2m^2 n}} G_{m,n}\left(\frac{y_1\pi}{m^2 n}, \frac{y_2\pi}{m^2 n}, \ldots, \frac{y_n\pi}{m^2 n}\middle| -\frac{1}{m^2 n\tau}\right). \end{aligned}$$

And also,

$$\begin{aligned} &F_{m,n}(y_1, y_2, \ldots, y_n|\tau) \\ =&\sum_{k=0}^{mn-1} \sum_{\substack{s_1, s_2, \ldots, s_n = -\infty \\ m(s_1+s_2+\cdots+s_n)=k}}^{\infty} q^{m^2 n(s_1^2+s_2^2+\cdots+s_n^2) - 2(s_1 y_1 + s_2 y_2 + \cdots + s_n y_n) - k^2}. \end{aligned} \tag{1.1.7}$$

Here, we find (1.1.7) by equating the terms that are independent of z on both sides of (1.1.6). Our formula for $F_{m,n}$ in (1.1.7) corrects the corresponding formula in [23]. When $y_1 = y_2 = \cdots = y_n = 0$ and $m = 1$, Theorem 1.1.4 reduces to Ramanujan's circular summation formula, Theorem 1.1.3.

If we use Ramanujan's notation, Theorem 1.1.4 is equivalent to

$$\begin{aligned} &\sum_{k=0}^{mn-1} U_k^n \prod_{j=1}^{n} f\left(\frac{U_{mn+k}}{U_k}(ab)^{y_j/m}, \frac{V_{mn-k}}{U_k}(ab)^{-y_j/m}\right) \\ &\qquad\qquad = F_{m,n}(y_1, y_2, \ldots, y_n) f(a^{1/m}, b^{1/m}), \end{aligned}$$

where

$$a = q^m e^{2imz},\ b = q^m e^{-2imz},\ U_k^{mn} = a^{k(k+1)/2} b^{k(k-1)/2},\ V_k^{mn} = a^{k(k-1)/2} b^{k(k+1)/2}.$$

In [48], J. M. Zhu proved an alternating general circular summation formula.

Theorem 1.1.5. *Suppose that mn is any even positive integer and $y_1, y_2, \ldots, y_n$ are n complex numbers* *such that* $y_1 + y_2 + \cdots + y_n = 0$*. Then we have*

$$\sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^{n} \theta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \tag{1.1.8}$$
$$= H_{m,n}(y_1, y_2, \ldots, y_n|\tau)\theta_2(mnz|m^2 n\tau),$$

where

$$H_{m,n}(y_1, y_2, \ldots, y_n|\tau)$$
$$= mnq^{-\frac{m^2 n}{4}} \sum_{\substack{r_1, r_2, \ldots, r_n = -\infty \\ r_1 + r_2 + \cdots + r_n = mn/2}}^{\infty} q^{r_1^2 + r_2^2 + \cdots + r_n^2} e^{2i(r_1 y_1 + r_2 y_2 + \cdots + r_n y_n)}. \tag{1.1.9}$$

In Chapter 2, proofs will be given for Theorems 1.1.2, 1.1.3, 1.1.4 and 1.1.5. Although we can obtain Theorem 1.1.4 by applying the imaginary transformation formula to Theorem 1.1.3, we can also prove it directly in an elementary way. Some applications will be given.

1.2 The evaluation of two-dimensional lattice sums via Ramanujan's theta functions

In general, elementary evaluations are rare for higher-dimensional lattice-type sums. They have been studied for many years in the mathematical physics community. The most famous higher-dimensional sum is Madelung's constant from crystallography. In this paper, we analyze various generalized two-dimensional lattice sums, one of which arose from the solution to a certain Poisson equation. We evaluate certain lattice sums in closed form using results from Ramanujan's theory of theta functions, continued fractions and class invariants. For instance,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(8m)^2 + (4n+1)^2} = -\frac{\sqrt{2}\pi}{16} \log \frac{(\sqrt{2}-1) - (\sqrt{\sqrt{2}+1} - \sqrt[4]{2})}{1 + (\sqrt{2}-1)(\sqrt{\sqrt{2}+1} - \sqrt[4]{2})}.$$

In [15], Berndt, Lamb and Rogers evaluated in closed form the sum

$$F_{(a,b)}(q) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{(xm)^2 + (an+b)^2}, \qquad q = e^{-\pi/x}, \tag{1.2.1}$$

for any positive rational value of x, and for certain values of $(a, b) \in \mathbb{N}^2$. They used the notation $F_{(a,b)}(x)$ instead of $F_{(a,b)}(q)$. We use $F_{(a,b)}(q)$ here so that we can state Theorem 3.2.3 more easily and clearly. The authors of [15] first proved the following theorem.

Theorem 1.2.1. *Suppose that a and b are integers with $a \geq 2$ and $(a,b)=1$, and assume that* $\mathtt{Re}\ x > 0$. *Then*

$$F_{(a,b)}(q) = -\frac{2\pi}{ax}\sum_{j=0}^{a-1}\omega^{-(2j+1)b}\log\prod_{m=0}^{\infty}(1-\omega^{2j+1}q^{2m+1})(1-\omega^{-2j-1}q^{2m+1}), \tag{1.2.2}$$

where $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

Now, for any positive rational number x and positive integers a and b, in addition to (1.2.1), we consider two new types of sums,

$$G_{(a,b)}(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{(xm)^2+(an+b)^2}, \tag{1.2.3}$$

$$H_{(a,b)}(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{(xm)^2+(an+b)^2}, \tag{1.2.4}$$

where $q = e^{-\pi/x}$.

In analogy with Theorem 1.2.1 for $F_{(a,b)}(q)$, we are able to prove results for $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$, respectively. Moreover, it can be shown that $G_{(a,b)}(q)$ can be recast in the theory of $F_{(a,b)}$, which is proved in Section 3.2. In Section 3.5, we study the theory of $G_{(a,b)}(q)$ with the aid of the results from $F_{(a,b)}$ and derive many explicit examples afterwards.

The authors of [15] simplified Theorem 1.2.1 for $a \in \{3,4,5,6\}$ and $b = 1$ using classical results from theta functions and q-series, and evaluated in closed form certain classes of double series. When $a > 6$, the situation is much more complicated and we study the case for $(a,b) = (8,1)$ in detail in Section 3.4 and derive an explicit example afterwards. The case $a = 12$ can be derived in a similar fashion. Similarly, we can also derive the theory of $H_{(a,b)}$, and more double series can be evaluated. Although the main theorems of all these three types of sums are similar and not difficult to prove, the examination of special cases of $H_{(a,b)}$ is quite different from those of $F_{(a,b)}$. In Section 3.3, we examine $H_{(a,b)}$ for the cases where $a \in \{3,4,5,6\}$ and $b = 1$. The resulting formulas are closely related to continued fractions including the famous Rogers-Ramanujan continued fraction, Ramanujan's cubic continued fraction, the Ramanujan-Göllnitz-Gordon continued fraction and continued fractions of order 12. We are able to produce many explicit examples from the values of these continued fractions. In these instances, we assume $b = 1$ without loss of generality, because other possible values of the lattice sums can be easily recovered from the case when $b = 1$.

Inspired by all the nice results from $F_{(a,b)}(q)$, $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$, we consider a generalization of these lattice sums that is defined by

$$J_{(a,b,s,t)}(q) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{\pi i m s} e^{\pi i n t}}{(xm)^2 + (an+b)^2}, \tag{1.2.5}$$

where $q = e^{-\pi/x}$. In Section 3.2, we prove the main theorem for $J_{(a,b,s,t)}(q)$ in analogy with Theorem 1.2.1, and the main theorems for $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$ follow easily. We then specialize it in the case $(a, b) = (2, 1)$ in Section 3.6. For certain s and t, we are able to obtain very nice evaluations. Ramanujan's cubic continued fraction plays an important role in determining explicit examples. For instance,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{\frac{2}{3}\pi i m} e^{\frac{2}{3}\pi i n}}{2m^2 + (2n+1)^2} = \frac{\pi}{4\sqrt{2}} e^{-\pi i/3} \log 3.$$

Recently, Bailey, Borwein, Crandall and Zucker studied a class of lattice sums arising from solutions to Poisson equation in [6]. In their paper, they determined some closed-form evaluations using Jacobi theta functions for series $\psi_2(x, y)$ defined by

$$\psi_2(x, y) = \frac{1}{\pi^2} \sum_{m,n \texttt{ even}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2 + n^2}.$$

As graphically illustrated in [26], $\psi_2(x, y)$ is the 'natural' potential of the 2-dimensional jellium crystal, that is, the solution to the Poisson equation of the physical model of the jellium, with the Poisson equation

$$\nabla^2 \psi_2(\mathbf{r}) = 1 - \sum_{\mathbf{m} \in \mathbb{Z}^2} \delta^2(\mathbf{r} - \mathbf{m}),$$

where $\mathbf{r} = (x, y)$ and $\delta^2(\mathbf{r}) = \delta(x)\delta(y)$ is the Dirac delta-function, with the integral of this δ^2 over $\mathbb{R}^2$ being unity. The Dirac delta function is a non-traditional function which can only be defined by its action on continuous functions.

$$\int_{\mathbb{R}^n} \delta(r) f(r) = f(0).$$

And the Laplacian operator can be written as

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right).$$

One of the primary results in [6] is that for rational numbers x and y, the 'natural' potential $\psi_2(x, y)$ is $\frac{1}{4\pi} \log \beta$ where β is an algebraic number.

We are thus motivated to consider the class of sums when $b = 0$ in $J_{(a,b,s,t)}(q)$, that is,

$$J_{(a,0,s,t)}(q) := \sum_{\substack{m,n\in \mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{\pi ims}e^{\pi int}}{(xm)^2 + (an)^2}. \tag{1.2.6}$$

Clearly,

$$Re\{J_{(1,0,2x,2y)}(e^{-\pi})\} = 4\pi^2\psi_2(x, y).$$

After we prove the main theorem for $J_{(a,0,s,t)}(q)$ in Section 3.2, we then examine $J_{(a,0,s,t)}(q)$ when $a \in \{1, 2\}$ in Section 3.7. While the authors of [6] are mainly interested in applying numerical methods to first deduce the values of lattice sums, our rigorous determinations focus from the start on Ramanujan's theory of theta functions. We can not only derive all the evaluations of $\psi_2(x, y)$ rigorously established in [6], but also produce further nice results such as

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{m^2 + n^2} = \frac{\pi}{6}\log\frac{2-\sqrt{3}}{3\sqrt{3}}.$$

The explicit values of the two class invariants and Ramanujan's cubic continued fractions are frequently applied during the examinations.

1.3 Integral analogue of theta functions and Gauss sums

In Chapter 4, we examine an integral analogue of Gauss sums from one point of view or of theta functions from another point of view. In two papers [35], [36], [37, pp. 59–67, 202–207], Ramanujan examined the properties of two integrals

$$\phi_w(t) := \int_0^\infty \frac{\cos(\pi tx)}{\cosh(\pi x)} e^{-\pi wx^2}\,dx, \tag{1.3.1}$$
$$\psi_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\sinh(\pi x)} e^{-\pi wx^2}\,dx, \tag{1.3.2}$$

which can be regarded as continuous integral analogues of Gauss sums and theta functions. On Page 198 of [33], Ramanujan records theorems, much in the spirit of those for $\phi_w(t)$ and $\psi_w(t)$, for the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\tanh(\pi x)} e^{-\pi wx^2}\,dx. \tag{1.3.3}$$

The formulas claimed by Ramanujan on page 198 without proofs are difficult to read. The objective of this project is to prove them and to show that $F_w(t)$ is a beautiful continuous integral analogue of either theta functions or Gauss sums. For instance, we have the following entry.

Entry 1.3.1. *We have*

$$F_w(t) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w). \tag{1.3.4}$$

This beautiful transformation formula shows that $F_w(t)$ is an integral analogue of theta functions.

Chapter 2

An elementary proof of Ramanujan's circular summation formula and its generalizations

2.1 Main theorem

We will start this section by proving Theorem 1.1.2 in an elementary way.

Theorem. *For any positive integer $n \geq 2$,*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \theta_3^n(z + k\pi\tau | n\tau) = F_n(\tau)\theta_3(z|\tau).$$

When $n \geq 3$,

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots .$$

Proof. Let $x = e^{2iz}$. In the notation (1.1.2), we find that (1.1.3) is equivalent to

$$\sum_{k=0}^{n-1} q^{k^2} x^k \left(\sum_{m=-\infty}^{\infty} q^{m^2 n + 2km} x^m \right)^n = F_n(\tau) \sum_{m=-\infty}^{\infty} q^{m^2} x^m. \tag{2.1.1}$$

Define $F_{m,n}(\tau)$ by the following equation

$$\sum_{k=0}^{n-1} q^{k^2} x^k \left(\sum_{m=-\infty}^{\infty} q^{m^2 n + 2km} x^m \right)^n = \sum_{m=-\infty}^{\infty} F_{m,n}(\tau) q^{m^2} x^m. \tag{2.1.2}$$

Then equating the coefficient of x^m on both sides of (2.1.2), we find that

$$\begin{aligned}
F_{m,n}(\tau) &= \sum_{k=0}^{n-1} \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=k-m}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-2k(m_1+m_2+\cdots+m_n)+k^2-m^2} \\
&= \sum_{k=0}^{n-1} \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=k-m}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-(k-m)^2} \\
&= \sum_{k'=-m}^{n-m-1} \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=k'}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-k'^2} \\
&= \sum_{k'=-m}^{n-m-1} \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=k'}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-(m_1+m_2+\cdots+m_n)^2}.
\end{aligned} \tag{2.1.3}$$

Now we claim that $F_{m,n}(\tau) = F_n(\tau)$, i.e., $F_{m,n}$ is independent of m. First, we show that for any $1 \leq s \leq m$,

$$\begin{aligned}
&\sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=-s}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-(m_1+m_2+\cdots+m_n)^2} \\
&= \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=n-s}}^{\infty} q^{n\{(m_1-1)^2+(m_2-1)^2+\cdots+(m_n-1)^2\}-(m_1-1+m_2-1+\cdots+m_n-1)^2} \\
&= \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=n-s}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-(m_1+m_2+\cdots+m_n)^2}.
\end{aligned} \tag{2.1.4}$$

It we use (2.1.4) in (2.1.3), it follows that

$$\begin{aligned}
F_{m,n}(\tau) &= \sum_{k'=-m}^{n-m-1} \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=k'}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-(m_1+m_2+\cdots+m_n)^2} \\
&= \sum_{k=0}^{n-1} \sum_{\substack{m_1,m_2,\ldots,m_n=-\infty \\ m_1+m_2+\cdots+m_n=k}}^{\infty} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-(m_1+m_2+\cdots+m_n)^2}.
\end{aligned}$$

Therefore, $F_{m,n}$ is independent of m. It remains to show that $F_n(\tau) = 1 + 2nq^{n-1} + \cdots$. An elementary proof is given in [22]. This completes the proof of Theorem 1.1.2.

Now we give the proof of Theorem 1.1.3 here.

Theorem. *Let m, n be any positive integers. Suppose $y_1, y_2, \ldots, y_n$ are n complex numbers such that $y_1 + y_2 + \cdots + y_n = 0$. Then we have*

$$\sum_{k=0}^{mn-1} \prod_{j=1}^{n} \theta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \, \tau \right) = G_{m,n}(y_1, y_2, \ldots, y_n|\tau)(mnz|m^2n\tau),$$

where

$$G_{m,n}(y_1, y_2, \ldots, y_n|\tau) = mn \sum_{\substack{r_1, r_2, \ldots, r_n = -\infty \\ r_1 + r_2 + \cdots + r_n = 0}}^{\infty} q^{r_1^2 + r_2^2 + \cdots + r_n^2} e^{2i(r_1 y_1 + r_2 y_2 + \cdots + r_n y_n)}.$$

Proof. From (1.1.2), we have

$$\theta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = \sum_{s=-\infty}^{\infty} q^{s^2} e^{2is(z + y_j + \frac{k\pi}{mn})},$$

$$\theta_3(mnz|m^2n\tau) = \sum_{s=-\infty}^{\infty} q^{m^2ns^2} e^{2imnsz}.$$

If $x = e^{2iz}$, then (1.1.4) becomes

$$\sum_{k=0}^{mn-1} \prod_{j=1}^{n} \sum_{s=-\infty}^{\infty} q^{s^2} e^{2is(y_j + \frac{k\pi}{mn})} x^s = G_{m,n}(y_1, y_2, \ldots, y_n|\tau) \sum_{s=-\infty}^{\infty} q^{m^2ns^2} x^{mns}.$$

This is equivalent to the identity to be proved. Equate the coefficients of x^N on both sides to obtain

$$\sum_{k=0}^{mn-1} \sum_{\substack{s_1, s_2, \ldots, s_n = -\infty \\ s_1 + s_2 + \cdots + s_n = N}}^{\infty} q^{s_1^2 + s_2^2 + \cdots + s_n^2} e^{2is_1y_1 + 2is_2y_2 + \cdots + 2is_ny_n} e^{2i\frac{k\pi N}{mn}}$$
$$= G_{m,n}(y_1, y_2, \ldots, y_n|\tau) q^{\frac{N^2}{n}}.$$

Define $G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau)$ by

$$\sum_{k=0}^{mn-1} \sum_{\substack{s_1, s_2, \ldots, s_n = -\infty \\ s_1 + s_2 + \cdots + s_n = N}}^{\infty} q^{s_1^2 + s_2^2 + \cdots + s_n^2} e^{2is_1y_1 + 2is_2y_2 + \cdots + 2is_ny_n} e^{2i\frac{k\pi N}{mn}}$$
$$= G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) q^{\frac{N^2}{n}}. \qquad (2.1.5)$$

To prove (1.1.4), it suffices to show that $G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau)$ is independent of N and has the representation in (1.1.5).

Now we claim that

$$
\begin{aligned}
&G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) \\
&\qquad = \begin{cases} G_{m,n}(y_1, y_2, \ldots, y_n|\tau), & \texttt{if } mn|N, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

From (2.1.5), we have

$$
\begin{aligned}
&G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) \\
&= \sum_{\substack{s_1, s_2, \ldots, s_n = -\infty \\ s_1+s_2+\cdots+s_n=N}}^{\infty} q^{s_1^2+s_2^2+\cdots+s_n^2-\frac{N^2}{n}} e^{2is_1y_1+2is_2y_2+\cdots+2is_ny_n} \sum_{k=0}^{mn-1} e^{2i\frac{k\pi N}{mn}}.
\end{aligned} \tag{2.1.6}
$$

Since

$$
\sum_{k=0}^{mn-1} e^{2i\frac{k\pi N}{mn}} = \begin{cases} mn, & \texttt{if } mn|N, \\ 0, & \text{otherwise,} \end{cases}
$$

$G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) = 0$ unless N is a multiple of mn. It follows that we only need to consider the case when $N = lmn,\ l \in \mathbb{Z}$. So (2.1.6) becomes

$$
\begin{aligned}
&G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) \\
&= mn \sum_{\substack{s_1, s_2, \ldots, s_n = -\infty \\ s_1+s_2+\cdots+s_n=lmn}}^{\infty} q^{s_1^2+s_2^2+\cdots+s_n^2-\frac{l^2m^2}{n}} e^{2is_1y_1+2is_2y_2+\cdots+2is_ny_n} \\
&= mn \sum_{\substack{s'_1, s'_2, \ldots, s'_n = -\infty \\ s'_1+s'_2+\cdots+s'_n=0}}^{\infty} q^{(s'_1+lm)^2+(s'_2+lm)^2+\cdots+(s'_n+lm)^2-l^2m^2n} e^{2is'_1y_1+2is'_2y_2+\cdots+2is'_ny_n} \\
&= mn \sum_{\substack{s'_1, s'_2, \ldots, s'_n = -\infty \\ s'_1+s'_2+\cdots+s'_n=0}}^{\infty} q^{s_1'^2+s_2'^2+\cdots+s_n'^2} e^{2is'_1y_1+2is'_2y_2+\cdots+2is'_ny_n}.
\end{aligned}
$$

Thus, $G_{m,n,N}$ is independent of N, and so if we set

$$
G_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) = G_{m,n}(y_1, y_2, \ldots, y_n|\tau),
$$

above, we deduce (1.1.5). This completes our proof.

Now we also use only an elementary method to prove Theorem 1.1.4.

Theorem. *Let m, n be any positive integers. Suppose that $y_1, y_2, \ldots, y_n$ are n complex numbers such that $y_1 + y_2 + \cdots + y_n = 0$. Then we have*

$$\sum_{k=0}^{mn-1} q^{k^2} e^{2kiz} \prod_{j=1}^{n} \theta_3(mz + (y_j + km)\pi\tau | m^2 n\tau) = F_{m,n}(y_1, y_2, \ldots, y_n|\tau)\theta_3(z|\tau),$$

where

$$\begin{aligned} &F_{m,n}(y_1, y_2, \ldots, y_n|\tau) \\ =&\frac{(-i\tau)^{(1-n)/2}}{(m^2 n)^{n/2}} q^{-\frac{y_1^2+y_2^2+\cdots+y_n^2}{2m^2 n}} G_{m,n}\left(\frac{y_1\pi}{m^2 n}, \frac{y_2\pi}{m^2 n}, \ldots, \frac{y_n\pi}{m^2 n} \middle| -\frac{1}{m^2 n\tau}\right). \end{aligned}$$

And also,

$$\begin{aligned} &F_{m,n}(y_1, y_2, \ldots, y_n|\tau) \\ =&\sum_{k=0}^{mn-1} \sum_{\substack{s_1, s_2, \ldots, s_n = -\infty \\ m(s_1+s_2+\cdots+s_n)=k}}^{\infty} q^{m^2 n(s_1^2+s_2^2+\cdots+s_n^2)-2(s_1 y_1+s_2 y_2+\cdots+s_n y_n)-k^2}. \end{aligned}$$

Proof. From (1.1.2), we have

$$\theta_3(mz + (y_j + km)\pi\tau | m^2 n\tau) = \sum_{s=-\infty}^{\infty} q^{m^2 ns^2} e^{2imsz+2is(y_j+km)\pi\tau}.$$

If $x = e^{2iz}$, then (1.1.6) becomes

$$\sum_{k=0}^{mn-1} q^{k^2} x^k \prod_{j=1}^{n} \sum_{s=\infty}^{\infty} q^{m^2 ns^2} e^{2is(y_j+km)\pi\tau} x^{ms} = F_{m,n}(y_1, y_2, \ldots, y_n|\tau) \sum_{s=-\infty}^{\infty} q^{s^2} x^s. \quad (2.1.7)$$

Define $F_{m,n,N}(y_1, y_2, \ldots, y_n|\tau)$ by

$$\sum_{k=0}^{mn-1} q^{k^2} x^k \prod_{j=1}^{n} \sum_{s=\infty}^{\infty} q^{m^2 ns^2} e^{2is(y_j+km)\pi\tau} x^{ms} = F_{m,n,N}(y_1, y_2, \ldots, y_n|\tau) \sum_{s=-\infty}^{\infty} q^{s^2} x^s. \quad (2.1.8)$$

Equate the coefficients of x^N on both sides to find that

$$
\begin{aligned}
&F_{m,n,N}(y_1,y_2,\dots,y_n)\\
&= q^{-N^2}\sum_{k=0}^{mn-1} q^{k^2} \sum_{\substack{s_1,s_2,\dots,s_n=-\infty\\ m(s_1+s_2+\dots+s_n)+k=N}}^{\infty} q^{m^2n(s_1^2+s_2^2+\dots+s_n^2)+2km(s_1+s_2+\dots+s_n)+2(s_1y_1+s_2y_2+\dots+s_ny_n)}\\
&= q^{-N^2}\sum_{k=N}^{N-mn+1} \sum_{\substack{s_1,s_2,\dots,s_n=-\infty\\ m(s_1+s_2+\dots+s_n)=k}}^{\infty} q^{m^2n(s_1^2+s_2^2+\dots+s_n^2)+2(N-k)k+(N-k)^2+2(s_1y_1+s_2y_2+\dots+s_ny_n)}\\
&= \sum_{k=N}^{N-mn+1} \sum_{\substack{s_1,s_2,\dots,s_n=-\infty\\ m(s_1+s_2+\dots+s_n)=k}}^{\infty} q^{m^2n(s_1^2+s_2^2+\dots+s_n^2)-k^2+2(s_1y_1+s_2y_2+\dots+s_ny_n)}\\
&= \sum_{k=mn-N-1}^{-N} \sum_{\substack{s_1,s_2,\dots,s_n=-\infty\\ m(s_1+s_2+\dots+s_n)=k}}^{\infty} q^{m^2n(s_1^2+s_2^2+\dots+s_n^2)-k^2-2(s_1y_1+s_2y_2+\dots+s_ny_n)}.
\end{aligned}
\tag{2.1.9}
$$

Now we write

$$
S_{m,n}(k) = \sum_{\substack{s_1,s_2,\dots,s_n=-\infty\\ m(s_1+s_2+\dots+s_n)=k}}^{\infty} q^{m^2n(s_1^2+s_2^2+\dots+s_n^2)-k^2-2(s_1y_1+s_2y_2+\dots+s_ny_n)}.
$$

We claim that

$$
\sum_{k'=0}^{mn-1} S_{m,n}(k') = \sum_{k'=-N}^{mn-N-1} S_{m,n}(k'). \tag{2.1.10}
$$

It follows that

$$
F_{m,n,N}(y_1,y_2,\dots,y_n) = F_{m,n}(y_1,y_2,\dots,y_n)
$$

is independent of N and has the form in (1.1.7). To prove the claim, it suffices to show that

$$
S_{m,n}(-k) = S_{m,n}(mn-k) \qquad \texttt{for any } 1 \le k \le N. \tag{2.1.11}
$$

From (2.1.9) , we have

$$S_{m,n}(-k) = \sum_{\substack{s_1,s_2,\ldots,s_n=-\infty \\ m(s_1+s_2+\ldots+s_n)=-k}}^{\infty} q^{m^2n(s_1^2+s_2^2+\ldots+s_n^2)-k^2-2(s_1y_1+s_2y_2+\ldots+s_ny_n)}$$

$$= \sum_{\substack{s_1,s_2,\ldots,s_n=-\infty \\ m(s_1+s_2+\ldots+s_n)=mn-k}}^{\infty} q^{m^2n((s_1-1)^2+(s_2-1)^2+\ldots+(s_n-1)^2)-k^2-2(s_1y_1+s_2y_2+\ldots+s_ny_n)}$$

$$= \sum_{\substack{s_1,s_2,\ldots,s_n=-\infty \\ m(s_1+s_2+\ldots+s_n)=mn-k}}^{\infty} q^{m^2n(s_1^2+s_2^2+\ldots+s_n^2)-(mn-k)^2-2(s_1y_1+s_2y_2+\ldots+s_ny_n)}$$

$$= S_{m,n}(mn-k).$$

The proof is complete.

We conclude this section by proving Theorem 1.1.5.

Theorem. *Suppose that mn is any even positive integer and $y_1, y_2, \ldots, y_n$ are n complex numbers such that $y_1 + y_2 + \cdots + y_n = 0$. Then we have*

$$\sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^{n} \theta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right)$$

$$= H_{m,n}(y_1, y_2, \ldots, y_n|\tau)\theta_2(mnz|m^2n\tau),$$

where

$$H_{m,n}(y_1, y_2, \ldots, y_n|\tau)$$

$$= mnq^{-\frac{m^2n}{4}} \sum_{\substack{r_1,r_2,\ldots,r_n=-\infty \\ r_1+r_2+\cdots+r_n=mn/2}}^{\infty} q^{r_1^2+r_2^2+\cdots+r_n^2} e^{2i(r_1y_1+r_2y_2+\cdots+r_ny_n)}.$$

Proof. From (1.1.1) and (1.1.2), we have

$$\theta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = \sum_{s=-\infty}^{\infty} q^{s^2} e^{2is(z+y_j+\frac{k\pi}{mn})},$$

$$\theta_2(mnz|m^2n\tau) = q^{\frac{m^2n}{4}} \sum_{s=-\infty}^{\infty} q^{m^2ns(s+1)} e^{(2s+1)imnz}.$$

Similar to the proof of Theorem 1.1.3, we set $x = e^{2iz}$. And thus (1.1.8) becomes

$$\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\sum_{s=-\infty}^{\infty} q^{s^2}e^{2is(y_j+\frac{k\pi}{mn})}x^s = H_{m,n}(y_1,y_2,\ldots,y_n|\tau)q^{\frac{m^2n}{4}}\sum_{s=-\infty}^{\infty} q^{m^2ns(s+1)}x^{\frac{mn}{2}(2s+1)}.$$

Equate the coefficients of x^N on both sides to obtain

$$\sum_{k=0}^{mn-1}(-1)^k\sum_{\substack{s_1,s_2,\ldots,s_n=-\infty\\ s_1+s_2+\cdots+s_n=N}}^{\infty} q^{s_1^2+s_2^2+\cdots+s_n^2}e^{2is_1y_1+2is_2y_2+\cdots+2is_ny_n}e^{2i\frac{k\pi N}{mn}}$$
$$=H_{m,n}(y_1,y_2,\ldots,y_n|\tau)q^{\frac{N^2}{n}}.$$

Define $H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)$ by

$$\sum_{k=0}^{mn-1}(-1)^k\sum_{\substack{s_1,s_2,\ldots,s_n=-\infty\\ s_1+s_2+\cdots+s_n=N}}^{\infty} q^{s_1^2+s_2^2+\cdots+s_n^2}e^{2is_1y_1+2is_2y_2+\cdots+2is_ny_n}e^{2i\frac{k\pi N}{mn}}$$
$$=H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)q^{\frac{N^2}{n}}. \tag{2.1.12}$$

Now it suffices to show that $H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)$ is independent of N and has the representation in (1.1.9). From (2.1.12), we have

$$H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)$$
$$=\sum_{\substack{s_1,s_2,\ldots,s_n=-\infty\\ s_1+s_2+\cdots+s_n=N}}^{\infty} q^{s_1^2+s_2^2+\cdots+s_n^2-\frac{N^2}{n}}e^{2is_1y_1+2is_2y_2+\cdots+2is_ny_n}\sum_{k=0}^{mn-1}\left(-e^{\frac{2i\pi N}{mn}}\right)^k. \tag{2.1.13}$$

If mn is even, then we obtain

$$\sum_{k=0}^{mn-1}\left(-e^{\frac{2i\pi N}{mn}}\right)^k = \begin{cases} 0, & \texttt{if } e^{\frac{2i\pi N}{mn}} \neq -1,\\ mn, & \texttt{otherwise}, \end{cases}$$

which implies that $H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau) = 0$ unless $\frac{2\pi N}{mn}$ is an odd integer. It follows that we only need to consider the case when $N=(2l+1)mn/2,\ l\in\mathbb{Z}$. So (2.1.13) reduces to

$$
\begin{aligned}
&H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)\\
=\ & mn \sum_{\substack{s_1,s_2,\ldots,s_n=-\infty\\ s_1+s_2+\cdots+s_n=(2l+1)mn/2}}^{\infty} q^{s_1^2+s_2^2+\cdots+s_n^2-\frac{N^2}{n}} e^{2is_1y_1+2is_2y_2+\cdots+2is_ny_n}\\
=\ & mn \sum_{\substack{s_1',s_2',\ldots,s_n'=-\infty\\ s_1'+s_2'+\cdots+s_n'=mn/2}}^{\infty} q^{(s_1'+lm)^2+(s_2'+lm)^2+\cdots+(s_n'+lm)^2-l^2m^2n} e^{2is_1'y_1+2is_2'y_2+\cdots+2is_n'y_n}\\
=\ & mn \sum_{\substack{s_1',s_2',\ldots,s_n'=-\infty\\ s_1'+s_2'+\cdots+s_n'=mn/2}}^{\infty} q^{s_1'^2+s_2'^2+\cdots+s_n'^2-\frac{m^2n}{4}} e^{2is_1'y_1+2is_2'y_2+\cdots+2is_n'y_n}.
\end{aligned}
$$

Therefore, $H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)$ is independent of N. If we set $H_{m,n,N}(y_1,y_2,\ldots,y_n|\tau)=H_{m,n}(y_1,y_2,\ldots,y_n|\tau)$, then the proof is completed.

2.2 Application

In [23], Chan and Liu defined the multiple theta series $a(y_1,y_2|\tau)$ by

$$
a(y_1,y_2|\tau)=\sum_{r_1,r_2=-\infty}^{\infty} q^{2r_1^2+2r_1r_2+2r_2^2} e^{2i(r_1(2y_1+y_2)+r_2(2y_2+y_1))}. \tag{2.2.1}
$$

With $\omega=e^{\frac{2\pi i}{3}}$, the well-known cubic theta functions $a(\tau), b(\tau)$ and $c(\tau)$ are defined in [19] as

$$
\begin{aligned}
a(\tau) &= \sum_{r_1,r_2=-\infty}^{\infty} q^{2r_1^2+2r_1r_2+2r_2^2} \ =\ a(0,0|\tau),\\
b(\tau) &= \sum_{r_1,r_2=-\infty}^{\infty} q^{2r_1^2+2r_1r_2+2r_2^2}\omega^{r_1-r_2} \ =\ a\left(\frac{\pi}{3},-\frac{\pi}{3}\middle|\ \tau\right),\\
c(\tau) &= \sum_{r_1,r_2=-\infty}^{\infty} q^{2(r_1+1/3)^2+2(r_1+1/3)(r_2+1/3)+2(r_2+1/3)^2} \ =\ q^{2/3}a\left(\frac{\pi\tau}{3},\frac{\pi\tau}{3}\middle|\ \tau\right).
\end{aligned}
$$

A direct computation gives that $G_{m,3}=3ma(y_1,y_2|\tau)$. Thus Theorem 1.1.3 reduces to the following corollary by taking $n=3$ [23].

Corollary 2.2.1. *With $a(y_1, y_2|\tau)$ defined above and $y_3 = -y_1 - y_2$, we have*

$$\sum_{k=0}^{3m-1} \theta_3\left(z + y_1 + \frac{k\pi}{3m}\middle|\, \tau\right)\theta_3\left(z + y_2 + \frac{k\pi}{3m}\middle|\, \tau\right)\theta_3\left(z + y_3 + \frac{k\pi}{3m}\middle|\, \tau\right)$$
$$= 3ma(y_1, y_2|\tau)(3mz|3m^2\tau). \tag{2.2.2}$$

Letting $m = 1$, we have the identity

$$\begin{aligned}3a(y_1, y_2|\tau)\theta_3(3z|3\tau) &= \theta_3\left(z + y_1|\tau\right)\theta_3\left(z + y_2|\tau\right)\theta_3\left(z + y_3|\tau\right)\\ &+\theta_3\left(z + y_1 + \frac{\pi}{3}\middle|\, \tau\right)\theta_3\left(z + y_2 + \frac{\pi}{3}\middle|\, \tau\right)\theta_3\left(z + y_3 + \frac{\pi}{3}\middle|\, \tau\right)\\ &+\theta_3\left(z + y_1 + \frac{2\pi}{3}\middle|\, \tau\right)\theta_3\left(z + y_2 + \frac{2\pi}{3}\middle|\, \tau\right)\theta_3\left(z + y_3 + \frac{2\pi}{3}\middle|\, \tau\right).\end{aligned} \tag{2.2.3}$$

In [23], the authors proved that

$$b(\tau) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^3}{(1 - q^{6n})}$$

by taking $y_1 = \frac{\pi}{3}$ and $y_2 = -\frac{\pi}{3}$ in (2.2.3). Now taking $y_1 = \frac{\pi\tau}{3} + \frac{\pi}{3}$ and $y_2 = \frac{\pi\tau}{3} - \frac{\pi}{3}$, we have the following corollary.

Corollary 2.2.2. *With $\omega = e^{\frac{2\pi i}{3}}$,*

$$d(\tau) := \sum_{r_1, r_2=-\infty}^{\infty} q^{2r_1^2 + 2r_1r_2 + 2r_2^2 + 2r_1 + 2r_2} \omega^{r_1 - r_2} = 0.$$

Proof. Putting $y_1 = \frac{\pi\tau}{3} + \frac{\pi}{3}$, $y_2 = \frac{\pi\tau}{3} - \frac{\pi}{3}$ and $z = 0$ in (2.2.3), we have

$$\begin{aligned}3d(\tau)\theta_3(0|3\tau) &= \theta_3\left(\frac{\pi\tau}{3} + \frac{\pi}{3}\middle|\, \tau\right)\theta_3\left(\frac{\pi\tau}{3} - \frac{\pi}{3}\middle|\, \tau\right)\theta_3\left(-\frac{2\pi\tau}{3}\middle|\tau\right)\\ &+ \theta_3\left(\frac{\pi\tau}{3} - \frac{\pi}{3}\middle|\, \tau\right)\theta_3\left(\frac{\pi\tau}{3}\middle|\, \tau\right)\theta_3\left(-\frac{2\pi\tau}{3} + \frac{\pi}{3}\middle|\, \tau\right)\\ &+ \theta_3\left(\frac{\pi\tau}{3}\middle|\, \tau\right)\theta_3\left(\frac{\pi\tau}{3} + \frac{\pi}{3}\middle|\, \tau\right)\theta_3\left(-\frac{2\pi\tau}{3} - \frac{\pi}{3}\middle|\, \tau\right).\end{aligned} \tag{2.2.4}$$

By direct computation, we have

$$\begin{aligned}
\theta_3\left(-\frac{2\pi\tau}{3}\middle|\,\tau\right) &= \frac{1+q^{-\frac{1}{3}}}{1+q^{\frac{1}{3}}}\theta_3\left(\frac{\pi\tau}{3}\middle|\,\tau\right),\\
\theta_3\left(-\frac{2\pi\tau}{3}+\frac{\pi}{3}\middle|\,\tau\right) &= \frac{1+q^{-\frac{1}{3}}\omega}{1+q^{\frac{1}{3}}\omega^{-1}}\theta_3\left(\frac{\pi\tau}{3}+\frac{\pi}{3}\middle|\,\tau\right),\\
\theta_3\left(-\frac{2\pi\tau}{3}-\frac{\pi}{3}\middle|\,\tau\right) &= \frac{1+q^{-\frac{1}{3}}\omega^{-1}}{1+q^{\frac{1}{3}}\omega}\theta_3\left(\frac{\pi\tau}{3}-\frac{\pi}{3}\middle|\,\tau\right).
\end{aligned}$$

So (2.2.4) becomes

$$\left(\frac{1+q^{-\frac{1}{3}}}{1+q^{\frac{1}{3}}}+\frac{1+q^{-\frac{1}{3}}\omega}{1+q^{\frac{1}{3}}\omega^{-1}}+\frac{1+q^{-\frac{1}{3}}\omega^{-1}}{1+q^{\frac{1}{3}}\omega}\right)\theta_3\left(\frac{\pi\tau}{3}\middle|\,\tau\right)\theta_3\left(\frac{\pi\tau}{3}+\frac{\pi}{3}\middle|\,\tau\right)\theta_3\left(\frac{\pi\tau}{3}-\frac{\pi}{3}\middle|\,\tau\right)=0,$$

since

$$\frac{1+q^{-\frac{1}{3}}}{1+q^{\frac{1}{3}}}+\frac{1+q^{-\frac{1}{3}}\omega}{1+q^{\frac{1}{3}}\omega^{-1}}+\frac{1+q^{-\frac{1}{3}}\omega^{-1}}{1+q^{\frac{1}{3}}\omega}=0.$$

This completes our proof.

Chapter 3

The evaluation of two-dimensional lattice sums via Ramanujan's theta functions

3.1 Preliminary results

Let us recall the standard notation, as given in (1.0.1),

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

Let us also recall the Ramanujan's general theta function $f(a,b)$ and the famous Jacobi triple product identity for $f(a,b)$, as defined in (1.0.2) and (1.0.7), respectively. For $|ab|<1$,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} = (-a,ab)_\infty(-b,ab)_\infty(ab,ab)_\infty.$$

Following Ramanujan's notation for theta functions, as given in (1.0.3), (1.0.4), (1.0.5) and (1.0.6), we have the definitions and product representations for $\varphi(q)$, $\psi(q)$, $f(-q)$ and $\chi(q)$. From Entry 24 in Chapter 16 of Ramanajan's third Notebook [10, p. 39], we have

$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}, \tag{3.1.1}$$

and

$$\chi(q)\chi(-q) = \chi(-q^2). \tag{3.1.2}$$

If n is any positive rational number and $q=\exp(-\pi\sqrt{n})$, the two class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \texttt{and} \qquad g_n := -2^{-1/4}q^{-1/24}\chi(-q). \tag{3.1.3}$$

In the notation of Weber [42], $G_n = 2^{-1/4}\mathfrak{f}(\sqrt{-n})$ and $g_n = 2^{-1/4}\mathfrak{f}_1(\sqrt{-n})$. The term invariant is due to Weber. From the definitions, it follows easily that $G_n = G_{1/n}$ is equivalent to the following identity [10, p.

43, Entry 27 (v)]

$$e^{\alpha/24}\chi(e^{-\alpha}) = e^{\beta/24}\chi(e^{-\beta}), \tag{3.1.4}$$

where $\alpha\beta = \pi^2$.

There are four continued fractions that play important roles in this chapter. Their explicit values are frequently used to deduce explicit examples of lattice sums. First of all, let us recall the famous Rogers-Ramanujan continued fraction and its product representation

$$\begin{aligned} R(q) :=& \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \cdots \\ =& q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}, \qquad |q|<1. \end{aligned}$$

Ramanujan was interested in determining exact formulas for $R(e^{-2\pi\sqrt{n}})$ and $R(-e^{\pi\sqrt{n}})$ for rational values of n. For instance, he gave the values for $R(e^{-2\pi})$ and $R(-e^{-\pi})$, which were first established by G. N. Watson. Using modular equations, more precisely, eta-function identities discovered by Ramanujan, Berndt and Chan established some particular values of $R(q)$ in [12], [11, p. 20–30]. In another paper [13], Berndt, Chan and Zhang also used Ramanujan's eta-function identities to obtain general formulas for evaluating $R(e^{-2\pi\sqrt{n}})$ and $R(-e^{\pi\sqrt{n}})$ in terms of class invariants. More evaluations can be found in [7], [29], [40], [43] and [44]. Comprehensive discussions can be found in [4, Chap. 2].

The second one is Ramanujan's cubic continued fraction, which is defined by

$$G(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, \qquad |q|<1. \tag{3.1.5}$$

From Ramanujan's lost notebook, we have, as given in [4, p. 94, Eq. (3.3.1 a)], [4, p. 95, Eq. (3.3.6)]

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{3.1.6}$$

In [14], Berndt, Chan and Zhang derived a general method for evaluating $G(\pm q)$. In Yi's thesis [43], she systematically exploited modular equations, in particular eta-function identities, to find 22 new values for $G(e^{-\pi\sqrt{n}})$ and $G(e^{\pi\sqrt{n}})$. Further evaluations can be found in [1], [2] and [8].

Thirdly, the Ramanujan-Göllnitz-Gordon continued fraction is defined as

$$T(q) := \frac{q^{1/2}}{1+q} \underset{+}{} \frac{q^2}{1+q^3} \underset{+}{} \frac{q^4}{1+q^5} \underset{+}{} \frac{q^6}{1+q^7} \underset{+}{} \cdots, \qquad |q|<1. \tag{3.1.7}$$

Ramanujan recorded a product representation of $T(q)$ on p. 229 of his second notebook [31], namely,

$$T(q) = q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}. \tag{3.1.8}$$

Chan and Huang evaluated explicitly $T(q)$ at $q = e^{-\pi\sqrt{n}/2}$ for various positive integers n in [21].

The last one is a continued fraction of order 12 defined by

$$K(q) := \frac{q(1-q)}{1-q^3} + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} + \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})} + \cdots, \qquad |q| < 1. \tag{3.1.9}$$

The continued fraction $K(q)$ is a special case of one of the fascinating continued fraction identities recorded by Ramanujan in his second notebooks [31] [10, Entry 12, p. 24]. Indeed, replacing q by q^3 and letting $a = q$ and $b = q^2$ in [10, Entry 12, p. 24], we can obtain the product representation

$$K(q) = q\frac{f(-q;-q^{11})}{f(-q^5;-q^7)} = q\frac{(q,q^{12})_\infty (q^{11},q^{12})_\infty}{(q^5,q^{12})_\infty (q^7,q^{12})_\infty}. \tag{3.1.10}$$

The addition formula for theta functions [10, p. 48, Entry 31] is stated below.

Lemma 3.1.1. *Let $U_n = a^{n(n+1)/2} b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2} b^{n(n+1)/2}$. Then, for each positive integer n,*

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{3.1.11}$$

We also need the following two lemmas [10, p. 36, Entry 20], [10, p. 45, Entry 29].

Lemma 3.1.2. *If $\alpha\beta = \pi$, $Re(\alpha^2) > 0$, and n is any complex number, then*

$$\sqrt{\alpha} f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) = e^{n^2/4}\sqrt{\beta} f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}). \tag{3.1.12}$$

Lemma 3.1.3. *If $ab = cd$, then*

$$\begin{aligned} &(i)\ f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \\ &(ii)\ f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \end{aligned} \tag{3.1.13}$$

As special cases of the above lemma [10, p. 51, Example(iv)], we have

$$\varphi(-q) + \phi(q^2) = 2\frac{f^2(q^3, q^5)}{\psi(q)}, \tag{3.1.14}$$

$$\varphi(-q) - \phi(q^2) = -2\frac{f^2(q, q^7)}{\psi(q)}. \tag{3.1.15}$$

3.2 Main theorems

We begin this section by proving the main theorem for $J_{(a,b,s,t)}(q)$ defined by (1.2.5).

Theorem 3.2.1. *Suppose that a and b are integers with $a \geq 2$ and $(a,b) = 1$, s and t are any real numbers with at least one not being an even number, and* `Re` *$x > 0$. Then*

$$J_{(a,b,s,t)}(q) = -\frac{\pi}{ax}\sum_{j=0}^{a-1}\omega^{-(2j+t)b}\log\prod_{m=-\infty}^{\infty}(1-\omega^{2j+t}q^{|2m+s|})(1-\omega^{-(2j+t)}q^{|2m+s|}),$$

where $J_{(a,b,s,t)}(q)$ is defined in (1.2.5), $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

Proof. Suppose that N is a positive integer. Since the series is not absolutely convergent, we adopt the convention $\sum_n = \lim_{N\to\infty}\sum_{-N<n<N}$. Then we have

$$\sum_{-N<n<N}\sum_{m\in\ \mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an+b)^2}$$
$$= \pi\sum_{-N<n<N}e^{\pi int}\int_0^\infty e^{-\pi(an+b)^2u}\left(\sum_{m\in\ \mathbb{Z}}e^{\pi ims}e^{-\pi m^2x^2u}\right)du. \tag{3.2.1}$$

Now we may apply (3.1.12) with $\alpha = \sqrt{\pi}/(x\sqrt{u})$, $\beta = x\sqrt{u\pi}$, and $n = \pi s/(x\sqrt{u})$ to deduce that

$$\sum_{m\in\ \mathbb{Z}}e^{\pi ims-\pi m^2x^2u} = \frac{1}{x\sqrt{u}}\sum_{m\in\ \mathbb{Z}}e^{-\frac{\pi(m+s/2)^2}{x^2u}}. \tag{3.2.2}$$

Using this standard form of theta inversion (3.2.2) and inverting the order of summation and integration twice by absolute convergence in (3.2.1), we obtain

$$\begin{aligned}
&\sum_{-N<n<N}\sum_{m\in\ \mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an+b)^2}\\
&=\frac{\pi}{x}\sum_{-N<n<N}e^{\pi int}\int_0^\infty e^{-\pi(an+b)^2u}\sum_{m\in\ \mathbb{Z}}e^{-\frac{\pi(m+s/2)^2}{x^2u}}\frac{du}{\sqrt{u}}\\
&=\frac{\pi}{x}\sum_{m\in\ \mathbb{Z}}\sum_{-N<n<N}e^{\pi int}\int_0^\infty e^{-\pi(an+b)^2u-\frac{\pi(m+s/2)^2}{x^2u}}\frac{du}{\sqrt{u}}.
\end{aligned}$$

Applying the elementary formula [28, p. 384, formula (3.471), no. 9] [19, p. 39]

$$\int_0^\infty e^{-\pi(A^2u+B^2/u)}\frac{du}{\sqrt{u}}=\frac{e^{-2\pi|A||B|}}{|A|}, \tag{3.2.3}$$

we have

$$\sum_{-N<n<N}\sum_{m\in\ \mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an+b)^2}=\frac{\pi}{x}\sum_{m\in\ \mathbb{Z}}\sum_{-N<n<N}\frac{e^{\pi int}q^{|2m+s||an+b|}}{|an+b|}. \tag{3.2.4}$$

We now introduce a variable r and establish the following claim by comparing Taylor series coefficients in r and letting $N\to\infty$.

$$\sum_{-\infty<n<\infty}\frac{e^{\pi int}r^{|an+b|}}{|an+b|}=-\frac{1}{a}\sum_{j=0}^{a-1}\omega^{-(2j+t)b}\log\{(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r)\}. \tag{3.2.5}$$

Indeed, we have

$$\begin{aligned}
&-\frac{1}{a}\sum_{j=0}^{a-1}\omega^{-(2j+t)b}\log\{(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r)\}\\
&=\frac{1}{a}\sum_{j=0}^{a-1}e^{-\frac{\pi i(2j+t)b}{a}}\sum_{k=1}^{\infty}\left(\frac{r^k}{k}e^{\pi ik(2j+t)/a}+\frac{r^k}{k}e^{-\pi ik(2j+t)/a}\right)\\
&=\frac{1}{a}\sum_{k=1}^{\infty}\frac{r^k}{k}\left(e^{\pi it(k-b)/a}\sum_{j=0}^{a-1}e^{2\pi ij(k-b)/a}+e^{-\pi it(k+b)/a}\sum_{j=0}^{a-1}e^{-2\pi ij(k+b)/a}\right)\\
&=\frac{1}{a}\sum_{na+b\geq 1}\frac{r^{|na+b|}}{|na+b|}e^{\pi int}\cdot a+\frac{1}{a}\sum_{na-b\geq 1}\frac{r^{|na-b|}}{|na-b|}e^{-\pi int}\cdot a\\
&=\sum_{na+b\geq 1}\frac{r^{|na+b|}}{|na+b|}e^{\pi int}+\sum_{na+b\leq -1}\frac{r^{|na+b|}}{|na+b|}e^{\pi int}\\
&=\sum_{n=-\infty}^{\infty}\frac{e^{\pi int}r^{|na+b|}}{|na+b|},
\end{aligned}$$

where we used the fact that $na+b\neq 0$ since $(a,b)=1$ in the last identity above. Note that if both s and t

are even numbers, then we have $\log 0$ at $m=0$ on the right-hand side of the above identity. Therefore we exclude the case in the assumption of the theorem to ensure the convergence of the series. Similar to [15, Eq. (2.5)], we use a crude error estimate to bound the terms where $n \geq N$ and $n \leq -N$ as follows:

$$\sum_{-N<n<N} \frac{e^{\pi int} r^{|an+b|}}{|an+b|} = -\frac{1}{a}\sum_{j=0}^{a-1} \omega^{-(2j+t)b} \log(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r) + O\left(\frac{r^N}{(1-r)N}\right). \tag{3.2.6}$$

To complete the proof, we substitute (3.2.6) into (3.2.4) and take the limit as $N \to \infty$.

Note that we have $F_{(a,b)}(q) = J_{(a,b,1,1)}(q)$, $G_{(a,b)}(q) = J_{(a,b,1,0)}(q)$ and $H_{(a,b)}(q) = J_{(a,b,0,1)}(q)$. Thus we have the following corollary.

Corollary 3.2.2. *Suppose that a and b are integers with $a \geq 2$, $(a,b)=1$, and assume that $\mathtt{Re}\ x > 0$. Then we have*

$$G_{(a,b)}(q) = -\frac{2\pi}{ax}\sum_{j=0}^{a-1} \omega^{-2jb} \log \prod_{m=0}^{\infty} (1-\omega^{2j}q^{2m+1})(1-\omega^{-2j}q^{2m+1}), \tag{3.2.7}$$

$$H_{(a,b)}(q) = -\frac{\pi}{ax}\sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \prod_{m\in\ \mathbb{Z}} (1-\omega^{2j+1}q^{2|m|})(1-\omega^{-2j-1}q^{2|m|}), \tag{3.2.8}$$

where $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$ are defined in (1.2.3) and (1.2.4), $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

The following theorem shows that $G_{(a,b)}(q)$ can be placed within the theory of $F_{(a,b)}$.

Theorem 3.2.3. *(i) Suppose that a and b are integers with $a \geq 2$ and $(2a,b)=1$, and assume that $\mathtt{Re}\ x > 0$. Then*

$$G_{(2a,b)}(q) = \frac{1}{2}F_{(a,b)}(q) + \frac{1}{2}G_{(a,b)}(q). \tag{3.2.9}$$

(ii) If we further assume that a is any odd integer, then we have

$$G_{(a,b)}(q) = \begin{cases} F_{(a,b)}(-q), & \text{if } b \text{ is even,} \\ -F_{(a,b)}(-q), & \text{if } b \text{ is odd.} \end{cases} \tag{3.2.10}$$

Proof. We begin with proving (3.2.10). Note that $\sin(2jb\pi/a)$ does not appear in the summation below since $G_{(a,b)}(q)$ is real-valued and the imaginary terms sum to 0. Now from (3.2.7), we have

$$G_{(a,b)}(q) = -\frac{2\pi}{ax}\sum_{j=0}^{a-1} \cos\frac{2jb\pi}{a} \log \prod_{m=0}^{\infty} (1-2\cos\frac{2j\pi}{a}q^{2m+1}+q^{4m+2})$$

$$= -\frac{4\pi}{ax}\sum_{j=0}^{\frac{a-1}{2}}\cos\frac{2jb\pi}{a}\log\prod_{m=0}^{\infty}(1-2\cos\frac{2j\pi}{a}q^{2m+1}+q^{4m+2})+\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= \frac{4\pi}{ax}\sum_{j=0}^{\frac{a-1}{2}}\cos\frac{(a-2jb)\pi}{a}\log\prod_{m=0}^{\infty}(1+2\cos\frac{(a-2j)\pi}{a}q^{2m+1}+q^{4m+2}) + \frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= \frac{4\pi}{ax}\sum_{j=0}^{\frac{a-1}{2}}\cos\frac{\left[a-2(\frac{a-1}{2}-j)b\right]\pi}{a}\log\prod_{m=0}^{\infty}(1+2\cos\frac{[a-2(\frac{a-1}{2}-j)]\pi}{a}q^{2m+1}+q^{4m+2}) + \frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= \frac{4\pi}{ax}\sum_{j=0}^{\frac{a-1}{2}}\cos\frac{[a(1-b)+(2j+1)b]\pi}{a}\log\prod_{m=0}^{\infty}(1+2\cos\frac{(2j+1)\pi}{a}q^{2m+1}+q^{4m+2}) + \frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= -\frac{4\pi}{ax}\sum_{j=0}^{\frac{a-1}{2}}(-1)^b\cos\frac{(2j+1)b\pi}{a}\log\prod_{m=0}^{\infty}(1+2\cos\frac{(2j+1)\pi}{a}q^{2m+1}+q^{4m+2}) + \frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= -(-1)^b\frac{4\pi}{ax}\sum_{j=0}^{\frac{a-1}{2}}\cos\frac{(2j+1)b\pi}{a}\log\prod_{m=0}^{\infty}(1+2\cos\frac{(2j+1)\pi}{a}q^{2m+1}+q^{4m+2})$$
$$+(-1)^b\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}\cos\frac{(2\cdot\frac{a-1}{2}+1)\pi b}{a}\left(1+2\cos\frac{(2\cdot\frac{a-1}{2}+1)\pi}{a}q^{2m+1}+q^{4m+2}\right)$$
$$=(-1)^bF_{(a,b)}(-q).$$

This completes the proof of (3.2.10). It remains to prove (3.2.9) from (3.2.7). Now,

$$G_{(2a,b)}(q)$$
$$= -\frac{\pi}{ax}\sum_{j=0}^{2a-1}e^{\frac{\pi ibj}{a}}\log\prod_{m=0}^{\infty}(1-e^{\frac{\pi ij}{a}}q^{2m+1})(1-e^{-\frac{\pi ij}{a}}q^{2m+1})$$
$$= -\frac{\pi}{ax}\Bigg\{\sum_{j=0}^{a-1}e^{\frac{\pi ib(2j+1)}{a}}\log\prod_{m=0}^{\infty}(1-e^{\frac{\pi i(2j+1)}{a}}q^{2m+1})(1-e^{-\frac{\pi i(2j+1)}{a}}q^{2m+1})$$
$$+\sum_{j=0}^{a-1}e^{\frac{\pi ib(2j)}{a}}\log\prod_{m=0}^{\infty}(1-e^{\frac{\pi i(2j)}{a}}q^{2m+1})(1-e^{-\frac{\pi i(2j)}{a}}q^{2m+1})\Bigg\}.$$
$$= \frac{1}{2}F_{(a,b)}(q)+\frac{1}{2}G_{(a,b)}(q).$$

To finish this section, we prove the main theorem for $J(a,0,s,t)$, defined by (1.2.6).

Theorem 3.2.4. *Suppose that a is a positive integer, s and t are any real numbers with at least one not being an even number, and* $\texttt{Re}\, x > 0$. *Then*

$$J_{(a,0,s,t)}(q) = -\frac{\pi}{ax}\sum_{j=0}^{a-1}\log\prod_{m=-\infty}^{\infty}(1-\omega^{2j+t}q^{|2m+s|})(1-\omega^{-(2j+t)}q^{|2m+s|}) + \sum_{m\neq 0}\frac{e^{\pi ims}}{(xm)^2},$$

where $J_{(a,0,s,t)}(q)$ is defined in (1.2.6), $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

Proof. The proof is similar to the proof of Theorem 3.2.1. The main difference is that the index (m,n) can not be $(0,0)$. Therefore, we need to separate the sum when $n=0$ at the very beginning, and thus we have

$$\sum_{\substack{-N<n<N\\ n\neq 0}}\sum_{m\in\,\mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an)^2} = \frac{\pi}{x}\sum_{m\in\,\mathbb{Z}}\sum_{\substack{-N<n<N\\ N\neq 0}}\frac{e^{\pi int}q^{|2m+s||an|}}{|an|}. \tag{3.2.11}$$

Then we claim that for $|r|<1$,

$$\sum_{n\neq 0}\frac{e^{\pi int}r^{|na|}}{|na|} = -\frac{1}{a}\sum_{j=0}^{a-1}\log(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r). \tag{3.2.12}$$

Let us briefly prove the claim here. By expanding the right side of (3.2.12) as Taylor series in r, we have

$$\begin{aligned}
&-\frac{1}{a}\sum_{j=0}^{a-1}\log(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r)\\
&\qquad = \frac{1}{a}\sum_{j=0}^{a-1}\sum_{k=1}^{\infty}\left(\frac{r^k}{k}e^{\pi ik(2j+t)/a} + \frac{r^k}{k}e^{-\pi ik(2j+t)/a}\right)\\
&\qquad = \frac{1}{a}\sum_{k=1}^{\infty}\frac{r^k}{k}\left(e^{\pi ikt/a}\sum_{j=0}^{a-1}e^{2\pi ijk/a} + e^{-\pi ikt/a}\sum_{j=0}^{a-1}e^{-2\pi ijk/a}\right)\\
&\qquad = \frac{1}{a}\sum_{na\geq 1}\frac{r^{|na|}}{|na|}e^{\pi int}\cdot a + \frac{1}{a}\sum_{na\geq 1}\frac{r^{|na|}}{|na|}e^{-\pi int}\cdot a\\
&\qquad = \sum_{na\geq 1}\frac{r^{|na|}}{|na|}e^{\pi int} + \sum_{na\leq 1}\frac{r^{|na|}}{|na|}e^{\pi int}\\
&\qquad = \sum_{n\neq 0}\frac{e^{\pi int}r^{|na|}}{|na|}.
\end{aligned}$$

To finish the proof of Theorem 3.2.4, we use the same idea as in the proof of Theorem 3.2.1.

3.3 Examinations of $H_{(a,b)}$ for $a \in \{3,4,5,6\}$ and explicit examples

Although the proofs of (3.2.8) and Theorem 1.2.1 are similar, the examinations of special cases of $H_{(a,b)}$ are quite different from those of $F_{(a,b)}$, and they are actually more difficult, because we have even powers of q instead of odd powers in the evaluation. In this section, we examine the cases where $a \in \{3,4,5,6\}$ and $b = 1$.

Let us prove a couple of lemmas before the examinations.

Lemma 3.3.1. *For $|q| < 1$, we have*

$$\prod_{m\geq 1}(1-\sqrt{2}q^{2m}+q^{4m}) = \frac{qf(-q^{32})}{f(-q^2)}\sqrt{\frac{f(-q^4)}{f(-q^8)}}\left(\frac{1}{\sqrt{T(q^4)}}-(\sqrt{2}+1)\sqrt{T(q^4)}\right), \tag{3.3.1}$$

$$\prod_{m\geq 1}(1+\sqrt{2}q^{2m}+q^{4m}) = \frac{qf(-q^{32})}{f(-q^2)}\sqrt{\frac{f(-q^4)}{f(-q^8)}}\left(\frac{1}{\sqrt{T(q^4)}}+(\sqrt{2}-1)\sqrt{T(q^4)}\right), \tag{3.3.2}$$

$$\prod_{m\geq 1}\frac{1-\sqrt{2}q^{2m}+q^{4m}}{1+\sqrt{2}q^{2m}+q^{4m}} = \frac{1-(\sqrt{2}+1)T(q^4)}{1+(\sqrt{2}-1)T(q^4)}, \tag{3.3.3}$$

where $T(q)$ is the Ramanujan-Göllnitz-Gordon continued fraction, defined by (3.1.7).

Proof. The equality (3.3.3) can be easily derived from (3.3.1) and (3.3.2). Here we give the proof of (3.3.1) only, as the proof of (3.3.2) is similar. Letting $\omega = e^{\pi i/4}$ and using the Jacobi triple product identity (1.0.7) for Ramanujan's general theta function $f(a,b)$, we have

$$\begin{aligned}\prod_{m\geq 1}(1-\sqrt{2}q^{2m}+q^{4m}) &= \prod_{m\geq 1}(1-(\omega+\omega^{-1})q^{2m}+q^{4m})\\ &= \prod_{m\geq 1}(1-\omega q^{2m})(1-\omega^{-1}q^{2m})\\ &= (\omega q^2;q^2)_\infty(\omega^{-1}q^2;q^2)_\infty\\ &= \frac{1}{(q^2;q^2)_\infty}\frac{f(-\omega,-\omega^{-1}q^2)}{1-\omega}. \end{aligned}\tag{3.3.4}$$

Applying the addition formula (3.1.11) with $n=4$, $a=-\omega$ and $b=-\omega^{-1}q^2$, we obtain

$$f(-\omega,-\omega^{-1}q^2) = (1-\omega)f(-q^{12},-q^{20}) + (\omega^2+\omega^3)q^2f(-q^4,-q^{28}).$$

It follows that

$$\frac{f(-\omega,-\omega^{-1}q^2)}{1-\omega} = f(-q^{12},-q^{20}) - (\sqrt{2}+1)q^2f(-q^4,-q^{28}). \tag{3.3.5}$$

To complete the proof, we substitute (3.3.5) into (3.3.4), divide both the denominator and numerator by $\sqrt{f(-q^{12},-q^{20})f(-q^4,-q^{28})}$, use the product representation of the Ramanujan-Göllnitz-Gordon continued fraction (3.1.8), and manipulate theta products to deduce that

$$f(-q^{12};-q^{20})f(-q^4;-q^{28}) = f^2(-q^{32})\frac{f(-q^4)}{f(-q^8)}.$$

Lemma 3.3.2. *For $|q|<1$, we have*

$$\prod_{m\geq 1}(1-\sqrt{3}q^{2m}+q^{4m}) = \frac{f(-q^{30},-q^{42})}{f(-q^2)}(1+(\sqrt{3}+1)J(q^6)+(2+\sqrt{3})K(q^6)), \tag{3.3.6}$$

$$\prod_{m\geq 1}(1+\sqrt{3}q^{2m}+q^{4m}) = \frac{f(-q^{30},-q^{42})}{f(-q^2)}(1+(\sqrt{3}-1)J(q^6)+(2-\sqrt{3})K(q^6)), \tag{3.3.7}$$

$$\prod_{m\geq 1}\frac{1-\sqrt{3}q^{2m}+q^{4m}}{1+\sqrt{3}q^{2m}+q^{4m}} = \frac{1-(\sqrt{3}+1)J(q^6)+(2+\sqrt{3})K(q^6)}{1+(\sqrt{3}-1)J(q^6)+(2-\sqrt{3})K(q^6)}, \tag{3.3.8}$$

where $J(q) := q^{1/3}\frac{f(-q^3,-q^9)}{f(-q^5,-q^7)}$ and $K(q) = q\frac{f(-q,-q^{11})}{f(-q^5,-q^7)}$ is the continued fractions of order 12 defined by (3.1.9).

Proof. The proof of Lemma 3.3.2 is similar to the proof of Lemma 3.3.1. Here we give the proof of (3.3.7) only. Letting $\omega = e^{\pi i/6}$ and using the Jacobi triple product identity (1.0.7) for Ramanujan's general theta function $f(a,b)$, we have

$$\begin{aligned}\prod_{m\geq 1}(1+\sqrt{3}q^{2m}+q^{4m}) &= \prod_{m\geq 1}(1+(\omega+\omega^{-1})q^{2m}+q^{4m})\\ &= \prod_{m\geq 1}(1+\omega q^{2m})(1+\omega^{-1}q^{2m})\\ &= (-\omega q^2;q^2)_\infty(-\omega^{-1}q^2;q^2)_\infty\\ &= \frac{1}{(q^2;q^2)_\infty}\frac{f(\omega,\omega^{-1}q^2)}{1+\omega}.\end{aligned} \tag{3.3.9}$$

Applying the addition formula (3.1.11) with $n=6$, $a=\omega$ and $b=\omega^{-1}q^2$, we obtain

$$f(\omega,\omega^{-1}q^2) = (1+\omega)f(-q^{30},-q^{42}) + (\omega^2-\omega^5)q^2 f(-q^{18},-q^{54}) + (\omega^3-\omega^4)q^6 f(-q^6,-q^{66}).$$

It follows that

$$\frac{f(\omega,\omega^{-1}q^2)}{1+\omega} = f(-q^{30},-q^{42}) + (\sqrt{3}-1)q^2 f(-q^{18},-q^{54}) + (2-\sqrt{3})q^6 f(-q^6,-q^{66}). \tag{3.3.10}$$

Substituting (3.3.10) into (3.3.9) and dividing through by $f(-q^{30}, -q^{42})$, we obtain (3.3.7).

Theorem 3.3.3. *Suppose that $q = e^{-\pi/x}$. Let $G(q)$, $T(q)$ and $K(q)$ be the continued functions defined in (3.1.5), (3.1.7), and (3.1.9), respectively. Let $J(q)$ be the function defined in Lemma 3.3.2. Then*

$$H_{(3,1)}(q) = \frac{2\pi}{9x} \log \frac{8(1+G^3(q^2))}{1-8G^3(q^2)}, \tag{3.3.11}$$

$$H_{(4,1)}(q) = -\frac{\pi}{\sqrt{2}x} \log \frac{\sqrt{2}-1-T(q^4)}{1+(\sqrt{2}-1)T(q^4)}, \tag{3.3.12}$$

$$\begin{aligned} H_{(5,1)}(q) = \frac{2\pi}{5x}\log 2 + \frac{\pi}{5x}\log\frac{\chi(-q^{10})}{\chi^5(-q^2)} - \frac{\pi}{\sqrt{5}x}\log\frac{\sqrt{5}\varphi(q^5)-\varphi(q)}{\sqrt{5}\varphi(q^5)+\varphi(q)} \\ - \frac{\pi}{5\sqrt{5}x}\log\frac{(1-\alpha^5R^5(q))(1-\beta^5R^5(q^2))}{(1-\beta^5R^5(q))(1-\alpha^5R^5(q^2))}, \end{aligned} \tag{3.3.13}$$

$$H_{(6,1)}(q) = \frac{\pi}{\sqrt{3}x} \log(2+\sqrt{3}) \frac{1+(\sqrt{3}-1)J(q^6)+(2-\sqrt{3})K(q^6)}{1+(\sqrt{3}+1)J(q^6)+(2+\sqrt{3})K(q^6)}. \tag{3.3.14}$$

Proof. We begin by proving (3.3.11). If we set $(a,b) = (3,1)$, then (3.2.8) immediately reduces to

$$\begin{aligned} &H_{(3,1)}(q) \\ &= -\frac{\pi}{3x}\sum_{j=0}^{2}\cos\left(\frac{(2j+1)\pi}{3}\right)\log\prod_{m\in\mathbb{Z}}\left(1-2\cos\left(\frac{(2j+1)\pi}{3}\right)q^{2|m|}+q^{4|m|}\right) \\ &= -\frac{\pi}{3x}\log\prod_{m\in\mathbb{Z}}\frac{1-q^{2|m|}+q^{4|m|}}{1+2q^{2|m|}+q^{4|m|}} \\ &= -\frac{\pi}{3x}\log\frac{1}{4}\prod_{m\geq 1}\frac{(1-q^{2m}+q^{4m})^2}{(1+2q^{2m}+q^{4m})^2} \\ &= \frac{2\pi}{3x}\log 2 - \frac{2\pi}{3x}\log\prod_{m\geq 1}\frac{1+q^{6m}}{(1+q^{2m})^3} \\ &= \frac{2\pi}{3x}\log 2 - \frac{2\pi}{3x}\log\frac{(-q^6;q^6)_\infty}{(-q^2;q^2)_\infty^3} \\ &= \frac{2\pi}{3x}\log 2 - \frac{2\pi}{3x}\log\frac{\chi^3(-q^2)}{\chi(-q^6)}. \end{aligned} \tag{3.3.15}$$

Notice that we used $\chi(-q) = 1/(-q;q)_\infty$ in the last equality above. To finish the calculation, let us take $\alpha = 1 - \phi^4(-q)/\phi^4(q)$ and $\beta = 1 - \phi^4(-q^3)/\phi^4(q^3)$ so that β has degree 3 over α in the theory of modular equations. Then using [10, p. 124, Entry 12],

$$\chi(-q) = 2^{1/6}(1-x)^{1/12}(xq)^{-1/24},$$

we have

$$q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} = 2^{-1/3}\frac{(1-\alpha)^{1/12}\beta^{1/8}}{(1-\beta)^{1/4}\alpha^{1/24}}, \qquad \frac{\chi^3(-q)}{\chi(-q^3)} = 2^{1/3}\frac{(1-\alpha)^{1/4}\beta^{1/24}}{(1-\beta)^{1/12}\alpha^{1/8}}.$$

It is known that α and β admit birational parameterizations $\alpha = p(2+p)^3/(1+2p)^3$ and $\beta = p^3(2+p)/(1+2p)$ [10, p. 230, Entry 5(vi)]. Thus we deduce that

$$q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} = \left(\frac{p}{2(1+p)}\right)^{1/3}, \tag{3.3.16}$$

$$\frac{\chi^3(-q)}{\chi(-q^3)} = \left(\frac{2(1-p)^2}{(2+p)(1+2p)}\right)^{1/3}. \tag{3.3.17}$$

Now if $v = q^{1/3}\chi(-q)/\chi^3(-q^3) = G(q)$, as given in (3.1.6), then we solve for p from (3.3.16) to obtain

$$p = \frac{1-4v^3+\sqrt{1-8v^3}}{4v^3}, \tag{3.3.18}$$

and it follows that

$$\frac{\chi^3(-q)}{\chi(-q^3)} = \left(\frac{1-8v^3}{1+v^3}\right)^{1/3} \tag{3.3.19}$$

by substituting (3.3.18) into (3.3.17). Replacing q by q^2 and substituting (3.3.19) into (3.3.15) completes the proof of (3.3.11).

Notice that if $(a,b) = (4,1)$, then (3.2.8) becomes

$$\begin{aligned}
&H_{(4,1)}(q)\\
&= -\frac{\pi}{4x}\sum_{j=0}^{3}\cos\left(\frac{(2j+1)\pi}{4}\right)\log\prod_{m\in\mathbb{Z}}\left(1-2\cos\left(\frac{(2j+1)\pi}{4}\right)q^{2|m|}+q^{4|m|}\right)\\
&= -\frac{\sqrt{2}\pi}{4x}\log\prod_{m\in\mathbb{Z}}\frac{1-\sqrt{2}q^{2|m|}+q^{4|m|}}{1+\sqrt{2}q^{2|m|}+q^{4|m|}}\\
&= -\frac{\sqrt{2}\pi}{2x}\log\left((\sqrt{2}-1)\prod_{m\geq 1}\frac{1-\sqrt{2}q^{2m}+q^{4m}}{1+\sqrt{2}q^{2m}+q^{4m}}\right).
\end{aligned}$$

Using (3.3.3), we are led to the closed form

$$H_{(4,1)} = -\frac{\sqrt{2}\pi}{2x}\log\frac{\sqrt{2}-1-T(q^4)}{1+(\sqrt{2}-1)T(q^4)}.$$

We set $\alpha = 2\cos\frac{3}{5}\pi = \frac{1-\sqrt{5}}{2}$, and $\beta = 2\cos\frac{1}{5}\pi = \frac{1+\sqrt{5}}{2}$. With $(a,b) = (5,1)$ in (3.2.8), we have

$$\begin{aligned}
&H_{(5,1)}(q)\\
&= -\frac{\pi}{5x}\sum_{j=0}^{4}\cos\left(\frac{(2j+1)\pi}{5}\right)\log\prod_{m\in\mathbb{Z}}\left(1-2\cos(\frac{(2j+1)\pi}{5})q^{2|m|}+q^{4|m|}\right)\\
&= -\frac{\pi}{5x}\log\prod_{m\in\mathbb{Z}}\frac{(1-\alpha q^{2|m|}+q^{4|m|})^{\alpha}(1-\beta q^{2|m|}+q^{4|m|})^{\beta}}{(1+q^{2|m|})^{2}}\\
&= -\frac{\pi}{5x}\log\prod_{m\in\mathbb{Z}}\frac{(1+q^{10|m|})^{\frac{1}{2}}}{(1+q^{2|m|})^{\frac{5}{2}}}\left(\frac{1-\beta q^{2|m|}+q^{4|m|}}{1-\alpha q^{2|m|}+q^{4|m|}}\right)^{\frac{\sqrt{5}}{2}}\\
&= -\frac{\pi}{5x}\log\frac{1}{4}\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{\sqrt{5}}-\frac{\pi}{5x}\log\frac{\chi^{5}(-q^{2})}{\chi(-q^{10})}-\frac{\pi}{\sqrt{5}x}\log\prod_{m\ \mathrm{even}}\frac{1-\beta q^{m}+q^{2m}}{1-\alpha q^{m}+q^{2m}}.
\end{aligned} \tag{3.3.20}$$

Factorizations of certain theta-function identities of degree 5 are given by [4, p. 30, Entry 1.7.2 (i),(ii)]

$$\varphi(q)+\sqrt{5}\varphi(q^{5}) = \frac{(1+\sqrt{5})f(-q^{2})}{\prod_{n\ \mathrm{odd}}(1+\alpha q^{n}+q^{2n})\prod_{n\ \mathrm{even}}(1-\beta q^{n}+q^{2n})},$$

$$\varphi(q)-\sqrt{5}\varphi(q^{5}) = \frac{(1-\sqrt{5})f(-q^{2})}{\prod_{n\ \mathrm{even}}(1-\alpha q^{n}+q^{2n})\prod_{n\ \mathrm{odd}}(1+\beta q^{n}+q^{2n})},$$

from which we deduce that

$$\prod_{m\ \mathrm{even}}\frac{1-\beta q^{m}+q^{2m}}{1-\alpha q^{m}+q^{2m}}\prod_{m\ \mathrm{odd}}\frac{1+\alpha q^{m}+q^{2m}}{1+\beta q^{m}+q^{2m}} = \frac{(\sqrt{5}+1)(\sqrt{5}\varphi(q^{5})-\varphi(q))}{(\sqrt{5}-1)(\sqrt{5}\varphi(q^{5})+\varphi(q))}. \tag{3.3.21}$$

Now we use the formulas of the factorizations of two of the most important formulas for the Rogers-Ramanujan continued fraction from Ramanujan's lost notebook [4, p. 21–22, Entry 1.4.1],

$$\left(\frac{1}{\sqrt{t}}\right)^{5}-\left(\alpha\sqrt{t}\right)^{5} = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^{5})}}\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^{n}+q^{2n})^{5}},$$

$$\left(\frac{1}{\sqrt{t}}\right)^{5}-\left(\beta\sqrt{t}\right)^{5} = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^{5})}}\prod_{n=1}^{\infty}\frac{1}{(1+\beta q^{n}+q^{2n})^{5}},$$

to obtain

$$\prod_{m\ \mathrm{odd}}\frac{1+\beta q^{m}+q^{2m}}{1+\alpha q^{m}+q^{2m}} = \sqrt[5]{\frac{(1-\alpha^{5}R^{5}(q))(1-\beta^{5}R^{5}(q^{2}))}{(1-\beta^{5}R^{5}(q))(1-\alpha^{5}R^{5}(q^{2}))}}. \tag{3.3.22}$$

To complete the proof of (3.3.13), we substitute (3.3.21) and (3.3.22) into (3.3.20).

Now if $(a,b) = (6,1)$, then we can easily prove (3.3.14) by applying Lemma 3.3.2. From (3.2.8), we have

$$
\begin{aligned}
H_{(6,1)}(q) &= -\frac{\pi}{6x}\sum_{j=0}^{5}\cos\left(\frac{(2j+1)\pi}{6}\right)\log\prod_{m\in\mathbb{Z}}\left(1-2\cos\left(\frac{(2j+1)\pi}{6}\right)q^{2|m|}+q^{4|m|}\right)\\
&= -\frac{\sqrt{3}\pi}{6x}\log\prod_{m\in\mathbb{Z}}\frac{1-\sqrt{3}q^{2|m|}+q^{4|m|}}{1+\sqrt{3}q^{2|m|}+q^{4|m|}}\\
&=\frac{\pi}{\sqrt{3}x}\log(2+\sqrt{3})\prod_{m\geq 1}\frac{1+\sqrt{3}q^{2m}+q^{4m}}{1-\sqrt{3}q^{2m}+q^{4m}}.
\end{aligned}
\tag{3.3.23}
$$

Applying (3.3.8), we are led to the closed form (3.3.14). This completes our proof.

Next we consider explicit examples of $H_{(a,b)}(q)$ from Theorem 3.3.3. We first derive examples for $H_{(3,1)}(q)$ from (3.3.11). It is clear that the formulas for $G(q)$ can also be used to evaluate $H_{(3,1)}(q)$. When $x = 1/\sqrt{2}$ we appeal to [4, p. 100, Eq. (3.4.4)]. We have

$$
G(e^{-\sqrt{2}\pi}) = \frac{-2+\sqrt{6}}{2}. \tag{3.3.24}
$$

Thus we obtain

$$
\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{2m^2+(3n+1)^2} = \frac{\sqrt{2}\pi}{9}\log(4+2\sqrt{6}).
$$

Similarly, set $x = 3\sqrt{2}$. We use the fact from [4, p. 100, Eq. (3.4.5)] to find that

$$
G^3(e^{-\sqrt{2}\pi/3}) = \frac{-2+\sqrt{6}}{4}. \tag{3.3.25}
$$

Therefore we have

$$
\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{18m^2+(3n+1)^2} = \frac{\sqrt{2}\pi}{27}\log(44+18\sqrt{6}).
$$

As another example, when $x = 1$ we appeal to [4, p. 100, Eq. (3.4.3)] to find that

$$
G(e^{-2\pi}) = \frac{-(1+\sqrt{3})+\sqrt{6\sqrt{3}}}{4},
$$

and thus

$$
\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{m^2+(3n+1)^2} = \frac{\sqrt{2}\pi}{9}\log\frac{\sqrt[4]{3}(3\sqrt{2}-2\sqrt[4]{3}+\sqrt{6})}{(2-\sqrt{2\sqrt{3}})(3+\sqrt{3})}.
$$

Now we derive explicit examples from (3.3.12), which include the Ramanujan-Göllnitz-Gordon continued fraction on the right-hand side. When $x = 8$ we appeal to [21, p. 84, Eq. (4.2)] to find that

$$T(e^{-\pi/2}) = \sqrt{\sqrt{2}+1} - \sqrt[4]{2},$$

which yields

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(8m)^2 + (4n+1)^2} = -\frac{\sqrt{2}\pi}{16} \log \frac{(\sqrt{2}-1) - \left(\sqrt{\sqrt{2}+1} - \sqrt[4]{2}\right)}{1 + (\sqrt{2}-1)\left(\sqrt{\sqrt{2}+1} - \sqrt[4]{2}\right)}.$$

When $x = \frac{8}{3}\sqrt{3}$ we appeal to [21, p. 86]. We have

$$T(e^{-\pi\sqrt{3}/2}) = \sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}},$$

and hence

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(8m)^2 + 3(4n+1)^2}$$
$$= -\frac{\sqrt{6}\pi}{48} \log \frac{(\sqrt{2}-1) - \left(\sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}}\right)}{1 + (\sqrt{2}-1)\left(\sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}}\right)}.$$

We can find further evaluations for $H_{(4,1)}(q)$ by applying other formulas in [21, p. 84, Eq. (4.3), (4.4)] and [21, p. 86, Examples].

Now we examine the more difficult case of $H_{(6,1)}$. We consider $x = 6$, which yields $q^6 = e^{-\pi}$. We appeal to [30, Theorem 5.1] to find that

$$K(e^{-\pi}) = \frac{(6\sqrt{3}-9)^{1/4} - 1}{(6\sqrt{3}-9)^{1/4} + 1}. \tag{3.3.26}$$

We still need to examine $J(e^{-\pi})$. We apply [30, Lemma 3.1] first to obtain

$$J(q) = \frac{2q^{1/3}\chi(q)\psi(-q^3)}{\varphi(q) + \varphi(q^3)}.$$

So we can evaluate $J(e^{-\pi})$ from formulas for $\varphi(e^{-\pi})$, $\varphi(e^{-3\pi})$, $\psi(-e^{-3\pi})$ and $\chi(e^{-\pi})$. We appeal to [45, Lemma 5.1, Theorem 5.5], [46, Theorem 5.6] and [11, p. 326, Entry 2 (viii)] respectively. We have for

$a = \pi^{-1/4}/\Gamma(3/4)$,

$$\varphi(e^{-\pi}) = a,$$
$$\varphi(e^{-3\pi}) = a2^{-1}3^{-3/8}\sqrt{\sqrt{3}+1},$$
$$\psi(-e^{-3\pi}) = a2^{-3/4}3^{-1/2}e^{3\pi/8}(2\sqrt{3}-3)^{1/4},$$
$$\chi(e^{-\pi}) = e^{-\pi/24}2^{1/4}.$$

Simplify the resulting quotient to obtain

$$J(e^{-\pi}) = \frac{\sqrt{2}(2-\sqrt{3})^{1/4}}{3^{3/8}+2^{-1/4}\sqrt{\sqrt{3}+1}}. \tag{3.3.27}$$

To finish the calculation, we just need to insert (3.3.26) and (3.3.27) into (3.3.14) and simplify. Therefore we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{(6m)^2+(6n+1)^2}$$
$$= \frac{\pi}{6\sqrt{3}}\log\left(\frac{2+\sqrt{3}}{2}\left(5-\sqrt{3}+\sqrt{2}\ 3^{3/4}+\sqrt[4]{6\sqrt{3}-9}\left(1+\sqrt{3}+\sqrt{2}\ 3^{3/4}\right)\right)\right).$$

3.4 Simplification of $F_{(8,1)}$ and $F_{(12,1)}$ and explicit example

Theorem 3.4.1. *Suppose that $q = e^{-\pi/x}$. Then we have*

$$\begin{aligned} F_{(8,1)}(q) = &-\frac{\pi}{4x}\sqrt{2+\sqrt{2}}\ \log\left(\frac{A(q)-\sqrt{1+\frac{\sqrt{2}}{2}q\sqrt{C(q)}}-\sqrt{1-\frac{\sqrt{2}}{2}q\sqrt{D(q)}}}{A(q)+\sqrt{1+\frac{\sqrt{2}}{2}q\sqrt{C(q)}}+\sqrt{1-\frac{\sqrt{2}}{2}q\sqrt{D(q)}}}\right) \\ &-\frac{\pi}{4x}\sqrt{2-\sqrt{2}}\ \log\left(\frac{B(q)-\sqrt{1-\frac{\sqrt{2}}{2}q\sqrt{C(q)}}+\sqrt{1+\frac{\sqrt{2}}{2}q\sqrt{D(q)}}}{B(q)+\sqrt{1-\frac{\sqrt{2}}{2}q\sqrt{C(q)}}-\sqrt{1+\frac{\sqrt{2}}{2}q\sqrt{D(q)}}}\right), \end{aligned} \tag{3.4.1}$$

where

$$
\begin{aligned}
A(q) &= \frac{\varphi(-q^{64})}{\psi(-q^{16})} + \sqrt{2}q^4 \frac{\psi(-q^{32})}{\psi(-q^{16})}, \\
B(q) &= \frac{\varphi(-q^{64})}{\psi(-q^{16})} - \sqrt{2}q^4 \frac{\psi(-q^{32})}{\psi(-q^{16})}, \\
C(q) &= \frac{\varphi(q^{16})}{\psi(-q^{16})} + \frac{\varphi(q^{32})}{\psi(-q^{16})}, \\
D(q) &= \frac{\varphi(q^{16})}{\psi(-q^{16})} - \frac{\varphi(q^{32})}{\psi(-q^{16})}.
\end{aligned}
$$

Proof. If $(a,b) = (8,1)$, then (1.2.2) reduces to

$$
\begin{aligned}
&F_{(8,1)}(q) \\
&= -\frac{\pi}{4x}\sum_{j=0}^{7}\cos\left(\frac{(2j+1)\pi}{8}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\frac{(2j+1)\pi}{8}q^{2m+1}+q^{4m+2}\right) \\
&= -\frac{\pi}{4x}\log\prod_{m=0}^{\infty}\left(\frac{1-2\cos\frac{\pi}{8}q^{2m+1}+q^{4m+2}}{1+2\cos\frac{\pi}{8}q^{2m+1}+q^{4m+2}}\right)^{2\cos\frac{\pi}{8}} \\
&\qquad\qquad \times\left(\frac{1-2\cos\frac{3\pi}{8}q^{2m+1}+q^{4m+2}}{1+2\cos\frac{3\pi}{8}q^{2m+1}+q^{4m+2}}\right)^{2\cos\frac{3\pi}{8}} \\
&= -\frac{\pi}{4x}\log\prod_{m=0}^{\infty}\left(\frac{(1-2\cos\frac{\pi}{8}q^{2m+1}+q^{4m+2})(1-q^{2m+2})}{(1+2\cos\frac{\pi}{8}q^{2m+1}+q^{4m+2})(1-q^{2m+2})}\right)^{2\cos\frac{\pi}{8}} \\
&\qquad\qquad \times\left(\frac{(1-2\cos\frac{3\pi}{8}q^{2m+1}+q^{4m+2})(1-q^{2m+2})}{(1+2\cos\frac{3\pi}{8}q^{2m+1}+q^{4m+2})(1-q^{2m+2})}\right)^{2\cos\frac{3\pi}{8}}.
\end{aligned}
\tag{3.4.2}
$$

Letting $\xi = e^{\pi i/8}$ and using the Jacobi triple product identity (1.0.7), we find that

$$
\begin{aligned}
F(q) &:= \prod_{m=0}^{\infty}(1+2\cos\frac{\pi}{8}q^{2m+1}+q^{4m+2})(1-q^{2m+2}) \\
&= (-\xi q;q^2)_\infty(-\bar{\xi}q;q^2)_\infty(q^2;q^2)_\infty \\
&= \sum_{n=-\infty}^{\infty}\xi^n q^{n^2} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n\left[q^{(8n)^2}+\xi q^{(8n+1)^2}+\xi^2 q^{(8n+2)^2}+\cdots+\xi^7 q^{(8n+7)^2}\right].
\end{aligned}
$$

Note that $F(q)$ is real-valued, so the imaginary terms above sum to 0. Therefore we only need to consider the real parts. First, we have

$$\begin{aligned}
\mathtt{Re}\left(\sum_{n=-\infty}^{\infty}(-1)^n(\xi^2q^{(8n+2)^2}+\xi^6q^{(8n+6)^2})\right) =&2\mathtt{Re}\left(\sum_{n=-\infty}^{\infty}(-1)^n\xi^2q^{(8n+2)^2}\right)\\
=&\sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^nq^{4(4n+1)^2}\\
=&\sqrt{2}\left(\sum_{n=0}^{\infty}(-1)^nq^{4(4n+1)^2}-\sum_{n=0}^{\infty}(-1)^nq^{4(4n+3)^2}\right)\\
=&\sqrt{2}\sum_{n=0}^{\infty}(-1)^{n(n+1)/2}q^{4(2n+1)^2}\\
=&\sqrt{2}q^4\psi(-q^{32}).
\end{aligned}$$

Now we consider

$$\begin{aligned}
&\mathtt{Re}\left(\sum_{n=-\infty}^{\infty}(-1)^n\left[\xi q^{(8n+1)^2}+\xi^7q^{(8n+7)^2}\right]\right)\\
&=\mathtt{Re}\left(\xi\sum_{n=-\infty}^{\infty}(-1)^n\left[q^{(8n+1)^2}-q^{(8n+7)^2}\right]\right)\\
&=2\cos\left(\frac{\pi}{8}\right)\sum_{n=-\infty}^{\infty}(-1)^nq^{(8n+1)^2}\\
&=2\cos\left(\frac{\pi}{8}\right)q\sum_{n=-\infty}^{\infty}(-1)^nq^{16n(4n+1)}\\
&=2\cos\left(\frac{\pi}{8}\right)qf(-q^{16\cdot 3},-q^{16\cdot 5})\\
&=\sqrt{2}\cos\left(\frac{\pi}{8}\right)\sqrt{\psi(-q^{16})[\varphi(q^{16})+\varphi(q^{32})]},
\end{aligned}$$

where we apply (3.1.14) in the last identity. If we apply (3.1.15), then we can find that

$$\mathtt{Re}\left(\sum_{n=-\infty}^{\infty}(-1)^n\left[\xi^3q^{(8n+3)^2}+\xi^5q^{(8n+5)^2}\right]\right)=\sqrt{2}\cos\left(\frac{3\pi}{8}\right)\sqrt{\psi(-q^{16})[\varphi(q^{16})-\varphi(q^{32})]}.$$

Combining the results above, we are led to the closed form

$$\begin{aligned}
F(q)=\varphi(-q^{64})+\sqrt{2}q^4\psi(-q^{32})+\sqrt{2}\cos\left(\frac{\pi}{8}\right)q\sqrt{\psi(-q^{16})[\varphi(q^{16})+\varphi(q^{32})]}&\\
+\sqrt{2}\cos\left(\frac{3\pi}{8}\right)q\sqrt{\psi(-q^{16})[\varphi(q^{16})-\varphi(q^{32})]}.&
\end{aligned}\tag{3.4.3}$$

Similarly, we can derive a formula for the other factor in the denominator of (3.4.2). Thus,

$$\begin{aligned} G(q) &:= \prod_{m=0}^{\infty}(1+2\cos\frac{3\pi}{8}q^{2m+1}+q^{4m+2})(1-q^{2m+2}) \\ &= \varphi(-q^{64})-\sqrt{2}q^4\psi(-q^{32})+\sqrt{2}\cos\left(\frac{3\pi}{8}\right)q\sqrt{\psi(-q^{16})[\varphi(q^{16})+\varphi(q^{32})]} \\ &\quad -\sqrt{2}\cos\left(\frac{3\pi}{8}\right)q\sqrt{\psi(-q^{16})[\varphi(q^{16})-\varphi(q^{32})]}. \end{aligned} \tag{3.4.4}$$

To complete the proof, we just need to apply (3.4.3), (3.4.4) and the facts that

$$\cos\left(\frac{\pi}{8}\right)=\frac{\sqrt{2+\sqrt{2}}}{2} \qquad \text{and} \qquad \cos\left(\frac{3\pi}{8}\right)=\frac{\sqrt{2-\sqrt{2}}}{2}.$$

The formula for $a = 12$ can be deduced in a similar fashion. However, the formula is more complicated and thus we give the theorem without rigorous proof here.

Theorem 3.4.2. *Suppose that $q = e^{-\pi/x}$. Then we have*

$$\begin{aligned} &F_{(12,1)}(q) = -\frac{\pi}{6x} \\ &\times\Bigg\{\sqrt{2+\sqrt{3}}\ \log\frac{\varphi(-q^{144})-\sqrt{2}\psi(-q^{72})+\sqrt{3}q^4f(-q^{96})+q^{32}\varphi(-q^{288})-\sqrt{2+\sqrt{3}}X(q^{24})-\sqrt{2-\sqrt{3}}Y(q^{24})}{\varphi(-q^{144})+\sqrt{2}\psi(-q^{72})+\sqrt{3}q^4f(-q^{96})+q^{32}\varphi(-q^{288})+\sqrt{2+\sqrt{3}}X(q^{24})+\sqrt{2-\sqrt{3}}Y(q^{24})} \\ &+\sqrt{2-\sqrt{3}}\ \log\frac{\varphi(-q^{144})+\sqrt{2}\psi(-q^{72})-\sqrt{3}q^4f(-q^{96})+q^{32}\varphi(-q^{288})-\sqrt{2+\sqrt{3}}X(q^{24})-\sqrt{2-\sqrt{3}}Y(q^{24})}{\varphi(-q^{144})-\sqrt{2}\psi(-q^{72})-\sqrt{3}q^4f(-q^{96})+q^{32}\varphi(-q^{288})+\sqrt{2+\sqrt{3}}X(q^{24})+\sqrt{2-\sqrt{3}}Y(q^{24})} \\ &\qquad +\sqrt{2}\log\frac{\varphi(-q^{16})-\sqrt{2}q\psi(-q^{8})}{\varphi(-q^{16})+\sqrt{2}q\psi(-q^{8})}\Bigg\}, \end{aligned}$$

where

$$\begin{aligned} X(q) &= \sqrt{\frac{f(-q^6)\varphi(q^3)\chi(-q^6)+f(-q^2)\varphi(-q^3)\chi(-q^2)}{2\chi(-q^2)}}, \\ Y(q) &= \sqrt{\frac{f(-q^6)\varphi(q^3)\chi(-q^6)-f(-q^2)\varphi(-q^3)\chi(-q^2)}{2\chi(-q^2)}}. \end{aligned}$$

We first obtain the following identity from Theorem 1.2.1.

$$\begin{aligned} F_{(12,1)}(q) = -\frac{\pi}{6x}\Bigg\{&2\cos\frac{\pi}{12}\log\frac{f(-e^{\pi i/12}q,-e^{-\pi i/12}q)}{f(e^{\pi i/12}q,e^{-\pi i/12}q)}+2\cos\frac{5\pi}{12}\log\frac{f(-e^{5\pi i/12}q,-e^{-5\pi i/12}q)}{f(e^{5\pi i/12}q,e^{-5\pi i/12}q)} \\ &+\sqrt{2}\log\frac{f(-e^{\pi i/4}q,-e^{-\pi i/4}q)}{f(e^{\pi i/4}q,e^{-\pi i/4}q)}\Bigg\}. \end{aligned}$$

We also need to obtain the following formulas similar to (3.1.14) and (3.1.15), namely,

$$f(q^5,q^7)=\sqrt{\frac{f(-q^6)\varphi(q^3)\chi(-q^6)+f(-q^2)\varphi(-q^3)\chi(-q^2)}{2\chi(-q^2)}}, \tag{3.4.5}$$

$$f(q,q^{11})=\sqrt{\frac{f(-q^6)\varphi(q^3)\chi(-q^6)-f(-q^2)\varphi(-q^3)\chi(-q^2)}{2q^2\chi(-q^2)}}. \tag{3.4.6}$$

For the proof of (3.4.5) and (3.4.6), we apply Lemma 3.1.13 (i) and (ii) with $a=q^2$, $b=q^4$ and $c=d=q^3$.

We conclude this section by proving a formula for $F_{(8,1)}$ from (3.4.1). In principle, these calculations are straightforward exercises if the values of $\varphi(q)$, $\varphi(q^2)$, $\varphi(-q^2)$, $\varphi(-q^4)$, $\psi(-q)$ and $\psi(-q^2)$ are known. However, (3.4.1) is a long equation, so in practice, we only identify one instance where q^{16} is reasonably simple, that is when $q^{16}=e^{-\pi}$. We appeal to [45, Theorem 5.5, Theorem 5.7] and [46, Theorem 5.6, Theorem 5.7], respectively. We have for $a=\pi^{-1/4}/\Gamma(3/4)$,

$$\begin{aligned}
\varphi(e^{-\pi})&=a,\\
\varphi(e^{-2\pi})&=a2^{-1}(\sqrt{2}+2)^{1/2},\\
\varphi(-e^{-2\pi})&=a2^{-1/8},\\
\varphi(-e^{-4\pi})&=a2^{-7/16}(\sqrt{2}+1)^{1/2},\\
\psi(-e^{-\pi})&=a2^{-3/4}e^{\pi/8},\\
\psi(-e^{-2\pi})&=a2^{-15/16}e^{\pi/4}(\sqrt{2}-1)^{1/4}.
\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}
A(e^{-\pi})&=2^{15/16}e^{-\pi/8}\{(\sqrt{2}+1)^{1/4}+(\sqrt{2}-1)^{1/4}\},\\
B(e^{-\pi})&=2^{15/16}e^{-\pi/8}\{(\sqrt{2}+1)^{1/4}-(\sqrt{2}-1)^{1/4}\},\\
C(e^{-\pi})&=e^{-\pi/8}(2^{3/4}+(\sqrt{2}+1)^{1/2}),\\
D(e^{-\pi})&=e^{-\pi/8}(2^{3/4}-(\sqrt{2}+1)^{1/2}).
\end{aligned}$$

With all the calculations above, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+n}}{(16m)^2+(8n+1)^2}$$
$$=-\frac{\pi}{64}(\sqrt{2+\sqrt{2}})\log\frac{\sqrt{2}a-\sqrt{2+\sqrt{2}}c-\sqrt{2-\sqrt{2}}d}{\sqrt{2}a+\sqrt{2+\sqrt{2}}c+\sqrt{2-\sqrt{2}}d}$$
$$-\frac{\pi}{64}(\sqrt{2-\sqrt{2}})\log\frac{\sqrt{2}b-\sqrt{2-\sqrt{2}}c+\sqrt{2+\sqrt{2}}d}{\sqrt{2}b+\sqrt{2-\sqrt{2}}c-\sqrt{2+\sqrt{2}}d},$$

where

$$a=2^{15/16}\{(\sqrt{2}+1)^{1/4}+(\sqrt{2}-1)^{1/4}\},$$
$$b=2^{15/16}\{(\sqrt{2}+1)^{1/4}-(\sqrt{2}-1)^{1/4}\},$$
$$c=2^{3/4}+(\sqrt{2}+1)^{1/2},$$
$$d=2^{3/4}-(\sqrt{2}+1)^{1/2}.$$

Curiously, $\sqrt{2+\sqrt{2}}$ is the connective constant of the honeycomb lattice, see [27].

3.5 Examinations of $G_{(a,b)}$ and explicit examples

The authors of [15] examined $F_{(a,b)}$ for the cases where $a\in\{3,4,5,6\}$, and we just examined the case $a=8$ in the previous section. Applying Theorem 3.2.3, we can easily examine $G_{(a,b)}$ for $a\in\{3,5,6,10,12\}$ in terms of $F_{(a,b)}$. Moreover, it can easily be derived from (3.2.7) that

$$G_{(2,1)}(q)=-\frac{\pi}{x}\log\prod_{m=-\infty}^{\infty}\frac{(1-q^{2m+1})^2}{(1+q^{2m+1})^2}$$
$$=-\frac{2\pi}{x}\log\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}=-\frac{2\pi}{x}\log\frac{\chi(-q)}{\chi(q)}, \tag{3.5.1}$$

where $\chi(q)$ is defined in (1.0.6). Combining (3.2.9), (3.5.1) and the fact that $F_{(2,1)}=0$, we can examine the case when $a=4$. Indeed, we have

$$G_{(4,1)}(q)=-\frac{\pi}{x}\log\frac{\chi(-q)}{\chi(q)}. \tag{3.5.2}$$

By iterating, we can now examine the cases where $a\in\{8,16\}$.

Now we produce explicit examples for $G_{(a,b)}(q)$. We first consider the simple case when $(a,b)=(3,1)$.

We appeal to [15, Eq. (3.3)] to obtain

$$F_{(3,1)}(q) = -\frac{2\pi}{9x}\log\frac{1+G^3(-q)}{1-8G^3(-q)}, \tag{3.5.3}$$

where $G(q)$ is Ramanujan's cubic continued fraction given in (3.1.5). Applying (3.5.3) and (3.2.10), we can easily derive that

$$G_{(3,1)}(q) = \frac{2\pi}{9x}\log\frac{1+G^3(q)}{1-8G^3(q)}. \tag{3.5.4}$$

We can use formulas for $G(q)$ to evaluate $G_{(3,1)}(q)$. When $x = 1/\sqrt{2}$ we appeal to (3.3.24). After simplification, it follows from (3.5.4) that

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{m^2+2(3n+1)^2} = \frac{\sqrt{2}\pi}{9}\log\frac{2+\sqrt{6}}{4}.$$

Similarly, when $x = \sqrt{3}$, we appeal to [4, p. 105] to find that

$$G(e^{-\pi/\sqrt{3}}) = \frac{\sqrt{3}-1}{4^{1/3}}.$$

After simplification, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{3m^2+(3n+1)^2} = \frac{2\pi}{9\sqrt{3}}\log\frac{5+3\sqrt{3}}{2}.$$

Now we examine $G_{(6,1)}(q)$ from (3.2.9). It follows from [15, Eq. (3.3)] and (3.5.4) that

$$G_{(6,1)}(q) = \frac{\pi}{9x}\log\frac{(1+v^3)(1-8u^3)}{(1-8v^3)(1+u^3)},$$

where $u = G(-q)$ and $v = G(q)$. When $x = 1$ we appeal to [23, p. 350, Eq. (4.1) and Eq. (4.2)] to find that

$$G(-e^{-\pi}) = \frac{1-\sqrt{3}}{2}, \quad G(e^{-\pi}) = \frac{(1+\sqrt{3})(-(1+\sqrt{3})+\sqrt{6\sqrt{3}})}{4},$$

which yield

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{m^2+(6n+1)^2} = \frac{\pi}{9}\log\frac{(\sqrt{3}-1)(21-12\sqrt{2}\ 3^{1/4}+13\sqrt{3}-7\sqrt{2}\ 3^{3/4})}{2(-45+24\sqrt{2}\ 3^{1/4}-26\sqrt{3}+14\sqrt{2}\ 3^{3/4})}.$$

Next we consider $G_{(4,1)}(q)$ and then $G_{(8,1)}(q)$. Recall from (3.5.2) that

$$G_{(4,1)}(q) = -\frac{\pi}{x}\log\frac{\chi(-q)}{\chi(q)} = -\frac{\pi}{x}\log\frac{\psi(-q)}{\psi(q)} = -\frac{\pi}{2x}\log\frac{\varphi(-q)}{\varphi(q)}. \tag{3.5.5}$$

The formulas for $\varphi(q)$ and $\psi(q)$ can be used to evaluate $G_{(4,1)}$. For example, many explicit evaluations can be found in [11, p. 325], [45] and [46]. Set $x = 4$. By [11, Entry 1, p. 325], we have for $a = \pi^{1/4}/\Gamma(3/4)$,

$$\varphi(e^{-\pi/4}) = a(1 + 2^{-1/4}),$$
$$\varphi(-e^{-\pi/4}) = a(1 - 2^{-1/4}).$$

It follows from (3.5.5) that

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{16m^2 + (4n+1)^2} = \frac{\pi}{8}\log\frac{\sqrt[4]{2}+1}{\sqrt[4]{2}-1}. \tag{3.5.6}$$

Using (3.2.9), we can obtain the formula for $G_{(8,1)}(e^{-\pi/4})$ from $F_{(4,1)}(e^{-\pi/4})$ and $G_{(4,1)}(e^{-\pi/4})$. We first evaluate $F_{(4,1)}(q)$ at $x = 4$ from [15, Eq. (3.4)]. Similar to the evaluation of [15, Eq. (3.18)], we appeal to [11, Examples 9.4] to find that $\alpha_4 = (\sqrt{2}-1)^4$, and thus

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+n}}{16m^2 + (4n+1)^2} = \frac{\pi}{4\sqrt{2}}\log\frac{1+\sqrt{\sqrt{2}-1}}{1-\sqrt{\sqrt{2}-1}}. \tag{3.5.7}$$

Substituting (3.5.6) and (3.5.7) into (3.2.9) leads to

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{16m^2 + (8n+1)^2} = \frac{\pi}{16}\log\frac{\sqrt[4]{2}+1}{\sqrt[4]{2}-1} + \frac{\pi}{8\sqrt{2}}\log\frac{1+\sqrt{\sqrt{2}-1}}{1-\sqrt{\sqrt{2}-1}}.$$

3.6 Examinations of $J_{(a,b,s,t)}(q)$ and explicit examples

Simplifications of $J_{(a,b,s,t)}(q)$ are difficult and complicated when $(s,t) \neq (1,0), (0,1)$ or $(1,1)$. However, we can get several nice evaluations in the case when $(a,b) = (2,1)$.

Theorem 3.6.1. *Assume that s and t are any real numbers with at least one not being an even number and*

Re $x>0$. *With* $q=e^{-\pi/x}$, *we have*

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(2n+1)^2}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}\log\prod_{m\geq 0}\frac{(1-2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+s})(1-2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s})}{(1+2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+s})(1+2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s})}. \tag{3.6.1}$$

Proof. The proof is straightforward. We apply (3.6.1) with $a=2$ and $b=1$. Then we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(2n+1)^2}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}\log\prod_{m\in\mathbb{Z}}\frac{(1-e^{\frac{\pi it}{2}}q^{|2m+s|})(1-e^{-\frac{\pi it}{2}}q^{|2m+s|})}{(1+e^{\frac{\pi it}{2}}q^{|2m+s|})(1+e^{-\frac{\pi it}{2}}q^{|2m+s|})}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}$$
$$\times\log\prod_{m\geq 0}\frac{(1-e^{\frac{\pi it}{2}}q^{2m+s})(1-e^{-\frac{\pi it}{2}}q^{-(2m+s)})(1-e^{\frac{\pi it}{2}}q^{2m+2-s})(1-e^{-\frac{\pi it}{2}}q^{-(2m+2-s)})}{(1+e^{\frac{\pi it}{2}}q^{2m+s})(1+e^{-\frac{\pi it}{2}}q^{-(2m+s)})(1+e^{\frac{\pi it}{2}}q^{2m+2-s})(1+e^{-\frac{\pi it}{2}}q^{-(2m+2-s)})}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}\log\prod_{m\geq 0}\frac{(1-2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+2s})(1-2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s})}{(1+2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+2s})(1+2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s})}.$$

This completes the proof.

Theorem 3.6.2. *With* Re $x>0$ *and* $q=e^{-\pi/x}$, *we have*

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{\psi(q)\psi(-q^3)}{\psi(-q)\psi(q^3)}, \tag{3.6.2}$$
$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{1}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\left(-\frac{G(q^{-1/3})\psi(q)\psi(q^3)}{G(-q^{1/3})\psi(-q)\psi(-q^3)}\right), \tag{3.6.3}$$
$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{\varphi^2(-q^2)}{\varphi(-q^{\frac{2}{3}})\varphi(-q^6)}, \tag{3.6.4}$$

where $G(q)$ *is Ramanujan's cubic continued fraction defined by* (3.1.5).

Proof. If we set $t=\frac{2}{3}$, then (3.6.1) immediately reduces to

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\pi ims}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\prod_{m\geq 0}\frac{(1-q^{2m+s}+q^{4m+2s})(1-q^{2m+2-s}+q^{4m+4-2s})}{(1+q^{2m+s}+q^{4m+2s})(1+q^{2m+2-s}+q^{4m+4-2s})}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{(q^s;q^2)_\infty(q^{2-s};q^2)_\infty(-q^{3s};q^6)_\infty(-q^{6-3s};q^6)_\infty}{(-q^s;q^2)_\infty(-q^{2-s};q^2)_\infty(q^{3s};q^6)_\infty(q^{6-3s};q^6)_\infty}. \tag{3.6.5}$$

We begin by proving (3.6.2). If $s=1$, then (3.6.5) becomes

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2} &= -\frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{(q;q^2)_\infty(-q^3;q^6)_\infty}{(-q;q^2)_\infty(q^3;q^6)_\infty}\\ &= \frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{\chi(q)\chi(-q^3)}{\chi(-q)\chi(q^3)}\\ &= \frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{\psi(q)\psi(-q^3)}{\psi(-q)\psi(q^3)},\end{aligned}$$

where we applied (3.1.1) in the last identity.

Next, we prove (3.6.3). Notice that if we set $s=\frac{1}{3}$, then (3.6.5) becomes

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{1}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}\\ &\quad= -\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{(q^{1/3};q^2)_\infty(q^{5/3};q^2)_\infty(-q;q^6)_\infty(-q^5;q^6)_\infty}{(-q^{1/3};q^2)_\infty(-q^{5/3};q^2)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}.\end{aligned} \tag{3.6.6}$$

To manipulate the q-products on the right side of (3.6.6), we first replace q by q^3 to obtain

$$\begin{aligned}&\frac{(q;q^6)_\infty(q^5;q^6)_\infty(-q^3;q^{18})_\infty(-q^{15};q^{18})_\infty}{(-q;q^6)_\infty(-q^5;q^6)_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty}\\ &=\frac{(q;q^2)_\infty(-q^3;q^6)_\infty^2(q^9;q^{18})_\infty}{(-q;q^2)_\infty(q^3;q^6)_\infty^2(-q^9;q^{18})_\infty}\\ &=\frac{\chi(-q)\chi^2(q^3)\chi(-q^9)}{\chi(q)\chi^2(-q^3)\chi(q^9)}.\end{aligned}$$

We recall from (3.1.6) that

$$G(-q)=-q^{\frac{1}{3}}\chi(q)/\chi^3(q^3), \qquad G(q)=q^{\frac{1}{3}}\chi(-q)/\chi^3(-q^3). \tag{3.6.7}$$

After replacing q by $q^{1/3}$ in the above identity and simplifying, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{1}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\left(-\frac{G(-q^{1/3})\psi(q)\psi(q^3)}{G(q^{1/3})\psi(-q)\psi(-q^3)}\right).$$

Now it remains to prove (3.6.4). Similar to the proof of (3.6.3), we set $t=\frac{2}{3}$ in (3.6.5), manipulate the resulting q-products and simplify to obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{f^2(-q^2)\chi^2(-q^2)}{f(-q^{2/3})f(-q^6)\chi(-q^{2/3})\chi(-q^6)}.$$

Finally, we use (3.1.2) to complete the proof.

Now we derive some explicit examples from Theorem 3.6.2. All of our identities follow from well-known q-series evaluations. We first examine an example from (3.6.2). This case is relatively easy to evaluate. When $x = 1$ we appeal to [46, Theorem 5.6, Theorem 5.7]. We have for $a = \pi^{-1/4}/\Gamma(3/4)$,

$$\psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8}, \tag{3.6.8}$$

$$\psi(e^{-\pi}) = a2^{-5/8}e^{\pi/8}, \tag{3.6.9}$$

$$\psi(-e^{-3\pi}) = a2^{-3/4}3^{-1/2}e^{3\pi/8}(2\sqrt{3}-3)^{1/4}, \tag{3.6.10}$$

$$\psi(e^{-3\pi}) = \frac{ae^{3\pi/8}}{2^{1/8}3^{3/8}\sqrt{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}}. \tag{3.6.11}$$

With all the evaluations above, we have, from (3.6.2),

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi in}}{m^2+(2n+1)^2} = \pi e^{-\pi i/3}\log\frac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})}. \tag{3.6.12}$$

If we equate the real and imaginary parts of (3.6.12), then lattice sums involving sine and cosine functions can be evaluated. We obtain, respectively,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m \cos(\frac{2}{3}\pi n)}{m^2+(2n+1)^2} = \frac{\pi}{2}\log\frac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})},$$

and

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m \sin(\frac{2}{3}\pi n)}{m^2+(2n+1)^2} = -\frac{\sqrt{3}\pi}{2}\log\frac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})}.$$

Similarly, when $x = 1/\sqrt{3}$, we have [46, Theorem 4.7 (iii), Theorem 4.10 (x)]

$$\frac{\psi(-e^{-\sqrt{3}\pi})}{\psi(-e^{-3\sqrt{3}\pi})} = 3^{1/4}e^{-\sqrt{3}\pi/4}\frac{\sqrt{3}}{\sqrt[3]{4}-1},$$

and

$$\frac{\psi(e^{-\sqrt{3}\pi})}{\psi(e^{-3\sqrt{3}\pi})} = 3^{1/4}e^{-\sqrt{3}\pi/4}\frac{3^{1/6}(1-\sqrt{3}+\sqrt{3}\sqrt[3]{4})^{1/3}}{2^{1/12}(\sqrt[3]{2}-1)^{2/3}(1+\sqrt{3})^{1/6}}.$$

Thus we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi in}}{m^2+3(2n+1)^2} = \frac{\sqrt{3}}{9}\pi e^{-\pi i/3}\log\frac{1-\sqrt{3}+\sqrt{3}\sqrt[3]{4}}{\sqrt[4]{2}(\sqrt{3}+1)^{1/2}}. \tag{3.6.13}$$

To calculate further examples from (3.6.2), we rewrite the ψ-quotient on the right-hand side of (3.6.2) in terms of Ramanujan's cubic continued fraction. We appeal to (3.1.1), (3.6.7) and [11, p. 330, Eq. (4.6)] to deduce that

$$\frac{\varphi(q)}{\varphi(q^3)} = \left(1 + 8q\frac{\chi^3(q)}{\chi^9(q^3)}\right)^{1/4} = \left(1 - 8qG^3(-q)\right)^{1/4},$$

and therefore we can rewrite (3.6.2) as

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{\frac{2}{3}\pi i n}}{(xm)^2 + (2n+1)^2} = \frac{\pi}{8x} e^{-\frac{\pi i}{3}} \log \frac{1 - 8G^3(-q)}{1 - 8G^3(q)}. \tag{3.6.14}$$

Besides (3.6.12) and (3.6.13), we can derive more examples from (3.6.2) by applying formulas for $G(q)$ and $G(-q)$. For instance, when $x = 3$, we have [4, p. 105] and [16, Corollary 4.6]

$$G(-e^{-\frac{\pi}{3}}) = -\left(\frac{1+\sqrt{3}}{4}\right)^{1/3}, \tag{3.6.15}$$

$$G(e^{-\frac{\pi}{3}}) = \frac{1}{2}\left(1 - \left(\frac{3-b}{1+b}\right)^2\right)^{1/3}, \quad \texttt{with} \ \ b = \sqrt{2\sqrt{3}+3}, \tag{3.6.16}$$

and thus we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{\frac{2}{3}\pi i n}}{9m^2 + (2n+1)^2} = \frac{\pi}{12} e^{-\frac{\pi i}{3}} \log \frac{\sqrt{3+2\sqrt{3}}\left(1 + \sqrt{3+2\sqrt{3}}\right)}{3 - \sqrt{3+2\sqrt{3}}}. \tag{3.6.17}$$

Now we examine (3.6.3). Similarly, when $x = 1$, we have the evaluations for $\psi(-e^{-\pi})$, $\psi(e^{-\pi})$, $\psi(-e^{-3\pi})$, $\psi(e^{-3\pi})$, $G(-e^{-\frac{\pi}{3}})$ and $G(e^{-\frac{\pi}{3}})$ as in the previous example, namely, (3.6.8), (3.6.9), (3.6.10), (3.6.11), (3.6.15) and (3.6.16). By a direct computation, we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{e^{\frac{1}{3}\pi i m} e^{\frac{2}{3}\pi i n}}{m^2 + (2n+1)^2}$$
$$= \frac{\pi}{2} e^{-\frac{\pi i}{3}} \log 2^{\frac{1}{4}} (2+\sqrt{3})^{\frac{1}{4}} \left(\sqrt{2\sqrt{3}+3} + 1\right)\left(1 + \sqrt{2}\sqrt[4]{3} + \sqrt{3}\right)^{-\frac{1}{2}}.$$

After further simplification, we obtain the very neat and nice formula

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{e^{\frac{1}{3}\pi i m} e^{\frac{2}{3}\pi i n}}{m^2 + (2n+1)^2} = \frac{\pi}{4} e^{-\pi i/3} \log(2+\sqrt{3}). \tag{3.6.18}$$

Equate the real and imaginary parts of the above identity to obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\cos\left(\frac{1}{3}\pi m+\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2}=\frac{\pi}{8}\log(2+\sqrt{3}),$$
$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin\left(\frac{1}{3}\pi m+\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2}=-\frac{\sqrt{3}\pi}{8}\log(2+\sqrt{3}).$$

Next, we examine (3.6.4). Theoretically, the calculation is also straightforward if the values of these φ-functions on the right side are known. In practice, it is actually difficult to simultaneously obtain the values of $\varphi(-q^{2/3})$, $\varphi(-q^2)$ and $\varphi(-q^6)$. However, we can rewrite the right side in terms of cubic continued fractions. From Ramanujan's Lost Notebook, we have [4, p. 96, Eq. (3.3.10)]

$$\frac{\varphi(-q^{1/3})}{\varphi(-q^3)}=1-2G(q).$$

In [45, Theorem 4.3], J. Yi proved that for $P=\varphi(q)/\varphi(q^3)$ and $Q=\varphi(q^3)/\varphi(q^9)$,

$$\left(\frac{Q}{P}\right)^2=PQ+\frac{3}{PQ}-3. \tag{3.6.19}$$

Now we apply (3.6.19) with $P=\varphi(-q^{2/3})/\varphi(-q^2)$ and $Q=\varphi(-q^2)/\varphi(-q^6)$ to find that

$$\frac{Q}{P}=\frac{\varphi^2(-q^2)}{\varphi(-q^{\frac{2}{3}})\varphi(-q^6)},\qquad PQ=\frac{\varphi(-q^{2/3})}{\varphi(-q^6)}=1-2G(q^2).$$

To evaluate Q/P, we only need to know the value of the relative cubic continued fraction. We give a couple of examples here. If we set $x=\sqrt{2}$, then $G(q^2)=G(e^{-\sqrt{2}\pi})=(-1+\sqrt{6})/2$, as given in (3.3.24). It follows that $PQ=3-\sqrt{6}$. Applying (3.6.19), we have

$$\frac{\varphi^2(-e^{-\sqrt{2}\pi})}{\varphi(-e^{-\sqrt{2}\pi/3})\varphi(-e^{-3\sqrt{2}\pi})}=\sqrt{3}, \tag{3.6.20}$$

which implies

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi i m}e^{\frac{2}{3}\pi i n}}{2m^2+(2n+1)^2}=\frac{\pi}{4\sqrt{2}}e^{-\pi i/3}\log 3. \tag{3.6.21}$$

If we equate the real and imaginary parts of (3.6.21), then we obtain, respectively,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\cos\left(\frac{2}{3}\pi m+\frac{2}{3}\pi n\right)}{2m^2+(2n+1)^2}=\frac{\pi}{8\sqrt{2}}\log 3,$$

and

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin\left(\frac{2}{3}\pi m+\frac{2}{3}\pi n\right)}{2m^2+(2n+1)^2}=-\frac{\sqrt{3}\pi}{8\sqrt{2}}\log 3.$$

When $x=1$ we appeal to [4, p. 100, Eq. (3.4.5)] to find that

$$G(e^{-2\pi})=\frac{-(1+\sqrt{3})+\sqrt{6\sqrt{3}}}{4},$$

which yields

$$\frac{\varphi(-e^{-2\pi/3})}{\varphi(-e^{-2\pi})}\frac{\varphi(-e^{-2\pi})}{\varphi(-e^{-6\pi})}=1-2G(e^{-2\pi})=\frac{3+\sqrt{3}+\sqrt{6\sqrt{3}}}{2}.$$

Substituting this last result into (3.6.19) and simplifying, we deduce that

$$\frac{\varphi^2(-e^{-2\pi})}{\varphi(-e^{-2\pi/3})\varphi(-e^{-6\pi})}=3^{1/4}. \tag{3.6.22}$$

Thus we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi im}e^{\frac{2}{3}\pi in}}{m^2+(2n+1)^2}=\frac{\pi}{8}e^{-\pi i/3}\log 3. \tag{3.6.23}$$

Again, if we equate the real parts and imaginary parts of (3.6.23), then we obtain, respectively,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\cos\left(\frac{2}{3}\pi m+\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2}=\frac{\pi}{16}\log 3,$$

and

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin\left(\frac{2}{3}\pi m+\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2}=-\frac{\sqrt{3}\pi}{16}\log 3.$$

3.7 Examinations of $J(a,0,s,t)$ and explicit examples

Now we examine $J(a,0,s,t)$ from Theorem 3.2.4 in two cases: $a=1$ and $a=2$. From calculation, it easily follows that

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+n^2}=\sum_{m\neq 0}\frac{e^{\pi ims}}{(xm)^2}-\frac{\pi}{x}\log\prod_{m=-\infty}^{\infty}\left(1-2\cos(\pi t)q^{|s+2m|}+q^{2|s+2m|}\right), \tag{3.7.1}$$

and

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(2n)^2} = \sum_{m\neq 0}\frac{e^{\pi ims}}{(xm)^2} - \frac{\pi}{2x}$$

$$\times \log\prod_{m=-\infty}^{\infty}\left(1-2\cos\left(\frac{\pi t}{2}\right)q^{|s+2m|}+q^{2|s+2m|}\right)\left(1+2\cos\left(\frac{\pi t}{2}\right)q^{|s+2m|}+q^{2|s+2m|}\right). \qquad (3.7.2)$$

Before we derive explicit examples from (3.7.1) and (3.7.2), let us recall the definition of the Weber-Ramanujan class invariants G_n and g_n, as defined in (3.1.3). The table at the end of Weber's book [42, p. 721–726] contains the values of 105 class invariants. Without the knowledge of class field theory, Ramanujan calculated class invariants independently for different reasons. His table of 46 class invariants in his paper does not contain any values that are in Weber's book. As G. N. Watson [41] remarked, "For reasons which had commended themselves to Weber and Ramanujan independently, it is customary to determine G_n for odd values of n; and g_n for even values of n." With the help of the properties of χ, i.e., (3.1.1), (3.1.2) and (3.1.4), we can calculate many values of χ functions using the values of class invariants in the table [42, p. 721–726][11, p. 189–204]. For instance,

$$\chi(e^{-\pi}) = 2^{1/4}e^{-\pi/24}G_1 = e^{-\pi/24}2^{1/4},$$

$$\chi(-e^{-2\pi}) = 2^{1/4}e^{-\pi/12}g_4 = e^{-\pi/12}2^{3/8},$$

$$\chi(e^{-3\pi}) = 2^{1/4}e^{-\pi/8}G_9 = e^{-\pi/8}2^{1/12}(1+\sqrt{3})^{1/3},$$

$$\chi(-e^{-6\pi}) = 2^{1/4}e^{-\pi/4}g_{36} = e^{-\pi/4}2^{1/8}\left(2+\sqrt{3}+\sqrt{9+6\sqrt{3}}\right)^{1/3},$$

$$\chi(e^{-\pi/3}) = e^{\pi/9}\chi(e^{-3\pi}) = e^{-\pi/72}2^{1/12}(1+\sqrt{3})^{1/3},$$

$$\chi(e^{-\pi/2}) = e^{3\pi/48}\chi(e^{-2\pi}) = e^{3\pi/48}\frac{\psi(e^{-2\pi})\chi(-e^{-2\pi})}{\psi(-e^{-2\pi})} = e^{-\pi/48}2^{1/16}(\sqrt{2}+1)^{1/4}.$$

Example 3.7.1.

$$\sum_{(m,n)\neq(0,0)}\frac{(-1)^{m+n}}{m^2+n^2} = -\pi\log 2, \qquad (3.7.3)$$

$$\sum_{(m,n)\neq(0,0)}\frac{(-1)^{m+n}}{(2m)^2+n^2} = -\frac{\pi}{4}\log(4+3\sqrt{2}), \qquad (3.7.4)$$

$$\sum_{(m,n)\neq(0,0)}\frac{(-1)^{n}}{m^2+(2n)^2} = \frac{\pi}{4}\log\frac{4+3\sqrt{2}}{2}, \qquad (3.7.5)$$

$$\sum_{(m,n)\neq(0,0)}\frac{(-1)^{m+n}}{3m^2+7n^2} = -\frac{\pi}{\sqrt{21}}\log\left(2^{-1/3}(\sqrt{7}-\sqrt{3})(3+\sqrt{7})^{2/3}\right), \qquad (3.7.6)$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}\right)\cos\left(\frac{\pi n}{3}\right)}{m^2+n^2} = \frac{\pi}{3}\log(2+\sqrt{3}), \tag{3.7.7}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{m^2+n^2} = \frac{\pi}{6}\log\frac{2-\sqrt{3}}{3\sqrt{3}}, \tag{3.7.8}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{2m^2+n^2} = -\frac{\pi}{2\sqrt{2}}\log 3, \tag{3.7.9}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{m^2+(2n)^2} = \frac{\pi}{6}\log\frac{2-\sqrt{3}}{3^{3/4}}. \tag{3.7.10}$$

Note that (3.7.3) is the classical lattice evaluation [19, Eq. (9.2.4)]. By interchanging the order of m and n and using the special values of the cosine function, we very easily see that

$$\sum_{(m,n)\neq(0,0)} \frac{(-1)^n\cos\left(\frac{\pi m}{2}\right)}{m^2+(2n)^2} = \sum_{(m,n)\neq(0,0)} \frac{(-1)^m\cos\left(\frac{\pi n}{2}\right)}{(2m)^2+n^2}$$
$$= \sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{2}\right)\cos\left(\frac{\pi n}{2}\right)}{m^2+n^2} = \frac{1}{4}\sum_{(m,n)\neq(0,0)} \frac{(-1)^{m+n}}{m^2+n^2} = -\frac{\pi}{4}\log 2.$$

Identities (3.7.3), (3.7.4), (3.7.5), (3.7.7) and (3.7.8) can be found in [6, Ex. 18] and [6, Appendix C]. However, they can only rigorously establish (3.7.3), (3.7.4), (3.7.5) and (3.7.7). The authors of [6] obtain (3.7.8) experimentally, and moreover, they have a misprint in their evaluation. They have $\frac{\pi}{6}\log\left(\frac{2-\sqrt{3}}{\sqrt{3}}\right)$ instead of $\frac{\pi}{6}\log\left(\frac{2-\sqrt{3}}{3\sqrt{3}}\right)$ on the right-hand side of (3.7.8). Using (3.7.1) and (3.7.2), we can derive all these identities from well-known q-series evaluations.

We derive some explicit formulas from (3.7.1) first. If we set $s=t=1$, then (3.7.1) immediately reduces to

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{(-1)^{m+n}}{(xm)^2+n^2} = -\frac{4\pi}{x}\log\chi(q) + \sum_{m\neq 0}\frac{(-1)^m}{(xm)^2}.$$

Note that $\sum_{m\neq 0}(-1)^m/m^2 = -\pi^2/6$. When $x=1$, then $\chi(q)=\chi(e^{-\pi}) = e^{-\pi/24}2^{1/4}$, and therefore we have (3.7.3). Similarly, when $x=2$, we have $\chi(q)=\chi(e^{-\pi/2}) = e^{-\pi/48}2^{1/16}(\sqrt{2}+1)^{1/4}$. Thus we obtain (3.7.4). We can obtain many additional formulas using the explicit values of the class invariants G_n and g_n. For instance, when $x=\sqrt{3}/\sqrt{7}$, we have $G_{7/3} = 2^{-1/3}(\sqrt{7}-\sqrt{3})^{1/4}(3+\sqrt{7})^{1/6}$[11, p. 341]. This completes the evaluation of (3.7.6).

If we set $s=t=1/3$ and then $s=1/3$, $t=-1/3$, we obtain

$$\sum_{(m,n)\neq(0,0)} \frac{e^{\pi i m/3}e^{\pi i n/3}}{m^2+n^2} = \sum_{(m,n)\neq(0,0)} \frac{e^{\pi i m/3}e^{-\pi i n/3}}{m^2+n^2} = -\frac{\pi}{x}\log\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} + \sum_{m\neq 0}\frac{e^{\pi i m/3}}{(xm)^2}.$$

Equate the real parts of each side to find that

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}+\frac{\pi n}{3}\right)}{m^2+n^2} = \sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}-\frac{\pi n}{3}\right)}{m^2+n^2} = -\frac{\pi}{x}\log\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} + \sum_{m\neq 0}\frac{\cos\left(\frac{\pi m}{3}\right)}{(xm)^2}.$$

Therefore we also have

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}\right)\cos\left(\frac{\pi n}{3}\right)}{m^2+n^2} = -\frac{\pi}{x}\log\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} + \sum_{m\neq 0}\frac{\cos\left(\frac{\pi m}{3}\right)}{(xm)^2}.$$

When $x=1$, we have

$$\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} = \frac{\chi^2(e^{-\pi})}{\chi(e^{-\pi/3})\chi(e^{-3\pi})} = e^{\pi/18}2^{-1/3}(\sqrt{3}-1)^{2/3}.$$

Note that $\sum_{m\neq 0}\frac{\cos(\pi m/3)}{m^2} = \pi^2/18$. The last two identities lead to (3.7.7).

Now we examine a more complicated case when $s=t=2/3$. Similar to the previous case, we find that

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{(xm)^2+n^2} = -\frac{\pi}{x}\log\frac{f^2(-q^2)}{f(-q^{2/3})f(-q^6)} + \sum_{m\neq 0}\frac{\cos\left(\frac{2\pi m}{3}\right)}{(xm)^2}. \tag{3.7.11}$$

To calculate the theta function quotient on the right side above, we first apply (3.1.1) to obtain

$$\frac{f^2(-q^2)}{f(-q^{2/3})f(-q^6)} = \frac{\varphi^2(-q^2)}{\varphi(-q^{2/3})\varphi(-q^6)}\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)}.$$

Consider the case $x=1$. Recall that we have (3.6.22) for the φ-quotient. So it remains to calculate the χ-quotient

$$\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)} = \frac{\chi(-q^{2/3})\chi(-q^2)\chi(-q^6)}{\chi^3(-q^2)} = q^{-2/9}G(q^{2/3})\chi(-q^2)\chi(-q^6).$$

We appeal to [4, p. 39, Eq. (3.3.9)], (3.6.15) and (3.6.16) to find that

$$G(e^{-2\pi/3}) = -G(e^{-\pi/3})G(-e^{-\pi/3}) = 2^{-5/3}(\sqrt{3}-1)^{1/3}(\sqrt{2\sqrt{3}+3}-1),$$

which yields

$$\frac{\chi(-e^{-2\pi/3})\chi(-e^{-6\pi})}{\chi^2(-e^{-2\pi})} = e^{-\pi/9}2^{-1/6}(\sqrt{3}-1)^{1/3}(2+\sqrt{3})^{1/3}. \tag{3.7.12}$$

Notice that $\sum_{m\neq 0}\frac{\cos(2\pi m/3)}{m^2} = -\pi^2/9$. Substituting all these results into (3.7.11) and simplifying, we complete the proof of (3.7.8). Smililarly, when $x=\sqrt{2}$, we first appeal to [11, p. 200] and (3.3.25) to find that

$$g_2 = 1, \quad g_{18} = (\sqrt{2}+\sqrt{3})^{1/3} \ \texttt{and} \ G(e^{-\sqrt{2}\pi/3}) = \frac{1}{\sqrt{2}}(-\sqrt{2}+\sqrt{3})^{1/3},$$

and therefore,

$$\frac{\chi(-e^{-\sqrt{2}\pi/3})\chi(-e^{-3\sqrt{2}\pi})}{\chi^2(-e^{-\sqrt{2}\pi})} = e^{\sqrt{2}\pi/9}G(e^{-\sqrt{2}\pi/3})\chi(-e^{-\sqrt{2}\pi})\chi(-e^{-3\sqrt{2}\pi}) = e^{-\sqrt{2}\pi/18}.$$

Substituting (3.6.20) and the result above into (3.7.11), and simplifying, we complete the proof of (3.7.9).

We conclude this section by deriving (3.7.10) from (3.7.2). If we set $s = t = 2/3$, then (3.7.2) reduces to

$$\begin{aligned}
&\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{(xm)^2+(2n)^2} \\
&= -\frac{\pi}{2x}\log\left(\frac{f^2(-q^2)}{f(-q^{2/3})f(-q^6)}\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)}\right) + \sum_{m\neq 0}\frac{\cos\left(\frac{2\pi m}{3}\right)}{(xm)^2} \\
&= -\frac{\pi}{2x}\log\left(\frac{\varphi^2(-q^2)}{\varphi(-q^{2/3})\varphi(-q^6)}\left(\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)}\right)^2\right) + \sum_{m\neq 0}\frac{\cos\left(\frac{2\pi m}{3}\right)}{(xm)^2}.
\end{aligned}$$

Set $x = 1$. Substituting (3.6.22) and (3.7.12) into the identity above, and simplifying, we obtain (3.7.10).

Chapter 4

Integral analogues of theta functions and Gauss sums

4.1 Introduction

In two papers [35], [36], [37, pp. 59–67, 202–207], Ramanujan examined the properties of two integrals defined by (1.3.1) and (1.3.2), that is,

$$\phi_w(t) := \int_0^\infty \frac{\cos(\pi tx)}{\cosh(\pi x)} e^{-\pi wx^2} dx,$$
$$\psi_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\sinh(\pi x)} e^{-\pi wx^2} dx,$$

which can be regarded as continuous integral analogues of Gauss sums from one point of view or of theta functions from another point of view. These integrals are also briefly discussed in a two-page fragment copied by G. N. Watson from Ramanujan's "loose papers" and published with Ramanujan's lost notebook [33, pp. 221–222]. A thorough discussion of this two-page fragment can be found in the fourth book by G. E. Andrews and B. C. Berndt on Ramanujan's lost notebook [5].

Page 198 of [33] is an isolated page that is actually part of the original lost notebook, and its contents are related to the two aforementioned papers by Ramanujan and the fragment on pages 221–222 of [33]. On this page, Ramanujan records theorems, much in the spirit of those for $\phi_w(t)$ and $\psi_w(t)$, for the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\tanh(\pi x)} e^{-\pi wx^2} dx,$$

as given in (1.3.3). The formulas claimed by Ramanujan on page 198 without proofs are difficult to read, partly because the original page was perhaps a thin, colored piece of paper, for example, a piece of blue airmail stationery, that was difficult for the photographers of [33] to photocopy. Since Ramanujan never discussed the results on page 198 in any of his papers and since no one else has apparently ever discussed them as well, it is the objective of this chapter to prove them and to also convince readers that $F_w(t)$ is a beautiful continuous integral analogue of either theta functions or Gauss sums.

4.2 Some theorems about $\phi_w(t)$ and $\psi_w(t)$

Recall that $\phi_w(t)$ and $\psi_w(t)$ are defined by (1.3.1) and (1.3.2), respectively. We first note that

$$\phi_w(t) = \phi_w(-t) \qquad \text{and} \qquad \psi_w(t) = -\psi_w(-t). \tag{4.2.1}$$

In this brief section we recall some of Ramanujan's theorems for these two functions from [35] and [36]. More detailed proofs can be found in [5].

Theorem 4.2.1. *[37, p. 202, eq. (1)] For $w > 0$,*

$$\phi_w(t) = \frac{1}{\sqrt{w}} e^{-\pi t^2/(4w)} \phi_{1/w}(it/w). \tag{4.2.2}$$

Theorem 4.2.2. *[37, p. 202] We have*

$$e^{\pi(t+w)^2/(4w)} \phi_w(t+w) = e^{\pi t^2/(4w)} \left(\frac{1}{2} + \psi_w(t) \right). \tag{4.2.3}$$

Theorem 4.2.3. *[37, p. 203, eq. (4)] We have*

$$\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w}\left(\frac{it}{w} + i \right) \right\}. \tag{4.2.4}$$

Theorem 4.2.4. *[37, p. 203, eq. (5)] We have*

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}} e^{-\pi t^2/(4w)}. \tag{4.2.5}$$

Theorem 4.2.5. *[37, p. 203, eq. (8)] We have*

$$e^{\pi(t+w)^2/(4w)} \left\{ \frac{1}{2} - \psi_w(t+w) \right\} = e^{\pi(t-w)^2/(4w)} \left\{ \frac{1}{2} + \psi_w(t-w) \right\}. \tag{4.2.6}$$

Theorem 4.2.6. *[37, p. 203, eq. (10)] If n is any positive integer,*

$$\psi_w(t) - \psi_w(t+2ni) = -\frac{i}{\sqrt{w}} \sum_{k=0}^{n-1} e^{-\pi(t+(2k+1)i)^2/(4w)}. \tag{4.2.7}$$

Observe that Theorem 4.2.6 indicates that the function $\psi_w(t)$ possesses a "quasi-period" $2i$, and that the right-hand side of (4.2.7) is an analogue of a Gauss sum.

4.3 The claims on Page 198 in the Lost Notebook

We note immediately from the definition (1.3.3) that

$$F_w(t) = -F_w(-t). \tag{4.3.1}$$

Entry 4.3.1. *We have*

$$F_w(t) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w). \tag{4.3.2}$$

The beautiful transformation formula (4.3.2) shows that $F_w(t)$ is an integral analogue of theta functions.

Proof. Write

$$\begin{aligned}
F_w(t) &= \int_0^\infty \frac{\sin(\pi t x)\cosh(\pi x)}{\sinh(\pi x)} e^{-\pi w x^2} dx \\
&= \int_0^\infty \frac{\sin(\pi t x)\cos(i\pi x)}{\sinh(\pi x)} e^{-\pi w x^2} dx \\
&= \frac{1}{2}\int_0^\infty \frac{\sin(t+i)\pi x + \sin(t-i)\pi x}{\sinh(\pi x)} e^{-\pi w x^2} dx \\
&= \frac{1}{2}\left\{\psi_w(t+i) + \psi_w(t-i)\right\}, \qquad (4.3.3)
\end{aligned}$$

by (1.3.2). Recall from (4.2.4) that

$$\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{\frac{1}{2} - \psi_{1/w}\left(\frac{it}{w} + i\right)\right\}. \tag{4.3.4}$$

Since $\psi(t)$ is odd, we find from (4.3.4) that

$$-\frac{1}{2} + \psi_w(t-i) = -\frac{1}{2} - \psi_w(-t+i) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{\frac{1}{2} - \psi_{1/w}\left(-\frac{it}{w} + i\right)\right\}. \tag{4.3.5}$$

Hence, from (4.3.3)–(4.3.5),

$$\begin{aligned}
F_w(t) &= \frac{1}{2}\left\{\frac{1}{2}+\psi_w(t+i)-\frac{1}{2}+\psi_w(t-i)\right\}\\
&= \frac{1}{2}\left(\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(\frac{it}{w}+i\right)\right\}\right.\\
&\quad \left.-\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(-\frac{it}{w}+i\right)\right\}\right)\\
&= \frac{i}{2\sqrt{w}}e^{-\pi t^2/(4w)}\left(-\psi_{1/w}\left(\frac{it}{w}+i\right)+\psi_{1/w}\left(-\frac{it}{w}+i\right)\right)\\
&= -\frac{i}{2\sqrt{w}}e^{-\pi t^2/(4w)}\left(\psi_{1/w}\left(\frac{it}{w}+i\right)+\psi_{1/w}\left(\frac{it}{w}-i\right)\right)\\
&= -\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}F_{1/w}(it/w),
\end{aligned}$$

by (4.3.3), and this completes the proof.

Entry 4.3.2. *If n is any positive integer, then*

$$F_w(t)-F_w(t+2ni) = -\frac{i}{\sqrt{w}}\sum_{j=0}^{n}{}' e^{-\pi(t+2ji)^2/(4w)}, \tag{4.3.6}$$

where the prime $'$ on the summation sign indicates that the terms with $j=0,n$ are to be multiplied by $\frac{1}{2}$.

Entry 4.3.2 is an analogue of Theorem 4.2.6 and demonstrates that $F_w(t)$ has a quasi-period $2i$.

Proof. Recall from (4.3.3) that

$$F_w(t) = \frac{1}{2}\{\psi_w(t+i)+\psi_w(t-i)\}, \tag{4.3.7}$$

and so

$$\begin{aligned}
&F_w(t)-F_w(t+2ni)\\
&= \frac{1}{2}\{\psi_w(t+i)-\psi_w(t+(2n+1)i)\}+\frac{1}{2}\{\psi_w(t-i)-\psi_w(t+(2n-1)i)\}.
\end{aligned}$$

Applying Entry 4.2.6 on the right side above, we see that

$$\begin{aligned}
&F_w(t)-F_w(t+2ni)\\
&= \frac{1}{2}\left\{-\frac{i}{\sqrt{w}}\sum_{k=0}^{n-1}e^{-\pi(t+(2k+2)i)^2/(4w)}-\frac{i}{\sqrt{w}}\sum_{k=0}^{n-1}e^{-\pi(t+2ki)^2/(4w)}\right\}\\
&= -\frac{i}{\sqrt{w}}\sum_{j=0}^{n}{}' e^{-\pi(t+2ji)^2/(4w)}.
\end{aligned}$$

This concludes the proof.

Entry 4.3.3. *If n is a positive integer, then*

$$F_w(t) - e^{\pi n(t+nw)} F_w(t+2nw) = -e^{-\pi t^2/(4w)} {\sum_{j=0}^{n}}' e^{\pi(t+2jw)^2/(4w)}, \tag{4.3.8}$$

where the prime on the summation sign has the same meaning as in Entry 4.3.2.

Observe that Entry 4.3.3 indicates that $2w$ is a quasi-period for $F_w(t)$. The functions $\phi_w(t)$ and $\psi_w(t)$ also possess the same quasi-period.

Proof. Replacing t by $t+i$ and $t-i$ in Theorem 4.2.5, we deduce, respectively, that

$$e^{\pi(t+i+w)^2/4w}\psi_w(t+i+w) + e^{\pi(t+i-w)^2/4w}\psi_w(t+i-w) = \frac{1}{2}(e^{\pi(t+i+w)^2/4w} - e^{\pi(t+i-w)^2/4w}), \tag{4.3.9}$$

and

$$e^{\pi(t-i+w)^2/4w}\psi_w(t-i+w) + e^{\pi(t-i-w)^2/4w}\psi_w(t-i-w) = \frac{1}{2}(e^{\pi(t-i+w)^2/4w} - e^{\pi(t-i-w)^2/4w}). \tag{4.3.10}$$

Now observe that $e^{4\pi i(t+w)/4w} = e^{4\pi i(t-w)/4w}$. We multiply $e^{\pi(t-i+w)^2/4w}$ in its two appearances in (4.3.10) by $e^{4\pi i(t+w)/4w}$, and we multiply $e^{\pi(t-i-w)^2/4w}$ in its two appearances in (4.3.10) by $e^{4\pi i(t-w)/4w}$. Thus, (4.3.10) can be recast in the form

$$e^{\pi(t+i+w)^2/4w}\psi_w(t-i+w) + e^{\pi(t+i-w)^2/4w}\psi_w(t-i-w) = \frac{1}{2}(e^{\pi(t+i+w)^2/4w} - e^{\pi(t+i-w)^2/4w}). \tag{4.3.11}$$

Using (4.3.7) and (4.3.11), we find that

$$\begin{aligned} &e^{\pi(t+i+w)^2/(4w)} F_w(t+w) + e^{\pi(t+i-w)^2/(4w)} F_w(t-w) \\ &= \frac{1}{2}\{e^{\pi(t+i+w)^2/4w}\psi_w(t+i+w) + e^{\pi(t+i+w)^2/4w}\psi_w(t-i+w) \\ &\quad + e^{\pi(t+i-w)^2/4w}\psi_w(t+i-w) + e^{\pi(t+i-w)^2/4w}\psi_w(t-i-w)\} \\ &= \frac{1}{2}(e^{\pi(t+i+w)^2/4w} - e^{\pi(t+i-w)^2/4w}). \end{aligned} \tag{4.3.12}$$

We now apply (4.3.12) with t successively replaced by $t+w$, $t+3w$, $\dots$, $t+(2n-1)w$ to deduce the n

equations

$$
\begin{aligned}
&e^{\pi(t+i+2w)^2/(4w)}F_w(t+2w)+e^{\pi(t+i)^2/(4w)}F_w(t)\\
&\qquad=\frac{1}{2}(e^{\pi(t+i+2w)^2/(4w)}-e^{\pi(t+i)^2/(4w)}),\\
&e^{\pi(t+i+4w)^2/(4w)}F_w(t+4w)+e^{\pi(t+i+2w)^2/(4w)}F_w(t+2w)\\
&\qquad=\frac{1}{2}(e^{\pi(t+i+4w)^2/(4w)}-e^{\pi(t+i+2w)^2/(4w)}),\\
&\qquad\vdots\\
&e^{\pi(t+i+2nw)^2/(4w)}F_w(t+2nw)+e^{\pi(t+i+(2n-2)w)^2/(4w)}F_w(t+(2n-2)w)\\
&\qquad=\frac{1}{2}(e^{\pi(t+i+2nw)^2/(4w)}-e^{\pi(t+i+(2n-2)w)^2/(4w)}).
\end{aligned}
$$

Alternately adding and subtracting the identities above, we conclude that

$$
e^{\pi(t+i)^2/(4w)}F_w(t)+(-1)^{n+1}e^{\pi(t+i+2nw)^2/(4w)}F_w(t+2nw)={\sum_{j=0}^{n}}'(-1)^{j+1}e^{\pi(t+i+2jw)^2/(4w)},
$$

that is to say,

$$
F_w(t)-e^{\pi n(t+nw)}F_w(t+2nw)=-e^{-\pi t^2/(4w)}{\sum_{j=0}^{n}}'e^{\pi(t+2jw)^2/(4w)},
$$

which completes our proof.

Entry 4.3.4. *Let $s=t+2\eta_1 mw+2\eta_2 ni$, where $\eta_1^2=\eta_2^2=1$, and where m and n are positive integers. Then*

$$
\begin{aligned}
F_w(s)+(-1)^{mn-1}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}F_w(t)&=\eta_1 e^{-\pi s^2/(4w)}{\sum_{j=0}^{m}}'e^{\pi(s-2j\eta_1 w)^2/(4w)}\\
&\quad+\eta_2(-1)^{mn}\frac{i}{\sqrt{w}}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}{\sum_{j=0}^{n}}'e^{-\pi(t+2\eta_2 ji)^2/(4w)}, \qquad (4.3.13)
\end{aligned}
$$

where the prime on the summation signs has the same meaning as in the two previous entries.

Proof. If we examine the proof of Entry 4.3.3, we see that we can similarly obtain an expression for $F_w(t)-e^{-\pi n(t-nw)}F_w(t-2nw)$, but with the right-hand side multiplied by -1 and the exponents j in the summands being replaced by $-j$. Thus, we shall apply Entry 4.3.3 and its just described analogue with n replaced by m and t replaced by $t+2\eta_2 ni$. Note that the right-hand side will be multiplied by η_1, and so we obtain

$$F_w(t+2\eta_2 ni) - e^{\pi\eta_1 m(t+2\eta_2 ni+\eta_1 mw)} F_w(t+2\eta_2 ni+2\eta_1 mw)$$
$$= -\eta_1 e^{-\pi(t+2\eta_2 ni)^2/(4w)} \sideset{}{'}\sum_{j=0}^{m} e^{\pi(t+2\eta_2 ni+2\eta_1 jw)^2/(4w)}.$$

Using the definition of s, we can reformulate the foregoing equality as

$$F_w(t+2\eta_2 ni) - e^{\pi\eta_1 m(s-\eta_1 mw)} F_w(s)$$
$$= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)} \sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2\eta_1 mw+2\eta_1 jw)^2/(4w)}$$
$$= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)} \sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2\eta_1 jw)^2/(4w)}.$$

If we examine the proof of Entry 4.3.2 with n replaced by $-n$, we see that the identity still holds except that the right-hand side must now be multiplied by -1. Since in the present notation $n > 0$, then if we apply Entry 4.3.2 to $F_w(t+2\eta_2 ni)$ above, we must multiply the right-hand side by η_2. Hence,

$$F_w(t) - e^{\pi\eta_1 m(s-\eta_1 mw)} F_w(s)$$
$$= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)} \sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2\eta_1 jm)^2/(4w)} - \eta_2 \frac{i}{\sqrt{w}} \sideset{}{'}\sum_{j=0}^{n} e^{-\pi(t+2\eta_2 ji)^2/(4w)}.$$

Upon multiplying both sides above by $e^{-\pi\eta_1 m(s-\eta_1 mw)}$ and simplifying, we find that

$$F_w(s) + (-1)^{mn-1} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} F_w(t) = -\eta_1 e^{-\pi s^2/(4w)} \sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2j\eta_1 w)^2/(4w)}$$
$$+\eta_2(-1)^{mn} \frac{i}{\sqrt{w}} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} \sideset{}{'}\sum_{j=0}^{n} e^{\pi(t+2\eta_2 ji)^2/(4w)},$$

where we used the fact that

$$(-1)^{mn} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} = e^{-\pi\eta_1 m(s-\eta_1 mw)}.$$

This completes our proof.

4.4 Examples

If we set $s = t$ in Entry 4.3.4, it follows that $w = -(\eta_2 ni)/(\eta_1 m)$. If we further suppose that both m and n are odd, then (4.3.13) reduces to the identity

$$(1+e^{-\pi\eta_1 mt})F_w(t) = \eta_1 e^{-\pi t^2/4w}\sum_{j=0}^{m}{}' e^{\pi(t-2j\eta_1 w)^2/(4w)} - \eta_2 \frac{i}{\sqrt{w}} e^{-\pi\eta_1 mt}\sum_{j=0}^{n}{}' e^{-\pi(t+2\eta_2 ji)^2/(4w)}.$$

In the identity above, first let $\eta_1 = 1, \eta_2 = -1$ and multiply both sides by e^{mt}. Secondly, let $\eta_1 = -1, \eta_2 = 1$ and multiply both sides by e^{-mt}. Replace t by $2t/\pi$ in each identity. We then respectively obtain the two identities

$$\begin{aligned} &2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} e^{-\frac{\pi n x^2}{m}i}dx \qquad (4.4.1)\\ &= \frac{1}{2}e^{mt} + e^{(m-2)t+\frac{\pi n}{m}i} + e^{(m-4)t+\frac{4\pi n}{m}i} + \cdots + \frac{1}{2}e^{-mt+\pi mni}\\ &+\sqrt{\frac{m}{n}}\left\{\frac{1}{2}e^{-mt+(\frac{mt^2}{\pi n}+\frac{\pi}{4})i} + e^{(\frac{2}{n}-1)mt+[(\frac{t^2}{\pi^2}-1)\frac{\pi m}{n}+\frac{\pi}{4}]i} + \cdots + \frac{1}{2}e^{mt+[(\frac{t^2}{\pi^2}-n^2)\frac{\pi m}{n}+\frac{\pi}{4}]i}\right\} \end{aligned}$$

and

$$\begin{aligned} &2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} e^{-\frac{\pi n x^2}{m}i}dx \qquad (4.4.2)\\ &= -\frac{1}{2}e^{-mt} - e^{(2-m)t+\frac{\pi n}{m}i} - e^{(4-m)t+\frac{4\pi n}{m}i} + \cdots - \frac{1}{2}e^{mt+\pi mni}\\ &-\sqrt{\frac{m}{n}}\left\{\frac{1}{2}e^{mt+(\frac{mt^2}{\pi n}+\frac{\pi}{4})i} + e^{(1-\frac{2}{n})mt+[(\frac{t^2}{\pi^2}-1)\frac{\pi m}{n}+\frac{\pi}{4}]i} + \cdots + \frac{1}{2}e^{-mt+[(\frac{t^2}{\pi^2}-n^2)\frac{\pi m}{n}+\frac{\pi}{4}]i}\right\}. \end{aligned}$$

Next add (4.4.1) and (4.4.2), divide both sides by 2, and equate the real and imaginary parts on both sides to obtain the two identities

$$\begin{aligned} &2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} \cos\frac{\pi n x^2}{m}dx\\ &= \frac{1}{2}\sinh\{mt\} + \sinh\{(m-2)t\}\cos\frac{\pi n}{m} + \sinh\{(m-4)t\}\cos\frac{4\pi n}{m}\\ &+\cdots+\frac{1}{2}\sinh\{-mt\}\cos(\pi mn)\\ &+\sqrt{\frac{m}{n}}\left\{\frac{1}{2}\sinh\{-mt\}\cos\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right) + \sinh\left\{\left(\frac{2}{n}-1\right)mt\right\}\cos\left(\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right.\\ &\left.+\cdots+\frac{1}{2}\sinh\{mt\}\cos\left(\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right\} \qquad (4.4.3) \end{aligned}$$

and

$$
\begin{aligned}
& -2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\sin\frac{\pi n x^2}{m}dx \\
= \quad & \sinh\{(m-2)t\}\sin\frac{\pi n}{m}+\sinh\{(m-4)t\}\sin\frac{4\pi n}{m}+\cdots+\frac{1}{2}\sinh\{-mt\}\sin(\pi mn) \\
& +\sqrt{\frac{m}{n}}\left\{\frac{1}{2}\sinh\{-mt\}\sin\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right)+\sinh\left\{\left(\frac{2}{n}-1\right)mt\right\}\sin\left(\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right. \\
& \left.+\cdots+\frac{1}{2}\sinh\{mt\}\sin\left(\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right\}. \qquad (4.4.4)
\end{aligned}
$$

Using (4.4.3) and (4.4.4), we can evaluate several definite integrals. For example, if we set $m=n=1$ in (4.4.3) and (4.4.4), we find that, respectively,

$$
\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\cos(\pi x^2)dx=\frac{\sinh t}{2\cosh t}\left(1-\cos\left(\frac{t^2}{\pi}+\frac{\pi}{4}\right)\right),
$$

and

$$
\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\sin(\pi x^2)dx= \frac{\sinh t}{2\cosh t}\sin\left(\frac{t^2}{\pi}+\frac{\pi}{4}\right).
$$

These evaluations can be found in [28, p. 542, formulas 3.991, nos. 1, 2], respectively. No further cases of (4.4.3) and (4.4.4) can be found in [28].

4.5 One further integral

There is one further integral, namely,

$$
G_w(t):=\int_0^\infty \frac{\sin(\pi tx)}{\coth(\pi x)}e^{-\pi w x^2}dx,
$$

which can be placed in the theory of $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$. First,

$$
\begin{aligned}
G_w(t) &= \int_0^\infty \frac{\sin(\pi tx)\sinh(\pi x)}{\cosh(\pi x)}e^{-\pi w x^2}dx \\
&= -i\int_0^\infty \frac{\sin(\pi tx)\sin(i\pi x)}{\cosh(\pi x)}e^{-\pi w x^2}dx \\
&= \frac{i}{2}\int_0^\infty \frac{\cos\{\pi x(t+i)\}-\cos\{\pi x(t-i)\}}{\cosh(\pi x)}e^{-\pi w x^2}dx \\
&= \frac{i}{2}\{\phi_w(t+i)-\phi_w(t-i)\}, \qquad (4.5.1)
\end{aligned}
$$

by (1.3.1). The formula (4.5.1) should be compared with (4.2.5).

The next result shows that the theory of $G_w(t)$ can be recast in the theory of $\psi_w(t)$.

Theorem 4.5.1. *If $w' := 1/w$, then*

$$G_w(t) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \psi_{w'}(-itw').$$

Proof. Using, in order, (4.5.1), Theorem 4.2.1, (4.2.1), Theorem 4.2.2, and (4.2.1), we find that

$$\begin{aligned}
G_w(t) &= \frac{i}{2\sqrt{w}} e^{-\pi(t+i)^2/(4w)} \phi_{1/w}\left(\frac{it}{w} - \frac{1}{w}\right) - \frac{i}{2\sqrt{w}} e^{-\pi(t-i)^2/(4w)} \phi_{1/w}\left(\frac{it}{w} + \frac{1}{w}\right) \\
&= \frac{i}{2\sqrt{w}} e^{-\pi(t^2-1)/(4w)} \left\{ e^{-i\pi t/(2w)} \phi_{1/w}\left(-\frac{it}{w} + \frac{1}{w}\right) - e^{i\pi t/(2w)} \phi_{1/w}\left(\frac{it}{w} + \frac{1}{w}\right) \right\} \\
&= \frac{i}{2\sqrt{w}} e^{-\pi(t^2-1)/(4w)} \left\{ e^{-\pi/(4w)} \left(\frac{1}{2} + \psi_{1/w}\left(-\frac{it}{w}\right)\right) - e^{-\pi/(4w)} \left(\frac{1}{2} + \psi_{1/w}\left(\frac{it}{w}\right)\right) \right\} \\
&= \frac{i}{2\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \psi_{w'}(-itw') - \psi_{w'}(itw') \right\} \\
&= \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \psi_{w'}(-itw'),
\end{aligned}$$

which completes the proof.

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