FORBIDDEN SUBSTRUCTURES: INDUCED SUBGRAPHS, RAMSEY GAMES, AND SPARSE HYPERGRAPHS

BY

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DISSERTATION

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Abstract

We study problems in extremal combinatorics with respect to forbidden induced subgraphs, forbidden colored subgraphs, and forbidden subgraphs. In Chapter 2, we determine exactly which graphs H have the property that almost every H-free graph has a vertex partition into k cliques and independent sets and provide a characterization. Such graphs contain homogeneous sets of size linear in the number of vertices, and so this result provides a strong partial result toward proving the Erdős-Hajnal conjecture.

In Chapter 3, we study a Ramsey-type game in an online and random setting. The player must color edges of K_n in an order chosen uniformly at random, and loses when she has created a monochromatic triangle. We provide upper bounds on the threshold for the number of edges the player is almost surely able to paint before losing in the k-color game. When $k > 2$, these upper bounds provide the first separation from the offline threshold.

In Chapter 4, we consider the family of 3-uniform hypergraphs that do not contain a copy of F_5, sometimes called the *generalized triangle*. We extend known extremal results to the sparse random setting, proving that with probability tending to 1 the largest subgraph of the random 3-uniform hypergraph that does not contain F_5 is tripartite.

Dedicated to Camellia Sinensis, without which none of this would have been possible.

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Chapter 1

Introduction

Graph theory is a comparatively young and extremely active branch of mathematics. Extremal graph theory is one area that has recently been recognized globally, with Endre Szemerédi being named as the 2012 Abel Prize recipient. A well-studied example of a question in extremal graph theory is what is the maximum number of edges a graph on n vertices can have without containing a copy of some fixed graph H? Denote this number by $\mathrm{ex}(n, H)$, called the *Turán number* of H.

In this thesis we present results in three areas of extremal graph theory, all of which consider the structure of graphs (or hypergraphs) that do not contain some fixed structure. An r-uniform hypergraph is a hypergraph whose edges are all sets of size r. For an r-uniform hypergraph H, we let $Forb(n, H)$ denote the family of r-uniform hypergraphs on n vertices that do not contain H. Notice that $\mathrm{ex}(n, H)$ is therefore equal to $\max\{|E(G)| : G \in Forb(n, H)\}$. Many other characteristics of $Forb(n, H)$ have been studied, such as its size or the typical structure of its members.

In the 1990s, attention shifted to an induced version of $Forb(n, H)$. Call a graph *H-free* if it does not contain an induced copy of H as a subgraph. A *homogenous set* is a set of vertices in $V(G)$ that is either an independent set or a clique. There are many interesting questions to be asked about the family of H-free graphs. For example, Erdős and Hajnal [34] conjectured that for any graph H every H-free graph must contain a large homogeneous set. In Chapter 2 we present results related to this conjecture, which are also of independent interest.

We can also consider a colored version of $Forb(n, H)$. For a fixed graph H, we say that $G \to H$ if any 2-coloring of the edges of G produces a *monochromatic* copy of H as a subgraph. The classical problem in Ramsey theory seeks to determine for a fixed graph H which graphs G have the property that $G \to H$. More generally, for an integer k we say that $G \stackrel{k}{\to} H$ if any k-coloring of the edges of G contains a monochromatic copy of H. This parameter can be considered in a game setting: one player, Builder, presents a graph G and the other player, Painter, must k-color the edges of G. If Painter creates a monochromatic copy of H, then Painter loses. This game is rather simple, since Builder need only find some G for which $G \stackrel{k}{\to} H$ in order to guarantee a win; in Chapter 3 we consider a more interesting variation.

An important recent trend in combinatorics is the study of sparse versions of classical problems. Following this trend, we can consider a sparse notion of $Forb(n, H)$. That is, think of $Forb(n, H)$ as the set of all subgraphs of K_n (or more generally the complete r-uniform hypergraph on n vertices) that do not contain H. Instead of K_n, consider a random r-uniform hypergraph $G^r(n, p)$, in which each r-set of $[n]$ is included as an edge with probability p. In Chapter 4 we consider the structure of subgraphs of $G^r(n, p)$ that do not contain some fixed hypergraph H.

1.1 Excluding Induced Subgraphs

For a graph G and a set of vertices S, let $G[S]$ denote the subgraph of G that has vertex set S and contains exactly those edges of G whose endpoints both lie in S. We say that H is an *induced* subgraph of G if there is some $S \subseteq V(G)$ for which $G[S]$ is isomorphic to H; if H is not an induced subgraph of G, then we say that G is H-free.

Several different choices of notation have been used to denote the family of H-free graphs and the family of graphs that do not contain H as a subgraph. Unfortunately, some notation has been used by different authors to refer to both families. Prömel and Steger [64], [62] used $\mathcal{F}orb_n(H)$ to denote the family of graphs on n vertices that do not contain H as a subgraph, and $\mathcal{F}orb_n^*(H)$ to denote the family of H-free graphs on n vertices. Loebl, Reed, Scott, Thomason, and Thomassé [54] use $Forb(H)^n$ for the latter family and do not consider the former. Kang, McDiarmid, Reed, and Scott [48] follow the notation of [54]. Because we will discuss both families frequently, we will roughly follow the convention of Prömel and Steger [64], [62], but while avoiding unnecessary subscripts. For a fixed r-uniform hypergraph H, let $Forb(n, H)$ denote the set of r-uniform hypergraphs with n vertices that do not contain H as a subgraph and let $Forb(H)$ be the union of $Forb(n, H)$ over all n. Let $Forb^*(n, H)$ denote the set of r-uniform H-free hypergraphs with n vertices and let $Forb^*(H)$ be the union of $Forb^*(n, H)$ over all n.

A *homogeneous set* is a set of vertices forming either an independent set or a clique. We say that a family $\mathcal{F}$ of graphs has the *Erdős-Hajnal property* if there exists ϵ depending only on $\mathcal{F}$ such that every graph G in $\mathcal{F}$ contains a homogeneous set of size at least $|V(G)|^\epsilon$. In 1989, Erdős and Hajnal [34] made the following famous conjecture.

Conjecture 1.1.1 (Erdős-Hajnal Conjecture)**.** *For any graph H, the family $Forb^*(H)$ has the Erdős-Hajnal property.*

This conjecture is in general still open. Erdős and Hajnal [34] proved their conjecture when H is obtained recursively from K_1 by a series of disjoint unions and edge complementations. Alon, Pach, and Solymosi [4] considered a different graph operation: suppose G has k vertices and suppose that $H_1, \ldots, H_k$ are graphs such that $Forb^*(G)$ and $Forb^*(H_i)$ for $1 \leq i \leq k$ all have the Erdős-Hajnal property. Let G' be the graph with vertex set $V_1 \cup \cdots \cup V_k$ and edge set $\bigcup_{ij \in E(G)} (V_i \times V_j)$. Alon, Pach, and Solymosi [4] proved that $Forb^*(G')$ also has the Erdős-Hajnal property.

If G is a graph with n vertices and chromatic number at most k, then G must contain an independent set of size at least n/k. In particular, if n is large, k is a constant, and every graph in a family $\mathcal{F}$ has chromatic number at most k, then $\mathcal{F}$ has the Erdős-Hajnal property. Prömel and Steger [64], proved that if $\chi(H) = k + 1$ then almost every graph in $Forb(n, H)$

is k-colorable if and only if there is some edge e in H whose removal reduces the chromatic number. In other words, H contains a *critical edge*. This result implies that if H contains such an edge then $Forb(n, H)$ "almost" has the Erdős-Hajnal property. Specifically, there is a subfamily $\mathcal{Q}'(n, H) \subseteq Forb(n, H)$ such that $\mathcal{Q}'(n, H)$ has the Erdős-Hajnal property and $\lim_{n\to\infty} \frac{|Forb(n,H)|}{|\mathcal{Q}'(n,H)|} = 1$.

Recently Loebl, Reed, Scott, Thomason, and Thomassé [54] proved a more general result for the family $Forb^*(n, H)$: for every H there exists a subfamily $\mathcal{Q}'(n, H)$ of $Forb^*(n, H)$ such that $\mathcal{Q}'(n, H)$ has the Erdős-Hajnal property and $\lim_{n\to\infty} \frac{|Forb^*(n,H)|}{|\mathcal{Q}'(n,H)|} = 1$. We say then that $Forb^*(n, H)$ has the *asymptotic Erdős-Hajnal property.* For a certain class of graphs considered in Chapter 2, the result of [54] follows from our main theorem, Theorem 2.1.10. Our theorem is analogous to the critical edge result of Prömel and Steger [64], but concerns induced subgraphs.

Just as the chromatic number of H was key to the result of Prömel and Steger [64], an analogous parameter, binary chromatic number of H, is key to ours. Whereas small chromatic number implies the existence of a large independent set, small binary chromatic number implies the existence of a large homogeneous set. We will define and discuss this parameter in Chapter 2. We let $\mathcal{Q}(n, H)$ denote the family of graphs on n vertices that are H-free and can be covered by a certain number of homogeneous sets (this definition will be made precise later) and then ask for which H almost every H-free graph lies in $\mathcal{Q}(n, H)$. We characterize the class of graphs H for which this holds and call them "critical" graphs. Every graph in $\mathcal{Q}(n, H)$ has the Erdős-Hajnal property, and so when H is a critical graph our theorem implies the [54] result. It in fact implies a stronger result, which is that almost every graph in $Forb^*(n, H)$ contains a homogeneous set of *linear* size. Loebl, Reed, Scott, Thomason, and Thomassé [54] suggested that it would be interesting to determine for which graphs H the family $Forb^*(n, H)$ has this stronger property.

Let us say that a graph family has the *linear Erdős-Hajnal property* if every member of the family contains a homogeneous set of linear size. Say that a graph family $\mathcal{F}$ has the *asymptotic linear Erdős-Hajnal property* if there is a subfamily $\mathcal{F}'$ having the linear Erdős-Hajnal property for which the following is true. Let $\mathcal{F}_n$ be the family of n-vertex graphs in $\mathcal{F}$ and let $\mathcal{F}'_n$ be the family of n-vertex graphs in $\mathcal{F}'$; then $|\mathcal{F}_n|/|\mathcal{F}'_n| \to 1$ as $n \to \infty$. Kang, McDiarmid, Reed, and Scott [48] proved that for almost every graph H it is the case that $Forb^*(n, H)$ has the asymptotic linear Erdős-Hajnal property.

1.2 Online Ramsey Games in Random Graphs

In classical Ramsey theory we are given a target graph F and seek a host graph H such that any k-coloring of the edges of H results in a monochromatic copy of F. In such a case we say that $H \xrightarrow{k} F$. Ramsey's Theorem [67] states that for any graph F and integer k there exists such a graph. The *Ramsey number* of a graph, $R(F)$, is the least n such that there exists a graph H having n vertices for which $H \to F$.

The number of vertices in a host graph H is not the only interesting parameter to consider. For example, let $\Delta(H)$ denote the maximum degree of the graph H. The *degree Ramsey number* of a graph F, denoted $R_\Delta(F; k)$, is the least D such that there exists a graph H for which $\Delta(H) \leq D$ and $H \xrightarrow{k} F$. The degree Ramsey number is known for K_n and $K_{1,n}$ [26], for the double-star [50], and for cycles and blowups of trees [47].

The classical Ramsey theory problem can be viewed as a two-player game in which one player, Builder, presents a graph H to the second player, Painter, who must then provide a k-coloring of the edges of H. If the resulting coloring contains a monochromatic copy of the target graph, F, then Painter has lost. This game is not particularly interesting: if $n = R(F; k)$, then obviously $K_n \xrightarrow{k} F$.

Many classical two-player games are famously intractable to analyze, notable examples being Chess and Go. Such games involve so many cases that analysis is difficult, a characteristic that Beck [19] calls "Combinatorial Chaos". Traditional game theory studies games of *incomplete information*: for example, in the Prisoner's Dilemma neither participant knows what choice the other has made. In games of *complete information*, each player knows what moves the other player has made at that stage of the game. Tic-tac-toe is an example of a game with complete information. Of course, the players do not know each other's *strategies*. Beck [19] defines a Combinatorial Game to be a 2-player zero-sum game of skill (no chance moves) with complete information such that the game can end only in one of three possible states for each player: win, draw, lose.

In 1963, Hales and Jewett [45] considered a class of combinatorial games that they called *positional games*. A positional game is played with a set of elements and a family of subsets of these elements. The two players take turns claiming elements, and whoever first claims all elements of a subset from the family wins. The well-known game tic-tac-toe is one such positional game: the elements are the squares of a 3×3 grid, and the family is the collection of all lines that intersect 3 squares. Hales and Jewett [45] studied generalizations of "tick-tack-toe"[1], but since then many other positional games have been studied.

If we let the game board be the edges (or vertices) of a graph, then many positional games arise naturally from graph parameters. For example, the winning family could be the set of all graphs with minimum degree at least d, the set of all 2-connected spanning graphs, or the set of all spanning trees. If the board is the set of vertices of a graph, then the winning family could be the set of all large independent sets, all large cliques, or all induced graphs with girth exactly 4. If the game board is $\mathbb{N}$, then the winning family could be the set of all arithmetic progressions of length at least 3.

In 1973, Erdős and Selfridge [38] mentioned a positional game they called *the Ramsey game*, in which two players take turns claiming edges of K_n and whoever first claims all edges of a k-clique wins. This game is different from the Builder/Painter Ramsey game above; in the Erdős-Selfridge [38] game, each of the two players is in some sense both a builder

[1] Also known as "noughts and crosses" in the UK. *Littell's Living Age*, in 1864, refers to a game called "noughts and crosses" or "tit-tat-to", but this may have been a different game. The term "tick-tack-toe" dates from 1884 [60]. In modern American English it is also spelled "tic-tac-toe" [32]. In this work we will respect the spelling choices of the authors in question.

and a painter, and each of them wishes to create a monochromatic copy of a fixed target graph. Many other positional games have been studied, including Maker/Breaker games and Avoider/Enforcer games. See Beck's book *Combinatorial Games: Tic-Tac-Toe* [19] for a fairly recent overview of the field. Beck also applies probabilistic methods to bridge the gap between Ramsey theory and Random graphs, via positional games.

The positional games described above are more interesting than the Builder/Painter game because the players take turns making moves and can therefore respond to each other's decisions in an "online" fashion. We too can consider an "online" version of our Builder/Painter game. In this game, Builder presents edges of a graph one by one, and Painter colors each as it is presented with one of k colors. Painter loses when she has created a monochromatic copy of F. To keep Builder from winning too easily, we will require him to play in a family $\mathcal{H}$ of graphs to which his underlying graph must belong. For a fixed graph F and fixed family $\mathcal{H}$, we call this the *k-color online Ramsey game* $(F, \mathcal{H})$. This model was introduced by Beck [18] and was further explored by Grytczuk, Hałuszczak, and Kierstead [44]. For example, $\mathcal{H}$ could be the family of graphs with maximum degree D. Every parameter Ramsey number has an online analogue; Butterfield, Grauman, Kinnersley, Milans, Stocker, and West [28] studied the online degree Ramsey number of trees and cycles.

Rather than making our Builder/Painter game into an online game, we could replace Builder with a random opponent. That is, for fixed n and some $M \leq \binom{n}{2}$, Builder chooses randomly a graph on n vertices that has M edges. If Painter can k-color the resulting graph without producing a monochromatic copy of F, then she wins. If n is large relative to $|V(F)|$, then classical Ramsey theory implies that Painter will lose the game for large enough M. The threshold for the number of edges, M, for which there is almost surely k-coloring of $G(n, M)$ with no monochromatic copy of F is known (see for example [55], [68], [69]). This threshold is $n^{2-\frac{1}{m_2(F)}}$, where $m_2(F) = \max_{G \subseteq F} \frac{e(G)-1}{v(G)-2}$. When the target graph is K_3, this threshold is $n^{3/2}$.

In Chapter 3 we will consider an online version of the random game. It is related to the online Ramsey game $(F, \mathcal{H}_d)$, where $\mathcal{H}_d$ is the set of all graphs G that have no subgraph G for which $|E(G)|/|V(G)| > d$. In the *k-color online F-avoidance edge-coloring game*, a graph on n vertices is generated by at each stage randomly adding a new edge. Painter must irrevocably color each new edge as it appears while trying to avoid a monochromatic copy of F. Let $N_0(F, k, n)$ be the threshold for the number of edges that Painter is almost surely able to paint before she loses; this number is known to exist for all F, k, n [57]. The random game described above is the offline version of this game, and Painter can do no better in the online game than in the offline game. Even when the target graph is K_3, the order of magnitude for $N_0(K_3, k, n)$ is unknown for $k \geq 3$.

Prior to the results in Chapter 3, the only known upper bound on $N_0(K_3, k, n)$ when $k \geq 3$ was from the offline threshold. We provide the first separation from this upper bound, proving that

$$N_0(K_3, k, n) \leq n^{\frac{3}{2} - c_k} \text{ for some } cf_k > 0$$

$$N_0(K_3, 3, n) \leq n^{\frac{3}{2} - \frac{1}{42}}.$$

Our result supports a conjecture of Marciniszyn, Spöhel, and Steger [58] that $N_0(K_3, k, n) = n^{\frac{3}{2}\left(1 - \frac{1}{3^k}\right)}$.

1.3 Extremal subgraphs of $G^3(n, p)$

The earliest result in the area of extremal graph theory is Mantel's Theorem [56], from 1907, which states that any K_3-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges. The complete bipartite graph whose partite sets differ in size by at most one is the K_3-free graph that achieves this bound. In fact, Erdős-Kleitman-Rosthchild [36] proved in 1975 that almost all K_3-free graphs are bipartite.

The Turán graph $T_k(n)$ is the complete k-partite graph whose part sizes differ by at most 1. Turán's Theorem states that among all K_{k+1}-free graphs on n vertices, the Turán graph has the most edges. Mantel's Theorem can be seen as a special case of Turán's theorem.

One of the most important developments in combinatorics over the past twenty years has been the study of "sparse random" analogues of various classical extremal results. This study was begun by Babai, Simonovits, and Spencer [8] for graphs and by Kohayakawa, Łuczak, and Rödl [51] for additive structures. Recent breakthroughs by Conlon and Gowers [30] and by Schacht [71] develop a general theory for attacking these problems. It is now known that many classical theorems, such as Turán's Theorem [77], the Erdős-Stone Theorem [39], and Szemerédi's Theorem [75] about arithmetic progressions have extensions for sparse random structures. The online Ramsey game described above is another translation of a classical problem into a sparse setting.

Szemerédi [75] proved that for any $\epsilon > 0$ and any integer k there exists an integer $N = N(\epsilon, k)$ such that if $n \geq N$, then every set of at least ϵn integers from $\{1, \ldots, n\}$ must contain an arithmetic progression of length k. Conlon and Gowers [30] and Schacht [71] proved that this result extends to sparse random sets with density asymptotically greater than $n^{-1/(k-1)}$ but not to those with density asymptotically less than $n^{-1/(k-1)}$.

Schur's Theorem [73] states that for any integer k there exists an integer N such that if $n \geq N$, then any k-coloring of $[n]$ contains a monochromatic solution to $x + y = z$. A sparse analogue of Schur's theorem was proven in 1996 by Graham, Rödl, and Ruciński [43], but the extremal version was open for another 15 years before being resolved [30], [71]. Balogh, Morris, and Samotij [17] sharpened this result by characterizing the structure of the largest sum-free subsets of a random subset of an abelian group.

Recall that almost all triangle-free graphs are bipartite [36]. Osthus, Prömel and Taraz [61] proved a sparse version of this result, proving that if $m \geq (\sqrt{3}/4 + \epsilon)n^{3/2}\sqrt{\log n}$, then almost all triangle-free graphs with n vertices and m edges are bipartite. It is natural to consider a sparse version of Mantel's Theorem, and in order to do so we will first rephrase Mantel's Theorem. For a fixed graph G, let $B(G)$ be a bipartite subgraph of G having the most edges of any bipartite subgraph. Let $F(G)$ be a K_3-free subgraph of G that has the most edges of any K_3-free subgraph. Because every bipartite graph is K_3-free, it is obvious that

$|E(B(G))| \leq |E(F(G))|$ for any G. Using this notation, Mantel's Theorem says that $B(K_n)$ and $F(K_n)$ have the same number of edges.

In the sparse version of this problem, rather than considering extremal subgraphs of K_n we will consider extremal subgraphs of a random graph. Babai, Simonovits, and Spencer [8] considered this sparse version. We will follow the definitions of Alon and Spencer [5] regarding random graphs. For a positive integer n and some p in the interval $[0, 1]$, the *random graph* $G(n, p)$ is a probability space over the set of graphs with vertex set $\{1, \ldots, n\}$ determined by

$$\mathbb{P}[\{i, j\} \in E(G)] = p,$$

with these events being mutually independent. Although it is, strictly speaking, a misnomer, it is common to refer to "the random graph" and to refer to $G(n, p)$ as a graph. For a fixed graph H with chromatic number 3 and any graph G let $F(G; H)$ be a largest (with respect to number of edges) subgraph of G that does not contain H. Babai, Simonovits, and Spencer [8] proved that with probability approaching 1 as n approaches infinity, $B(G(n, p))$ and $F(G(n, p); H)$ have almost the same number of edges. They also proved that if H is K_3, then $F(G(n, 1/2); H)$ is almost surely bipartite.

Brightwell, Panagiotou, and Steger [25] proved the existence of a constant c, depending only on ℓ, such that whenever $p \geq n^{-c}$, almost surely every maximum K_ℓ-free subgraph of $G(n, p)$ is $(\ell - 1)$-partite [25]. Kahn and DeMarco [31] recently solved the sparse problem for K_3, proving that $B(G(n, p))$ and $F(G(n, p); K_3)$ almost surely have the same number of edges if $p > Cn^{-1/2} \log^{1/2}(n)$, for some suitable constant C. Moreover, this threshold on p is best possible up to the choice of C.

In Chapter 4 we consider a hypergraph version of this problem. We say that a hypergraph is k-partite if its vertex set can be partitioned into k sets such that every edge in the hypergraph contains at most one vertex from each set. The Turán graph then naturally extends to hypergraphs; let $T_k^r(n)$ be the complete n-vertex r-uniform k-partite hypergraph with partite sets as equally-sized as possible. We will consider a random r-uniform hypergraph, $G^r(n,p)$, which is a hypergraph formed by including each element of $\binom{[n]}{r}$ with probability p. For a fixed r-uniform hypergraph H, let F be a subgraph of $G^r(n,p)$ that contains the most edges among all subgraphs that do not contain H. The natural question is "is F k-partite?"

In Chapter 4, we consider this question when H is a particular 3-uniform hypergraph, F_5, sometimes called the *generalized triangle*. We prove that for appropriately chosen p the largest subgraph of $G^3(n,p)$ that does not contain F_5 is almost surely 3-partite.

1.4 Background material

This section provides some basic terms and definitions. For a positive integer n, let $[n]$ denote the set $\{1,\ldots,n\}$.

1.4.1 Graphs

A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs from $V(G)$, called *edges*. Sometimes we will let $v(G)=|V(G)|$ and $e(G)=|E(G)|$. If $\{x,y\}$ is an edge then we say that x and y are *adjacent*, that x is *incident* to the edge $\{x,y\}$, and that x and y are the *endpoints* of the edge $\{x,y\}$. We will often denote the edge $\{x,y\}$ by xy. If V_1 and V_2 are disjoint sets of vertices then we let $E_G(V_1,V_2)$ denote the set of edges of G with one endpoint in V_1 and one endpoint in V_2. If G is understood we will omit the subscript. Sometimes we will let $e(V_1,V_2)=|E(V_1,V_2)|$.

A graph H is said to be a *subgraph* of a graph G if there exists an injection $f\colon V(H)\to V(G)$ such that if $xy\in E(H)$ then $f(x)f(y)\in E(G)$. If H is a subgraph of G, then we write $H\subseteq G$. If in fact $xy\in E(H)$ if and only if $f(x)f(y)\in E(G)$ then H is an *induced subgraph* of G. For example, if H has n vertices then H is a subgraph of K_m whenever $m\geq n$, but H is an induced subgraph of K_m only if H is a complete graph. For a set of vertices $S\subseteq V(G)$, let $G[S]$ be the subgraph of G whose vertex set is S and whose edge set is $\{xy : xy \in E(G) \text{ and } \{x, y\} \subseteq S\}$. We call $G[S]$ the subgraph *induced* by S.

An *isomorphism* from a graph G to a graph G' is a bijection $f\colon V(G) \to V(G')$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(G')$. We say that two graphs are *isomorphic* if there exists a bijection from one to the other. For example, H is an induced subgraph of G if and only if H is isomorphic to $G[S]$ for some $S \subseteq V(G)$. Isomorphism therefore defines an equivalence relation between graphs; we will often use the same notation for a graph and its isomorphism class. For an isomorphism class H, we say that H is a subgraph of G if there exists some H' in H such that $H' \subseteq G$.

A *path* is a graph whose vertices can be labelled $x_1, \ldots, x_\ell$ such that $x_i x_{i+1}$ is an edge for every $i \in [\ell - 1]$. We say that two vertices x and y are *connected* if there exists a path that starts at x and ends at y. We say that a graph is *connected* if every pair of vertices in the graph is connected. The *components* of a graph are its maximally connected subgraphs.

The graph that has n vertices, every pair of which form an edge, is the *complete graph*, denoted K_n. The graph that has n vertices, no two of which are adjacent, is the *empty graph*. If G is neither complete nor empty, then we say that G is *nontrivial*. The following are some nontrivial graphs that are frequently referred to in this thesis.

Definition 1.4.1. A *cycle* is a graph whose vertices can be labelled $x_1, \ldots, x_n$ (for some n) so that its edge set is $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}$ and $\{x_n, x_1\}$. Let C_n denote the isomorphism class of n-vertex cycles. Notice that if $n = 3$ then the complete graph and the cycle are the same graph, which we will often call a *triangle*.

Definition 1.4.2. A *complete bipartite graph* is a graph whose vertices can be partitioned into 2 sets, called *partite sets*, such that x and y are adjacent if and only if they are not contained in the same set. Let $K_{s,t}$ denote the isomorphism class of complete bipartite graphs whose partite sets have size s and t.

Definition 1.4.3. A *complete k-partite graph* is a graph whose vertices can be partitioned into k sets such that x and y are adjacent if and only if they are not contained in the same set. Let $K_{s_1,\ldots,s_k}$ denote the isomorphism class of complete k-partite graphs whose sets have size $s_1, \ldots, s_k$.

The *degree* of a vertex x, denoted $d(x)$, is the number of edges to which x is incident. The set of vertices adjacent to x is the *neighborhood* of x, denoted $N(x)$. Note that always $d(x) = |N(x)|$ under our definition of a graph. The degree of a vertex gives rise to a graph parameter:

Definition 1.4.4. The *minimum degree* of a graph G, written $\delta(G)$, is $\min\{d(v)\colon v \in V(G)\}$. The *maximum degree* of a graph G, written $\Delta(G)$, is $\max\{d(v)\colon v \in V(G)\}$.

If a graph G contains no cycle as a subgraph then G is a *forest*. If in addition G is connected, then G is a *tree*. Notice that the components of a forest are trees. Every tree with n vertices has exactly $n-1$ edges. A vertex of degree 1 is called a *leaf*, and the edge to which a leaf is incident is called a *pendant edge*.

In this thesis we frequently consider the family of graphs that do not contain some fixed H as a subgraph. We also frequently consider the family of graphs that do not contain some fixed H as an *induced* subgraph, also known as the family of H*-free* graphs.

Definition 1.4.5. For a fixed graph H and an integer n, let $Forb(n, H)$ denote the family of graphs on n vertices that do not contain H as a subgraph. Let $Forb(H) = \bigcup_{n=1}^{\infty} Forb(n, H)$. For a fixed graph H and an integer n, let $Forb^*(n, H)$ denote the family of graphs on n vertices that do not contain H as an induced subgraph. Let $Forb^*(H) = \bigcup_{n=1}^{\infty} Forb^*(n, H)$.

The *Turán number* of a graph H, denoted $\mathrm{ex}(n, H)$, is the maximum number of edges a graph on n vertices can contain without containing H as a subgraph. In other words,

$$\mathrm{ex}(n, H) = \max\{|E(G)|\colon G \in Forb(n, H)\}.$$

Definition 1.4.6. A set S of vertices is *independent* in G if $G[S]$ is empty. A set S of vertices is a *clique* in G if $G[S]$ is complete. A set of vertices is *homogeneous* in G if it is either an independent set or a clique.

Both independent sets and cliques give rise to graph parameters:

Definition 1.4.7. The *independence number* of a graph G, denoted $\alpha(G)$, is the size of the largest independent set in G. The *clique number* of a graph G, denoted $\omega(G)$, is the size of the largest clique in G.

Suppose $V(G)$ can be partitioned into two independent sets, V_1 and V_2. Such a graph is called *bipartite*, and we call the two independent sets *partite sets* of G. Notice that there could be multiple ways in which to partition $V(G)$ into two independent sets. A complete bipartite graph, defined above, is of course bipartite. More generally, suppose that the vertex set of G admits a partition into k sets $V_1, \ldots, V_k$ such that V_i is independent for each $i \in [k]$. To put it another way, suppose there is a function $f\colon V(G) \to [k]$ such that if $xy \in E(G)$ then $f(x) \neq f(y)$. Such a function is a *proper k-coloring* of the vertices of G, and when such a coloring exists G is *k-colorable*. A complete k-partite graph, defined above, is k-colorable. It is natural to wonder for what values of k a given graph is k-colorable.

Definition 1.4.8. The *chromatic number* of G, denoted $\chi(G)$, is the least k such that there exists a proper k-coloring of the vertices of G.

Note that the chromatic number of any graph G is at least $|V(G)|/\alpha(G)$.

Edges can also be colored; an edge coloring of G is *proper* if no two incident edges have the same color. For a fixed graph H, we say that $G \to H$ if every 2-coloring of the edges of G contains a *monochromatic* copy of H as a subgraph. More generally, for an integer k we say that $G \xrightarrow{k} H$ if any k-coloring of the edges of G contains a monochromatic copy of H. Note that $G \xrightarrow{1} H$ is equivalent to $H \subseteq G$. By Ramsey's Theorem, for every graph H and integer k there exists n such that $K_n \xrightarrow{k} H$.

Recall that $e_G(V_1, V_2)$ is the number of edges of G that connect vertices in V_1 to vertices in V_2. Of course, $e_G(V_1, V_2)$ is at most $|V_1||V_2|$ for any G.

Definition 1.4.9. If V_1, V_2 are disjoint vertex sets in G, then the *edge density* of a pair of disjoint vertex sets, written $d(V_1, V_2)$, is defined by

$$d(V_1, V_2) = \frac{e(V_1, V_2)}{|V_1||V_2|}.$$

For a graph G, let $m_2(G) = \max_{F \subseteq G} \frac{e(F)-1}{v(F)-2}$. We say that a graph G is *strictly* 2*-balanced* if $e(G), v(G) \geq 3$ and $\frac{e(F)-1}{v(F)-2} < \frac{e(G)-1}{v(G)-2}$ for every proper subgraph F of G. Cycles and complete graphs with at least three vertices are strictly 2-balanced.

The following graph operations will frequently be used to obtain a new graph from an existing graph. If G is a graph and e is an edge of the graph, then $G - e$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. The *complement* of the graph G, denoted $\overline{G}$, has vertex set $V(G)$ and edge set $\{uv : u \neq v \text{ and } uv \notin E(G)\}$.

1.4.2 Hypergraphs

Recall that a graph consists of a vertex set and a collection of pairs of vertices called edges. We can generalize this idea by not requiring edges to have exactly two vertices.

Definition 1.4.10. A *hypergraph* H consists of a set $V(H)$ of *vertices* and a set $E(H)$ of subsets of $V(H)$, called *edges*. An r*-uniform* hypergraph is a hypergraph in which every edge consists of exactly r vertices.

A graph is then a 2-uniform hypergraph. Many graph parameters have natural extensions to hypergraphs. We list some below.

Definition 1.4.11. The *degree* of a vertex, written $d(x)$, is the number of edges containing x. The *minimum degree* of a hypergraph, $\delta(H)$, is $\min\{d(x) \colon x \in V(H)\}$, and the *maximum degree* of a hypergraph, $\Delta(H)$, is $\max\{d(x) \colon x \in V(H)\}$. The *co-degree* of a pair of vertices x and y is the number of edges containing $\{x, y\}$. A vertex coloring $f \colon V(H) \to [k]$ is *proper* if no edge of H is monochromatic. A set of vertices, S, is *independent* if no edge of H is contained in S and is *strongly independent* if no edge of H contains more than one vertex of S. A hypergraph is *k-partite* if its vertex set can be partitioned into k strongly independent sets.

A *complete r-uniform hypergraph* H has edge set $\{S \subseteq V(H)\colon |S| = r\}$. A *complete r-uniform k-partite hypergraph* is a hypergraph whose vertex set can be partitioned into k sets $V_1, \ldots, V_k$ such that $\{x_1, \ldots, x_r\}$ is an edge if and only if $i \neq j$ implies that x_i and x_j are in different sets. Notice that V_ℓ is strongly independent for every $1 \leq \ell \leq k$.

The family $Forb(n, H)$ also has a hypergraph extension.

Definition 1.4.12. A hypergraph H is a *subgraph* of G if there exists an injection $f\colon V(H) \to V(G)$ such that whenever $\{v_1, \ldots, v_i\}$ is an edge of H the set $\{f(v_1), \ldots, f(v_i)\}$ is an edge of G. If H is an r-uniform hypergraph, then $Forb(n, H)$ is the family of all n-vertex r-uniform hypergraphs that do not contain H as a subgraph and $Forb(H) = \bigcup_{n=1}^{\infty} Forb(n, H)$.

If H is r-uniform and F is r'-uniform, where $r \neq r'$, then H cannot be a subgraph of F. If H is not k-partite, then H is not a subgraph of a complete r-uniform k-partite hypergraph. This brings us to the hypergraph extension of the Turán graph:

Definition 1.4.13. The *Turán hypergraph*, written $T_k^r(n)$, is the r-uniform k-partite hypergraph on n vertices whose vertex set be partitioned into k independent sets, $V_1, \ldots, V_k$, such that every r-set with at most one vertex in each V_i is an edge and such that $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$.

1.4.3 Asymptotics

In a paper published in 1960, Erdős and Rényi [37] introduced the notion of a random graph. For integers n and M, they let $G(n, M)$ be a probability space over the set of graphs that have n vertices and M edges, with each graph being equally likely. We will also consider a different model for generating random graphs, following the definitions of Alon and Spencer [5].

Definition 1.4.14. For a positive integer n and some p in the interval $[0, 1]$, $G(n, p)$ is a probability space over the set of graphs with vertex set $[n]$ determined by

$$\mathbb{P}[\{i, j\} \in E(G)] = p,$$

with these events being mutually independent.

Although $G(n, p)$ is a probability space, it is common to refer to $G(n, p)$ as a graph. We will often abuse notation and say, for example, that $G(n, p)$ has a certain property when in fact we mean that the probability that a graph in $G(n, p)$ has that property is equal to 1. Note that in the Erdős-Rényi model $G(n, M)$, the constant M is always an integer, whereas p is always in the interval $[0, 1]$, and so the notation is not ambiguous.

For an integer r with $r \geq 2$ and some real number p in the interval $[0, 1]$ we can also consider a *random r-uniform hypgraph*, $G^r(n, p)$. This model is similar to the above model, with each r-set of vertices included as an edge independently with probability p. The expected number of edges in $G^r(n, p)$ is $p\binom{n}{r}$, and $G^2(n, p) = G(n, p)$.

If $f(n)$ and $g(n)$ are functions of n, then we say that $f(n) \ll g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. If $f(n) \ll n$, then we write $f(n) = o(n)$.

An event holds *almost surely* if, over a sequence of sets, the probability of the event holding converges to 1.[2] For example, if A is a family of graphs then we say that $G(n, p)$ is almost surely in A if $\lim_{n\to\infty} \mathbb{P}[G(n, p) \in A] = 1$.

A graph property is a family of graphs that is closed under isomorphism. For example, "H is a subgraph" and "H is an induced subgraph" are graph properties. Erdős and Rényi [37] observed that many natural graph properties are true for a very narrow range of p (where p depends on n). They made the following definition.

Definition 1.4.15. For a graph property P, $t(n)$ is called a *threshold function* for P if

[2]It should be noted that "almost surely" is often defined to mean "with probability 1", while what we are calling "almost surely" is often called "asymptotically almost surely".

1. When $p(n) \ll t(n)$, $\lim_{n\to\infty} \mathbb{P}[G(n,p) \in P] = 0$,

2. When $p(n) \gg t(n)$, $\lim_{n\to\infty} \mathbb{P}[G(n,p) \in P] = 1$,

or vice versa.

Suppose that $P(n)$ and $Q(n)$ are two properties of graphs on n vertices, and so $P(n)$ and $Q(n)$ are families of n-vertex graphs. Then we say that *almost every* graph in $P(n)$ has property $Q(n)$ if

$$\lim_{n\to\infty} \frac{|Q(n)|}{|P(n)|} = 1.$$

If a graph property is closed under the operation of taking subgraphs, then we call it a *hereditary graph property.* Note that "H is a subgraph" is a hereditary property, but "H is an induced subgraph" is not. To see this, consider that K_4 contains no induced copy of C_4.

Chapter 2

Excluding Induced Subgraphs: Critical Graphs

The structure and enumeration of graphs not containing a fixed subgraph H have been well-studied for many choices of H. In the 1990s, Prömel and Steger proved that for a graph H with chromatic number $k+1$, almost all graphs not containing H as a subgraph are k-colorable if and only if H contains a color-critical edge. We consider the analogous problem with induced subgraph containment, proving that if H has binary chromatic number $k+1$ then almost all H-free graphs can be covered by k graphs that are cliques or independent sets if and only if H is in some well-defined sense critical.[1] The family of critical graphs includes C_4 and C_{2k+1} for $k \geq 3$.

2.1 Introduction

2.1.1 Definitions and motivation

It is obvious that if $G \in Forb(n, H)$, then every subgraph of G is also in $Forb(n, H)$, and so $|Forb(n, H)| \geq 2^{\mathrm{ex}(n,H)}$. Being H-free (that is, containing no induced copy of H) is *not* closed under the operation of taking subgraphs, however. Also, if $\chi(H) = k+1$ then every k-colorable graph on n vertices is in $Forb(n, H)$, but not necessarily in $Forb^*(n, H)$.

Erdős, Kleitman, and Rothschild [35] proved that almost all K_3-free graphs are bipartite, and that if $k > 3$ then the number of $K_{\ell+1}$-free graphs on n vertices is approximated by $2^{\mathrm{ex}(n,K_{\ell+1})}$. Kolaitis, Prömel and Rothschild [53] also studied K_ℓ-free graphs, proving that almost all $K_{\ell+1}$-free graphs are ℓ-colorable.

[1]This work appeared in Random Structures & Algorithms in 2011 [14].

Erdős, Frankl, and Rödl [33] partially proved a conjecture of Erdős by proving that if H contains a cycle and $\chi(H) > 2$, then $|Forb(n, H)| = 2^{(1+o(1))\mathrm{ex}(n,H)}$. It remains open whether this is true when H is bipartite. It is much more difficult to determine the structure of graphs in $Forb(n, H)$ or $Forb^*(n, H)$. For example, although Erdős, Kleitman, and Rothschild [35] proved that almost all K_3-free graphs are bipartite, there are many bipartite graphs that are not subgraphs of the extremal graph $K_{n/2,n/2}$. Much work has been done to characterize for a fixed graph H the structure of graphs that do not contain H as a subgraph [64] or as an *induced* subgraph [11], [23], and [65].

Prömel and Steger [66], Alekseev [2], and Bollobás and Thomason [24] defined the parameter in Definition 2.1.1, which will be important in what follows.

Definition 2.1.1. A graph H is (s,t)*-colorable* if the vertices of H can be partitioned into $s+t$ (possibly empty) sets, t of which are independent sets while the remaining s are cliques. The *binary chromatic number* of H, denoted $\chi_B(H)$, is the least integer k such that for every (s,t) satisfying $s+t=k$ the graph H is (s,t)-colorable.

To see why Axenovich, Kézdy, and Martin [7] called this parameter the binary chromatic number, note that any (s,t)-coloring of G is a coloring with $s+t$ colors such that s of the color classes are colored "clique" and t of the color classes are colored "independent set".

If H is (s,t)-colorable, then we call (s,t) a *covering pair*. If $s+t = \chi_B(H) - 1$ and H is not (s,t)-colorable, then we call (s,t) a *witnessing pair*. Note that if $\chi(G) = k$ then G is $(0,k)$-colorable, $(0,k-1)$ is a witnessing pair. Therefore, $\chi_B(G) > \chi(G) - 1$.

Example: C_7 is not $(3,0)$-colorable or $(2,1)$-colorable, so $\chi_B(C_7) > 3$. On the other hand, each of $(0,4), (1,3), (2,2), (3,1)$, and $(4,0)$ is a covering pair, so $\chi_B(C_7) = 4$. Because $(1,2)$ and $(0,3)$ are covering pairs, the only witnessing pairs are $(3,0)$ and $(2,1)$. In general, $\chi_B(C_{2k+1}) = k+1$ for $k \geq 1$.

This parameter has a long history, and was re-discovered and re-named by several different authors. To avoid confusion, we will briefly mention the terms and notation that were in use before the term "binary chromatic number" was introduced. Recall that a hereditary graph property is an infinite family of graphs closed under isomorphism and induced subgraphs. Bollobás and Thomason [24] defined their parameter on a hereditary family of graphs $\mathcal{P}$ and called it the 'coloring number' of the family. They defined $r(\mathcal{P})$ to be the maximum r such that there exist integers s and r with $0 \leq s \leq r$ for which every (r, s)-colorable graph is in $\mathcal{P}$.

Because the coloring number of a graph is usually used to mean the least k for which there exists an ordering of $V(G)$ in which no vertex has more than $k-1$ neighbors earlier than it in the ordering, we do not follow the terminology of Bollobás and Thomason. Prömel and Steger [65] defined an equivalent parameter for a graph H; they gave it no name and denoted it by $\tau(H)$, where $\tau(H)$ is the largest integer k for which there exists an integer $0 \leq s \leq k-1$ such H is not $(s, k-s-1)$-colorable. Later, Balogh and Martin [16] and Axenovich, Kézdy, and Martin [7] reintroduced the parameter under the name 'binary chromatic number' while studying the edit distance of graphs. Alon and Stav [6] adopted this notation and also defined it on a hereditary family, as Bollobás and Thomason had. This is the notation and terminology given in Definition 2.1.1, and which we will use in this thesis.

If (s, t) is a witnessing pair for H, then any graph whose vertex set can be partitioned into s cliques and t independent sets is necessarily H-free. We can use this to build many H-free graphs in the following manner. Partition n vertices into $\chi_B(H) - 1$ classes of equal size. For some witnessing pair (s, t), add edges to make exactly s of those classes cliques. Then no matter how edges are placed between vertices from different classes the resulting graph is H-free. This proves that

$$|Forb^*(n, H)| \geq 2^{\left(1-\frac{1}{\chi_B(H)-1}\right)\binom{n}{2}},$$

and this is close to the actual size of $Forb^*(n, H)$.

Theorem 2.1.2 (Prömel and Steger [65])**.**

$$|Forb^*(n,H)| = 2^{\left(1-\frac{1}{\chi_B(H)-1}+o(1)\right)\binom{n}{2}}.$$

Bollobás and Thomason [24], and independently Alekseev [2], proved a more general result for a hereditary family of graphs (recall that $Forb^*(n,H)$ is not generally a hereditary family). Recent sharper results for hereditary families of graphs appear in [3].

Theorem 2.1.3 ([2])**.** *If $\mathcal{P}$ is a hereditary family of graphs with $\chi_B(\mathcal{P}) = k$ and $\mathcal{P}_n$ is the subfamily of graphs in $\mathcal{P}$ with n vertices then*

$$\lim_{n\to} \frac{\log_2(|\mathcal{P}_n|)}{\binom{n}{2}} = 1 - \frac{1}{k}.$$

Theorem 2.1.4 ([3])**.** *Let $\mathcal{P}$ be a hereditary family of graphs, with $\chi_B(\mathcal{P}) = k$. Let $\mathcal{P}_n$ denote the subfamily of graphs in $\mathcal{P}$ that have n vertices. Then there exists a positive constant $\epsilon = \epsilon(\mathcal{P})$ such that*

$$2^{\left(1-\frac{1}{k}\right)\frac{n^2}{2}} \leq |\mathcal{P}_n| \leq 2^{\left(1-\frac{1}{k}\right)\frac{n^2}{2}+n^{2-\epsilon}}.$$

Definition 2.1.5. For two sequences $\mathcal{A}$ and $\mathcal{B}$ of families of graphs, where $\mathcal{A}_n$ and $\mathcal{B}_n$ denote the families on n vertices and $\mathcal{B}_n \subseteq \mathcal{A}_n$, we say that *almost every* graph in $\mathcal{A}$ is in $\mathcal{B}$ if

$$\lim_{n\to\infty} \frac{|\mathcal{A}_n|}{|\mathcal{B}_n|} = 1.$$

Definition 2.1.6. For a graph H, let $\mathcal{Q}(n,\mathcal{H})$ denote the family of graphs G on n vertices that are (s,t)-colorable for some s and t such that (s,t) is a witnessing pair for H.

Recall that if (s,t) is a witnessing pair for H, then by definition $s+t = \chi_B(H) - 1$. We have shown that if (s,t) is a witnessing pair for H, then any graph G that is (s,t)-colorable is H-free. Therefore $\mathcal{Q}(n,H) \subset Forb^*(n,H)$. One might ask the following natural question: for what graphs H is almost every H-free graph in $\mathcal{Q}(n,H)$? A similar question was answered in the non-induced case by Prömel and Steger in [64]:

Theorem 2.1.7. *If $\chi(H) = k + 1$, then almost every graph G that does not contain H as a (not necessarily induced) subgraph is k-colorable $\Leftrightarrow$ there exists an edge e such that $\chi(H - e) = k$.*

In other words, H contains a color-critical edge. The aim of this chapter is to extend Theorem 2.1.7 to H-free graphs. Such an extension proves partial results regarding the well-known Erdős-Hajnal conjecture. A family $\mathcal{F}$ of graphs has the *Erdős-Hajnal property* if there exists $\epsilon = \epsilon(\mathcal{F})$ such that every graph G in $\mathcal{F}$ contains a homogeneous set of size at least $|V(G)|^{\epsilon}$.

Conjecture 2.1.8 (Erdős-Hajnal Conjecture)**.** *For any graph H, the family $Forb^{*}(H)$ has the Erdős-Hajnal property.*

A graph family $\mathcal{F}$ has the *asymptotic linear Erdős-Hajnal property* if there is a subfamily $\mathcal{F}'$ having the linear Erdős-Hajnal property such that almost every graph in $mathcal F$ is in $\mathcal{F}'$.

Prömel and Steger [62], [63], described the typical structure of graphs in $Forb^{*}(n, H)$ when H is a cycle on four vertices or H is a cycle on five vertices. In particular, they proved that almost all C_4-free graphs admit a partition of their vertices into two sets, one of which is a clique and one of which is an independent set [62]. Such a graph is called a *split graph.* They also proved that almost all C_5-free graphs are *generalized split graphs* [63]. A graph G is a generalized split graph if the following is true of either G or its complement: the vertex set can be partitioned into two sets so that one is a clique and the other induces a disjoint union of cliques. These results imply that $Forb^{*}(C_4)$ and $Forb^{*}(C_5)$ have the linear asymptotic Erdős-Hajnal property (almost every generalized split graph contains a linear-size homogeneous set).

Recently, Kang, McDiarmid, Reed, and Scott [48] proved that for almost every graph H it is the case that $Forb^{*}(n, H)$ has the asymptotic linear Erdős-Hajnal property. Loebl, Reed, Scott, Thomason, and Thomassé [54] recently proved that for every H, the family $Forb^*(n, H)$ has the asymptotic Erdős-Hajnal property. The authors of [48] and [54] were aware of the results in this chapter.

2.1.2 New results

The characterization in Theorem 2.1.7 is fairly natural; in the induced case, the correct characterization is far from obvious. Prömel and Steger's cycle results suggest that C_4 is critical and C_5 is not, which may lead one to expect that in general C_{2k} is critical and C_{2k+1} is not. Interestingly, the truth is quite the opposite.

Definition 2.1.9. Let $\mathcal{F}(H, s, t)$ denote the set of minimal (by induced containment) graphs F such that the vertices of H can be partitioned into s cliques, t independent sets, and a set inducing F. A graph G is *critical* if $\chi_B(G) = k + 1$ and for all $s + t = k - 1$ and large enough n,

$$|Forb^*(n, \mathcal{F}(H, s, t))| \leq 2.$$

See below for examples of critical graphs. Our main result completely characterizes those graphs H for which almost every graph in $Forb^*(n, H)$ is in $\mathcal{Q}(n, H)$.

Theorem 2.1.10. *For $k \geq 2$ and $\chi_B(H) = k + 1$,*

$$\lim_{n\to\infty} \frac{Forb^*(n, H)}{\mathcal{Q}(n, H)} = 1$$

if and only if H is critical.

A direct consequence of Theorem 2.1.10 is that if H is critical then $Forb^*(H)$ has the asymptotic linear Erdős-Hajnal property. Alon, Balogh, Bollobás, and Morris [3] show that if $\mathcal{P}$ is a graph property that is closed under the operation of taking induced subgraphs then almost every graph in $\mathcal{P}$ has a certain structure. Note that the property of being H-free is closed under taking induced subgraphs and so their result applies, but the structure we characterize is much more specific.

Example: C_{2k+1} is critical for every $k \geq 3$. For example, let us consider C_7; recall that $\chi_B(C_{2k+1}) = k+1$ and so $\chi_B(C_7) = 4$. As Figure 2.1 illustrates, $\mathcal{F}(C_7, 2, 0)$ consists of K_3^- and $\overline{K_3^-}$. Now consider a graph on n vertices, where $n > 4$, that contains no induced copy of K_3^- or of $\overline{K_3^-}$. Such a graph must contain no isolated edge, otherwise it contains $\overline{K_3^-}$. If the graph is not $\overline{K_n}$, then it must contain two edges that share a vertex. In that case, the three vertices in those two edges must form K_3, otherwise the graph contains K_3^-, but this implies that the graph is K_n.

Figure 2.1 also illustrates that $\mathcal{F}(C_7, 1, 1) = \{\overline{K_2}\}$ and $\mathcal{F}(C_7, 0, 2) = \{K_1\}$. If a graph on n vertices contains no induced copy of $\overline{K_2}$, then it is a clique. No graph with any vertices can avoid containing a copy of K_1.

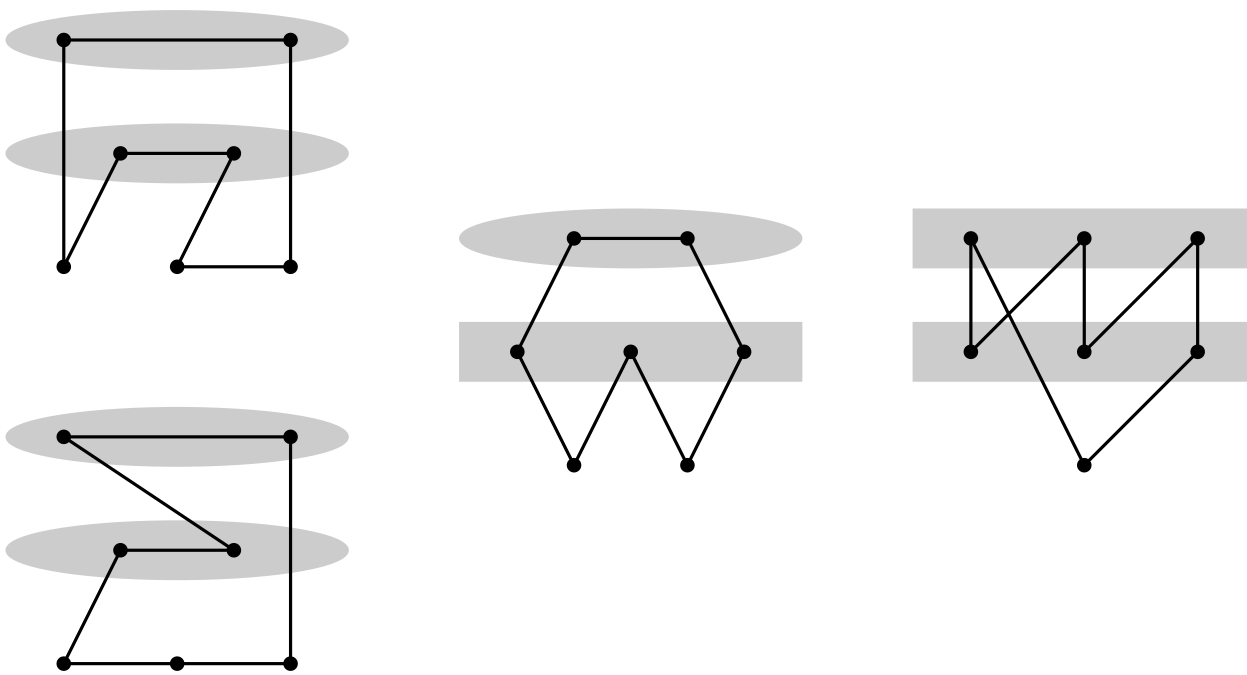

Figure 2.1: $\mathcal{F}(C_7, 2, 0) = \{\overline{K_3^-}, K_3^-\}$, $\mathcal{F}(C_7, 1, 1) = \{\overline{K_2}\}$, and $\mathcal{F}(C_7, 0, 2) = \{K_1\}$.

We have therefore shown that for any $s + t = \chi_B(C_7) - 2$ it is the case that

$$Forb^*(n, \mathcal{F}(C_7, s, t)) \subseteq \{K_n, \overline{K_n}\}$$

(in fact $Forb^*(n, \mathcal{F}(C_7, 0, 2)) = \emptyset$). Therefore, C_7 is critical. For $k > 3$, we find that $\mathcal{F}(C_{2k+1}, k-1, 0) = \{K_3^-, \overline{K_3^-}, \overline{K_3}\}$, which means that $Forb^*(n, \mathcal{F}(C_{2k+1}, k-1, 0)) = \{K_n\}$; because $\mathcal{F}(C_{2k+1}, k-2, 1) = \{K_1\}$ and for every other $s + t = k - 1$ the pair (s, t) is a covering pair this means that C_{2k+1} is critical. The following Proposition now follows from Theorem 2.1.10.

Proposition 2.1.11. *Almost every graph in $Forb^*(n, C_7)$ can be covered by 3 cliques or by 2 cliques and 1 independent set. In particular, $Forb^*(n, C_7)$ has the asymptotic linear Erdős-Hajnal property.*

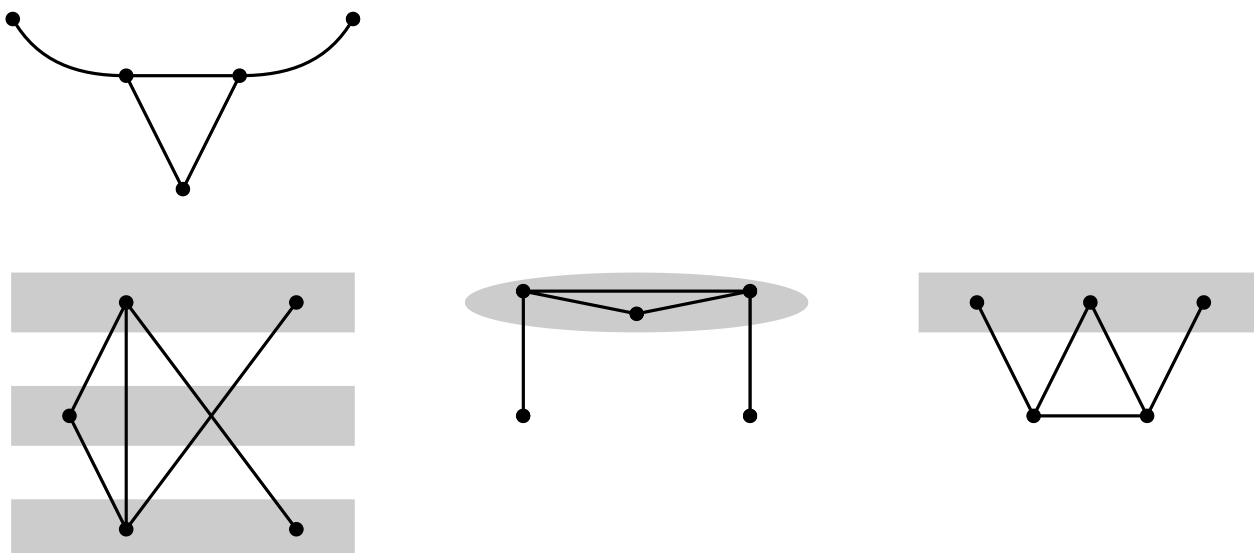

Figure 2.2: The bull graph, B, (top left). $\chi_B(B) = 3$, $\mathcal{F}(B, 1, 0) = \{\overline{K_2}\}$ and $\mathcal{F}(B, 0, 1) = \{K_2\}$.

Example: Let B be the so-called "bull graph", which is the 5-vertex graph formed by adding pendant edges at two different vertices of K_3. If we let the three vertices of the triangle be in one class and the two leaves each be in their own class, we see that $(3, 0)$,

$(2,1)$, and $(1,2)$ are covering pairs of G. We can also cover G with three independent sets, as Figure 2.2 shows. Therefore, $\chi_B(G) \leq 3$. Now, $(1,1)$ is not a witnessing pair because the two leaves form an independent set while the central triangle is a clique. However, $(2,0)$ and $(0,2)$ are both witnessing pairs: no two cliques (and no two independent sets) can cover more than four vertices of G. Therefore, $\chi_B(G) = 3$.

To determine whether the bull graph is critical, we therefore need only find $\mathcal{F}(B, s, t)$ for $s+t = 1$. Because $\mathcal{F}(B, 1, 0)$ contains $\overline{K_2}$ (take the triangle to be the clique) the only graph in $Forb^*(n, \mathcal{F}(B, 1, 0))$ is K_n. Because $\mathcal{F}(B, 0, 1)$ contains K_2 (take the two degree one vertices and the degree two vertex to be the independent set) the only graph in $Forb^*(n, \mathcal{F}(B, 0, 1))$ is $\overline{K_n}$. Therefore, the bull graph is critical. The following Proposition now follows from Theorem 2.1.10.

Proposition 2.1.12. *If B is the bull graph, then almost every graph in $Forb^*(n, B)$ can be covered by either two cliques or by two independent sets. In particular, $Forb^*(n, B)$ has the asymptotic linear Erdős-Hajnal property.*

In fact, Chudnovsky and Safra used perfect graphs to prove that the family of bull-free graphs has the Erdős-Hajnal property [29]. Specifically, they prove that every bull-free graph contains a homogeneous set of size $n^{1/4}$.

Recall that Prömel and Steger characterized the structure of almost all C_4-free graphs in [62] and characterized the structure of almost all C_5-free graphs in [63]. It would be natural to consider C_6-free graphs next, but C_6 is not critical.

Example: C_6 is not critical. Although $\chi_B(C_6) = 3$, $\mathcal{F}(C_6, 1, 0)$ consists of the path on four vertices: besides K_n and $\overline{K_n}$, any star forest also avoids this path, as does the complement of a star forest. Consequently, $Forb^*(n, C_6)$ contains any graph that can be covered by the complement of a star forest and a clique; this is a much larger family than $\mathcal{Q}(n, C_6)$.

It can be seen similarly that larger even cycles are not critical. For all $k > 5$, $\mathcal{F}(C_{2k}, k - 2, 0) = \{P_4, P_3 \cup K_1, K_2 \cup K_2, K_2 \cup \overline{K_2}, \overline{K_4}\}$, where $G \cup H$ indicates the disjoint union of the two graphs. The only graphs that avoid all of these induced subgraphs are those whose complements are disjoint copies of K_3 and a star forest. Every other $\mathcal{F}(C_{2k}, s, t)$ is either empty or consists of K_1, except that $\mathcal{F}(C_{12}, 3, 1) = \{\overline{K_2}\}$. Consequently, the following conjecture seems reasonable.

Conjecture 2.1.13. *For $k > 5$, almost every $G \in Forb^*(n, C_{2k})$ can be covered by $k-2$ cliques and a graph whose complement is disjoint copies of K_3 and a star forest.*

One could formulate a similar conjecture for C_6, C_8, or C_{10}; the statements, however, seem rather technical.

Remark: The following may seem like a natural definition for a "critical" graph. For each $x, y \in V(H)$ let $H(x, y)$ be the graph formed from H by removing the edge xy if $xy \in E(H)$ and by adding the edge uv if $xy \notin E(H)$. Then we might consider H critical if $\chi_B(H(x, y)) < \chi_B(H)$ for some $x, y \in V(H)$. This is not equivalent to Definition 2.1.9, however: we will show that K_4^- is critical by Definition 2.1.9 and is not critical by this definition. Note that $\chi_B(K_4^-) = 3$. Suppose xy is the edge missing from K_4^-; then $K_4^-(x, y)$ is K_4. It is obvious that $\chi_B(K_n) = n$ and that $(0, n-1)$ is a witnessing pair, so $\chi_B(K_4) = 4 \not< \chi_B(K_4^-)$.

For any other choice of x and y, the graph $K_4^-(x, y)$ is either C_4 or a triangle with a pendant edge. Because C_4 is not $(1, 1)$-colorable, $3 \leq \chi_B(C_4) \not< \chi_B(K_4^-)$. Because a triangle with a pendant edge is not $(0, 2)$-colorable, we see that $\chi_B(K_4^-(x, y))$ is not less than $\chi_B(K_4^-)$ for any choice of x, y.

On the other hand, to determine whether K_4^- is critical by Definition 2.1.9, we need only consider $\mathcal{F}(K_4^-, s, t)$ for $s + t = 1$. Because $\mathcal{F}(K_4^-, 1, 0) = \{K_1\}$ and $\mathcal{F}(K_4^-, 0, 1) = \{K_2\}$, it is obvious that $Forb^*(n, \mathcal{F}(K_4^-, s, t)) \subseteq \{K_n, \overline{K_n}\}$ for any $s + t = 1$. Therefore, K_4^- is critical.

Prömel and Steger proved [66] a Turán-type of statement in the induced case for the following definition of 'critical' graph.

Definition 2.1.14. A graph H is called *PS-critical*[2] if $\chi_B(H) = k+1$ and for all $0 \le \tilde{k} \le k$ the following is satisfied. Either $(\tilde{k}, k-\tilde{k})$ is a covering pair or there exist (possibly four different) partitions of $V(H)$, $\Pi = (P_1, \ldots, P_{k+1})$, such that $P_1, \ldots, P_{\tilde{k}}$ are cliques and $P_{\tilde{k}+1}, \ldots, P_{k+1}$ are independent sets and

1. $|P_{k+1}| = 1$ and $e(P_{k+1}, P_1) \le 1$,
2. $|P_{k+1}| = 1$ and $e(P_{k+1}, P_1) \ge |P_1| - 1$,
3. $|P_{k+1}| = 1$ and $e(P_{k+1}, P_k) \le 1$,
4. $|P_{k+1}| = 1$ and $e(P_{k+1}, P_k) \ge |P_k| - 1$,

and there exist (again, possibly two different) partitions of $V(H)$, $\Pi = (P_1, \ldots, P_{k+1})$, with either $P_1, \ldots, P_{\tilde{k}}$ or $P_1, \ldots, P_{\tilde{k}+1}$ cliques and the rest independent sets such that

5. (for $\tilde{k} \ge 1$) $|P_{\tilde{k}+1}| = 2$ and $P_{\tilde{k}+1}$ is not connected to P_1, and
6. (for $\tilde{k} \le k-1$) $|P_{\tilde{k}+1}| = 2$ and $P_{\tilde{k}+1}$ is completely connected to P_{k+1}.

We note that our definition of critical and the definition of PS-critical are both technical enough that it is unclear to us whether one is a generalization of the other. In particular, we know of no graph that is critical but not PS-critical, or that is PS-critical but not critical.

2.2 Observations

It is fairly easy to prove necessity in Theorem 2.1.10. We prove sufficiency by finding constants $0 < \sigma < 1$ and C such that if H is critical, then

$$|Forb^*(n, H) - \mathcal{Q}(n, H)| < C\sigma^n |\mathcal{Q}(n, H)| \tag{2.1}$$

[2]We say "PS-critical" to distinguish it from the "critical" in Definition 2.1.9 and because it was defined by Prömel and Steger.

for sufficiently large n. This we do by first showing in Section 2.3 that almost every H-free graph can be covered by k sets such that together they contain fewer than ϑn^2 "bad" edges, for some small $\vartheta > 0$. We also show that almost all H-free graphs have certain other random-like properties (Section 2.4).

In Section 2.5 we show that almost all H-free graphs have a partition into s almost-cliques and t almost-independent sets (for some $s + t = k$) such that no vertex has more than βn "bad" edges incident with it. Finally, in Section 2.6 we show that almost all H-free graphs with no "bad" vertex in fact contain no "bad" edge at all, and are consequently in $\mathcal{Q}(n, H)$.

We will make use of the well-known Szemerédi Regularity Lemma [76].

Definition 2.2.1. Fix a graph G with n vertices. If A and B are sets of vertices of G then their *edge density* $d(A, B)$ is given by $e(A, B)/(|A||B|)$.

A pair of vertex sets (A, B) is said to be γ*-regular* if for any $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$ it is the case that $|d(A', B') - d(A, B)| < \gamma$.

A γ*-regular partition* of $V(G)$ is a partition $V_0, V_1, \ldots, V_k$ such that $|V_0| \leq \gamma n$, and $|V_i| = |V_j|$ for all $1 \leq i < j \leq k$, and all but at most γk^2 pairs (V_i, V_j) with $1 \leq i < j \leq k$ are γ-regular.

Theorem 2.2.2 (Szemerédi's Regularity Lemma)**.** *For every $\gamma > 0$ and every $m \in \mathbb{N}$ there exists and integer M such that if G is a graph with at least m vertices then for some $k \in [m, M]$ there exists an γ-regular partition of $V(G)$ with sets $V_0, V_1, \ldots, V_k$.*

Such a partition gives rise to an auxiliary graph called a *cluster graph.* There are many different definitions of a cluster graph; one definition is that the cluster graph has vertex set $[k]$ and edge set $\{ij\colon (V_i, V_j) \text{ is an } \epsilon\text{-regular pair}\}$.

Definition 2.2.3 ([9])**.** The *binary entropy function* is $H(x) = -x\log(x) - (1-x)\log(1-x)$, where $\log(x)$ denotes $\log_2(x)$. It satisfies the following property: for x sufficiently small,

$$\sum_{i \leq xn} \binom{n}{i} \leq 2^{H(x)n}. \tag{2.2}$$

Lemma 2.2.4 (Chernoff bound [9]). *For $X_1, \ldots, X_\ell \in \{0,1\}$ with $Pr(X_i = 1) = p$,*

$$Pr\left(\sum_{i=1}^{\ell} X_i < p\ell/2\right) < \exp(-p\ell/8).$$

To avoid unnecessary technicalities, we will omit ceiling and floor functions whenever they are not crucial.

One part of Theorem 2.1.10 is easy to see. If H is not a critical graph, then

$$|Forb^*(n, H)| \gg |\mathcal{Q}(n, H)|. \tag{2.3}$$

Sketch of proof, Theorem 2.1.10 necessity. Suppose $Forb^*(m, \mathcal{F}(H, \tilde{s}, \tilde{t}))$ contains more than two graphs for some $\tilde{s}$ and $\tilde{t}$ satisfying $\tilde{s}+\tilde{t} = k-1$. Then there is a graph in $Forb^*(m, \mathcal{F}(H, \tilde{s}, \tilde{t})) \setminus \{K_n, \overline{K_n}\}$. This means that there are at least m labelled graphs in $Forb^*(m, \mathcal{F}(H, \tilde{s}, \tilde{t})) \setminus \{K_n, \overline{K_n}\}$ (see Balogh-Bollobás-Weinreich [12] and Scheinerman-Zito [72]). Fix some partition Π of $[n]$ into k classes, and suppose there are c possible edges between vertices in different classes of the partition. For each witnessing pair (s, t) there are $\binom{k}{s}2^c$ possible ways to form a graph in $\mathcal{Q}(n, H)$ that respects this partition, and each graph in $\mathcal{Q}(n, H)$ respecting this partition must be formed in such a way. Consequently, the number of graphs in $|\mathcal{Q}(n, H)|$ respecting Π is at most $\sum_{s=0}^{k} \binom{k}{s} 2^c \leq 2^k 2^c$.

On the other hand, if H is not critical then there exist $\tilde{s}$ and $\tilde{t}$ such that $\tilde{s} + \tilde{t} = k + 1$ and for sufficiently large n there are at least n/k labeled graphs in $Forb^*(n/k, \mathcal{F}(H, \tilde{s}, \tilde{t}))$. We may therefore generate many graphs that are in $Forb^*(n, H)$ and not in $\mathcal{Q}(n, H)$ by replacing the largest class in the partition with some such F and then choosing $\tilde{s}$ of the remaining classes to be cliques, letting the remaining $\tilde{t}$ be independent sets, and choosing the cross-edges arbitrarily. The number of graphs respecting Π that are in $Forb^*(n, H)$ but not $\mathcal{Q}(n, H)$ is therefore at least $\frac{n}{k}\binom{k}{\tilde{s}}2^c$.

Since k is a constant, the ratio of graphs in $Forb^*$ to graphs in $\mathcal{Q}$, that respect Π, is

$$\frac{\frac{n}{k}\binom{k}{\tilde{s}}2^c}{2^k 2^c} \geq \frac{n\binom{k}{\tilde{s}}}{k2^k},$$

which tends to infinity. If $|Forb^*(n/k, \mathcal{F}(H, \tilde{s}, \tilde{t}))| \gg k^n$ then this is more than there are choices of Π and we are done. Otherwise, "most" graphs in $Forb^*$ respect not many partitions of $[n]$; we omit the technical details. $\square$

For the remainder of the chapter, assume that H is a critical graph on h vertices with binary chromatic number$k+1$. It remains to show that there exist constants $\sigma < 1$ and C such that (2.1) is true.

We will prove sufficiency in Theorem 2.1.10 by induction on n; in what follows we will determine an integer n_0 such that $1/n_0$ is smaller than any other constant we use, and large enough to satisfy all of the following lemmas. From that, we will choose C large enough that (2.1) holds for every $n \leq n_0$; for $n > n_0$ we will use induction.

First, we observe some facts about $\mathcal{Q}(n, H)$.

Lemma 2.2.5. $|\mathcal{Q}(n, H)| \geq |\mathcal{Q}(n-1, H)|2^{(1-\frac{1}{k})(n-1)}$.

Proof. Consider a graph $G \in \mathcal{Q}(n-1, H)$; it can be covered by s cliques and t independent sets for some witnessing pair (s, t). Form a new graph $G' \in \mathcal{Q}(n, H)$ by adding a new vertex, labeled 'n', to the smallest clique or independent set (break ties by choosing whichever contains the least-indexed vertex). Then n must be adjacent to every vertex in its clique, or to no vertex in its independent set. It may or may not be adjacent to each other vertex in the graph; since we added n to the smallest clique or independent set there are at least $(1-\frac{1}{k})(n-1)$ vertices in other cliques and independent sets. Consequently, there are at least $2^{(1-\frac{1}{k})(n-1)}$ possible ways to form a new graph G' from each graph $G \in \mathcal{Q}(n-1, H)$. $\square$

Iterating Lemma 2.2.5 results in the following.

Corollary 2.2.6.

$$|\mathcal{Q}(n, H)| \geq |\mathcal{Q}(n-i, H)|2^{(1-\frac{1}{k})in-\binom{i+1}{2}(1-\frac{1}{k})}.$$

We can now bound $|\mathcal{Q}(n,H)|$ above and below by functions depending only on n and H.

Corollary 2.2.7.

$$2^{(1-\frac{1}{k})\binom{n}{2}} \leq |\mathcal{Q}(n,H)| < 2^k k^n 2^{(1-\frac{1}{k})\binom{n}{2}}.$$

Proof. It follows from Corollary 2.2.6 with $i = n$ that

$$|\mathcal{Q}(n,H)| \geq 2^{(1-\frac{1}{k})n^2 - \binom{n+1}{2}(1-\frac{1}{k})} = 2^{(1-\frac{1}{k})\binom{n}{2}},$$

because $n^2 - \frac{n^2+n}{2} = \binom{n}{2}$.

For the upper bound, there are fewer than k^n ways to partition n vertices into k classes, $\binom{k}{s}$ ways to let s of the classes be cliques, and $2^{(1-\frac{1}{k})\binom{n}{2}}$ ways to select edges between vertices in different classes. Since s may range from 0 to k, this gives

$$|\mathcal{Q}(n,H)| < \sum_{s=0}^{k} \binom{k}{s} k^n 2^{(1-\frac{1}{k})\binom{n}{2}} = 2^k k^n 2^{(1-\frac{1}{k})\binom{n}{2}}.$$

□

The following lemma is crucial in the proof of our main result. A *star* is a complete bipartite graph with one partite set having size 1.

Lemma 2.2.8. *If H is a critical graph then for any $s+t=k$ such that (s,t) is a witnessing pair of H:*

1. *If $s \geq 1$, then $\mathcal{F}(H, s-1, t)$ contains a graph that is the complement of a star.*

2. *If $t \geq 1$, then $\mathcal{F}(H, s, t-1)$ contains a star.*

Proof. Because H is critical, $Forb^*(n, \mathcal{F}(s-1,t,H)) \subseteq \{K_n, \overline{K_n}\}$ for all $n > n_0$. Consequently, any graph consisting of K_{n-1} and a vertex v such that v has no edges to the clique must contain some graph in $\mathcal{F}(s-1,t,H)$ as an induced subgraph. Since (s,t) is a witnessing pair, that graph cannot be a clique and so must contain the vertex v.

Similarly, $K_{1,n-1}$ must contain some graph in $\mathcal{F}(s-1, t, H)$ as an induced subgraph. Since (s, t) is a witnessing pair, that graph cannot be an independent set and so must contain the vertex of degree $n-1$.

$\square$

2.2.1 Constants

This section may be useful to clarify size and dependence. In theory, we should define the constants from largest to smallest; one can check, however, that the following are all well-defined.

$$\frac{1}{n_0} \ll \gamma \ll \mu \ll \epsilon \ll \omega \ll \lambda \ll \alpha \ll \vartheta \ll \delta,$$

$$\text{and } \rho \ll \delta \ll \beta \ll 1$$

1. We fix a graph H and let $h = |V(H)|$ and $k = \chi_B(H) - 1$.

2. We choose $\lambda > 0$ such that $\lambda < \frac{1}{(10kR(h))^2}$. This inequality is needed for Lemma 2.3.11.

3. We obtain $\omega = \omega(k, \lambda, R(h))$ from Theorem 2.3.4.

4. We choose $\epsilon > 0$ so that for all sufficiently large m, $\left(1 - \frac{1}{k}\right)\frac{m}{2} + \epsilon\binom{m}{2} < \omega m^2$. This will be needed in Lemma 2.3.11.

5. We choose $\mu > 0$ so that $H(2\mu) < \epsilon/6$. This is used in Lemma 2.3.8.

6. We obtain $\gamma_{2.3.5}(\mu, k)$ from Lemma 2.3.5, and then we choose $\gamma < \gamma_{2.3.5}$ such that $\gamma < \min\left\{\frac{\epsilon}{12}, \mu\right\}$. We need $\gamma < \frac{\epsilon}{12}$ for Lemma 2.3.8, and we need $\gamma < \mu$ for Lemma 2.3.3 and Lemma 2.3.8.

7. We obtain $m_0(k, R(h), \epsilon)$ from Theorem 2.3.7 and then if necessary make m_0 large enough to satisfy the condition in 2.2.1.4 and large enough that $m_0 \geq \frac{12}{\epsilon}$. We then obtain $m \geq m_0$ from Lemma 2.3.6. The inequality $m \geq \frac{12}{\epsilon}$ is needed in Lemma 2.3.8.

8. We choose $\alpha > 0$ such that $\alpha^2 > 100\lambda$ and $\alpha < \frac{1}{kR(h)}$ (this is possible because we chose λ small enough). Both inequalities are needed in Lemma 2.3.11.

9. Let $\vartheta = \frac{\gamma}{2} + \alpha h + \frac{\lambda}{m^2} + \frac{1}{2m}$. As a shorthand we let $\psi = \sqrt{\vartheta}\log(\frac{1}{\vartheta})$. We can choose γ, α, λ, and $1/m$ small enough that $2H(\vartheta) < \frac{1}{k^2}$ and $\vartheta < \frac{1}{4k^4}$ and $\psi < \min\{\frac{1}{4k(k-2)}, \frac{1}{12hk2^{h+3}}\}$. These inequalities are needed for an application of Lemma 2.5.2 in Lemma 2.4.2, an application of Lemma 2.5.2 in Lemma 2.4.6, Lemmas 2.5.5 and 2.6.1, and Lemma 2.6.2 respectively.

10. We find $\delta_0(h-1)$ and $\delta_0(kh2^h)$ from Lemma 2.5.2. We choose $\delta > 0$ so that $\delta < \min\{\frac{1}{k^2}, \delta_0(h-1), \delta_0(kh2^h)\}$ and $H(2\delta) < \min\{\frac{1}{4k^2}, \frac{1}{3k^2 2^{h+3}}\}$ and

$$\delta > \max\left\{2H(\vartheta), \frac{8k^2}{3}\vartheta, \sqrt{\frac{4\vartheta}{3}}\right\}$$

. The inequality $\delta < \frac{1}{k^2}$ is needed for Lemma 2.4.6. We need $\delta < \min\{\delta_0(h-1), \delta_0(k2^h)\}$ for Lemmas 2.5.3 and 2.6.1. Lemma 2.5.5 requires that $H(2\delta) < \frac{1}{4k^2}$ and Lemma 2.6.2 requires that $H(2\delta) < \frac{1}{3k^2 2^{h+3}}$. The final set of inequalities are needed for Lemma 2.4.2, Lemma 2.4.6, and Lemmas 2.5.3 and 2.6.1, respectively.

11. We obtain ρ_0 from Lemma 2.4.2. In Corollary 2.4.4 we find $\rho' > 0$ such that $\rho' < \min\{\rho_0, 2\vartheta\}$ and then find $\rho > 0$ such that $\rho < \rho' - \frac{1}{n_0^2}$.

12. We choose $\beta > 0$ so that $\frac{\beta}{2^h} \leq 1 - \frac{1}{4k-8}$ and $\beta \gg \max\left\{\frac{8k^2h}{3}\vartheta, \frac{8k^3h2^h}{3}\vartheta\right\}$ and $H(\beta k) < \min\{\frac{1}{8k}, \frac{1}{3hk2^{h3}}\}$. We need $2\beta \leq 1 - \frac{1}{4k-8}$ for Lemma 2.6.1. The second set of inequalities are needed for applications of Lemma 2.5.2 in Lemmas 2.5.3 and 2.6.1. Lemma 2.5.5 requires that $H(\beta k) < \frac{1}{8k}$ and Lemma 2.6.2 requires that $H(\beta k) < \frac{1}{3hk2^{h+3}}$.

13. We obtain $\hat{\rho} = \frac{1}{k} - H(2\delta)k - 2H(\beta k) - (k-2)\psi - \frac{2\log(n)}{n}$ from Lemma 2.5.5. Previous bounds guarantee that $\hat{\rho}$ is positive. We obtain $-\rho' = H(2\delta) + H(\beta)h + 4\psi h - \frac{1}{k2^{h+3}} - \frac{\psi}{2^{h+3}}$ during the proof of Lemma 2.6.2, and then let $\tilde{\rho} = \frac{9}{10}\rho'$. Previous bounds guarantee that $\tilde{\rho}$ is positive.

14. We obtain σ by induction such that $\max\{2^{-\hat{\rho}}, 2^{-\tilde{\rho}}\} < \sigma < 1$. The proof of Theorem 2.1.10 requires $\sigma > \max\{2^{-\hat{\rho}}, 2^{-\tilde{\rho}}\}$, and the statement of Equation 2.1 requires $\sigma < 1$.

15. We choose n_0 an integer such that $\frac{1}{n_0}$ is smaller than every preceding constant and larger than the n_0 in Lemma 2.5.2 for $m = h-1$ and for $m = kh2^h$. That is needed for Lemma 2.5.3 and for Lemma 2.6.1. We also require n_0 to be large enough that whenever $n > n_0$ the following inequalities are all true:

$$
\begin{aligned}
n2^{(1-\frac{1}{k}-\frac{\epsilon}{6})\frac{n^2}{2}} &< 2^{(1-\frac{1}{k}-\frac{\epsilon}{5})\frac{n^2}{2}} && \text{(for Lemma 2.3.3)},\\
2^{2\log(m)n+\log(3)\frac{m^2}{2}+m^2} &< 2^{\epsilon n^2/4} && \text{(for Lemma 2.3.8)},\\
\frac{1}{n}2\log(n) &< \frac{1}{4k} && \text{(for Lemma 2.5.3)},\\
(1-\tfrac{1}{k})\tbinom{h}{2} - \rho' n &< -\tfrac{9}{10}\rho' n && \text{(for Lemma 2.6.2)},\\
\frac{\log(n)}{n} &< \psi && \text{(for Lemma 2.6.2)},
\end{aligned}
$$

and $\min\{|\mathcal{Q}(n,H)|2^{-\hat{\rho}n}, |\mathcal{Q}(n,H)|2^{-\tilde{\rho}n}\} > \max\{2^{(1-\frac{1}{k})\frac{n^2}{2}-\frac{\epsilon}{10}n^2}, 2^{(1-\frac{1}{k})\frac{n^2}{2}-\rho n^2}\}$ (for Theorem 2.1.10).

16. We choose $C > 8$ large enough that for $n \leq n_0$ Equation (2.1) trivially holds.

2.3 Sparse and Dense classes

We will show that almost every graph in $Forb^*(n,H)$ can be covered by s "almost-cliques" and t "almost-independent sets," for some witnessing pair (s,t). We will make this statement precise later in this section. To this end, we will define a certain cluster graph and consider a partition of *its* vertices.

Definition 2.3.1. If $\mu, \gamma > 0$ and $\mathcal{B} = (U_1, \ldots, U_m)$ is an equipartition of a subset of $V(G)$ such that every pair (X_i, X_j) is γ-regular, then the *cluster graph* $A_\mu(G)$ has vertex set $[m]$ and edge set $\{ij : \mu < d(U_i, U_j) < 1 - \mu\}$.

Given a graph $G \in Forb^*(n, H)$ and a partition Π of $V(G)$, call a pair of vertices a *bad edge* if it appears as an edge in a class of Π that has more non-edges than edges, or if it is a non-edge inside a class of Π that has more edges than non-edges.

Definition 2.3.2. For $\vartheta > 0$, let $Forb^*(n, H, \vartheta)$ be the set of graphs $G \in Forb^*(n, H)$ that admit a partition of $V(G)$ into k parts which results in at most ϑn^2 bad edges.

The main lemma in this section is the following, which says that almost all graphs in $Forb^*(n, H)$ admit such a partition.

Lemma 2.3.3. *For n large enough there exist $\vartheta > 0$ and $\epsilon > 0$ such that*

$$|Forb^*(n, H) - Forb^*(n, H, \vartheta)| < 2^{(1-\frac{1}{k}-\frac{\epsilon}{5})\frac{n^2}{2}}.$$

Our main tools to prove Lemma 2.3.3 are the stability theorem of Simonovits [74], a variant of the Regularity Lemma that is due to Alon and Stav [6], and a corresponding embedding lemma.

Theorem 2.3.4. *[Stability Theorem] For any $\lambda > 0$ and any integers k, R, there exists $\omega = \omega(k, \lambda, R) > 0$ such that if H is a graph on m vertices, for m sufficiently large, that does not contain $K_{k+1}(R)$ and*

$$e(H) > e(T_{m,k}) - \omega m^2,$$

then we H differs from $T_{m,k}$ in at most λm^2 edges.

The following embedding lemma is from [6]:

Lemma 2.3.5. *For every real $0 < \mu < 1$ and integer $f \geq 1$ there exists $\gamma = \gamma_{2.3.5}(\mu, f)$ with the following property. Suppose that F is a graph on f vertices $v_1, \dots, v_f$ and that $U_1, \dots, U_f$ is an f-tuple of disjoint vertex sets of a graph G such that for every $1 \leq i < j \leq f$ the pair (U_i, U_j) is γ-regular. Moreover, suppose that whenever $(v_i, v_j) \in E(F)$ we have $d(U_i, U_j) \geq \mu$, and whenever $(v_i, v_j) \notin E(F)$ we have $d(U_i, U_j) \leq 1 - \mu$. Then some f-tuple $(u_1, \dots, u_f) \in U_1 \times \cdots \times U_f$ spans an* induced *copy of F in which v_i is mapped to u_i for every $i \in [f]$.*

The following lemma, also due to Alon and Stav [6], is a variant of Szemerédi's Regularity Lemma [76] that instead finds an equipartition in which *every* pair of classes is γ-regular, by restricting attention to an induced subgraph.

Lemma 2.3.6. *For every integer m_0 and every $\gamma > 0$ there exists an integer $M = M(m_0, \gamma)$ that satisfies the following. For any graph G on $n \geq M$ vertices there exists an equipartition $\mathcal{A} = \{V_i | 1 \leq i \leq m\}$ of $V(G)$ and a set of vertices $U \in V(G)$, with an equipartition $\mathcal{B} = \{U_i | 1 \leq i \leq m\}$ of the vertices of U, that satisfy:*

1. *$m_0 \leq m \leq M$.*
2. *$U_i \subset V_i$ for all $i \in [m]$, and $|U_i| \geq n/M$.*
3. All *pairs of classes of $\mathcal{B}$ are γ-regular in $G[U]$.*
4. *All but at most $\gamma\binom{m}{2}$ of the pairs $1 \leq i < j \leq m$ satisfy $|d(V_i, V_j) - d(U_i, U_j)| < \gamma$.*

Note that $\mathcal{B}$ satisfies the hypothesis of Lemma 2.3.5. We will also use the Erdős-Stone Theorem [39].

Theorem 2.3.7 (Erdős-Stone Theorem)**.** *For any integers k and t and real number $\epsilon > 0$ there exists $m_0(k, t, \epsilon)$ such that for $m > m_0$ any graph on m vertices with at least*

$$\left(1-\frac{1}{k}+\epsilon\right)\binom{m}{2}$$

edges contains a copy of $K_{k+1}(t)$*.*

In Section 2.2.1 we fixed λ; note that λ can be chosen to be arbitrarily small and still satisfy the conditions in 2.2.1.2. From Theorem 2.3.4 we obtain $\omega = \omega(k, \lambda, R(h))$, where $R(h)$ is the Ramsey number. Throughout what follows, we will have to make sure our constants remain small enough relative to ω that the cluster graph we consider has at least $e(T_{m,k}) - \omega m^2$ edges. We also fixed ϵ and μ in Section 2.2.1 and obtained $\gamma_{2.3.5}(\mu, k)$ from Lemma 2.3.5, then fixed $\gamma < \gamma_{2.3.5}$. Note that if (U_i, U_j) is a γ-regular pair then it is also a $\gamma_{2.3.5}(\mu, k)$-regular pair. For m_0 defined by 2.2.1.7, consider the partition $\mathcal{B}$ guaranteed by Lemma 2.3.6 and let A be the cluster graph defined for that partition. Note that by the definition of m in 2.2.1.7, $|V(A)| = m$ and $\frac{12}{\epsilon} < m_0 < m < M(\gamma, m_0)$.

We call a pair *sparse* if $d(U_i, U_j) \leq \mu$, *average* if $\mu < d(U_i, U_j) < 1 - \mu$, and *dense* if $d(U_i, U_j) \geq 1 - \mu$. When it will not cause confusion, we will often refer to a pair of vertices (i, j) in the cluster graph A as "sparse", "average", or "dense" if the corresponding pair (U_i, U_j) is respectively sparse, average, or dense.

Lemma 2.3.8. *For sufficiently large* n*, all but at most* $2^{(1-\frac{1}{k}-\frac{\epsilon}{6})\frac{n^2}{2}}$ *graphs* $G \in Forb^*(n, H)$ *have cluster graph* A *on* m *vertices that satisfies*

$$\left(1-\frac{1}{k}-\epsilon\right)\binom{m}{2} < e(A) < \left(1-\frac{1}{k}+\epsilon\right)\binom{m}{2}. \tag{2.4}$$

Proof. Let $r = R(h)$. If $K_{k+1}(r)$ is a subgraph of the cluster graph A it follows by Lemma 2.3.5 that G contains an induced copy of H; that is because each class of $K_{k+1}(r)$ contains either h mutually sparse pairs or h mutually dense pairs, which correspond to, respectively, an independent set or a clique in G. The edges in A are average pairs, and so may correspond to an edge or non-edge of G as needed. Since $\chi_B(H) = k+1$, any collection of $k+1$ cliques and independent sets cover H, and so by Lemma 2.3.5 G contains an induced copy of H. By the Erdős-Stone theorem, because $|V(A)| = m$, if $e(F) \geq (1 - \frac{1}{k} + \epsilon)\binom{m}{2}$ then $K_{k+1}(r)$ is a subgraph of A. Consequently, $e(A) < (1 - \frac{1}{k} + \epsilon)\binom{m}{2}$.

Now we will count the number of graphs whose cluster graph has fewer than $(1 - \frac{1}{k} - \epsilon)\binom{m}{2}$ edges. Partition n vertices into m classes to determine a partition $\mathcal{A} = \{V_1, \ldots, V_m\}$, then choose a partition $\mathcal{B} = \{U_1, \ldots, U_m\}$ satisfying $U_i \subset V_i$ for all i. There are at most $m^{2n} = 2^{2\log(m)n}$ ways to do this. There are $3^{\binom{m}{2}} \leq 2^{\log(3)m^2/2}$ ways to three-color the edges of K_m, which corresponds to determining which pairs of classes (U_i, U_j) are sparse, average, or dense.

There are at most $2^{\binom{n/m}{2}}$ ways to place edges inside each class V_i, so there are at most $2^{m\binom{n/m}{2}} \leq 2^{n^2/2m}$ ways to place edges inside all m classes. If (U_i, U_j) is sparse or dense and $|d(V_i, V_j) - d(U_i, U_j)| < \gamma$, then there are at most $\sum_{i \leq 2\mu \frac{n^2}{m^2}} \binom{n^2/m^2}{i}$ ways to place edges between V_i and V_j (recall from 2.2.1.6 that $\gamma < \mu$, so $\mu + \gamma < 2\mu$). There are at most $\binom{m}{2}$ such pairs, so there are at most

$$\left[2 \sum_{i \leq 2\mu \frac{n^2}{m^2}} \binom{n^2/m^2}{i}\right]^{\binom{m}{2}} \leq 2^{m^2 + H(2\mu)\frac{n^2}{2}}$$

ways to place edges between such pairs (V_i, V_j).

There are at most $(1 - \frac{1}{k} - \epsilon)\binom{m}{2}$ average pairs (U_i, U_j) with $|d(V_i, V_j) - d(U_i, U_j)| < \gamma$, so there are at most $2^{n^2/m^2(1-\frac{1}{k}-\epsilon)\binom{m}{2}} \leq 2^{(1-\frac{1}{k}-\epsilon)\frac{n^2}{2}}$ ways to place edges between such pairs (V_i, V_j). Finally, there are at most $\gamma\binom{m}{2}$ pairs (V_i, V_j) for which $|d(V_i, V_j) - d(U_i, U_j)| \geq \gamma$, so there are at most $2^{\frac{n^2}{m^2}\gamma\binom{m}{2}} \leq 2^{\gamma\frac{n^2}{2}}$ ways to place edges between such pairs.

Consequently, the number of graphs $G \in Forb^*(n, H)$ whose cluster graph A on m vertices has fewer than $(1 - \frac{1}{k} - \epsilon)\binom{m}{2}$ edges is at most

$$2^{2\log(m)n + \log(3)\frac{m^2}{2} + m^2} 2^{\frac{n^2}{2m} + H(2\mu)\frac{n^2}{2} + (1-\frac{1}{k}-\epsilon)\frac{n^2}{2} + \gamma\frac{n^2}{2}}.$$

For large enough n, this is at most

$$2^{(1-\frac{1}{k}+\frac{1}{m}+H(2\mu)+\gamma-\frac{\epsilon}{2})\frac{n^2}{2}},$$

and n is chosen in Section 2.2.1 to be large enough. Recall that $H(2\mu) < \frac{\epsilon}{6}$ (from 2.2.1.5), $\frac{1}{m} < \frac{\epsilon}{12}$ (from 2.2.1.7), and $\gamma < \frac{\epsilon}{12}$ (from 2.2.1.6), and so $\frac{1}{m} + H(2\mu) + \gamma - \frac{\epsilon}{2} < -\frac{\epsilon}{6}$. Consequently, at most $2^{(1-\frac{1}{k}-\frac{\epsilon}{6})\frac{n^2}{2}}$ graphs $G \in Forb^*(n, H)$ have a cluster graph with fewer than $(1-\frac{1}{k}-\epsilon)\binom{m}{2}$ edges.

□

Note that

$$e(A) > \left(1-\frac{1}{k}-\epsilon\right)\binom{m}{2} = e(T_{m,k}) - \left[\left(1-\frac{1}{k}\right)\frac{m}{2} + \epsilon\binom{m}{2}\right] > e(T_{m,k}) - \omega m^2,$$

and so by the Stability Theorem (Theorem 2.3.4) A differs from $T_{m,k}$ in at most λm^2 edges.

Given a partition of the vertices of a graph, we will call an edge *interior* both of its endpoints are in the same class, and a *cross-edge* otherwise. Fix some k-partition $\Pi = (P_1, \ldots, P_k)$ of the vertices of A that maximizes the number of cross-edges. By Theorem 2.3.4, the number of missing cross-edges in A is at most λm^2 and the number of present interior edges is at most λm^2.

Lemma 2.3.9. *Let q be a prime number and k any integer. Then we may pack q^2 edge-disjoint copies of K_k into $K_k(q)$.*

Proof. Order the k classes and label the vertices in each class with $\{0, 1, \ldots, q-1\}$. For any $a \in \{0, 1, \ldots, q-1\}$ and any $b \in \{0, 1, \ldots, q-1\}$ let v_i be the vertex in the ith class with label $a+(i-1)b$, modulo q. Let $F_{a,b}$ be the resulting induced copy of K_k; there are q^2 such copies. We claim that if $(a, b) \neq (a', b')$ then $F_{a,b}$ and $F_{a',b'}$ are edge-disjoint. Note that because q

is prime, $\mathbb{Z}_q$ is a field. Suppose the edge $\{a+(i-1)b, a+(j-1)b\}$ appears in $F_{a',b'}$. Then $a+(i-1)b = a'+(i-1)b'$ and $a+(j-1)b = a'+(j-1)b'$. Consequently, $(i-j)b = (i-j)b'$. Because $\mathbb{Z}_q$ is a field, this implies that $b = b'$. Therefore, $a+(i-1)b = a'+(i-1)b$ and so $a = a'$. □

Definition 2.3.10. Call a subset $S \subset V(A)$ *sparse-homogeneous* if for each $ij \in E(A[S])$ the pair (U_i, U_j) is sparse or average. Call S *dense-homogeneous* if for each $ij \in E(A[S])$ the pair (U_i, U_j) is dense or average. Call S *non-homogeneous* otherwise.

Note that by Lemma 2.3.5 if $S = \{v_1, \ldots, v_\ell\}$ is a sparse-homogeneous set in $V(A)$ then there is an independent set in G with exactly one vertex from each set U_{v_i}. Similarly, if S is a dense-homogeneous set in $V(A)$ there is a clique in G with exactly one vertex from each set U_{v_i}.

The following two lemmas will help to prove that each class of $V(A)$ under the partition Π either contains many sparse pairs or many dense pairs. This in turn will help us to find a partition of the original graph G whose classes again contain either many or very few edges. Let α be defined as in 2.2.1.8.

Lemma 2.3.11. *For large enough m and G satisfying the restrictions in Lemma 2.3.8, no class of the partition of $V(A)$, $\Pi = (P_1, \ldots, P_k)$, contains more than αm vertex-disjoint non-homogeneous sets of size h.*

Proof. Suppose the statement is false, and without loss of generality assume that P_1 is a contradictory class. Let $r = R(h)$ (the Ramsey number), and suppose that α is small enough that each class of the k-partition Π of $V(A)$ contains at least $\alpha m r$ vertices (since A is close to the Turán graph $T_{m,k}$, no class in the partition is small). Then each class other than P_1 can be divided into αm vertex-disjoint subsets, each of size at least r. Therefore, each subset contains either an h-vertex sparse-homogeneous set or an h-vertex dense-homogeneous set. Call these αm homogeneous subsets the *relevant* subsets.

On the other hand, P_1 contains αm non-homogeneous subsets of size h. For each such non-homogeneous set there is an induced subgraph in G with h vertices that is neither a clique nor an independent set, obtained by applying Lemma 2.3.5. Since H is critical, none of these αm subgraphs are in $Forb^*(\frac{m}{k}, \mathcal{F}(H,s,t))$ for any $s+t=k-1$. Consequently, for any $s+t=k-1$, every one of the subgraphs contains some graph from $\mathcal{F}(H,s,t)$, and so any choice of one relevant subset from each P_i and one non-homogeneous set from P_1 results in a potential induced copy of H in G, if we can apply Lemma 2.3.5.

Bertrand's postulate states that there is a prime number between j and $2j$ for any integer j, so fix the smallest prime number q such that $\frac{2\alpha m}{4} \geq q \geq \lfloor \frac{\alpha m}{4} \rfloor$. Now by Lemma 2.3.9 and Lemma 2.3.5, if every cross-edge is present then we can embed q^2 edge-disjoint copies of H into A in such a way that each induces H in G. Since G is H-free, at least one cross-edge must be missing from each such copy of H; because they are edge-disjoint, there are at least q^2 missing cross-edges in A. Consequently, there are $q^2 \geq \frac{\alpha^2 m^2}{100}$ cross-edges missing from A. By 2.2.1.8, $\frac{\alpha^2 m^2}{100} > \lambda m^2$. Since there are at most λm^2 missing cross-edges, this is a contradiction. □

Now, after removing vertex-disjoint non-homogeneous sets from a class of Π what remains of the class is either a sparse-homogeneous set or a dense-homogeneous set. We will call a class of a partition of the cluster graph "sparse" if what remains is sparse-homogeneous, and "dense" if what remains is dense-homogeneous. Call a pair (i,j) "bad" if $d(U_i,U_j) < \mu$ in a dense class or if $d(U_i,U_j) > 1-\mu$ in a sparse class.

Lemma 2.3.12. *For a graph $G \in Forb^*(n,H)$ with a cluster graph A satisfying (2.4) and $\Pi = (P_1,\ldots,P_k)$ a k-partition of $V(A)$ maximizing the number of cross-edges, A contains at most $\alpha m^2 h$ bad pairs.*

Proof. Each class P_i contains at most $\binom{\alpha m h}{2} + (|P_i| - \alpha m h)(\alpha m h)$ bad pairs. This can be rewritten as $\alpha m h|P_i| - \binom{\alpha m h+1}{2}$. Consequently, the total number of bad edges in A is at most $\sum_{i=1}^{k}(\alpha m h|P_i|) - \binom{\alpha m h+1}{2} \leq \sum_{i=1}^{k}(\alpha m h|P_i|) = \alpha m^2 h$. □

We are now ready to prove Lemma 2.3.3.

Proof of Lemma 2.3.3. There are fewer than n ways to choose $m < n$. We have found that for all but at most $n2^{(1-\frac{1}{k}-\frac{\epsilon}{6})\frac{n^2}{2}}$ graphs $G \in Forb^*(n, H)$, the cluster graph A of G has a partition $\Pi = (P_1, \ldots, P_k)$ with no more than $\alpha h m^2$ bad pairs in each class. From Section 2.2.1, $n2^{(1-\frac{1}{k}-\frac{\epsilon}{6})\frac{n^2}{2}} < 2^{(1-\frac{1}{k}-\frac{\epsilon}{5})\frac{n^2}{2}}$. Form a partition of $V(G)$ by letting V_i be in class j if $i \in P_j$. In this partition of $V(G)$, call a class "sparse" if it contains more missing edges than edges and "dense" otherwise. Call an edge "bad" if it appears in a sparse class or is missing from a dense class.

Recall that $|V_i| = \frac{n}{m}$ for all i and that for all but at most $\gamma\binom{m}{2}$ pairs, $|d(V_i, V_j) - d(U_i, U_j)| < \gamma$. There are therefore at most $\gamma\binom{m}{2}\frac{n^2}{m^2} \leq \frac{\gamma n^2}{2}$ bad edges between all such pairs (V_i, V_j). Recall also that A contains at most λm^2 interior edges, which correspond to average pairs. There are therefore at most $\lambda(1 - \mu + \gamma)\frac{n^2}{m^2} \leq \frac{\lambda n^2}{m^2}$ bad edges between all such pairs (recall that $\gamma < \mu$, by 2.2.1.6). If (i, j) is a bad pair in A then it results in at most $\frac{n^2}{m^2}$ bad edges in G, so there are at most $\alpha h m^2 \frac{n^2}{m^2} = \alpha h n^2$ such bad edges. Edges within each V_i may be bad; there are at most $m\binom{n/m}{2} \leq \frac{n^2}{2m}$ such edges. In total, that is at most

$$\left(\frac{\gamma}{2} + \alpha h + \frac{\lambda}{m^2} + \frac{1}{2m}\right) n^2$$

bad edges. In other words, $G \in Forb^*(n, H, \vartheta)$, for $\vartheta = \frac{\gamma}{2} + \alpha h + \frac{\lambda}{m^2} + \frac{1}{2m}$.

□

2.4 More properties of $Forb^*(n, H, \vartheta)$

Recall that for some $G \in Forb^*(n, H, \vartheta)$ and a partition $\Pi = (P_1, \ldots, P_k)$ of the vertices of G, we call a pair of vertices a "bad edge" if it is missing from a dense class or if it is present in a sparse class. Call such a partition *optimal* if it minimizes the number of bad edges; recall that if $G \in Forb^*(n, H, \vartheta)$ then an optimal partition of G contains at most ϑn^2 bad edges.

Lemmas 2.4.2, 2.4.3, and 2.4.6 are from [10], although Lemma 2.4.2 was updated in [9]. Note that although those papers deal with non-induced subgraphs the proofs of these lemmas need only be slightly modified, if at all, to suit our needs. For the sake of completeness we include those modified proofs.

Definition 2.4.1. Let $Forb^*(n, H, \vartheta, \delta)$ be the set of all $G \in Forb^*(n, H, \vartheta)$ such that if $\Pi = (P_1, \ldots, P_k)$ is an optimal partition of $V(G)$ then

1. for any $A \subset P_i$ and $B \subset V(G_n) - P_i$ with $|A| = |B| = \lfloor \delta n \rfloor$, we have $\frac{1}{4}|A||B| < e(A, B) < \frac{3}{4}|A||B|$, and
2. $||P_i| - n/k| < (\sqrt{\vartheta} \log 1/\vartheta)n$.

In other words, $Forb^*(n, H, \vartheta, \delta)$ consists of those graphs having an optimal partition Π that is somewhat close to being an equipartition and whose cross-edges behave somewhat randomly.

Lemma 2.4.2 (Lemma 6.1 in [10])**.** *For any $\delta \geq 2H(\vartheta)$ there is a positive constant $\rho_0 = \rho_0(\vartheta) > 0$ such that for sufficiently large n all but at most $2^{(1-\frac{1}{k})\frac{n^2}{2} - \rho_0 n^2}$ graphs in $Forb^*(n, H, \vartheta)$ satisfy condition 1 of Definition 2.4.1.*

Proof. Say that a graph in $Forb^*(n, H, \vartheta)$ is "bad" if there exists an optimal partition $\Pi = (P_1, \ldots, P_k)$ of $V(G)$ and there exist $A \subseteq P_i$ and $B \subseteq U_j$ such that $|A| = |B| = \lfloor \delta n \rfloor$ and $e(A, B) \leq \frac{1}{4}|A||B|$ or $e(A, B) \geq \frac{3}{4}|A||B|$. We will count the number of bad graphs.

For a fixed graph $G \in Forb^*(n, H, \vartheta)$, the number of optimal partitions of $V(G)$ is at most k^n. There are at most $\binom{n}{\delta n}^2$ ways to choose A and B, and $\binom{n}{\delta n}^2 < 2^{2H(\delta)n}$. For each $\ell \in [k]$ there are at most $\binom{n^2}{\vartheta n^2}$ ways to choose edges within U_ℓ, because G is in $Forb^*(n, H, \vartheta)$ and so does not contain many bad edges.

Claim: If $i \neq j$ and $A \subseteq P_i$ and $B \subseteq P_j$ then the number of ways to choose edges between P_i and P_j is at most

$$2^{\frac{1}{2}(1-1/k)n^2} \cdot 2^{-(1-H(1/4))\delta^2 n^2}.$$

Proof of Claim. If $e(A,B) \leq \frac{1}{4}|A||B|$ then $e(A,B) \leq \frac{1}{4}\delta^2 n^2$. If $e(A,B) \geq \frac{3}{4}|A||B|$ then $\overline{e}(A,B) \leq \frac{1}{4}\delta^2 n^2$. Either way, there are $\binom{(\delta n)^2}{\frac{1}{4}(\delta n)^2}$ ways to choose edges between A and B. If there had been no restriction on the number of edges between A and B, there would have been $2^{\delta^2 n^2}$ ways to choose these edges. Because $\binom{(\delta n)^2}{\frac{1}{4}(\delta n)^2} \leq 2^{H(\frac{1}{4})(\delta n)^2}$, we find that there are

$$2^{\frac{1}{2}(1-1/k)n^2} \cdot 2^{-\delta^2 n^2} \cdot 2^{H(\frac{1}{4})\delta^2 n^2}$$

ways to choose edges between P_i and P_j. □

The number of bad graphs is therefore at most

$$2^{\frac{1}{2}\left(1-\frac{1}{k}\right)n^2} \cdot k^n \cdot 2^{2H(\delta)n} \cdot 2^{H(\vartheta)n^2} \cdot 2^{-\delta^2 n^2} \cdot 2^{H(1/4)\delta^2 n^2}.$$

If $H(\vartheta) < \frac{1}{3}\delta^2$ and n is large enough that

$$\log(k) + 2H(\delta) + \left(H\left(\frac{1}{4}\right) - \frac{2}{3}\right)\delta^2 n < -1,$$

then the number of bad graphs is at most $2^{\frac{1}{2}(1-1/k)n^2 - n}$. Therefore, almost all graphs in $Forb^*(n,H,\vartheta)$ have the desired property. □

Lemma 2.4.3 (Lemma 6.6 in [10]). *Given ϑ sufficiently small, all but at most $2^{(1-\frac{1}{k})\frac{n^2}{2} - 2\vartheta n^2}$ graphs in $Forb^*(n,H,\vartheta)$, satisfy condition 2 of Definition 2.4.1.*

Proof. Once again we will count the number of bad graphs, and show that there are at most $2^{\frac{1}{2}(1-1/k)n^2 - n}$ of them. Suppose $G \in Forb^*(n,H,\vartheta)$ but there exists $i_0 \in [k]$ for which

$$||P_{i_0}| - n/k| \geq (\sqrt{\vartheta}\log 1/\vartheta)n.$$

We will need the following claim (see [10], [74]):

Claim: If G is a k-partite graph with n vertices with k-partition $\Pi = (P_1, \ldots, P_k)$ then

$$e(G) \leq e(T_{n,k}) - \sum_{i=1}^{k} \binom{s_i}{2},$$

where $s_i = \lfloor |n/p - |P_i|| \rfloor$ for $i \in [k]$.

Now, $s_{i_0} = \lfloor \left(\sqrt{\vartheta}\log(1/\vartheta)\right) n \rfloor$. Applying the above claim we find

$$\sum_{i<j} e(P_i, P_j) \leq e(T_{n,k}) - \binom{s_{i_0}}{2} < e(T_{n,k}) - \frac{\vartheta}{3}\left(\log\left(\frac{1}{\vartheta}\right)\right)^2 n^2.$$

Since $H(x) \approx x\log(1/x)$,

$$\binom{\binom{n}{2}}{\vartheta n^2} < 2^{2\vartheta(\log(1/\vartheta))n^2}.$$

If $\log(1/\vartheta) > 12$, we find that the number of bad graphs is at most

$$\begin{aligned} k^n \cdot 2^{\frac{1}{2}(1-1/k)n^2} \cdot 2^{-\frac{\vartheta}{3}(\log(1/\vartheta))^2 n^2} \cdot \binom{\binom{n}{2}}{\vartheta n^2} &\leq k^n \cdot 2^{\frac{1}{2}(1-1/k)n^2} \cdot 2^{-\frac{\vartheta}{3}(\log(1/\vartheta))^2 n^2 + 2\vartheta(\log(1/\vartheta))n^2} \\ &\leq k^n \cdot 2^{\frac{1}{2}(1-1/k)n^2} \cdot 2^{-\frac{\vartheta}{6}(\log(1/\vartheta))^2 n^2} \\ &< 2^{\frac{1}{2}(1-1/k)n^2} \cdot 2^{-2\vartheta n^2} < 2^{\frac{1}{2}(1-1/k)n^2 - n}. \end{aligned}$$

□

Corollary 2.4.4. *For any $\delta \geq 2H(\vartheta)$ there exists $\rho = \rho(\vartheta)$ such that*

$$|Forb^*(n, H, \vartheta) - Forb^*(n, H, \vartheta, \delta)| < 2^{(1-\frac{1}{k})\frac{n^2}{2} - \rho n^2}.$$

Proof. By applying Lemmas 2.4.2 and 2.4.3 we obtain

$$|Forb^*(n, H, \vartheta) - Forb^*(n, H, \vartheta, \delta)| < 2^{(1-\frac{1}{k})\frac{n^2}{2} - \rho_0 n^2} + 2^{(1-\frac{1}{k})\frac{n^2}{2} - 2\vartheta n^2}.$$

Fix $\rho' < \min\{\rho_0, 2\vartheta\}$. Then $|Forb^*(n, H, \vartheta) - Forb^*(n, H, \vartheta, \delta)| < 2^{(1-\frac{1}{k})\frac{n^2}{2} - \rho' n^2 + 1}$. Fix some

$\rho < \rho' - \frac{1}{n_0}$; then

$$|Forb^*(n, H, \vartheta) - Forb^*(n, H, \vartheta, \delta)| < 2^{\left(1-\frac{1}{k}\right)\frac{n^2}{2} - \rho n^2}.$$

□

The following lemma is a variant of Lemma 6.10 in [10].

Lemma 2.4.5 (Lemma 6.10 in [10])**.** *For $\frac{4k^2}{3}\vartheta < \delta < \frac{1}{k^2}$, n sufficiently large (as defined by Lemma 2.4.2) and $G \in Forb^*(n, H, \vartheta, \delta)$, if $\Pi = (X_1, \ldots, X_k)$ and $\Pi' = (Y_1, \ldots, Y_k)$ are optimal partitions of $V(G)$ then for every $i \in [k]$ there exists a permutation π of $[k]$ such that for every $i \in [k]$ it is the case that $|X_i \Delta Y_{\pi(i)}| \leq 2\delta n$.*

Proof. We have already shown that all but at most $2^{(1-\frac{1}{k}+\frac{1}{m}+H(\gamma)-\epsilon)\frac{n^2}{2}}$ graphs in $Forb^*(n, k)$ have cluster graph H differing from $T_{m,k}$ in at most λm^2 edges. We have also shown that for all but at most $2^{(1-\frac{1}{k})\frac{n^2}{2}-n}$ graphs in $Forb^*(n, k)$ the conclusion of Lemma 2.4.2 holds, for $n \geq n_0(k, \delta)$.

If G is not one of these exceptions, then the number of bad edges in X_i is at most ϑn^2 and the number of bad edges in Y_i is also at most ϑn^2. Note that X_i is partitioned into $X_i \cap Y_1, X_i \cap Y_2, \ldots, X_i \cap Y_k$. Let X_i^* be the largest of these parts; of course then $|X_i^*| > \frac{n}{k^2}$. Define a function $\pi : [k] \to [k]$ to satisfy $X_i^* = X_i \cap Y_{\pi(i)}$.

Suppose π is not a permutation. That is, $\pi(i) = \pi(j)$ for some $j \neq i$. Then X_i^* and X_j^* are both contained in $Y_{\pi(i)}$. By Lemma 2.4.2, since $|X_i^*|, |X_j^*| \geq \frac{n}{k^2}$ and n is sufficiently large, $\frac{1}{4}|X_i^*||X_j^*| < e(X_i^*, X_j^*) < \frac{3}{4}|X_i^*||X_j^*|$. If $Y_{\pi(i)}$ is a dense class then this means that $\overline{e}(Y_{\pi(i)}) \geq \frac{3}{4}|X_i^*||X_j^*| \geq \frac{3}{4}\frac{n^2}{k^4} > \vartheta n^2$, which is a contradiction. If $Y_{\pi(i)}$ is a sparse class then this means similarly that $e(Y_{\pi(i)}) \geq \frac{1}{4}\frac{n^2}{k^4} > \vartheta n^2$, which is again a contradiction. Therefore, π is a permutation.

It remains to show that $|X_i \Delta Y_{\pi(i)}| \leq 2\delta n$. Suppose $|X_i - Y_{\pi(i)}| > \delta n$. Then by lemma 2.4.2, $\overline{e}(X_i) \geq \overline{e}(X_i^*, X_i - Y_{\pi(i)}) \geq \frac{3}{4}\frac{n}{k^2}\delta n > \vartheta n^2$, which as before is a contradiction. Since

X_i and $Y_{\pi(i)}$ are interchangeable in the above proof, $|Y_{\pi(i)} - X_i| \leq \delta n$ as well, and so $|X_i \Delta Y_{\pi(i)}| \leq 2\delta n$. □

As in [10], the above lemma immediately implies the following.

Lemma 2.4.6 (Lemma 6.11 in [10])**.** *Each $G \in Forb^*(n, H)$ has at most $2^{H(2\delta)kn}$ k-partitions $\Pi = (P_1, \ldots, P_k)$ such that Π results in at most $\vartheta(n+\ell)^2$ bad edges, for any $\ell \leq h$.*

Proof. This is because there are at most

$$\left(\sum_{i<2\delta n} \binom{n}{i} \right)^k \leq 2^{H(2\delta)kn}$$

partitions of $[n]$ at distance no greater than $2\delta n$ from a given partition. □

2.5 Bad vertices of a graph

For $\beta > 0$, we say that a vertex v in a graph is β-*bad* with respect to an optimal partition if there are at least βn bad edges between v and other vertices in its class. We shall show that for β chosen in accordance with 2.2.1.12 almost every $G \in Forb^*(n, H, \vartheta, \delta)$ contains no bad vertex.

Definition 2.5.1. Let $Forb^*(n, H, \vartheta, \delta, \beta)$ be the set of graphs G in $Forb^*(n, H, \vartheta, \delta)$ containing no β-bad vertex in any of the optimal partitions of $V(G)$.

If a graph $G \in Forb^*(n, H, \vartheta, \delta)$ with an optimal k-partition $\Pi = (P_1, \ldots, P_k)$ contains a β-bad vertex v, then by the optimality of the partition Π we know that $|N(v) \cap P_i| \geq \beta n$ for every sparse class P_i and $|\overline{N(v)} \cap P_j| \geq \beta n$ for every dense class P_j. The following lemmas say that if both $N(v) \cap P_i$ and $\overline{N(v)} \cap P_i$ are large for every $1 \leq i \leq k$, then we can find an induced copy of H in G. We need another embedding lemma first, Theorem 6.3 from [10]:

Lemma 2.5.2. *For every m there exists n_0, δ_0 such that if $n > n_0$ and $\delta < \delta_0 \ll \frac{\beta}{m}$ then for any graph G on n vertices and graph F on m vertices the following holds. Suppose*

$X_1, \ldots, X_m$ are disjoint sets in $V(G)$ with $|X_i| = \frac{\beta n}{m}$ such that for any pair $U_i \subset X_i$ and $U_j \subset X_j$ with $|U_i|, |U_j| \geq \delta n$, $e(U_i, U_j) \geq \frac{1}{4}|U_i||U_j|$ whenever $ij \in E(F)$ and $e(U_i, U_j) \leq \frac{3}{4}|U_i||U_j|$ whenever $ij \notin E(F)$. Then there is some choice of $x_i \in X_i$ such that $\{x_1, \ldots, x_m\}$ induces F in G.

Lemma 2.5.3. *Given $G \in Forb^*(n, H, \vartheta, \delta)$ with an optimal partition $\Pi = (P_1, \ldots, P_k)$, then G contains no bad vertex v such that $|N(v) \cap P_i| > \beta n$ and $|\overline{N(v)} \cap P_i| > \beta n$ for all P_i except perhaps the class containing v.*

Proof. Suppose there is such a vertex v and Π consists of s dense classes and t sparse classes. At least one of s, t is nonzero so without loss of generality suppose that v is in P_1 and P_1 is dense (a similar argument can be made if P_1 is sparse). Because H is critical and (s,t) is a witnessing pair with $s \geq 1$ by Lemma 2.2.8 there is some graph $F \in \mathcal{F}(s-1, t, H)$ such that F is the disjoint union of a clique and a single vertex. Given a copy of F, and $s-1$ cliques and t independent sets, each on h vertices, there is some way to place edges between them so as to induce a copy of H. Call such an embedding "proper". Note that $|V(F)| \leq h$.

Now, since $\overline{N(v)} \cap P_1$ is large we can divide it into h sets, each of size at least $\frac{\beta n}{h}$. For $1 < i \leq k$, $N(v) \cap P_i$ and $\overline{N(v)} \cap P_i$ are all large, so we can divide each of those $2k-2$ sets into h sets of size at least $\frac{\beta n}{h}$. Let these sets together with $P_1 \cap \overline{N(v)}$ be called $X_1, \ldots, X_{2hk-h}$. We can now use Lemma 2.5.2 with $m = h-1$ and $F = H[V(H) - \{v\}]$ to find an induced copy of H in G. That is, we will find a clique in $P_1 \cap \overline{N(v)}$, cliques in the remaining dense classes, and independent sets in the remaining sparse classes such that everything respects the "proper" embedding (and exactly one vertex is chosen from each X_i). Note that we will not be using every set X_i, only those that correspond to vertices in the covering of H by $s-1$ cliques, t independent sets, and F.

To apply Lemma 2.5.2 we require that $\delta \ll \frac{\beta}{h}$. Because Lemma 2.4.6 requires that $\frac{8k^2}{3}\vartheta < \delta$ this implies $\beta \gg \frac{8hk^2}{3}\vartheta$, which 2.2.1.12 guarantees. Recall from 2.2.1.10 that $\frac{3}{4}\delta^2 > \vartheta$. We suppose X_i and X_j are in the same dense class and that $U_i \subset X_i$, $U_j \subset X_j$

with $|U_i|, |U_j| = \delta n$. Then $e(U_i, U_j) \geq \frac{1}{4}|U_i||U_j| = \frac{1}{4}\delta^2 n^2$, otherwise there are more than ϑn^2 edges missing from that dense class, which is a contradiction. Similarly, if X_i and X_j are in the same sparse class then $e(U_i, U_j) \leq \frac{3}{4}|U_i||U_j|$. Finally, if X_i and X_j are in different classes then by the definition of $Forb^*(n, H, \vartheta, \delta)$, we have $\frac{1}{4}|U_i||U_j| \leq e(U_i, U_j) \leq \frac{3}{4}|U_i||U_j|$. Lemma 2.5.2 now provides the correct embedding, which is of course a contradiction. □

Corollary 2.5.4. *If $G \in Forb^*(n, H, \vartheta, \delta, \beta)$ and $\Pi = (P_1, \ldots, P_k)$ is an optimal partition of $V(G)$, then G contains no bad vertex x such that both $N(x) \cap P_i$ and $\overline{N(x)} \cap P_i$ have sizes at least βn for all but one i.*

Proof. Suppose there is such a vertex, v. Without loss of generality, suppose that v is in the dense class P_1. Call P_i *evenly partitioned* if $N(v) \cap P_i$ and $\overline{N(v)} \cap P_i$ both have size at least βn. If P_1 is the class that is not evenly partitioned, then Lemma 2.5.3 still applies. Suppose that $N(v) \cap P_i$ and $\overline{N(v)} \cap P_i$ are both large for $1 \leq i < k$; that is, P_k is the class that is not evenly partitioned. If P_k is a dense class then $|\overline{N(v)} \cap P_k| > \beta n$, otherwise Π was not an optimal partition. Since both $\overline{N(v)} \cap P_i$ and $N(v) \cap P_i$ are large for $i \neq k$ and v is a bad vertex if $v \in P_k$, we can temporarily move v into P_k and apply the same proof as in Lemma 2.5.3. A similar argument applies if P_k is a sparse class. □

We need therefore only consider those graphs G for which an optimal partition Π results in a bad vertex whose neighborhood leaves at least two classes of Π unevenly divided. The following lemma says that almost all $G \in Forb^*(n, H, \vartheta, \delta)$ do *not* meet that description.

Lemma 2.5.5. *There exists $\hat{\rho} = \hat{\rho}(\delta, \beta, \vartheta) > 0$ such that for all but at most $(C\sigma^{n-1} + 1)2^{-\hat{\rho}n}|\mathcal{Q}(n, H)|$ graphs $G \in Forb^*(n, H, \vartheta, \delta)$, optimal partition $\Pi = (P_1, \ldots, P_k)$, and bad vertex x, there exists at most one i such that $x \notin P_i$ and $|N(x) \cap P_i| < \beta n$ if P_i is sparse, or $|\overline{N(x)} \cap P_i| < \beta n$ if P_i is dense.*

Proof. We will count the number of graphs G_n for which there is a bad vertex having two or more classes satisfying the conclusion of the lemma. Note that if G is such a graph, Π is such a partition, and x is such a bad vertex then $G[V - \{x\}]$ is in $Forb^*(n-1, H)$. Moreover the partition Π when restricted to $V - \{x\}$ still contains at most ϑn^2 bad edges. In other words, each such graph G_n is formed from some graph $G' \in Forb^*(n-1, H)$ with a partition containing at most ϑn^2 bad edges.

There are at most n ways to choose x and by Lemma 2.4.6 there are at most $2^{H(2\delta)k(n-1)} < 2^{H(2\delta)kn}$ ways to partition $[n-1]$ that result in a k-partition with at most ϑn^2 bad edges. There are at most k^2 ways to choose the two classes, say P_1 and P_2. There are at most $\binom{\frac{n}{k} + (\sqrt{\vartheta}\log(\frac{1}{\vartheta}))n}{\beta n}^2 \leq 2^{2H(\beta)n}$ ways to place edges between x and the classes P_1 and P_2.

The remaining $k-2$ classes have no restrictions on their connections to x, so there are at most $2^{(k-2)(\frac{n}{k} + \sqrt{\vartheta}\log(\frac{1}{\vartheta})n)}$ ways to connect x with them. Let $\psi = \sqrt{\vartheta}\log(\frac{1}{\vartheta})$. The following is an upper bound for the number of graphs we are considering.

$$
\begin{aligned}
& nk^2|Forb^*(n-1, H)|2^{H(2\delta)kn + 2H(\beta k)n + (k-2)(\frac{n}{k} + \psi n)} \\
\leq\ & |Forb^*(n-1, H)|2^{H(2\delta)kn + 2H(\beta k)n + (k-2)(\frac{n}{k} + \psi n) + 2\log n} \\
\leq\ & (C\sigma^{n-1} + 1)|\mathcal{Q}(n, H)|2^{-(1-\frac{1}{k})n}2^{H(2\delta)kn + 2H(\beta k)n + (k-2)(\frac{n}{k} + \psi n) + 2\log(n)} \\
=\ & (C\sigma^{n-1} + 1)|\mathcal{Q}(n, H)|2^{-\hat{\rho}n},
\end{aligned}
$$

where $\hat{\rho} = \hat{\rho}(n) = \frac{1}{k} - H(2\delta)k - 2H(\beta k) - (k-2)\psi - \frac{2\log(n)}{n}$. We make use of (2.2) and the inductive hypothesis (2.1), as well as Lemma 2.2.5. Recall that $H(2\delta) < \frac{1}{4k^2}$ (from 2.2.1.10), $2H(\beta k) < \frac{1}{4k}$ (from 2.2.1.12), and $\psi < \frac{1}{4k(k-2)}$ (from 2.2.1.9), and n is large enough that $\frac{2\log(n)}{n} < \frac{1}{4k}$ (from 2.2.1.15), so $\hat{\rho}$ is positive.

□

The following is now immediate, after fixing $C, \sigma, \delta, \vartheta, \beta$ and n_0 from the preceding lemmas.

Corollary 2.5.6. *For all $n > n_0$,*

$$|Forb^*(n, H, \vartheta, \delta) - Forb^*(n, H, \vartheta, \delta, \beta)| < (C\sigma^{n-1} + 1)|Q(n, H)|2^{-\hat{\rho}n}.$$

2.6 Bad edges

It remains to show that for almost all $G \in Forb^*(n, H, \vartheta, \delta, \beta)$, there is some optimal k-partition of the vertex set of G that produces no bad edges at all. If Π is an optimal k-partition of $V(G)$ and F is a subgraph of some part, with $h = |V(F)|$, then there is a natural partition of each other part of Π into 2^h classes based on neighborhoods of the vertices of F. If each of these neighborhoods is large enough, then we can once again use Lemma 2.5.2 to find an induced copy of H in G.

Lemma 2.6.1. *Suppose $G \in Forb^*(n, H, \vartheta, \delta, \beta)$, where ϑ, δ, β are chosen in Section 2.2.1 and n is large enough. Let $\Pi = (P_1, \ldots, P_k)$ be an optimal partition of $V(G)$ and suppose it has s dense and t sparse classes. If $xy \in G[P_i]$ is a bad edge and P_i is a dense class, then the following holds. If F is an induced subgraph in P_i that has h vertices, including x and y, and contains some graph from $\mathcal{F}(H, s-1, t)$ as an induced subgraph then there exists some $j \neq i$ such that the natural partition of P_j induced by the vertices of F contains at least one part with size smaller than $\frac{|P_j|}{2^{h+1}}$. If P_i is a sparse class then an analogous statement is true for an $F \in \mathcal{F}(H, s, t-1)$.*

Proof. Suppose the statement is false. Assume without loss of generality that $i = 1$ and P_1 is a dense class and contains a missing edge (the proof is easily adapted if P_1 is sparse and contains an edge).

We will again apply Lemma 2.5.2 to obtain an embedding of H, which is a contradiction. Since H is critical and n is large, any non-trivial graph on $|P_1|$ vertices contains some graph in $\mathcal{F}(H, s-1, t)$ as an induced subgraph. Fix one, say $\tilde{F}$, and extend it to an induced subgraph of P_1 on h vertices, say F; because $\tilde{F} \in \mathcal{F}(H, s-1, t)$ it has at most h vertices, so this is possible. Now, there is some way to place edges between $s-1$ copies of K_h, t copies of $\overline{K_h}$, and F so as to induce H; call such an embedding "proper". Consider the natural neighbourhood partition induced in each P_j by the vertices of F; if we can find $\overline{K_h}$ in each sparse class and K_h in each dense class such that the vertices of each are in whichever neighbourhood respects the proper embedding, then we have found an induced copy of H in G. That would of course be a contradiction, since G is H-free.

If there is no $j \neq 1$ for which that the neighborhood partition of P_j contains a part with fewer than $|P_j|/2^{h+1}$ vertices, then we can find such an embedding. This is achieved through another application of Lemma 2.5.2 in which $X_1, \ldots, X_m$ are the neighborhood partition's parts, equally divided into h sub-parts, where $m = kh2^h$, and $\delta < \delta_0 \ll \frac{\beta}{kh2^h}$. Because $\frac{8k^2}{3}\vartheta < \delta$, this last inequalities implies that $\frac{8k^3h2^h}{3}\vartheta \ll \beta$, which 2.2.1.12 guarantees. Notice that $|P_i| \geq \frac{n}{k} - \psi n$ for every i. By 2.2.1.12 and 2.2.1.9, $2\beta \leq 1 - \frac{1}{4k-8} < 1 - k\psi$, and so $|X_i| \geq \frac{|P_i|}{h2^{h+1}} \geq \frac{\beta n}{kh2^h}$. Recall that $\delta^2 > \frac{4\vartheta}{3}$, by 2.2.1.10.

Consider a pair (U_i, U_j) such that $U_i \subset X_i$ and $U_j \subset X_j$ and $|U_i|, |U_j| > \delta n$. If X_i and X_j are in different classes of Π then $\frac{1}{4}|U_i||U_j| < e(U_i, U_j) < \frac{3}{4}|U_i||U_j|$. If X_i and X_j are in the same sparse class of Π then $e(U_i, U_j) < \frac{1}{4}|U_i||U_j| \leq \frac{1}{4}\delta^2 n^2$, otherwise there are more than $\frac{1}{4}\delta^2 n^2 > \vartheta n^2$ bad edges in that sparse class. Similarly, if X_i and X_j are in the same dense class of Π then $e(U_i, U_j) > \frac{3}{4}|U_i||U_j|$. Consequently, by Lemma 2.5.2, there is a copy of K_h in each dense class and a copy of $\overline{K_h}$ in each sparse class such that the vertices of each respect the proper embedding, and cross-edges can be chosen properly, which is a contradiction. □

Lemma 2.6.2. *There exists $\tilde{\rho} > 0$ such that for all but at most $(1 + C\sigma^{n-h})|\mathcal{Q}(n, H)|2^{-\tilde{\rho}n}$ graphs $G \in Forb^*(n, H, \vartheta, \delta, \beta)$, there exists an optimal partition $\Pi = (P_1, \ldots, P_k)$ of $V(G)$ containing no bad edge.*

Proof. Suppose there is some graph $G \in Forb^*(n, H, \vartheta, \delta, \beta)$ such that every optimal k-partition of $V(G)$ contains a bad edge. Fix some optimal k-partition Π; let s be the number of dense classes in Π and t be the number of sparse classes in Π. Without loss of generality, assume that P_1 is a dense class and that there is a bad edge inside of P_1. Since H is critical, P_1 contains some induced subgrah, F, on h vertices that contains a graph $\tilde{F} \in \mathcal{F}(H, s, t)$ (every graph in $\mathcal{F}(H, s, t)$ has at most h vertices). Removing the h vertices of F results in a graph $G' \in Forb^*(n - h, H)$; it remains to count how many graphs G that contain a bad edge can be built from such a graph G'.

By Lemma 2.4.6 there are at most $2^{H(2\delta)kn}$ partitions of $[n-h]$ that have at most ϑn^2 bad edges. There are fewer than n^h ways to choose the h vertices of F. It remains to count how many ways these h vertices can be added to G'; that is, in how many ways these h vertices can be connected to the other $n-h$.

There are fewer than βn bad edges incident with each of these vertices, so there are at most $\binom{n/k+\sqrt{\vartheta}\log 1/\vartheta n}{\beta n}^h \leq 2^{H(\beta k)nh/k}$ ways to connect them with vertices in their own class. For sufficiently small β, $H(\beta k)/k \leq 2H(\beta)$.

By Lemma 2.6.1 there is some class of the partition Π, say P_i, for which the partition induced by the neighborhoods of the h vertices contains a part with size less than $\frac{|P_i|}{2^{h+1}}$. Let N be the small class in P_i; by Chernoff's inequality,

$$Pr\left(|N| < \frac{|P_i|}{2^{h+1}}\right) < \exp\left(-\frac{|P_i|}{2^{h+3}}\right) < 2^{-|P_i|/2^{h+3}}.$$

Consequently, there are at most $2^{h|P_i|-\frac{|P_i|}{2^{h+3}}} = 2^{|P_i|(h-\frac{1}{2^{h+3}})}$ ways to connect the h vertices to P_i. There are at most $2^{(n-|P_1|-|P_i|)h}$ ways to connect the h new vertices with vertices outside of P_1 and P_i. Let $\psi = \sqrt{\vartheta}\log(1/\vartheta)$. In total, at most

$$2^{H(2\delta)kn+h\log(n)+2H(\beta)nh+(n-\frac{2n}{k}+2\psi n)h+(\frac{n}{k}+\psi n)(h-\frac{1}{2^{h+3}})} \tag{2.5}$$

"bad" graphs can be made from each graph in $Forb^*(n-h,H)$. Let $-\rho' = H(2\delta)k + 2H(\beta)h + 4\psi h - \frac{1}{k2^{h+3}} - \frac{\psi}{2^{h+3}}$, then (2.5) is at most

$$2^{(1-\frac{1}{k})nh-\rho' n}.$$

Because $H(2\delta) < \frac{1}{3k^2 2^{h+3}}$ (from 2.2.1.10), and $H(\beta) < \frac{1}{6hk2^{h+3}}$ (from 2.2.1.12), and $\psi < \frac{1}{12hk2^{h+3}}$ (from 2.2.1.9), and n is large enough that $h\log(n)/n < \psi h$ (2.2.1.15) we see that $\rho' > 0$. Consequently, there are at most

$$|Forb^*(n-h,H)|2^{(1-\frac{1}{k})nh-\rho' n}$$

"bad" graphs in $Forb^*(n,H,\vartheta,\delta,\beta)$. By induction and by Lemma 2.2.6, the number of "bad" graphs is therefore at most

$$\begin{aligned}(1+C\sigma^{n-h})|\mathcal{Q}(n-h,H)|2^{(1-\frac{1}{k})nh-\rho' n} &\leq (1+C\sigma^{n-h})|\mathcal{Q}(n,H)|2^{(1-\frac{1}{k})\binom{h}{2}-\rho' n}\\ &\leq (1+C\sigma^{n-h})|\mathcal{Q}(n,H)|2^{-\frac{9}{10}\rho' n},\end{aligned}$$

for large enough n. Let $\tilde{\rho}=\frac{9}{10}\rho'$. □

We are now ready to complete the proof of the main theorem.

Proof of Theorem 2.1.10. From the triangle inequality,

$$\begin{aligned}|Forb^*(n,H)-\mathcal{Q}(n,H)| &\leq |Forb^*(n,H)-Forb^*(n,H,\vartheta)|+\\ &+ |Forb^*(n,H,\vartheta)-Forb^*(n,H,\vartheta,\delta)|+\\ &+ |Forb^*(n,H,\vartheta,\delta)-Forb^*(n,H,\vartheta,\delta,\beta)|+\\ &+ |Forb^*(n,H,\vartheta,\delta,\beta)-\mathcal{Q}(n,H)|.\end{aligned}$$

We have, however, found bounds on the size of each of these sets:

$$\begin{aligned}|Forb^*(n,H)-Forb^*(n,H,\vartheta)| &< 2^{(1-\frac{1}{k})\frac{n^2}{2}-\frac{\epsilon}{10}n^2} && \text{(Lemma 2.3.3)},\\ |Forb^*(n,H,\vartheta)-Forb^*(n,H,\vartheta,\delta)| &< 2^{(1-\frac{1}{k})\frac{n^2}{2}-\rho n^2} && \text{(Corollary 2.4.4)},\\ |Forb^*(n,H,\vartheta,\delta)-Forb^*(n,H,\vartheta,\delta,\beta)| &< (C\sigma^{n-1}+1)|\mathcal{Q}(n,H)|2^{-\hat{\rho}n} && \text{(Corollary 2.5.6)},\\ |Forb^*(n,H,\vartheta,\delta,\beta)-\mathcal{Q}(n,H)| &< (C\sigma^{n-h}+1)|\mathcal{Q}(n,H)|2^{-\tilde{\rho}n} && \text{(Lemma 2.6.2)}.\end{aligned}$$

It is obvious now that $|Forb^*(n,H)-\mathcal{Q}(n,H)|<4T$, where we T is the largest of the four bounds above. Recall that $|\mathcal{Q}(n,H)|\geq 2^{(1-\frac{1}{k})\binom{n}{2}}$. We may therefore fix n_0 large enough so that if $n>n_0$ then

$$\min\{|\mathcal{Q}(n,H)|2^{-\hat{\rho}n}, |\mathcal{Q}(n,H)|2^{-\tilde{\rho}n}\} > \max\{2^{(1-\frac{1}{k})\frac{n^2}{2}-\frac{\epsilon}{10}n^2}, 2^{(1-\frac{1}{k})\frac{n^2}{2}-\rho n^2}\}.$$

In that case, the first two bounds above are smaller than the second two. It now suffices to show that

$$(C\sigma^{n-1}+1)2^{-\hat{\rho}n} + (C\sigma^{n-h}+1)2^{-\tilde{\rho}n} \leq \frac{1}{2}C\sigma^n, \tag{2.6}$$

because then $|Forb^*(n,H) - \mathcal{Q}(n,H)| < C\sigma^n|\mathcal{Q}(n,H)|$.

We now fix $\sigma > \max\{2^{-\hat{\rho}}, 2^{-\tilde{\rho}}\}$ such that $\sigma < 1$, as in 2.2.1.14. This is possible, since both $2^{-\hat{\rho}}$ and $2^{-\tilde{\rho}}$ are smaller than 1. Since C is chosen in Section 2.2.1 to be large enough to make the statement of Theorem 2.1.10 true for $n \leq n_0$, we may assume that $C > 8$. Now, for $n > n_0$,

$$\begin{array}{rclcl} C\sigma^{n-1}2^{-\hat{\rho}n} & < & C\sigma^{n-1}\frac{1}{8}\sigma & = & \frac{1}{8}C\sigma^n \\ C\sigma^{n-h}2^{-\tilde{\rho}n} & < & C\sigma^{n-h}\frac{1}{8}\sigma^h & = & \frac{1}{8}C\sigma^n \\ 2^{-\hat{\rho}n} & < & \sigma^n & < & \frac{1}{8}C\sigma^n \\ 2^{-\tilde{\rho}n} & < & \sigma^n & < & \frac{1}{8}C\sigma^n, \end{array}$$

so (2.6) is true. This completes the proof of Theorem 2.1.10. □

Chapter 3

Online Ramsey Games for Triangles in Random Graphs

In the k-color online F-avoidance edge-coloring game, a graph on n vertices is generated by at each stage randomly adding a new edge. The player must color each new edge as it appears; her goal is to avoid a monochromatic copy of F. Let $N_0(F, k, n)$ be the threshold function for the number of edges that the player is asymptotically almost surely able to paint before she loses. Even when $F = K_3$, the order of magnitude of $N_0(F, k, n)$ is unknown for $k \geq 3$. In particular, the only known upper bound prior to the results in this chapter is the threshold function for the number of edges in the offline version of the problem. In the offline game, an entire random graph on n vertices with M edges is presented to the player to be k edge-colored. In this chapter, we improve the upper bound for the k-color online triangle-avoidance game.[1] This is the first result that separates the online threshold function from the offline bound for $k \geq 3$. This also supports a conjecture of Marciniszyn, Spöhel, and Steger [57] that the known lower bound is tight for cliques and cycles for all k.

3.1 Introduction

Hales and Jewett studied a class of combinatorial game that they called "positional games" [45]. A positional game is played between two players on a board, which is some set. Winning is determined by a family of subsets of the board set. The two players take turns claiming elements (according to the rules of the game), and the first player to acquire all elements of a subset in the winning family wins the game. If all elements are claimed before either

[1] This work appeared in Discrete Mathematics in 2010 [13].

player wins then the game is a draw. Tic-tac-toe is an example of a positional game, and the Hales-Jewett paper focused mainly on variations of tic-tac-toe.

Remark: There are also combinatorial games played on a graph in which each player only temporarily claims an element. Pursuit games such as the *cops and robber* game are an example [1]. In the cops and robber game, one player (the robber) can claim one vertex at a time while the other player (the cops) can claim some fixed number of vertices, with repetition. In each turn, the robber can move his claim to an vertex adjacent to his previous claim, while the cop can similarly move each of her claims to an adjacent vertex. The cop wins if she ever claims a vertex currently claimed by the robber. This game was recently studied by Bollobás, Kun, and Leader [22] in a random graph setting. The author has studied another pursuit game, the *revolutionaries and spies* game, which was suggested by Beck and studied by Howard and Smyth [46] and by Butterfield, Cranston, Puleo, Zamani, and West [27].

Beck [19] defines a Combinatorial Game to be a 2-player zero-sum game of skill (no chance moves) with complete information such that the game can end only in one of three possible states for each player: win, draw, lose. Many of the winning families in the above graph parameter games have a Ramsey flavor. Erdős and Selfridge considered a positional game that they actually called *the Ramsey game*, in which two players take turns claiming one edge at a time from K_n and the first player to claim all edges of a copy of K_k wins [38]. In the more general class of combinatorial games we find games in which the two players have different goals. For example, Player 1 could be trying to claim all edges of a copy of K_k while Player 2 tries to prevent Player 1 from succeeding. We can also consider games in which the two players have different types of moves available to them. Beck introduced one such game [18], which was further explored by Grytczuk, Hałuszczak, and Kierstead [44]. This game is played between two players, Builder and Painter. In each turn, Builder chooses a new edge from K_n and then Painter must color the edge with one of k colors. Painter loses when she has created a monochromatic copy of some fixed target graph H. If n is sufficiently large and Builder's moves are unrestricted, then classical Ramsey theory implies that Builder can win the above game by presenting a sufficiently large complete graph. We therefore restrict Builder's available moves by requiring that the underlying subgraph of K_n that he presents is always an element of a fixed graph family $\mathcal{H}$. This is the *k-color online Ramsey game* $(F, \mathcal{H})$.

Recall that the *Ramsey number* of a graph, $R(F)$, is the least n such that there exists a graph H having n vertices for which $H \to F$. More generally, $R(F;k) = \min\{|V(H)|\colon\ H \xrightarrow{k} F\}$. In the special case when F is the complete graph K_n we write $R(n;k)$ instead of $R(K_n;k)$ and $R(n)$ instead of $R(K_n;2)$. Let ρ be some graph parameter, such as maximum degree, and let

$$\mathcal{H} = \{G\colon\ \rho(G) \le s\}$$

for some fixed s. For example, $\mathcal{H}$ could be the family of graphs with maximum degree D. We can then consider the k-color online Ramsey game $(F, \mathcal{H})$, obtaining an online analogue of any parameter Ramsey number. Butterfield, Grauman, Kinnersley, Milans, Stocker, and West studied the online degree Ramsey number of trees and cycles [28].

Recall that Beck's definition requires a combinatorial game to be a game of skill, by which he means there are no chance moves [19]. In recent years, many classical problems in combinatorics have been considered in a sparse (i.e. random) setting. Babai, Simonovits, and Spenser began this study for graphs [8], by asking how many edges an F-free subgraph of the random graph can contain. Kohayakawa, Łuczak, and Rödl [51] studied arithmetic progressions in sparse sets, and so began the study for additive structures. Recent results of Conlon and Gowers [30] and Schacht [71] provide a general theory for attacking such problems, and sparse analogues of many classical theorems are now known. In particular, Conlon and Gowers [30] proved the following Ramsey-type result about the random hypergraph $G^r(n,p)$. An r-uniform hypergraph F is said to be *strictly r-balanced* if for every subgraph G of F it is the case that $\dfrac{e(F)-1}{v(F)-r} > \dfrac{e(G)-1}{v(G)-r}$.

Theorem 3.1.1 ([30])**.** *For every $k \in \mathbb{N}$ and any strictly r-balanced r-uniform hypergraph F, there exists a positive constant C such that if $p > Cn^{-1/m_r(F)}$ then the following is almost surely true. Every k-coloring of the edges of $G^r(n,p)$ has a monochromatic copy of F.*

Friedgut, Kohayakawa, Rödl, Ruciński, and Tetali [41] similarly studied a sparse version of the online Ramsey game. They defined a one-player game in which edges are presented one by one in an order chosen uniformly at random. The player must color each edge as it is presented, trying to avoid a monochromatic triangle. Marciniszyn, Spöhel, and Steger [57] generalized this game:

Definition 3.1.2. The *k-color F-avoidance edge-coloring game* is a one-player game in which edges of K_n are presented one by one in an order chosen uniformly at random. The player colors each edge as it appears with one of k colors. The player loses when she has created a monochromatic copy of the target graph F.

In these one-player coloring games, we call the player *Painter*. If n is sufficiently large, then Painter will certainly lose the F-avoidance edge-coloring game after enough steps. Let k be a fixed positive integer, and let n approach infinity. We will call $N_0(F,k,n)$ a *threshold function* for the H-avoidance game if for $N \ll N_0(F,k,n)$ there exists a strategy such that the player almost surely wins the game played with N edges by following the strategy, and for $N \gg N_0(F,k,n)$ the player almost surely loses the game played with N edges. In [41] it was proved that if F is a triangle then $N_0(F,2,n) = n^{4/3}$ and $N_0(F,3,n) \geq n^{7/5}$, although the authors commented that their proof of the latter inequality can be improved to give the lower bound $n^{13/9}$.

To find an upper bound for $N_0(F,k,n)$, we shall consider an offline version of the game. Recall that $G(n,M)$ is the random graph on n vertices having M edges; Painter must color the edges of $G(n,M)$ with k colors. The following theorem provides a threshold function for the offline game; because Painter has more information available to her when choosing colors in the offline game, this yields an upper bound for $N_0(F,k,n)$. Recall that $m_2(F) =$

$\max_{G \subseteq F} \frac{e(G)-1}{v(G)-2}$.

Theorem 3.1.3 ([55],[68],[69])**.** *Fix an integer $k \geq 2$ and a graph F that is not a star forest or, if $k = 2$, a forest of stars and paths with three edges. Let $\mathcal{P}$ be the graph property that any k-coloring of the edges of H results in a monochromatic copy of F. Then there exist constants c and C, depending only on F and k, such that*

$$\lim_{n\to\infty} \mathbb{P}[G(n,M) \in \mathcal{P}] = \begin{cases} 1 & \text{if } M > Cn^{2-1/m_2(F)}, \\ 0 & \text{if } M < cn^{2-1/m_2(F)}. \end{cases}$$

Marciniszyn, Spöhel, and Steger [57] proved that for every graph F and integer $k \geq 1$, the threshold $N_0(F,k,n)$ exists and

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{\log N_0(F,k,n)}{\log n} = 2 - \frac{1}{m_2(F)}. \tag{3.1}$$

In particular, the following is known for $F = K_3$.

Theorem 3.1.4. *If $k \geq 1$, then the k-color online K_3-avoidance edge-coloring game has a threshold function $N_0(K_3,k,n)$ that satisfies*

$$N_0(K_3,2,n) = n^{\frac{4}{3}}, \tag{3.2}$$

$$n^{\frac{3}{2}\left(1-\frac{1}{3^k}\right)} \leq N_0(K_3,k,n) \leq n^{\frac{3}{2}}. \tag{3.3}$$

The lower bound in (3.2) is from [57], and the upper bound is from [41]. The upper bound in (3.3) is from Theorem 3.1.3. To prove the lower bound in (3.3), Marciniszyn, Spöhel, and Steger [57] consider a greedy strategy: as each edge is presented Painter colors it with color i if and only if for every j with $i < j \leq k$ using color j would close a monochromatic triangle, but using color i does not. In fact, they prove a more general lower bound for a general graph F, using a variant of a greedy strategy in which for each color Painter avoids a particular subgraph of F. To state this strategy they consider the following parameter, which is related to $m_2(F)$.

$$\overline{m}_2^k(F) = \begin{cases} \max_{G \subseteq F} \dfrac{e(G)}{v(G)} & \text{if } k = 1, \\ \max_{G \subseteq F} \dfrac{e(G)}{v(G) - 2 + 1/\overline{m}_2^{k-1}(F)} & \text{if } k \geq 2. \end{cases} \tag{3.4}$$

Now, given a non-empty graph F, let $G_r(F)$ be a subgraph of F that achieves equality in the above definition. That is, $e(G_1(F))/v(G_1(F)) = \overline{m}_2^1(F)$, and if $k \geq 2$ then $\frac{e(G_k(F))}{v(G_k(F))-2+1/\overline{m}_2^{k-1}(F)} = \overline{m}_2^k(F)$. Painter will play according to the following strategy, which we will call the "cunning" greedy strategy. First she orders the colors arbitrarily, calling them $c_1, \ldots, c_k$. As each edge appears, among $\{c_1, \ldots, c_k\}$ Painter will use the greatest-indexed color, c_i, that does not close a monochromatic copy of $G_i(H)$. The greatest color is used, rather than the least, because Marciniszyn, Spöhel, and Steger [57] use the definition of $\overline{m}_2^k(F)$ to determine the number of edges that Painter can color by using this strategy. Painter surrenders the game when she is forced to create a copy of $G_1(F)$ that is monochromatic in color c_1. Because $G_1(F)$ is not necessarily isomorphic to F, Painter could continue playing until she is actually forced to create a monochromatic copy of F, but it turns out that she will have colored asymptotically the same number of edges as in this strategy. The cunning greedy strategy results in the following bound on $N_0(F, k, n)$.

Theorem 3.1.5 ([57])**.** *Let F be a graph that is not a forest. For $k \geq 1$, the threshold for the k-color online F-avoidance edge-coloring game satisfies*

$$N_0(F, k, n) \geq n^{2 - \frac{1}{\overline{m}_2^k(F)}}.$$

In particular, if F is not a forest and $m_2(F) = (e(F) - 1)/(v(F) - 2)$, then

$$N_0(F, k, n) \geq n^{(2-1/m_2(F))\left(1-e(F)^{-k}\right)}.$$

Note that $\overline{m}_2^k(F)$ approaches $m_2(F)$ as k approaches infinity, consistent with Equation (3.1). If $m_2(F) = (e(F) - 1)/(v(F) - 2)$ then we call F *strictly balanced* and the above strategy can be simplified. Both cliques and cycles are strictly balanced. In a separate paper, Marciniszyn, Spöhel, and Steger [58] proved that if $k = 2$ then the bound in Theorem 3.1.5 is tight for F in a large class of graphs, which includes K_ℓ and C_ℓ for every $\ell \geq 2$.

Theorem 3.1.6. *Let F be a graph that is not a forest. If F has a subgraph F' with $e(F) - 1$ edges such that*

$$m_2(F') \leq \overline{m}_2^2(F),$$

then the threshold for the online F-avoidance edge-coloring game with 2 colors satisfies

$$N_0(F, 2, n) = n^{2-1/\overline{m}_2^2(F)}.$$

In particular, when $F = K_3$ Theorem 3.1.6 yields the bounds in Equations (3.2) and (3.3). The cunning greedy strategy is not optimal for every graph, however. For example, if $k = 2$ and F is the graph formed by two triangles sharing one vertex, then this strategy provides the bound $N_0(F, 2, n) \geq n^{25/18}$, while a different strategy improves it to $N_0(F, 2, n) \geq n^{17/12}$. In [57] it was conjectured that the above greedy strategy is optimal for K_3 and any number of colors. In fact, Marciniszyn, Spöhel, and Steger [58] conjecture that it is optimal for any clique and any cycle.

Conjecture 3.1.7. *For $\ell \geq 2$ and $k \geq 1$, the threshold for the k-color online K_ℓ-avoidance edge-coloring game is*

$$N_0(K_\ell, k, n) = n^{\left(2-\frac{2}{\ell+1}\right)\left(1-\binom{\ell}{2}^{-k}\right)}.$$

If $\ell \geq 3$, then in addition the threshold for the k-color online C_ℓ-avoidance edge-coloring game is

$$N_0(C_\ell, k, n) = n^{\left(2-\frac{\ell-2}{\ell-1}\right)\left(1-\ell^{-k}\right)}$$

Recall the two-player online Ramsey game $(F, \mathcal{H})$ of Grytczuk, Hałuszczak, and Kierstead [44]. Belfrage, Mütze, and Spöhel [20] connected the two-player online game with the one-player game. Let $\mathcal{H}_d$ denote the family of graphs for which every subgraph G satisfies $e(G)/v(G) \leq d$.

Theorem 3.1.8 ([20])**.** *Let F be a graph with at least one edge, and let $k \geq 2$. If $d > 0$ is a real number such that Builder has a winning strategy in the k-color online Ramsey game $(F, \mathcal{H}_d)$, then the threshold for the k-color F-avoidance edge-coloring game satisfies*

$$N_0(F, k, n) \leq n^{2-1/d}.$$

The density-constrained Builder is not equivalent to the random graph in the one-player game. However, Belfrage, Mütze, and Spöhel [20] prove that with high probability the random graph will "accidentally" play an optimal density-constrained Builder strategy. This is partially because low-density graphs are likely to appear in the random graphs, and with some additional work it can be proved that the low-density graph appears *in the correct order* to respond to Painter's color choices.

Remark: When F is a forest, Belfrage, Mütze, and Spöhel [20] proved that $N_0(F, k, n)$ is in fact equal to n^{2-1/d_0}, where d_0 is the least d such that Builder can win the k-color online Ramsey game $(F, \mathcal{H}_d)$. In general, however, it is not known whether the bound in Theorem 3.1.8 is tight in the edge-coloring game. Recently, Mütze, Rast, and Spöhel [59] proved a theorem analogous to Theorem 3.1.8 for a vertex-coloring version of the one-player game. In the vertex-coloring game, the vertices of $G(n, p)$ are revealed one-at-a-time along with the edges incident with already-revealed vertices. The player must color each vertex as it appears with one of k available colors, and she loses if she creates a monochromatic copy of some fixed target graph F. We then seek to determine the threshold, $p_0(F, k, n)$, for the value of p for which the player can win the vertex-coloring game. The vertex-coloring game is in general more approachable than the edge-coloring game, and Mütze, Rast, and

Spöhel [59] completely determined $p_0(F, k, n)$. Let $m_1^*(F, k)$ be the infimum over all d such that Builder has a winning strategy in the k-color online vertex-coloring game with density restriction d. Then Mütze, Rast, and Spöhel [59] proved that for any graph F with at least one edge and any fixed integer k with $k \geq 2$, the threshold $p_0(F, k, n)$ is equal to $n^{-1/m_1^*(F,k)}$. A similar result in the edge-coloring game would be very interesting.

In our main theorem, we use Theorem 3.1.8 to improve the upper bound from Theorem 3.1.4 in the K_3-avoidance edge-coloring game for any number of colors. This is the first result that separates $N_0(K_3, k, n)$ from the offline bound provided by Theorem 3.1.3 for $k \geq 3$. While there is still a gap between our upper bound and the lower bound in (3.3), this result supports Conjecture 3.1.7.

Theorem 3.1.9. *For $k \geq 3$, there exists a positive constant c_k such that*

$$N_0(K_3, k, n) \leq n^{\frac{3}{2} - c_k}.$$

For $k = 3$, we present a strategy for Builder in the online Ramsey game $(K_3, \mathcal{H}_{42/22})$ with three colors. Because c_3 from Theorem 3.1.9 is strictly greater than $42/22$, this results in an upper bound that is smaller than $n^{\frac{3}{2} - c_3}$. It is possible that Builder has no winning strategy in the online Ramsey game $(K_3, \mathcal{H}_d)$ with three colors for $d < 42/22$.

Theorem 3.1.

$$n^{\frac{3}{2} - \frac{1}{18}} \leq N_0(K_3, 3, n) \leq n^{\frac{3}{2} - \frac{1}{42}}.$$

The lower bound in Theorem 3.1 follows from (3.3) with $k = 3$.

We prove Theorem 3.1 in Section 3.2 and Theorem 3.1.9 in Section 3.3.

3.2 The game with three colors

We prove Theorem 3.1 by providing a strategy for Builder in the 3-color online Ramsey game $(K_3, \mathcal{H}_{42/22})$, which with Theorem 3.1.8 yields an upper bound of $n^{3/2-1/42}$.

Proof. First we present the strategy; we will check later that it satisfies the density restriction.

Algorithm:

- Phase I:
  - Step 1: Builder plays the edges of a star with center x and 25 leaves, and allows Painter to color the edges. There will be at least nine edges in one color, say blue. Label the non-x endpoints of nine of them $y_1, \ldots, y_9$.
  - Step 2: For each i, Builder gives 13 children to y_i, using new vertices, and lets Painter paint those edges. For each i, there is some 'majority color' such that y_i has at least five children whose edges to y_i receive the majority color. If there exist $y_{i_1}, y_{i_2}, y_{i_3}$ having the same majority color other than blue, then move to Phase III.
  - Step 3: If this step is reached, at most four of $\{y_1, \ldots, y_9\}$ do not have majority color blue, so there are five whose majority color (on edges to children) is blue. Without loss of generality, assume they are y_1, y_2, y_3, y_4, y_5. Move to Phase II.
- Phase II: Set $j = 1$.
  - Step 1: For $1 \leq j \leq 5$, Builder gives 13 children, using new vertices, to each of five children of y_j whose edges to y_j are blue. If some child of y_j has majority color blue, call that child z_j (if there is more than one such child of y_j, choose one arbitrarily). If for any $1 \leq j \leq 5$ there is no such z_j then there are three children

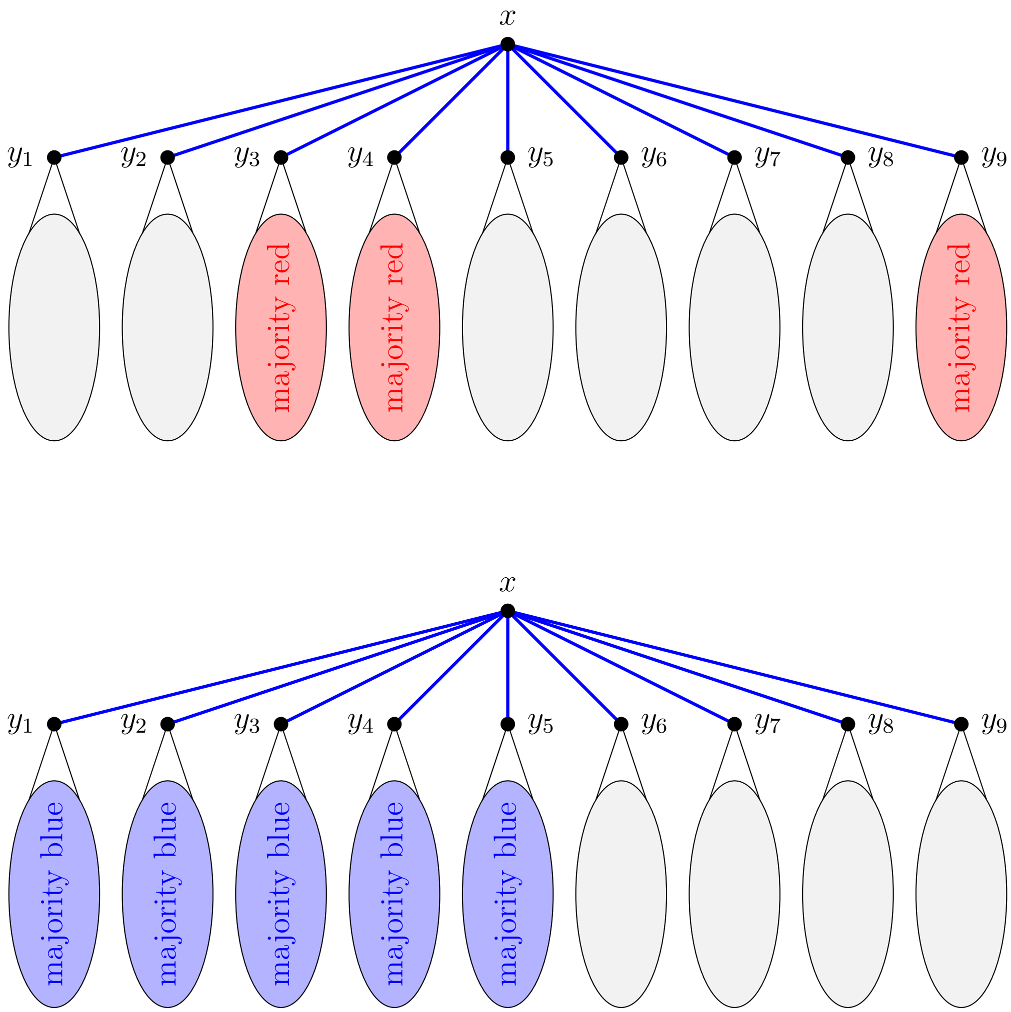

Figure 3.1: Phase I, step 3 of the 3-color algorithm; edges from x are blue and each y_i has 13 children.

of y_j whose majority color is red, or three whose majority color is green. In that case, move to Phase III.

– Step 2: If this step is reached, then Builder adds the edges $\{xz_j\}_{j=1}^5$ and lets Painter paint them. If any of those edges is painted blue, then $\{x, y_j, z_j\}$ forms a blue triangle. Consequently, at least three of the xz_j edges must have the same color, red or green. Move to Phase III.

- Phase III: When this phase is reached, there is some rooted tree with root r that has three children c_1, c_2, c_3, each with five children $\{a_{i,j}\}_{j=1}^5$ for $1 \leq i \leq 3$. Additionally,

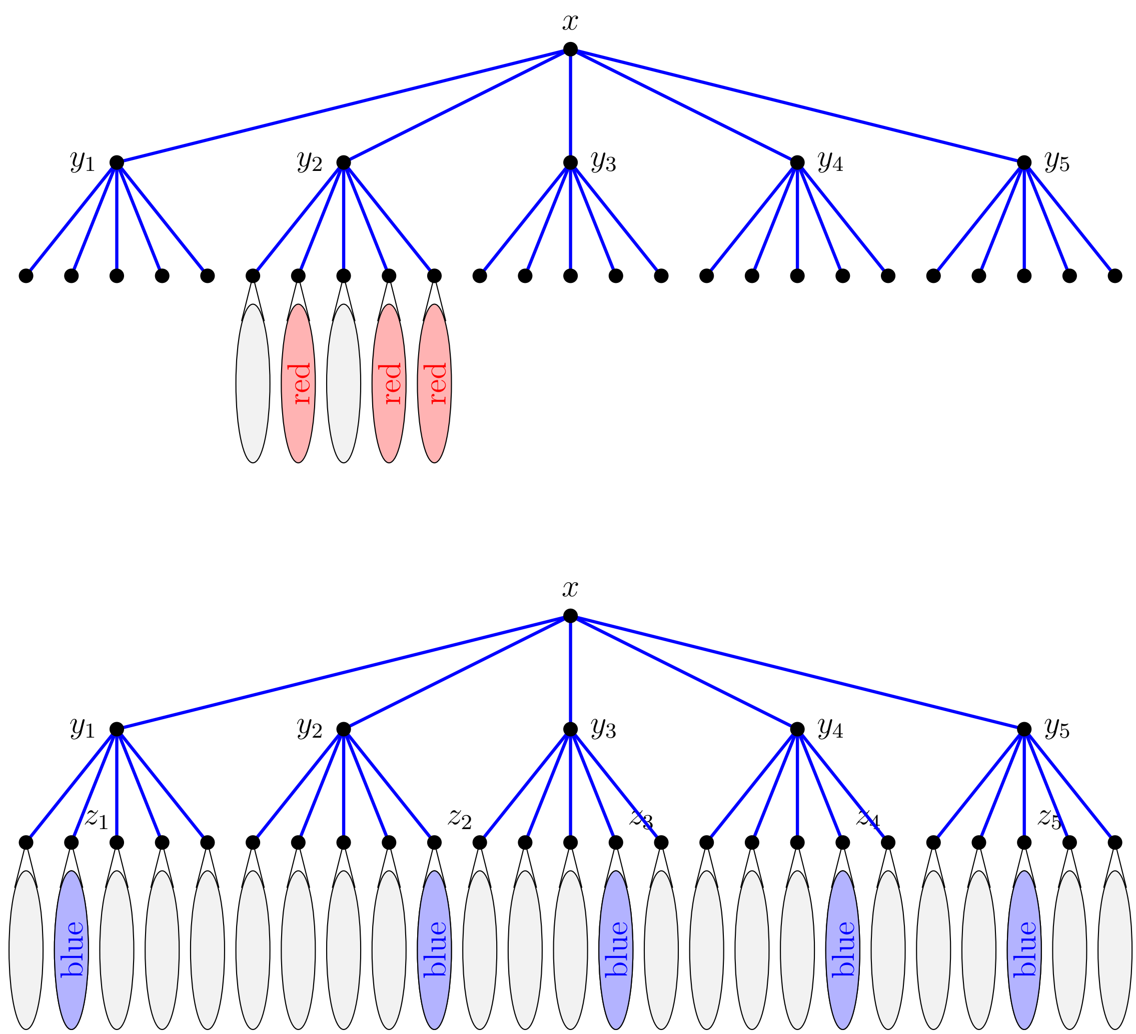

Figure 3.2: Phase II, step 1 of the 3-color algorithm. Edges from x are blue and each y_i has at least 5 children via blue edges. Pictured are the two possible outcomes.

edges of the form rc_i are all in one color (say red) and edges of the form $c_i a_{i,j}$ for any i, j are all in another color (say blue).

- Step 1: Builder adds the edges $\{c_1 a_{3,j}\}_{j=1}^5$, $\{c_2 a_{1,j}\}_{j=1}^5$, and $\{c_3 a_{2,j}\}_{j=1}^5$. This connects c_1 to each child of c_3, c_2 to each child of c_1, and c_3 to each child of c_2. If there is a blue edge from c_1 to a child of c_3, as well as a blue edge from c_3 to a child of c_2 and a blue edge from c_2 to a child of c_1, then move to step 2. Otherwise move to Phase IV.

- Step 2: If this step is reached, there is a blue cycle of the form $c_1 a_{1,j_1} c_2 a_{2,j_2} c_3 a_{3,j_3} c_1$.

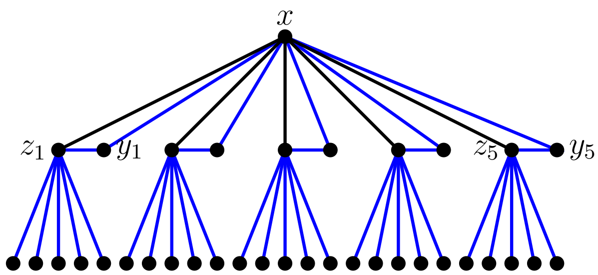

Figure 3.3: Phase II, step 2 of the 3-color algorithm; children of y_i other than z_i not pictured, for simplicity.

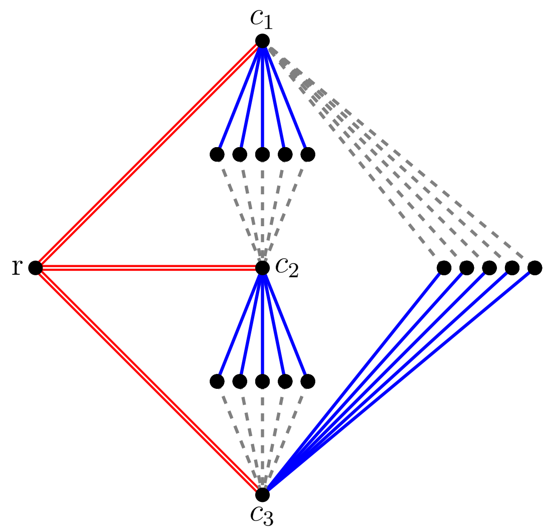

Figure 3.4: Phase III, step 1 of the 3-color algorithm; edges from r to c_1, c_2, c_3 are blue while edges from c_i to its children are red. Remaining edges are not yet colored.

In this case, Builder presents the edges c_1c_2, c_2c_3, c_1c_3. Painter now cannot avoid making a monochromatic triangle.

- Phase IV: If this phase is reached, then there exists i and j with $i \neq j$ and three vertices that are connected to c_i by blue edges and to c_j by red (or green) edges. In this case, Builder presents the edges of the triangle on those three vertices, and Painter now cannot avoid making a monochromatic triangle.

It remains to check that this strategy is permissible in the online Ramsey game $(K_3, \mathcal{H}_{42/22})$ with three colors. Let H be the graph at the end of the game. Among all densest subgraphs of H, let G be chosen to be inclusion-minimal. Obviously G is connected, otherwise some

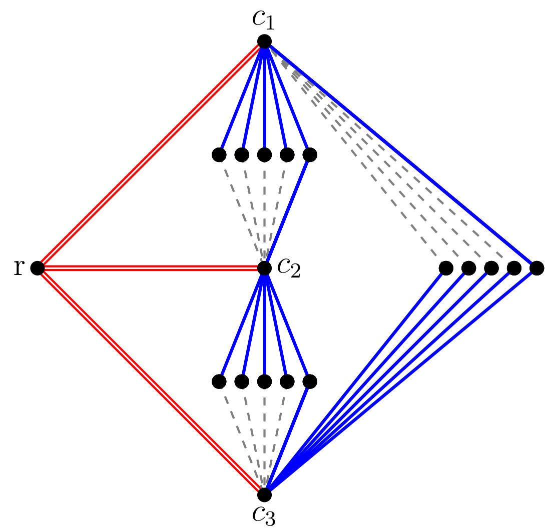

Figure 3.5: Beginning of Phase III, step 2 of the 3-color algorithm; a blue cycle is formed.

component of G has density at least as high, contradicting the minimality of G. With some case analysis it can be checked that H contains no subgraph G with $e(G)/v(G) \geq 2$; the idea is that if we iteratively remove a vertex with degree at most 2 then we obtain a single edge, which is a graph with density less than 2. If G is a forest then its density is strictly less than 1, which is less than $42/22$. If G_1 is a connected graph that is not a tree and G_1' is obtained from G_1 by adding a pendant edge, then $e(G_1')/v(G_1') = (e(G_1)+1)/(v(G_1)+1) \leq e(G_1)/v(G_1)$. Consequently, G contains no pendant edge.

On the other hand, if G_2 is a connected graph that is not a tree and G_2' is obtained from G_2 by adding a vertex that is connected to two distinct vertices in G_2, then $e(G_2')/v(G_2') = (e(G_2)+2)/(v(G_2)+1)$. If $e(G_2)/v(G_2) < 2$, then $(e(G_2)+2)/(v(G_2)+1) > e(G_2)/v(G_2)$. Consequently, if G is an inclusion-minimal densest subgraph of H, then every vertex outside of G has at most one neighbor in G.

If Phase II terminates before step 2, then the tree in Phase III has root y_j for some $1 \leq j \leq 5$. Now, step 1 of Phase III creates 15 vertices of degree 2 whose neighborhoods are connected. Consequently, G is densest if either Phase IV is never reached or Phase IV is reached at the end of step 1. In either case, G is a subgraph of one of the graphs in

Figure 3.6. It has 19 vertices: the root and its three children, and 15 vertices of degree 2. It has 36 edges: three from the root, three in the final triangle, and 30 from the vertices of degree 2. With some work, using the fact that if $x \notin V(G)$ then x has at most one neighbor in G, one can check that both graphs in Figure 3.6 are strictly balanced. The density of G is therefore at most $36/19$, which is less than $42/22$.

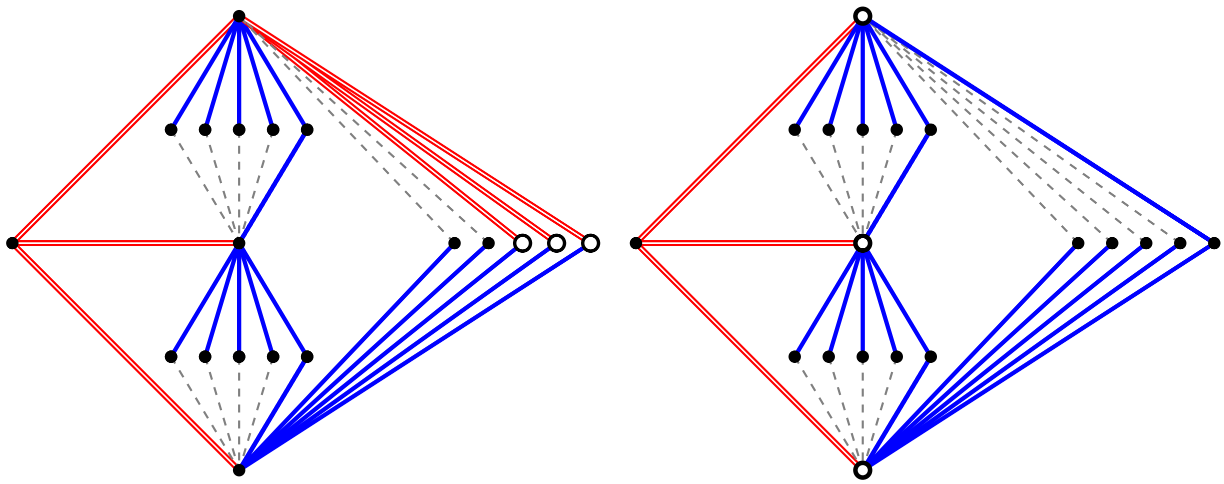

Figure 3.6: Larger vertices indicate position of final triangle.

If on the other hand Phase II reaches step 2, then the tree in Phase III has root x and children from $\{z_j\}_{j=1}^5$. Again, a densest subgraph F will occur if either Phase IV is never reached or Phase IV is reached at the end of step 1. This time, however, for each z_j that is a child in the tree there is a vertex y_j connected to both x and z_j. Note that if $e(G)/v(G) \leq 3/2$, then the density restriction is satisfied. If on the other hand $e(G)/v(G) > 3/2$, then there is no triangle in H having exactly one vertex in $V(G)$, as removing such a triangle would increase the density: $(e(G)-3)/(v(G)-2) > e(G)/v(G)$. Consequently, G is a subgraph of one of the graphs in Figure 3.7, both of which are strictly balanced.
A densest subgraph in this case therefore has density at most $42/22$. □

The following lemma is due to Grytczuk, Hałuszczak, and Kierstead [44]; they proved it only for the two-color game, but their proof easily generalizes to the k-color game.

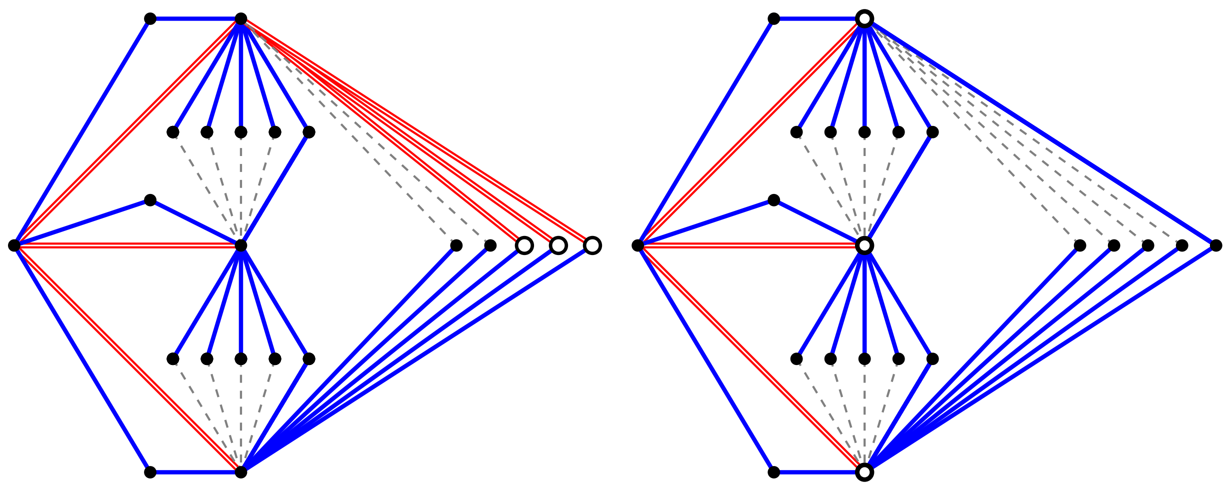

Figure 3.7: Larger vertices indicate position of final triangle.

Lemma 3.2.1. *If $\mathcal{F}$ is the family of all forests and $F \in \mathcal{F}$, then there exists a strategy for Builder to win the k-color online Ramsey game $(F, \mathcal{F})$.*

Proof. It suffices to prove the claim for trees. We will prove the claim using induction on the number of vertices in the tree and the number of colors available. The claim is obviously true for any number of colors and the only tree with two vertices, K_2: Builder need only present a star with $k+1$ edges. The claim is also obviously true for one color and any tree: Builder simply presents the tree, and Painter must make it monochromatic in the one available color. Suppose that T be a tree with n vertices, and let x be a leaf. Let y be the neighbor of x. Let T' be the tree formed from T by deleting x. By induction, Builder has a winning $(T', \mathcal{F})$ strategy in the k-color game. By induction, Builder also has a winning $(T, \mathcal{F})$ strategy in the $k-1$-color game.

By the pigeonhole principle, Builder can therefore force Painter to produce m disjoint copies of T', for any m, that are monochromatic in the same color, say red. Say the copies are $T'_1, T'_2, \ldots, T'_m$, and let y'_i be the vertex in T'_i that corresponds to y. Let m be the number of vertices needed in Builder's winning $(T, \mathcal{F})$ strategy with $k-1$ colors. Builder can play this strategy on the vertices $y'_1, \ldots, y'_m$. If Painter makes any of these edges red, then Builder wins. If Painter avoids using red, then Builder wins by using his $k-1$ strategy to force a monochromatic copy of T. Builder obviously introduced no cycles in this process, since he uses only one vertex from each tree T_i'. $\square$

An implication of Lemma 3.2.1 is that Builder can win the k-color online Ramsey game $(F, \mathcal{H}_1)$ when F is a forest, because every forest has density less than 1. This can be used to provide an alternative proof that Builder can win the 3-color online Ramsey game $(K_3, \mathcal{H}_{42/22})$: Builder can force a monochromatic tree with root x such that x has five children $y_1, \ldots, y_5$, each of which has one child which itself has five children. This is exactly the tree that our strategy requires for Phase II, step 2; Builder can therefore proceed with our strategy, starting with Phase II, step 2.

3.3 The game with many colors

We prove Theorem 3.1.9 by providing a strategy for Builder in the k-color online Ramsey game $(K_3, \mathcal{H}_d)$, for some $d < 2$. The strategy is along the lines of a method from [44].

Proof. Let S_i be Builder's strategy to force a triangle in the two-player game with i colors, and let m_i be the number of vertices Builder needs for strategy S_i. The strategy S_1 is obvious, and $m_1 = 3$. We will define S_k recursively in terms of S_{k-1}. Builder begins with two large sets of vertices, X_1 and Y_1 (their size will be determined by what follows). In Phase 1, Builder will place edges only between X_1 and Y_1.

For each step in Phase 1, Builder chooses $k+1$ vertices in Y_1 and one vertex in X_1 and presents the $k+1$ edges between them. Painter must paint at least two of these r edges with the same color; Builder will discard and never reuse the $k-1$ vertices from Y_1 that are not painted with the majority color (if more than two edges get the same color, Builder still discards $k-1$ vertices). Discarded vertices will never be used again; vertices from X_1 will also never be reused.

Builder maintains an auxiliary edge-colored graph $Aux(Y_1)$ whose vertex set is Y_1 and whose edges are the pairs of vertices kept at each stage, and whose color is the color of their edges to X_1. For example, if Builder's first move is to place the edges $xy_1, \dots, xy_{k+1}$ and Painter paints xy_1 and xy_2 red then Builder will discard $\{y_3, \dots y_{k+1}\}$ and will add the red edge y_1y_2 to his auxiliary graph. Builder does not present any edge from his auxiliary graph, and so these edges do not contribute to density. If yy' is a red edge in the auxiliary graph then there exists a vertex $x \in X_1$ such that xy and xy' are both red, so if Builder were to present the edge yy' then Painter would not be able to paint it red without creating a monochromatic triangle.

Claim 1: Builder can force an arbitrarily large, not necessarily monochromatic, star in the auxiliary graph.

Proof. This follows from induction on the number of leaves in the star. Builder can force a star with one leaf because he forces an edge in each step. Suppose he can force a star with $s-1$ leaves. He can therefore also force $k+1$ disjoint stars, each with $s-1$ leaves, by playing this star-forcing strategy repeatedly. Suppose the centers of these stars are $y_1, y_2, \dots, y_{r+1}$; Builder can then choose a new vertex $x \in X_1$ and present the edges $\{xy_1, \dots, xy_{k+1}\}$. At least two will be given the same color, say xy_1 and xy_2, and so Builder adds y_1y_2 to his auxiliary graph. Adding this edge to the star with center y_1 results in a star with s leaves. □

Claim 2: Builder can force an arbitrarily large, not necessarily monochromatic, clique in the auxiliary graph.

Proof. This follows from Claim 1 by induction on the number of vertices in the clique. We already verified that Builder can force K_2, which is a star with one edge. Suppose Builder can force K_{m-1}, and suppose his strategy to do so involves s_{m-1} vertices from Y_1. By Claim 1, Builder can force a star with s_{m-1} leaves. He can then play his strategy to force K_{m-1} on the leaves of the star, resulting in a copy of K_m. □

Let m be the Ramsey number $R_k(m_{k-1})$; by Claim 2 Builder can force a clique on m vertices in the $Aux(Y_1)$. This clique will contain a monochromatic (say, red) copy of $K_{m_{k-1}}$; let Y_1' be the vertices of this red $K_{m_{k-1}}$. See Figure 3.8. If Builder presents any edge among these vertices Painter may not color it red without creating a red triangle.

Builder may therefore play strategy S_{k-1} on these m_{k-1} vertices. Strategy S_{k-1} consists of $k-1$ phases, which will be Phases 2 through k in strategy S_k. He begins by splitting these m_{k-1} vertices into two sets, Y_2 and X_2, each sufficiently large for what follows; this is the beginning of Phase 2. At the end of Phase 2, Builder will have obtained a set of m_{k-2} vertices in Y_2 such that Painter can make no edge between them, say, blue (and Painter can still not make them red). Builder may therefore play strategy S_{k-2} among these m_{k-2} vertices; this is Phase 3. By the end of Phase $k-1$, Builder will have obtained a set of $m_1 = 3$ vertices in Y_{k-1} such that Painter can only make edges between them, say, green. These three vertices are Y_k, and X_k is an empty set. Builder therefore wins if he presents the three edges spanned by Y_k; this is Phase k. Phases 1 through k together form strategy S_k. Notice that $X_j, Y_j \subseteq Y_{j-1}$ for all $2 \leq j \leq k$ and that $X_j \cap Y_j = \emptyset$ for all $j \in [k]$.

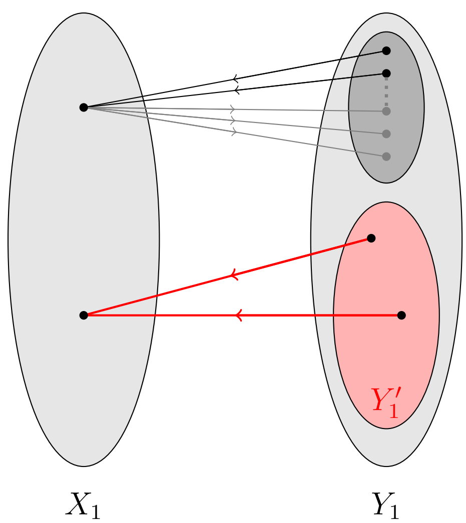

Figure 3.8: The k-color algorithm. Every vertex in X_1 has $k+1$ neighbours in Y_1, and two of the edges share a color. Every pair of vertices in Y_1' has a common neighbour in X_1 via red edges.

It remains to show that throughout the course of the game Builder never creates a graph with density 2 or higher. Note that the game, and therefore the graph, is finite. If $1 \le j \le k-1$ then in Phase j of the game Builder places edges only between X_j and Y_j. For the purpose of analysis, we will orient these edges in the following way. If $x \in X_j$ is used in phase j then x has exactly $k+1$ neighbours in Y_j, because x is used only once. Of these, $k-1$ are discarded for having minority colors and never used again; orient those edges from X_j to Y_j and orient the remaining two edges from Y_j to X_j (see Figure 3.8). If $y \in Y_j$ and y has non-zero in-degree then y is discarded at some point during Phase j. At the end of Phase j, therefore, vertices in X_j have in-degree 2 or 0 and vertices in Y_j have in-degree 1 or 0. Moreover, if $y \in Y_j$ has in-degree 1 then it is never used again in any phase.

Because X_j and Y_j are chosen from non-discarded vertices of Y_{j-1}, at the beginning of phase j each vertex in $X_j \cup Y_j$ has zero in-degree. By the end of Phase $k-1$, therefore, every vertex in the graph has in-degree at most 2, and vertices in Y_k have in-degree 0. Phase k consists of placing the edges of a triangle in Y_k; orient these to be a directed cycle. Now vertices in Y_k have in-degree exactly one.

If there is a subgraph F in the final graph whose average in-degree is at least 2, then every vertex in F has in-degree exactly 2. This implies that F contains an oriented cycle. The only oriented cycle in the graph is the final triangle, however, and these vertices have in-degree 1, which is a contradiction. Consequently, $e(F) < 2v(F)$, as desired.

□

3.4 Open problems

The strategy we present in Section 3.3 is unlikely to be optimal, which is why we do not compute the exact density. An improved strategy would yield a better bound on $N_0(K_3, k, n)$, but it is as yet unknown whether the upper bounds on $N_0(F, k, n)$ obtained from Theorem 3.1.8 are optimal.

It is also important to determine the least d for which there is a winning strategy for

Builder in the 3-color online Ramsey game $(K_3, \mathcal{H}_d)$. It is possible that the strategy we present in the proof of Theorem 3.1 is optimal, in the sense that $d = 42/22$ is the least d for which Builder can win. If this is the case, then either $N_0(K_3, 3, n) = n^{\frac{3}{2}-\frac{1}{42}}$ (in other words, the bound in Theorem 3.1 is tight), or the bounds given by Theorem 3.1.8 are not always tight. In the former case, this would disprove Conjecture 3.1.7. In the latter case, this would prove that the edge-coloring game is different from the vertex-coloring game in a significant way, since the vertex-coloring version of Theorem 3.1.8 is known to be tight [59].

Let $\mathcal{S}_k$ be the family of graphs whose maximum degree is at most k. Of course graphs with low maximum degree must have low density, and so it is natural to wonder whether Builder strategies in the 3-color online Ramsey game $(K_3, \mathcal{S}_r)$ may help to produce Builder strategies in the $(K_3, \mathcal{H}_d)$ game. Butterfield, Grauman, Kinnersley, Milans, Stocker, and West [28] considered the 2-color online Ramsey game $(H, \mathcal{S}_k)$. For triangles in particular, they proved that there is a strategy for Builder to win the $(K_3, \mathcal{S}_4)$ game but that for $k < 4$ Painter can always win the $(K_3, \mathcal{S}_k)$ game played with 2 colors. The graphs formed by this strategy, however, [28] have rather high density. Belfrage, Mütze, Spöhel [20] present a strategy for Builder in the $(K_3, \mathcal{H}_{3/2})$ game, but their strategy produces a graph with maximum degree 6 (not 4). It is therefore unlikely that good Builder strategies in the $(K_3, \mathcal{S}_r)$ game with 3 colors will lead to good Builder strategies in the $(K_3, \mathcal{H}_d)$ game with 3 colors.

Chapter 4

Extremal F_5-free subgraphs of $G^3(n,p)$

4.1 Introduction

Recall that $\text{ex}(n, H)$ is the maximum number of edges a graph in $Forb(n, H)$ can contain. Alternatively, this can be expressed as the maximum number of edges a subgraph of K_n can have without containing a copy of H. If $\chi(H) = k + 1$, then no k-partite subgraph of K_n can contain a copy of H. More generally, we can consider any graph on n vertices, not only K_n. We then ask: For fixed graphs G and H, where $\chi(H) = k + 1$, does the largest (with respect to number of edges) k-partite subgraph of G have the same number of edges as the largest subgraph of G that does not contain H?

As in Chapter 3, we will consider a sparse version of this question. Babai, Simonovits, and Spencer [8] considered the following sparse problem. For a fixed H with $\chi(H) = 3$, let $F(n, p; H)$ denote the largest (with respect to number of edges) subgraph of $G(n, p)$ that does not contain H, and let $B(n, p)$ denote the largest bipartite subgraph of $G(n, p)$. With $F = F(n, p; H)$ and $B = B(n, p)$, we have $|E(F)| \geq |E(B)|$, since H is not a subgraph of B. Babai, Simonovits, and Spencer [8] proved that for any 3-chromatic H and any fixed $p \in (0, 1)$,

$$|E(B)| \leq |E(F)| \leq |E(B)| + 2\text{ex}(n, \mathcal{S}) + O(n),$$

where $\mathcal{S}$ is the family of all induced subgraphs of H whose deletion results in an independent set. In the special case of $H = K_3$, the family $\mathcal{S}$ is simply $\{K_2\}$, and they were able to prove the more precise result that if H is the triangle and $p = 1/2$ then in fact the largest H-free subgraph of $G(n, 1/2)$ almost surely has the same number of edges as the largest bipartite subgraph, and in fact is almost surely bipartite itself. This is one of the earliest examples of a sparse version of a classical result.

Brightwell, Panagiotou, and Steger [25] improved the result of Babai, Simonovits, and Spencer [8], proving that there exists a constant $c > 0$ such that if $p \geq n^{-c}$ then the largest K_ℓ-free subgraph of $G(n, p)$ is almost surely $(\ell - 1)$-partite. DeMarco and Kahn [31] proved a precise result when $\ell = 3$:

Theorem 4.1.1 ([31])**.** *There exists $C > 0$ such that if $p > Cn^{-1/2} \log^{1/2}(n)$, then the largest K_3-free subgraph of $G(n, p)$ is almost surely bipartite.*

Moreover, this threshold on p is best possible up to the choice of C. Such precise statements are only known for K_3-free graphs: for K_ℓ, when $\ell \geq 4$, the problem is not completely solved.

Say that a graph G is (H, ϵ)-Turán if every subgraph of G that has at least

$$\left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) e(G)$$

edges contains a copy of H. Conlon and Gowers [30] proved an asymptotic result about the random graph $G(n, p)$, partially settling a conjecture of Kohayakawa, Łuczak, and Rödl [52]. Recall that $m_2(H) = \max_{F \subseteq H} \frac{e(F)-1}{v(F)-2}$ and that a graph is *strictly* 2*-balanced* if $m_2(H) = \frac{e(H)-1}{v(H)-2}$.

Theorem 4.1.2 ([30])**.** *Given $\epsilon > 0$ and a strictly 2-balanced graph H, there exists a positive constant C such that if $p > Cn^{-1/m_2(H)}$ then*

$$\lim_{n \to \infty} \mathbb{P}[G(n, p) \text{ is } (H, \epsilon)\text{-Turán}] = 1.$$

We continue the study of the sparse problem of Babai, Simonovits, and Spencer by considering hypergraphs. The Turán hypergraph $T_k^r(n)$ is the complete n-vertex r-uniform

k-partite hypergraph whose partite sets are as equally-sized as possible.

Definition 4.1.3. For $n \in \mathbb{Z}$ and $p \in [0,1]$, let $G^r(n,p)$ be a random r-uniform hypergraph with n vertices and probability p. That is, let each element of $\binom{[n]}{r}$ be an edge with probability p.

In particular, $G^2(n,p) = G(n,p)$. Let F_5 be the 3-uniform hypergraph on 5 vertices whose edges are $v_1v_2v_3$, $v_1v_4v_5$, and $v_2v_4v_5$. The vertices of F_5 cannot be partitioned into two strongly independent sets (that is, sets that contain at most one vertex from each edge), but can be partitioned into three strongly independent sets.

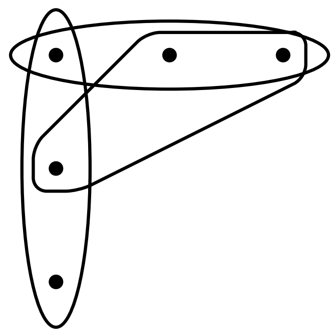

Figure 4.1: The 3-uniform hypergraph F_5

A 3-uniform hypergraph is said to be *cancellative* if for any three edges A, B, C in the hypergraph, $A \cup B = A \cup C$ implies that $B = C$. Equivalently, a 3-uniform hypergraph is cancellative if it contains no copy of F_5 or of the 4-vertex hypergraph with edge set $\{\{1,2,3\},\{1,2,4\},\{2,3,4\}\}$. Bollobás [21] showed that the most edges in a cancellative graph on n vertices is $s(n)$, where

$$s(n) = \lfloor n/3 \rfloor \lfloor (n+1)/3 \rfloor \lfloor (n+2)/3 \rfloor = n^3/27 + O(n^2).$$

Frankl and Füredi [40], however, strengthened this result by proving that $\mathrm{ex}(n, F_5) = s(n)$ if $n > 3000$ (this was improved to $n > 33$ by Keevash and Mubayi [49] and to $n \geq 1$ by Goldwasser [42]). The Frankl and Füredi [40] result implies that the largest hypergraph in $Forb(n, F_5)$ has the same number of edges as the largest cancellative hypergraph. The hypergraph F_5 is therefore a natural hypergraph to consider, particularly because it is one way to generalize K_3 to a 3-uniform hypergraph, and most of the precise sparse statements for graphs are about K_3.

Our main result is that for appropriately chosen p it is almost surely true that the largest subgraph of $G^3(n,p)$ that does not contain F_5 is 3-partite.

Theorem 4.1.4. *There exists a constant C such that if $p \geq C\frac{\omega(n)}{\sqrt{n}}$ for any $\omega(n)$ such that $\lim_{n\to\infty} \omega(n) = \infty$, then almost surely the following is true. Every subgraph of $G^3(n,p)$ that has the most edges among all subgraphs that do not contain F_5 is 3-partite.*

The threshold in Theorem 4.1.4 is a consequence of our proof techniques and is not likely to be optimal; the optimal threshold for p should be $C\log(n)/n$

The hypergraph F_5 is an example of what Balogh, Butterfield, Hu, Lenz, and Mubayi [15] call a "critical hypergraph"; they proved that if H is a critical hypergraph then for sufficiently large n the unique hypergraph in $Forb(n,H)$ with the most edges is the Turán hypergraph. We could prove results analogous to Theorem 4.1.4 for the family of critical hypergraphs, but extending these results to critical hypergraphs is likely to be very technical. It would therefore be more worthwhile to continue to improve the bound on p.

4.2 Tools

In order to prove Theorem 4.1.4, we need a sparse stability lemma. Conlon and Gowers [30] proved the following stability result.

Theorem 4.2.1 ([30])**.** *Given a strictly 2-balanced graph H with $\chi(H) \geq 3$ and a constant $\delta > 0$, there exist positive constants C and ϵ such that if $p \geq Cn^{-1/m_2(H)}$, then the following is almost surely true. Every H-free subgraph of $G(n,p)$ that has at least $\left(1 - \frac{1}{\chi(H)-1} - 2\epsilon\right) p\binom{n}{2}$ edges can be made $(\chi(H)-1)$-partite by removing at most δpn^2 edges.*

The above stability theorem leads to a sparse stability lemma, but only for $p \geq Cn^{-1+a}$ for some $C, a > 0$. However, Samotij [70] proved a stability version of a general extremal result of Schacht [71] that is true for $p \geq Cn^{-1}$. One consequence of this general stability theorem is the following theorem.

Theorem 4.2.2 ([70])**.** *For every $\delta > 0$ there exist positive constants C and ϵ such that if $p_n \geq Cn^{-1}$, then almost surely the following holds.*

For every subgraph of $G^3(n, p_n)$ with at least $(2/9 - \epsilon)\binom{n}{3}p_n$ edges that does not contain F_5, there exists a partition of $[n]$ into sets V_1, V_2, and V_3 such that all but at most $\delta n^3 p_n$ edges have one point in each V_i.

We will make use of the following Chernoff-type bound (see [5]).

Lemma 4.2.3. *Let Y be the sum of mutually independent indicator random variables, and let $\mu = E[Y]$. For all $\epsilon > 0$,*

$$P[|Y - \mu| > \epsilon\mu] < 2e^{-c_\epsilon \mu},$$

where $c_\epsilon = \min\{-\ln\left(e^\epsilon(1+\epsilon)^{-(1+\epsilon)}\right), \epsilon^2/2\}$.

Lemma 4.2.4. *For $\epsilon > 0$, c_ϵ given by Lemma 4.2.3, and $p > 3\log(n)/(c_\epsilon n)$, the following is almost surely true. For any pair of vertices x, y, the number of hyperedges in $G^3(n, p)$ containing both x and y is between $(1-\epsilon)pn$ and $(1+\epsilon)pn$ w.h.p.*

Proof. For each pair of vertices, x, y, let $X_{x,y}$ be the random variable given by the number of vertices $a \in V \setminus \{x, y\}$ such that axy is an edge. Then $X_{x,y}$ is the sum of the mutually independent indicator random variables $\{X^a_{x,y}\}_{a \in V \setminus \{x,y\}}$, and so Lemma 4.2.3 applies. Letting $\mu = E[X_{x,y}]$, we have $\mu = p(n-2)$, and

$$P[|X_{x,y} - \mu| > \epsilon\mu] < 2e^{-c_\epsilon(n-2)p} < 2e^{-2c_\epsilon np}).$$

Because $p > 3\log(n)/(c_\epsilon n)$, it follows that

$$\begin{aligned} p &> \tfrac{\log(2n^3)}{2c_\epsilon n} \\ 2c_\epsilon np &> \log(2n^3) \\ e^{2c_\epsilon np} &> 2n^3 \\ n^{-3} &> 2e^{-2c_\epsilon np}, \end{aligned}$$

so $P[|X_{x,y} - \mu| > \epsilon\mu] < n^{-3}$. By the union bound, it therefore follows that the probability that $|X_{x,y} - \mu| > \epsilon\mu$ for some $\{x, y\}$ is at most $n^2 n^{-3} = n^{-1}$. Since $\lim_{n\to\infty} n^{-1} = 0$, with high probability there is no such $\{x, y\}$. □

Lemma 4.2.5. *For $\epsilon > 0$, c_ϵ given by Lemma 4.2.3, and $p \geq \sqrt{300/c_\epsilon} n^{-1/2}$, the following is almost surely true. For every pair of disjoint vertex sets A, B in $G^3(n, p)$, each of size at least $n/10$, and for every pair of vertices x, y in $V(G^3(n, p)) \setminus A \setminus B$,*

$$(1 - \epsilon)p^2|A||B| \leq |\{(a, b) \in A \times B : xab, yab \in G\}| \leq (1 + \epsilon)p^2|A||B|.$$

Proof. For each pair of vertices x, y, let $X_{x,y}$ be the random variable given by the number of pairs (a, b) in $A \times B$ for which both xab and yab are edges in $G^3(n, p)$. Then $X_{x,y}$ is the sum of the mutually independent indicator random variables $\{X^{a,b}_{x,y}\}_{(a,b)\in A\times B}$, so Lemma 4.2.3 applies. Letting $\mu = E[X]$, we have $\mu = |A||B|p^2$, and

$$P[|X_{x,y} - \mu| > \epsilon\mu] < 2\exp\left(-\frac{c_\epsilon}{100} p^2 n^2\right),$$

where c_ϵ is defined in Lemma 4.2.3.

It therefore follows from the union bound that the probability that $|X_{x,y} - \mu| > \epsilon\mu$ for some $\{x, y\}$ is less than

$$2\exp\left(-\frac{c_\epsilon}{100} p^2 n^2\right) n^2.$$

Call a pair $\{x, y\}$ "bad" with respect to A, B if $|X_{x,y} - \mu| > \epsilon\mu$. We now take another union bound, over all choices of appropriate A and B, to see that the probability that for some choice of A, B there is a pair that is bad with respect to A, B is at most

$$2\exp\left(-\frac{c_\epsilon}{100}p^2n^2\right)n^2 2^{2n}.$$

Now, if $p > \sqrt{300/c_\epsilon}n^{-1/2}$ then

$$\begin{aligned}\log(2n^3 2^{2n}) < 3n &< c_\epsilon p^2 n^2/100\\ 2n^3 2^{2n} &< \exp\left(c_\epsilon p^2 n^2/100\right)\\ 2\exp\left(-c_\epsilon p^2 n^2\right)n^2 2^{2n} &< 1/n.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{1}{n} = 0$, almost surely there is no such A, B. □

Lemma 4.2.6. *For $\epsilon > 0$, c_ϵ given by Lemma 4.2.3, and $p > 100n^{-1}$, the following is almost surely true. For every triple of pairwise disjoint sets A, B, C in $G = G^3(n,p)$, each of size at least $n/10$,*

$$(1-\epsilon)p|A||B||C| \leq |\{(a,b,c) \in A \times B \times C : abc \in G\}| \leq (1+\epsilon)p|A||B||C|.$$

Proof. Let X be the number of triples in $A \times B \times C$ that are edges of G and let $\mu = E[X]$. By Lemma 4.2.3, $P[|X-\mu| > \epsilon\mu] < 2\exp\left(-c_\epsilon n^3 p/1000\right)$. There are fewer than 3^n choices of A, B, C, and if $p > 100/n$ then

$$2\exp\left(-c_\epsilon n^3 p/1000\right)3^n < 2\exp\left(-c_\epsilon n^2/10\right)2^{3n}.$$

It remains to check that this approaches 0 as $n \to \infty$:

$$\begin{aligned} 40/c_\epsilon &\ll n \\ 4n &\ll c_\epsilon n^2/10 \\ \log(2) + 3n\log(2) &\ll c_\epsilon n^2/10 \\ \log(2^{3n}2) &\ll c_\epsilon n^2/10 \\ 2^{3n}2 &\ll e^{c_\epsilon n^2/10} \\ 2e^{-c_\epsilon n^2/10}2^{3n} &\ll 1. \end{aligned}$$

□

Lemma 4.2.7. *For any $p \gg \log(n)/n$, any $\epsilon > 0$, any function $\omega(n)$ such that $\lim_{n\to\infty} \omega(n) = \infty$, and any integer $r > \omega(n)/p$, the following is almost surely true. For any $x \in V(G^3(n,p))$ and any set of vertices A_x with $|A_x| \leq \epsilon n$, there are at most r pairs $\{u,v\} \in \binom{V(G)}{2}$ such that xuv is an edge of $G^3(n,p)$ and $|N(u,v) \cap A_x| > 2\epsilon pn$.*

Proof. Fix a vertex x in $G^3(n,p)$ and some set A_x of size ϵn. We would like to show that there are at most r edges of the form xuv for which $|N(u,v) \cap A_x|$ is large. For each pair of vertices u, v, let $B(u,v)$ be the event that $xuv \in G^3(n,p)$ and $|N(u,v) \cap A_x| > 2\epsilon pn$. By Chernoff's inequality,

$$P[B(u,v)] < e^{-\epsilon pnc}$$

for some constant $c > 0$. If $\{u,v\} \neq \{u',v'\}$ then $B(u,v)$ and $B(u',v')$ are independent events. Consequently, the probability that $B(u,v)$ is true for at least r pairs is at most

$$\binom{pn^2}{r} e^{-\epsilon pncr}.$$

There are n choices of x in $V(G^3(n,p)$, and for each x there are $\binom{n}{\epsilon n}$ choices of A_x. It therefore remains to prove that

$$n \cdot \binom{n}{\epsilon n} \cdot \binom{pn^2}{r} \cdot \exp(-\epsilon pncr) = o(1).$$

This is true when $r > \omega(n)/p \gg 1/p$, using the fact that $\binom{n}{\epsilon n} \approx e^{H(\epsilon)n}$, where $H(x)$ is the entropy function. □

4.3 Proof of Theorem 4.1.4

We will begin with a sketch of the proof of Theorem 4.1.4, which will motivate the following lemmas. Recall that Theorem 4.1.4 states that there exists a constant C such that if $p \geq C\frac{\omega(n)}{\sqrt{n}}$ for some $\omega(n)$ such that $\lim_{n\to\infty} \omega(n) = \infty$ then the following is almost surely true: if H is a subgraph of $G^3(n,p)$ that does not contain F_5 and that has the most edges among all such subgraphs, then H is 3-partite.

Fix some H, a subgraph of $G^3(n,p)$ that does not contain F_5 and contains the maximum number of edges among all such subgraphs. Let X_1, X_2, X_3 be a partition of $V(H)$ that maximizes the number of edges in $H \cap (X_1 \times X_2 \times X_3)$. Let

$$\epsilon_1 = \frac{1}{2560}, \quad \epsilon_2 = \frac{1}{400}, \quad \delta = \frac{\epsilon_1^2 \epsilon_2}{64^2 \cdot 20}, \quad \epsilon_3 = \frac{96\delta}{\epsilon_1}.$$

Let M be the set of edges of $G^3(n,p)$ that are in $X_1 \times X_2 \times X_3$ and are not edges of H. This is the set of "missing" edges. For $1 \leq i \leq 3$, let B_i be the edges of H that have at least two vertices in X_i and let $B = \cup_i B_i$. By symmetry, we may assume that $|B_1| \geq \frac{1}{3}|B|$. This is the set of "bad" edges; if B is empty then H is tripartite. Notice that $B_i \cap B_j = \emptyset$ if $i \neq j$, because H is 3-uniform.

By Theorem 4.2.2, $|B| \leq \delta pn^3$. Since H has the maximum number of edges among all subgraphs of $G^3(n,p)$ that do not contain F_5, it follows that $|M| \leq |B|$. By Lemma 4.2.6, it is the case that for $1 \leq i \leq 3$

$$\frac{n}{4} < |X_i| < \frac{2n}{5}.$$

Now, suppose that B is not empty (and, as pointed out above, we may assume that $|B_1| \geq |B|/3$). For each $W \in B_1$ with $w_1, w_2 \in W \cap X_1$, there exist, by Lemma 4.2.5, at least $\frac{1}{32}p^2n^2$ choices of $y \in X_2$ and $z \in X_3$ such that W together with y and z form a copy of F_5 in $G^3(n,p)$. That is, w_1yz and w_2yz are both edges in $G^3(n,p)$. Since H contains no copy of F_5, at least one of w_1yz, w_2yz must be in M. We will count elements of M by counting the embeddings of F_5 in $G^3(n,p)$ that contain some $W \in B_1$. This will provide a lower bound on the size of M in terms of $|B_1|$, which in turn will lead to a contradiction with the fact that $|M| \leq |B|$. We therefore conclude that B is in fact empty, and so H is tripartite.

We will count copies of F_5 in $G^3(n,p)$ by considering several cases, based on the relative sizes of the sets C_1 and C_2, defined below.

We define the *shadow graph* of B_1 on the vertex set X_1. This is the graph L with vertex set X_1 and xy an edge if and only if there exists some edge of B_1 that contains both x and y. Let $C = \{x \in X_1 : d_L(x) \geq \epsilon_1 n\}$ and let $D = X_1 \setminus C$. Let C_1 be the set of all $x \in C$ such that there are at least $\epsilon_2 pn^2$ pairs (y,z) in $X_2 \times X_3$ for which xyz is an edge of H. Let $C_2 = C \setminus C_1$.

With these definitions in hand, we are ready to prove the following lemmas, which will lead to a proof of Theorem 4.1.4 at the end of the section.

Lemma 4.3.1. *If L' is a subgraph of L such that $\Delta(L') \leq \epsilon_3 n$, then*

$$|M| \geq \frac{1}{128\epsilon_3} pn|E(L')|.$$

Proof. For each $wx \in E(L')$, there are at least $\frac{1}{32}p^2n^2$ choices of $(y,z) \in X_2 \times X_3$ such that $wyz, xyz \in G^3(n,p)$, by Lemma 4.2.5. There are therefore at least $\frac{1}{32}|E(L')|p^2n^2$ copies of F_5 in $G^3(n,p)$ whose vertices are in $V(H)$. Because $wx \in E(L)$, at least one of wyz, xyz

must be in M for each of these copies of F_5.

Consider $R = xyz \in M$ with $x \in V(L')$. We will count how many of these copies of F_5 in $G^3(n,p)$ contain R. Say that R is *bad* if there exist at least $2\epsilon_3 pn$ vertices $w \in N_{L'}(x)$ with $wyz \in G^3(n,p)$. For each $x \in V(L')$, the degree of x is at most $\Delta(L') \leq \epsilon_3 n$. We may therefore apply Lemma 4.2.7 with $\epsilon = \epsilon_3$ and $A_x = N_{L'}(x)$ for each x to obtain that for every $x \in V(L')$ with $A_x \neq \emptyset$ there are at most $\log(n)/p$ pairs $\{y,z\}$ such that $\{x,y,z\}$ is bad (here we let $\omega(n) = \log(n)$). Therefore, for every $xw \in E(L')$ there are at most $2\log(n)/p$ bad edges containing x or w. Also, by Lemma 4.2.4, each bad edge is in at most $2pn$ of the above copies of F_5. Therefore, at least

$$\frac{1}{32}|E(L')|p^2n^2 - 2|E(L')|\frac{\log(n)}{p} \cdot 2pn \geq |E(L')|p^2n^2\left(\frac{1}{32} - \frac{2\log(n)}{n^2p^3}\right)$$

of the above copies of F_5 contain an edge from M that is not bad. For large n this is at least $\frac{1}{64}|E(L')|p^2n^2$. By definition, an edge that is not bad is in at most $2\epsilon_3 pn$ of the above copies of F_5. Therefore,

$$|M| \geq \frac{1}{64} \cdot \frac{|E(L')|p^2n^2}{2\epsilon_3 pn} = \frac{1}{128\epsilon_3} \cdot pn|E(L')|.$$

□

Lemma 4.3.2. *$|C| < 96\delta\epsilon_1^{-1}n$.*

Proof. Notice that $|E(L)| \leq 48\delta n^2$ because, by Lemma 4.2.5, for each edge $wx \in E(L)$ there are at least $\frac{p^2n^2}{32}$ choices of $y \in X_2$, $z \in X_3$ such that $xyz \in G^3(n,p)$ and $wyz \in G^3(n,p)$. One of these two edges must be in M, otherwise H contains a copy of F_5. Thus, by Lemma 4.2.4, $\frac{3}{2}pn|M| \geq |E(L)|\frac{1}{32}p^2n^2$. Because $\frac{3}{2}pn \cdot \delta pn^3 > \frac{3}{2}pn|M|$ it therefore follows that $48\delta n^2 > |E(L)|$.

Now, every vertex in C has degree in L at least $\epsilon_1 n$, so $\epsilon_1 n|C| < 2|E(L)| < 96\delta n^2$ implies that $|C| < 96\delta\epsilon_1^{-1}n$. □

Lemma 4.3.3.

$$|M| \geq \frac{\epsilon_1^2\epsilon_2}{2\cdot 3\cdot 2^{12}\delta}pn^2|C_1|.$$

Proof. For each $x \in C_1$, let $S_x = \{(y,z) \in X_2 \times X_3 \,|\, xyz \in H\}$. By the definition of C_1, each S_x has size at least $\epsilon_2 pn^2$. We will count the number of copies of F_5 in $G^3(n,p)$ of the following form. For $x \in C_1$ and $w \in N_L(x)$, there are $(y,z) \in X_2 \times X_3$ such that $xyz \in H$ and $wyz \in G^3(n,p)$. An application of Chernoff's bound shows that there are at least $\frac{1}{32}d_L(x)|S_x|p$ such copies of F_5.

The number of such copies of F_5 is therefore at least

$$\sum_{x\in C_1}\frac{1}{32}d_L(x)|S_x|p \geq \frac{1}{32}|C_1|p\cdot\epsilon_1 n\cdot\epsilon_2 pn^2 = \frac{1}{32}\epsilon_1\epsilon_2 p^2n^3|C_1|.$$

Say that an edge $wyz \in M$ is *bad* if $w \in X_1$, $y \in X_2$, $z \in X_3$, and there are at least $\delta\epsilon_1^{-1}pn$ vertices $x \in C_1$ for which xyz is in $G^3(n,p)$. We may now apply Lemma 4.2.7 with $A_x = C_1$ for each x and $\epsilon = 96\delta\epsilon_1^{-1}$, because $|A_x| = |C_1| \leq |C|$, which by Lemma 4.3.2 has size at most $96\delta\epsilon_1^{-1}n = \epsilon n$. For each vertex w that is a neighbor of some x in C, there are at most $(\omega(n))^2/p$ bad edges containing w, where $\omega(n)$ is some function of n such that $\lim_{n\to\infty}\omega(n) = \infty$. Therefore, the number of bad edges is at most $|X_1|(\omega(n))^2/p$. Each bad edge is contained in at most $2pn$ of the above copies of F_5, and so at least

$$\frac{1}{32}\epsilon_1\epsilon_2 p^2n^3|C_1| - |X_1|\cdot\frac{(\omega(n))^2}{p}\cdot 2pn$$

of the above copies of F_5 contain an edge from M that is not bad. Now,

$$\begin{aligned}\tfrac{(\omega(n))^2}{p}\cdot 2pn\cdot|X_1| &< 2(\omega(n))^2n^2\\ &< \tfrac{1}{64}\epsilon_1\epsilon_2p^2n^3\\ &\leq \tfrac{1}{32}\epsilon_1\epsilon_2p^2n^3|C_1|\end{aligned}$$

Where the second inequality follows from $p > C\omega(n)n^{-1/2}$ for some $C > 1/10$. There are therefore at least

$$\frac{1}{32}\epsilon_1\epsilon_2 p^2 n^3|C_1| - \frac{1}{64}\epsilon_1\epsilon_2 p^2 n^3|C_1| = \frac{1}{64}\epsilon_1\epsilon_2 p^2 n^3|C_1|$$

of the above copies of F_5 that contain an edge from M that is not bad. Each such edge is contained in at most $2\epsilon pn = 192\delta\epsilon_1^{-1}pn$ such copies of F_5, and so

$$|M| \geq \frac{\epsilon_1^2\epsilon_2 p^2 n^3|C_1|}{64 \cdot 192\delta pn} = \frac{\epsilon_1^2\epsilon_2}{64 \cdot 192\delta}pn^2|C_1|.$$

□

Lemma 4.3.4. $|M| \geq \frac{1}{20}pn^2\,|C_2|$.

Proof. For vertices $x \in C_2$, the number of edges from $(X_1 \times X_2 \times X_3) \cap H$ that contain x is at most $\epsilon_2 pn^2$ but w.h.p. the degree in $G^3(n,p)$ is at least $pn^2/16$. Thus there are at least $pn^2/20$ edges of M incident to x, so $|M| \geq |C_2|\,pn^2/20$. □

We now have three different lower bounds on the size of M. The final part of the proof is to show that $|M| > |B|$ by proving that no matter how the edges of B are arranged one of the above lower bounds on M is larger than $|B|$. To do this, we divide the edges of B_1 into three pieces. Let $B_1(1) = \{W \in B_1 : |W \cap C| \geq 2 \text{ or } |W \cap D| \geq 2\}$. Let $B_1(2) = \{W \in B_1 \setminus B_1(1) : |W \cap C_1| \geq 1\}$. Let $B_1(3) = B_1 \setminus B_1(1) \setminus B_1(2)$. Every edge in $B_1(3)$ contains a vertex in C_2 and is not completely contained in X_1.

The proof will be completed by showing that Lemma 4.3.1 implies that $|M| \geq 10\,|B_1(1)|$, Lemma 4.3.3 implies that $|M| \geq 10\,|B_1(2)|$, and Lemma 4.3.4 implies that $|M| \geq 10\,|B_1(3)|$. Since $|B_1| \geq |B|/3$ and one of $B_1(1)$, $B_1(2)$, or $B_1(3)$ has size at least $\frac{1}{3}\,|B_1|$, we obtain a contradiction.

Lemma 4.3.5. $|M| \geq 10\,|B_1(1)|$.

Proof. Let $L'' = L[C] \cup L[D]$. By definition, vertices $x \in D$ have degree at most $\epsilon_3 n$. For $x \in C$, Lemma 4.3.2 shows that x has degree in L'' at most $|C| \leq 96\epsilon_1^{-1}\delta n$. Pick $\epsilon_3 = 96\epsilon_1^{-1}\delta$ so that $\Delta(L') \leq \epsilon_3 n$. Lemma 4.2.4 shows that $|B_1(1)| \leq 2pn\,|E(L')|$. Combined with Lemma 4.3.1, this shows that $|M| \geq \frac{1}{256\epsilon_1}|B_1(1)|$. Pick $\epsilon_1 = \frac{1}{2560}$ so that $|M| \geq 10\,|B_1(1)|$. □

Lemma 4.3.6. $|M| \geq 10\,|B_1(2)|$.

Proof. For each vertex $x \in C_1$ and each $y \in D$, by Lemma 4.2.4 the co-degree of x and y is at most $\frac{3}{2}pn$. Since $|D| \leq n$, there are at most $2pn^2$ edges of $B_1 \setminus B_1(1)$ containing x. Thus $|B_1(2)| \leq 2\,|C_1|\,pn^2$ so Lemma 4.3.3 shows that $|M| \geq \frac{\epsilon_1^2\epsilon_2}{2\cdot 64^2\delta}|B_1(2)|$. Pick $\delta = \frac{\epsilon_1^2\epsilon_2}{64^2\cdot 20}$ so that $|M| \geq 10\,|B_1(2)|$. □

Lemma 4.3.7. $|M| \geq 10\,|B_1(3)|$.

Proof. Consider some $x \in C_2$, so that the cross-H-degree of x is at most $\epsilon_2 pn^2$. Note that every edge in $B_1(3)$ has at least one vertex in C_2 and is not completely contained in X_1 (edges completely contained in X_1 are in $B_1(1)$.) If there exist at least $\epsilon_2 pn^2$ edges of B which contain x and have a vertex in X_2, we could move x to X_3 and increase the number of edges across the partition. Similarly, there are at most $\epsilon_2 pn^2$ edges of B which contain x and have a vertex in X_3, since otherwise we could move x to X_2. Thus $|B_1(3)| \leq 2\epsilon_2\,|C_2|\,pn^2$. If $\epsilon_2 = \frac{1}{400}$, then Lemma 4.3.4 shows that $|M| \geq 10\,|B_1(3)|$. □

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