Transport relations for surface integrals arising in the formulation of balance laws for evolving fluid interfaces

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We establish transport relations for integrals over evolving fluid interfaces. These relations make it possible to localize integral balance laws over nonmaterial interfaces separating fluid phases and, therefore, obtain associated interface conditions in differential form.

1. Introduction

In formulating integral balance laws for a nonmaterial evolving interface $\mathcal{S}(t)$ separating two fluid phases, one often encounters terms of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}(t)} \varphi(\mathbf{x},t)\,\mathrm{d}a, \tag{1.1}$$

with $\varphi(\mathbf{x},t)$ a surface field on $\mathcal{S}(t)$ and $\mathcal{A}(t)$ an arbitrary evolving subsurface of $\mathcal{S}(t)$. To obtain the local differential consequences of such laws necessitates an appropriate transport relation. We here establish such relations.

To see the difficulties involved in deriving such transport relations it is useful to consider the analogous problem associated with the integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{R}(t)} \Phi(\mathbf{x},t)\,\mathrm{d}v \tag{1.2}$$

of a bulk scalar-field $\Phi(\mathbf{x},t)$ over a time-dependent region $\mathcal{R}(t)$ migrating through a fluid. Specifically, assume that the boundary $\partial\mathcal{R}(t)$ moves with (scalar) normal velocity $V_{\partial\mathcal{R}}(\mathbf{x},t)$ in the direction of its outward unit normal $\mathbf{m}(\mathbf{x},t)$ and write $V_{\partial\mathcal{R}}^{\mathrm{mig}} = V_{\partial\mathcal{R}} - \mathbf{u}\cdot\mathbf{m}$ for the normal velocity of $\partial\mathcal{R}$ relative to the fluid. Two well-known generalizations of the Reynolds (1903) transport relation then read

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{R}} \Phi\,\mathrm{d}v &= \int\limits_{\mathcal{R}} \left\{\frac{\partial\Phi}{\partial t} + \mathrm{div}\,(\Phi\mathbf{u})\right\}\mathrm{d}v + \int\limits_{\partial\mathcal{R}} \Phi V_{\partial\mathcal{R}}^{\mathrm{mig}}\,\mathrm{d}a,\\
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{R}} \Phi\,\mathrm{d}v &= \int\limits_{\mathcal{R}} \left\{\dot{\Phi} + \Phi\,\mathrm{div}\,\mathbf{u}\right\}\mathrm{d}v + \int\limits_{\partial\mathcal{R}} \Phi V_{\partial\mathcal{R}}^{\mathrm{mig}}\,\mathrm{d}a.
\end{aligned}\right\} \tag{1.3}$$

Here $\dot{\Phi}$ (often written $\mathrm{D}\Phi/\mathrm{D}t$) denotes the *material time-derivative* of Φ, and the second of (1.3) follows from the first using the standard identity

$$\dot{\Phi} = \frac{\partial \Phi}{\partial t} + \mathbf{u} \cdot \operatorname{grad} \Phi. \tag{1.4}$$

A difficulty in deriving counterparts for surfaces of the bulk relations (1.3) is associated with determining appropriate superficial analogs of the time derivatives $\dot{\Phi}$ and $\partial\Phi/\partial t$. In this regard, bear in mind that, for φ a surface field, if the surface is *not material*, then $\dot{\varphi}$ *is not well-defined*: since material points flow across $\mathcal{S}(t)$, it is not generally possible to compute a time-derivative holding material points fixed. Further, while one finds in the literature time-derivatives of surface fields φ expressed as conventional partial derivatives $\partial\varphi/\partial t$, such expressions without explanation are meaningless: the difference quotient $\{\varphi(\mathbf{x}, t+\tau) - \varphi(\mathbf{x}, t)\}/\tau$ underlying the partial derivative $\partial\varphi/\partial t$ of a surface field is generally undefined because there is no assurance that a point $\mathbf{x}$ on $\mathcal{S}(t)$ will also lie on $\mathcal{S}(t+\tau)$, even for sufficiently small τ. These observations hold even when $\mathcal{S}(t)$ is material. Of course, $\partial\varphi/\partial t$ may be defined using an extension of $\varphi(\mathbf{x}, t)$, at each t, to a three-dimensional region containing the surface; unfortunately, $\partial\varphi/\partial t$ so defined generally depends on the particular extension used.

The main results of this study are surface analogs of the transport relations (1.3) for the integral (1.1). To state these results, suppose that $\mathcal{S}(t)$ is oriented by a unit-normal field $\mathbf{n}(\mathbf{x}, t)$, let $V(\mathbf{x}, t)$ denote the (scalar) normal velocity of $\mathcal{S}(t)$ in the direction of $\mathbf{n}(\mathbf{x}, t)$, and let $K(\mathbf{x}, t)$ denotes the total (i.e., twice the mean) curvature of $\mathcal{S}(t)$. Further, let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity of the fluid and suppose that the tangential component $\mathbf{u}_{\tan}(\mathbf{x}, t)$ of $\mathbf{u}(\mathbf{x}, t)$ is continuous across $\mathcal{S}(t)$. Then for $\varphi(\mathbf{x}, t)$ a scalar field defined on $\mathcal{S}(t)$, and for $\mathcal{A}(t)$ an arbitrary evolving subsurface of $\mathcal{S}(t)$ with $V^{\mathrm{mig}}_{\partial\mathcal{A}}(\mathbf{x}, t)$ the normal velocity of $\partial\mathcal{A}(t)$ relative to the fluid,

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}} \varphi\,\mathrm{d}a &= \int\limits_{\mathcal{A}} \Big\{ \overset{\scriptscriptstyle\square}{\varphi} + \operatorname{div}_{\mathcal{S}}(\varphi\mathbf{u}_{\tan}) - \varphi K V \Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}} \varphi V^{\mathrm{mig}}_{\partial\mathcal{A}}\,\mathrm{d}s, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}} \varphi\,\mathrm{d}a &= \int\limits_{\mathcal{A}} \Big\{ \overset{\circ}{\varphi} + \varphi\operatorname{div}_{\mathcal{S}}\mathbf{u}_{\tan} - \varphi K V \Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}} \varphi V^{\mathrm{mig}}_{\partial\mathcal{A}}\,\mathrm{d}s. \end{aligned}\right\} \tag{1.5}$$

Here:

(i) $\overset{\scriptscriptstyle\square}{\varphi}$, the *normal time derivative*, is the time-derivative of φ following the motion of $\mathcal{S}$ along its normal trajectories; that is, along the trajectories associated with the vector normal-velocity $\mathbf{v}_n = V\mathbf{n}$ of $\mathcal{S}$;

(ii) $\overset{\circ}{\varphi}$, the *migrationally normal time-derivative*, is the time-derivative of φ along the trajectories associated with the velocity $\mathbf{v}$ for $\mathcal{S}$ defined by $\mathbf{v} - \mathbf{u} = (V - \mathbf{u}\cdot\mathbf{n})\mathbf{n}$.

We show that the time-derivatives $\overset{\circ}{\varphi}$ and $\overset{\scriptscriptstyle\square}{\varphi}$ are related through the identity

$$\overset{\circ}{\varphi} = \overset{\scriptscriptstyle\square}{\varphi} + \mathbf{u}_{\tan}\cdot \operatorname{grad}_{\mathcal{S}}\varphi, \tag{1.6}$$

which should be compared with its bulk counterpart (1.4).

A geometrically meaningful method of extending a surface field $\varphi(\mathbf{x}, t)$, at each t, to a three-dimensional region containing the surface is obtained by requiring that φ *be constant on normal lines*, where a normal line at time t is a line through a point $\mathbf{x}$ on $\mathcal{S}(t)$ parallel to $\mathbf{n}(\mathbf{x}, t)$. We refer to an extension $\hat{\varphi}$ obtained in this manner as a *normally constant extension* of φ and show that such an extension provides a simple relation for

$\overset{\square}{\varphi}$; namely,

$$\overset{\square}{\varphi} = \frac{\partial \hat{\varphi}}{\partial t}. \tag{1.7}$$

The normal time-derivative is therefore the conventional partial time-derivative of φ when φ is extended to be constant on normal lines.

Since the interface $\mathcal{S}$ need not be material, the transport relations (1.5) are applicable to the general description of phase transitions. On the other hand, when $\mathcal{S}$ is a material surface and $\mathcal{A}$ a material subsurface of $\mathcal{S}$,

$$\overset{\square}{\varphi} = \dot{\varphi} - \mathbf{u}_{\mathrm{tan}} \cdot \mathrm{grad}_{\mathcal{S}}\varphi, \qquad V = \mathbf{u}\cdot\mathbf{n}, \qquad V^{\mathrm{mig}}_{\partial\mathcal{A}} = 0, \tag{1.8}$$

and $(1.5)_1$ reduces to a classical transport relation for surfaces (e.g., Slattery 1990, eqt. (3-6)).

Our results apply, for instance, to the study of evaporating surfactant solutions. For a binary solution, with bulk surfactant concentration and flux c and $\mathbf{h}$ and surface concentration and flux Γ and $\mathbf{j}$, the balance of surfactant concentration for an arbitrary migrating subsurface $\mathcal{A}$ of the evaporation surface $\mathcal{S}$ has the form†

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}} \Gamma \,\mathrm{d}a = -\int\limits_{\partial\mathcal{A}} \big\{\mathbf{h}\cdot\boldsymbol{\nu} - \Gamma V^{\mathrm{mig}}_{\partial\mathcal{A}}\big\}\,\mathrm{d}s + \int\limits_{\mathcal{A}} \big\{\mathbf{j}\cdot\mathbf{n} - cV^{\mathrm{mig}}\big\}\,\mathrm{d}a. \tag{1.9}$$

Invoking the transport relation $(1.5)_2$ and the surface divergence theorem, we may localize (1.9) to yield the differential balance

$$\overset{\circ}{\Gamma} + \Gamma\mathrm{div}_{\mathcal{S}}\mathbf{u}_{\mathrm{tan}} - \Gamma K V = -\mathrm{div}_{\mathcal{S}}\mathbf{j} + \mathbf{h}\cdot\mathbf{n} - cV^{\mathrm{mig}}, \tag{1.10}$$

valid pointwise on the evaporation surface.

When evaporation is neglected, so that $\mathcal{S}$ is a material surface, then $\Gamma^{\circ} = \dot{\Gamma}$, $V = \mathbf{u}\cdot\mathbf{n}$, and the balance (1.10) becomes

$$\dot{\Gamma} + \Gamma\mathrm{div}_{\mathcal{S}}\mathbf{u}_{\mathrm{tan}} - \Gamma K\mathbf{u}\cdot\mathbf{n} = -\mathrm{div}_{\mathcal{S}}\mathbf{j} + \mathbf{h}\cdot\mathbf{n}. \tag{1.11}$$

The first two terms on the left side of (1.11) differ from those presented in the literature (e.g., Scriven 1960; Aris 1962; Slattery 1971; Waxman 1984; and Stone 1989). Instead, what one consistently finds are the terms

$$\frac{\partial \Gamma}{\partial t} + \mathrm{div}_{\mathcal{S}}(\gamma\mathbf{u}_{\mathrm{tan}}), \tag{1.12}$$

which agree with those on the left side of (1.10) only if the partial time-derivative of Γ is computed using the normally constant extension of Γ, since then, by (1.7) and (1.8), we have the identification

$$\frac{\partial \Gamma}{\partial t} = \dot{\Gamma} - \mathbf{u}_{\mathrm{tan}}\cdot\mathrm{grad}_{\mathcal{S}}\Gamma. \tag{1.13}$$

Without this interpretation the meaning of the equation that arises on replacing the first two terms on the left side of (1.11) by (1.12) is ambiguous. Further, the identification of $\partial\Gamma/\partial t$ with the partial time-derivative determined using the normally constant extension $\hat{\Gamma}$ of Γ provides a precise geometrical definition that may be useful for computations.

Returning to the topic of evaporating surfactant solutions, it is important to note that previous statements of the surfactant balance on the solution surface have been in error. In particular, consider equation (3b) of Danov, Alleborn, Raszillier & Durst (1998). To

† We neglect surfactant evaporation; it is generally assumed that only the fluid solvent evaporates.

clarify the comparison between (1.10) and that equation, suppose that the bulk and surface fluxes are given by $\mathbf{j} = -D\mathrm{grad}\,c$ and $\mathbf{h} = -D_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\Gamma$, in which case (1.10) specializes to

$$\overset{\circ}{\Gamma} + \Gamma\mathrm{div}_{\mathcal{S}}\mathbf{u}_{\mathrm{tan}} - \Gamma KV = \mathrm{div}_{\mathcal{S}}(D_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\Gamma) - \mathbf{n}\cdot(D\mathrm{grad}\,c) - cV^{\mathrm{mig}}. \tag{1.14}$$

In place of the left side of (1.14), Danov, Alleborn, Raszillier & Durst (1998) write

$$\frac{\partial\Gamma}{\partial t} + \mathrm{div}_{\mathcal{S}}(\Gamma\mathbf{u}_{\mathrm{tan}}). \tag{1.15}$$

By (1.6) and (1.7), the normally constant extension of Γ yields the identification

$$\frac{\partial\Gamma}{\partial t} = \overset{\circ}{\Gamma} - \mathbf{u}_{\mathrm{tan}}\cdot\mathrm{grad}_{\mathcal{S}}\Gamma, \tag{1.16}$$

which is the counterpart of (1.13) appropriate for a nonmaterial surface. Even with this identification, equation (3b) of Danov, Alleborn, Raszillier & Durst (1998) is missing the term $-\Gamma KV$ on its left side.

2. Surfaces

2.1. *Gradient and divergence on a surface*

Let $\mathcal{S}$ be a surface oriented by a *unit-normal field* $\mathbf{n}(\mathbf{x})$. A *surface field* is a field defined on $\mathcal{S}$. A *tangential vector-field* is a surface vector-field $\mathbf{f}(\mathbf{x})$ that satisfies $\mathbf{f}\cdot\mathbf{n} = 0$. We write $\mathrm{grad}_{\mathcal{S}}$ and $\mathrm{div}_{\mathcal{S}}$ for the surface gradient and surface divergence on $\mathcal{S}$.†

$$\mathrm{grad}_{\mathcal{S}}\varphi \quad \textit{is a tangential vector-field.} \tag{2.1}$$

SURFACE DIVERGENCE THEOREM *Let $\mathcal{A}$ be a subsurface of $\mathcal{S}$ with $\boldsymbol{\nu}$, a tangential vector-field, the outward unit normal to $\partial\mathcal{A}$. Then given any tangential vector-field* $\mathbf{f}$,

$$\int\limits_{\partial\mathcal{A}} \mathbf{f}\cdot\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\mathcal{A}} \mathrm{div}_{\mathcal{S}}\mathbf{f}\,\mathrm{d}a. \tag{2.2}$$

The field defined by

$$K = -\mathrm{div}_{\mathcal{S}}\mathbf{n} \tag{2.3}$$

is the *total* (i.e., twice the mean) *curvature.*

2.2. *Surface fields determined by limits of bulk fields*

Often in continuum mechanics a surface field is the restriction of a field that is well-defined and smooth up to the surface from one or both sides. In this case the surface gradient is simply the tangential component of the standard gradient; e.g., for such a bulk-field Φ,

$$\mathrm{grad}_{\mathcal{S}}\Phi = \mathrm{grad}\,\Phi - (\mathbf{n}\cdot\mathrm{grad}\,\Phi)\mathbf{n}. \tag{2.4}$$

(If a bulk field Φ is smooth up to the interface from each side, but not across the interface, then we would have two surface gradients $\mathrm{grad}_{\mathcal{S}}\Phi^{\pm}$, one for each of the limiting values $\Phi^{\pm}$ of Φ.)

† We omit smoothness assumptions associated with surfaces, evolving surfaces, and surface fields. We refer the reader to a treatise on differential geometry for precise definitions of $\mathrm{grad}_{\mathcal{S}}$ and $\mathrm{div}_{\mathcal{S}}$.

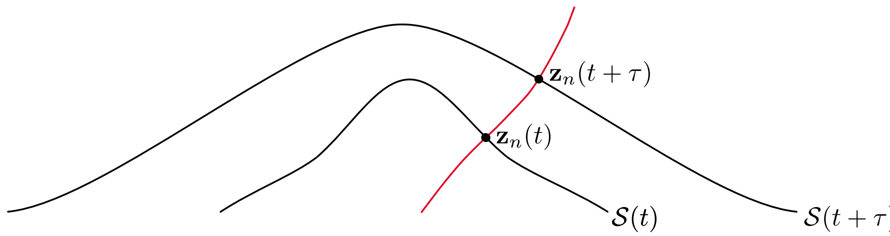

FIGURE 1. Two-dimensional schematic illustrating a normal trajectory passing through the points $\mathbf{z}_n(t)$ and $\mathbf{z}_n(t+\tau)$ on $\mathcal{S}(t)$ and $\mathcal{S}(t+\tau)$.

3. Evolving surfaces

3.1. *Local parametrization. Normal velocity*

We now consider an evolving surface $\mathcal{S}(t)$ oriented by a *unit-normal field* $\mathbf{n}(\mathbf{x},t)$. $\mathcal{S}(t)$ may be parametrized locally — that is, near any time t_0 and point $\mathbf{x}_0$ on $\mathcal{S}(t_0)$ — by a mapping

$$\mathbf{x} = \hat{\mathbf{x}}(\xi_1, \xi_2, t) \tag{3.1}$$

that, at each time t, establishes a one-to-one correspondence between points (ξ_1, ξ_2) — in an open set in a two-dimensional parameter space — and points $\mathbf{x}$ on $\mathcal{S}(t)$. Writing $(\xi_1, \xi_2) = \hat{\boldsymbol{\xi}}(\mathbf{x}, t)$ for the corresponding inverse map at fixed time, the function

$$\mathbf{v}(\mathbf{x},t) = \frac{\partial \hat{\mathbf{x}}}{\partial t}\Big|_{(\xi_1,\xi_2)=\hat{\boldsymbol{\xi}}(\mathbf{x},t)} \tag{3.2}$$

represents a local velocity field for $\mathcal{S}(t)$. This velocity field depends on the choice of parametrization: specifically, the normal component of $\mathbf{v}$, the *scalar normal-velocity*

$$V = \mathbf{v}\cdot\mathbf{n}, \tag{3.3}$$

is independent of the parametrization, but the tangential velocity is not.

The vectorial counterpart of V is the *vector normal-velocity*

$$\mathbf{v}_n = V\mathbf{n}. \tag{3.4}$$

3.2. *Velocity fields. Trajectories. Normal time-derivative*

Given *any* tangential vector-field $\mathbf{t}(\mathbf{x},t)$, consider the surface vector-field

$$\mathbf{v} \stackrel{\text{def}}{=} V\mathbf{n} + \mathbf{t}.$$

Any such $\mathbf{v}$ represents a *velocity field* for $\mathcal{S}$, in the sense that there exists a local parametrization (3.1) such that (3.2) holds. Fix $\mathbf{x}_0 \in \mathcal{S}(t_0)$ and write $\mathbf{x}_0 = \hat{\mathbf{x}}(\xi_1^0, \xi_2^0, t_0)$: the curve

$$\mathbf{z}(t) = \hat{\mathbf{x}}(\xi_1^0, \xi_2^0, t) \tag{3.5}$$

is referred to as the *trajectory corresponding to the velocity field* $\mathbf{v}$, since

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \mathbf{v}(\mathbf{z}(t),t), \qquad \mathbf{z}(t_0) = \mathbf{x}_0. \tag{3.6}$$

Trajectories corresponding to the vector normal-velocity $\mathbf{v}_n$ are called *normal trajectories* (Figure 1).

Let φ be a scalar surface-field. The notion of a *normal time-derivative* $\overset{\square}{\varphi}$ *of* φ *following*

$\mathcal{S}$ is basic. The field $\overset{\square}{\varphi}$ may be defined as follows: choose, arbitrarily, a time t_0 and a point $\mathbf{x}_0$ on $\mathcal{S}(t_0)$, and let $\mathbf{z}_n(t)$ denote the normal trajectory through $\mathbf{x}_0$ at t_0; then

$$\overset{\square}{\varphi}(\mathbf{x}_0, t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbf{z}_n(t), t)\Big|_{t=t_0}. \tag{3.7}$$

3.3. *Basic transport relation for surface integrals*

3.3.1. *Evolving subsurfaces of $\mathcal{S}(t)$*

Consider an arbitrary evolving subsurface $\mathcal{A}(t)$ of $\mathcal{S}(t)$ with boundary-curve $\partial\mathcal{A}(t)$ oriented by its *exterior* unit-normal field $\boldsymbol{\nu}(\mathbf{x}, t)$; $\boldsymbol{\nu}$ is normal to $\partial\mathcal{A}$, but tangent to $\mathcal{S}$.

The curve $\partial\mathcal{A}(t)$ evolves through space, and its motion is described by a velocity field $\mathbf{v}_{\partial\mathcal{A}}(\mathbf{x}, t)$ with $\mathbf{x} \in \partial\mathcal{A}(t)$. Only the component of $\mathbf{v}_{\partial\mathcal{A}}$ normal to the curve is independent of the parametrization of $\partial\mathcal{A}$ and, hence, intrinsic to the motion. On the other hand, since $\partial\mathcal{A}(t) \subset \mathcal{S}(t)$ for all t, $\mathbf{v}_{\partial\mathcal{A}} \cdot \mathbf{n} = V$.

Noting that, at each point of $\partial\mathcal{A}$, $\{\mathbf{n}, \boldsymbol{\nu}\}$ provides an orthonormal basis on the normal plane to the curve and writing $V_{\partial\mathcal{A}} = \mathbf{v}_{\partial\mathcal{A}} \cdot \boldsymbol{\nu}$, we may therefore express the intrinsic component of every velocity field for $\partial\mathcal{A}$ in the form

$$V\mathbf{n} + V_{\partial\mathcal{A}}\boldsymbol{\nu}. \tag{3.8}$$

We refer to $V_{\partial\mathcal{A}}(\mathbf{x}, t)$ as the scalar *normal-velocity of* $\partial\mathcal{A}(t)$; the field $V_{\partial\mathcal{A}}(\mathbf{x}, t)$ describes the intrinsic instantaneous motion of $\partial\mathcal{A}(t)$ in the tangent space to $\mathcal{S}(t)$ at $\mathbf{x}$.

3.3.2. *Transport relation*

In stating integral balance laws for an evolving phase interface $\mathcal{S}(t)$, one is often confronted with terms of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}} \varphi \,\mathrm{d}a, \tag{3.9}$$

with $\mathcal{A}(t)$ an arbitrary migrating subsurface of $\mathcal{S}(t)$. An essential ingredient in localizing such balances is the following *transport relation*:†

Let $\mathcal{A}(t)$ be an evolving subsurface of $\mathcal{S}(t)$ with $V_{\partial\mathcal{A}}(\mathbf{x}, t)$ the scalar normal-velocity of $\partial\mathcal{A}(t)$. Then given any scalar surface-field $\varphi(\mathbf{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}} \varphi \,\mathrm{d}a = \int_{\mathcal{A}} \Big\{\overset{\square}{\varphi} - \varphi K V\Big\} \,\mathrm{d}a + \int_{\partial\mathcal{A}} \varphi V_{\partial\mathcal{A}} \,\mathrm{d}s, \tag{3.10}$$

with $\overset{\square}{\varphi}$ the normal time-derivative of φ following $\mathcal{S}$.

An illustrative specialization of (3.10) arises on taking $\varphi \equiv 1$, in which case it follows that the area of $\mathcal{A}$ evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{area}(\mathcal{A}) = \underbrace{-\int_{\mathcal{A}} KV \,\mathrm{d}a}_{\substack{\text{area change}\\ \text{due to curvature}}} + \underbrace{\int_{\partial\mathcal{A}} V_{\partial\mathcal{A}} \,\mathrm{d}s}_{\substack{\text{area change due}\\ \text{to motion of } \partial\mathcal{A}}}. \tag{3.11}$$

† Established independently by Petryk & Mroz (1986) and Gurtin, Struthers & Williams (1989); see also Estrada & Kanwal (1991). A simple derivation of (3.10) for curves evolving in a planar domain is given by Angenent & Gurtin (1991).

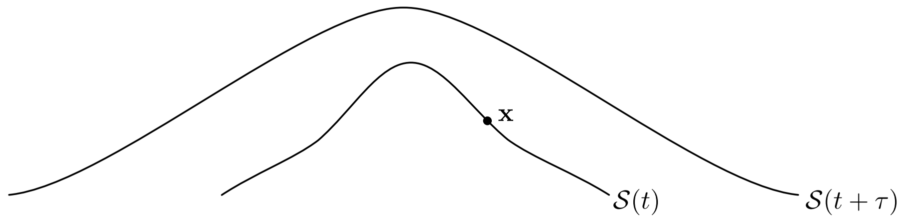

FIGURE 2. Two-dimensional schematic illustrating why a point $\mathbf{x}$ lying on an interface $\mathcal{S}$ at time t need not lie on the interface at a subsequent time $t+\tau$ and, thus, why the partial time-derivative of a surface field φ is generally undefined.

3.4. *Normally constant extension of a surface field. The derivative $\partial\varphi/\partial t$ for such an extension*

In applications one is often faced with the need to compute the time-derivative of, say, a scalar surface-field $\varphi(\mathbf{x},t)$. In the literature one typically finds such derivatives specified as standard partial derivatives $\partial\varphi/\partial t$, but such partial derivatives, *without explanation*, are meaningless: difference quotients of the form

$$\frac{\varphi(\mathbf{x},t+\tau)-\varphi(\mathbf{x},t)}{\tau} \tag{3.12}$$

are generally undefined because there is no assurance that $\mathbf{x}$ lies on $\mathcal{S}(t+\tau)$ when $\mathbf{x}$ lies on $\mathcal{S}(t)$, even for sufficiently small τ (Figure 2). In fact, there are an infinite number of partial derivatives $\partial\varphi/\partial t$ that one may compute, one for any given extension of φ, at each t, to a three three-dimensional neighborhood of $\mathcal{S}(t)$. We now give a natural method of extending φ.

A simple but useful method of *smoothly* extending a surface field $\varphi(\mathbf{x},t)$, at each time, to a three-dimensional region containing the surface is obtained by requiring that φ *be constant on normal lines*, where a normal line at time t is a line through a point $\mathbf{x}$ on $\mathcal{S}(t)$ parallel to $\mathbf{n}(\mathbf{x},t)$. The extension $\hat{\varphi}$ obtained in this manner is referred to as a *normally constant extension* of φ.† Since $\hat{\varphi}$ is constant on normal lines,

$$\mathbf{n}\cdot\operatorname{grad}\hat{\varphi}=0,$$

so that, by (2.4),

$$\operatorname{grad}_{\mathcal{S}}\varphi=\operatorname{grad}\hat{\varphi}. \tag{3.13}$$

Further, $\operatorname{grad}_{\mathcal{S}}\varphi\cdot\mathbf{v}_n=0$, so that, by (3.7),

$$\overset{\scriptscriptstyle\square}{\varphi}=\frac{\partial\hat{\varphi}}{\partial t}. \tag{3.14}$$

This identity asserts that *the normal time-derivative is the conventional partial time-derivative of φ when φ is extended to be constant on normal lines.*

4. Migrating surfaces in fluids

While valid for a surface migrating through a fluid, the transport relation (3.10) is peculiar in that it exhibits no influence of the flow field. We turn now to deriving alternative versions of (3.10) that account for that influence. In this regard, bear in mind

† Since normal lines may cross, such an extension is generally valid, at each t, at most in a neighborhood of $\mathcal{S}(t)$.

that, for φ a surface field, if the surface is *not material* then the material time-derivative $\dot{\varphi}$ *is not well-defined*: since material points flow across $\mathcal{S}(t)$, it is not generally possible to compute a time-derivative holding material points fixed.

4.1. *Fluid velocity. Migrational velocities*

We now suppose that the evolving surface $\mathcal{S}(t)$ is migrating through a fluid. We write $\mathbf{u}(\mathbf{x},t)$ for the velocity of the fluid and assume that this velocity has limiting values $\mathbf{u}^+(\mathbf{x},t)$ and $\mathbf{u}^-(\mathbf{x},t)$ on each side of $\mathcal{S}(t)$, where $\mathbf{u}^+$ denotes the limiting value from that side of $\mathcal{S}$ into which $\mathbf{n}$ points. *We assume that the tangential component* $\mathbf{u}_{\rm tan}$ *of* $\mathbf{u}$ *is continuous across* $\mathcal{S}$, so that

$$\mathbf{u}^+ - \mathbf{u}^- = (\mathbf{u}^+\cdot\mathbf{n} - \mathbf{u}^-\cdot\mathbf{n})\mathbf{n}. \tag{4.1}$$

We continue to write $V(\mathbf{x},t)$ and $\mathbf{v}_n(\mathbf{x},t)$ for the scalar and vector normal-velocities for $\mathcal{S}(t)$. In addition, we let $\mathbf{v}(\mathbf{x},t)$ denote a (for now arbitrary) velocity field for $\mathcal{S}(t)$. Then the fields

$$\mathbf{v} - \mathbf{u}^\pm \tag{4.2}$$

represent *migrational velocites of* $\mathcal{S}$ *relative to the fluid material on each of its sides.*

Consider an arbitrary migrating subsurface $\mathcal{A}(t)$ of $\mathcal{S}(t)$. The velocities $\mathbf{v}_{\partial\mathcal{A}}$ and $V_{\partial\mathcal{A}}$ of $\partial\mathcal{A}$ are as discussed in the paragraph containing (3.8). The motion of $\partial\mathcal{A}$ relative to the bulk material is described by the *migrational velocities* $\mathbf{v}_{\partial\mathcal{A}} - \mathbf{u}^\pm$. Further, bearing in mind that $\boldsymbol{\nu}$ is tangential,

$$\mathbf{u}^+\cdot\boldsymbol{\nu} = \mathbf{u}^-\cdot\boldsymbol{\nu} = \mathbf{u}_{\rm tan}\cdot\boldsymbol{\nu}, \tag{4.3}$$

and hence

$$V^{\rm mig}_{\partial\mathcal{A}} \stackrel{\rm def}{=} V_{\partial\mathcal{A}} - \mathbf{u}^\pm\cdot\boldsymbol{\nu} = V_{\partial\mathcal{A}} - \mathbf{u}_{\rm tan}\cdot\boldsymbol{\nu} \tag{4.4}$$

represents the *normal migrational velocity of* $\partial\mathcal{A}$; that is, the *normal velocity of* $\partial\mathcal{A}$ *relative to the fluid.*

4.2. *Migrationally normal velocity. Migrationally normal time-derivative*

In discussing the formulation of integral balance laws for a surface $\mathcal{S}(t)$ migrating through a fluid, what is needed is a velocity field for $\mathcal{S}$ that characterizes its *migration.* Specifically, we seek a velocity field $\mathbf{v}$ for $\mathcal{S}$ that renders each of the migrational velocities $\mathbf{v} - \mathbf{u}^\pm$ normal. With this in mind, note that

$$\mathbf{v} - \mathbf{u}^\pm = \mathbf{v} - (\mathbf{u}^\pm\cdot\mathbf{n})\mathbf{n} - \mathbf{u}_{\rm tan} = \underline{(V - \mathbf{u}^\pm\cdot\mathbf{n})\mathbf{n}} + (\mathbf{v}_{\rm tan} - \mathbf{u}_{\rm tan}),$$

so that, taking $\mathbf{v}_{\rm tan} = \mathbf{u}_{\rm tan}$, we arrive at a choice of velocity field $\mathbf{v}$ for $\mathcal{S}$ that renders its migrational velocities $\mathbf{v} - \mathbf{u}^\pm$ normal:

$$\mathbf{v} - \mathbf{u}^\pm = (V - \mathbf{u}^\pm\cdot\mathbf{n})\mathbf{n}. \tag{4.5}$$

The resulting velocity field $\mathbf{v}$, called the *migrationally normal velocity-field* for $\mathcal{S}$, has the specific form

$$\mathbf{v} = V\mathbf{n} + \mathbf{u}_{\rm tan} \tag{4.6}$$

and is important *because it is normal when computed relative to the material on either side of* $\mathcal{S}(t)$.

The *migrationally normal time-derivative* of $\varphi(\mathbf{x},t)$ following $\mathcal{S}(t)$ is defined — at an arbitrary time t_0 and point $\mathbf{x}_0$ on $\mathcal{S}(t_0)$ — as follows:

$$\overset{\circ}{\varphi}(\mathbf{x}_0, t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\mathbf{z}(t), t)\bigg|_{t=t_0}, \tag{4.7}$$

where $\mathbf{z}(t)$ is the trajectory through $\mathbf{x}_0$ at time t_0 corresponding to the migrationally normal velocity-field $\mathbf{v} = V\mathbf{n} + \mathbf{u}_{\rm tan}$ (cf. the paragraph containing (3.6)).

4.3. *Relation between the normal time-derivative and the migrationally normal time-derivative*

Let φ be a scalar surface-field and let $\hat{\varphi}$ denote its normally constant extension as defined in §3.4. Then, bearing in mind that the velocity field underlying the definition of $\overset{\circ}{\varphi}$ is the migrationally normal field $\mathbf{v} = V\mathbf{n}+\mathbf{u}_{\rm tan}$, we find, using (3.13), (4.7), and the chain-rule, that

$$\overset{\circ}{\varphi} = \mathbf{v}_{\rm tan}\cdot \mathrm{grad}_{\mathcal{S}}\varphi + \frac{\partial\hat{\varphi}}{\partial t}. \tag{4.8}$$

Thus, by (3.14), the time-derivatives $\overset{\circ}{\varphi}$ and $\overset{\Box}{\varphi}$ are related through the important identity

$$\overset{\circ}{\varphi} = \overset{\Box}{\varphi} + \mathbf{u}_{\rm tan}\cdot \mathrm{grad}_{\mathcal{S}}\varphi. \tag{4.9}$$

4.4. *Transport relations for a surface migrating through a fluid*

In stating integral balance-laws for a phase interface $\mathcal{S}(t)$ migrating through a fluid, one is again confronted with terms of the form (3.9). Of course, the transport relation (3.10) remains valid, but the more important results are obtained when (3.10) is combined with kinematical results that account, explicitly, for the migration of the surface.

Let $\mathcal{A}(t)$ be an evolving subsurface of $\mathcal{S}(t)$ with $V_{\partial\mathcal{A}}(\mathbf{x},t)$ the scalar normal-velocity of $\partial\mathcal{A}(t)$. Further, let $\varphi(\mathbf{x},t)$ be a scalar surface-field and let $\overset{\circ}{\varphi}(\mathbf{x},t)$ denote the migrationally normal time-derivative of $\varphi(\mathbf{x},t)$ following $\mathcal{S}(t)$. Then

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}}\varphi\,\mathrm{d}a &= \int\limits_{\mathcal{A}}\Big\{\overset{\Box}{\varphi} + \mathrm{div}_{\mathcal{S}}(\varphi\mathbf{u}_{\rm tan}) - \varphi K V\Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}}\varphi V^{\rm mig}_{\partial\mathcal{A}}\,\mathrm{d}s,\\
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}}\varphi\,\mathrm{d}a &= \int\limits_{\mathcal{A}}\Big\{\overset{\circ}{\varphi} + \varphi\,\mathrm{div}_{\mathcal{S}}\mathbf{u}_{\rm tan} - \varphi K V\Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}}\varphi V^{\rm mig}_{\partial\mathcal{A}}\,\mathrm{d}s.
\end{aligned}\right\} \tag{4.10}$$

To establish the first of (4.10), we use (3.10), (4.4), and the divergence theorem to justify the following chain of relations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{A}}\varphi\,\mathrm{d}a &= \int\limits_{\mathcal{A}}\Big\{\overset{\Box}{\varphi} - \varphi K V\Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}}\varphi V_{\partial\mathcal{A}}\,\mathrm{d}s,\\
&= \int\limits_{\mathcal{A}}\Big\{\overset{\Box}{\varphi} + \mathrm{div}_{\mathcal{S}}(\varphi\mathbf{u}_{\rm tan}) - \varphi K V\Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}}\varphi(V_{\partial\mathcal{A}} - \mathbf{u}_{\rm tan}\cdot\boldsymbol{\nu})\,\mathrm{d}s,\\
&= \int\limits_{\mathcal{A}}\Big\{\overset{\Box}{\varphi} + \mathrm{div}_{\mathcal{S}}(\varphi\mathbf{u}_{\rm tan}) - \varphi K V\Big\}\,\mathrm{d}a + \int\limits_{\partial\mathcal{A}}\varphi V^{\rm mig}_{\partial\mathcal{A}}\,\mathrm{d}s.
\end{aligned}$$

Further, granted $(4.10)_1$, the second of (4.10) follows upon noting that, by (4.6) and (4.9),

$$\overset{\Box}{\varphi} + \mathrm{div}_{\mathcal{S}}(\varphi\mathbf{u}_{\rm tan}) = \overset{\circ}{\varphi} - \mathbf{u}_{\rm tan}\cdot\mathrm{grad}_{\mathcal{S}}\varphi + \mathrm{div}_{\mathcal{S}}(\varphi\mathbf{u}_{\rm tan}) = \overset{\circ}{\varphi} + \varphi\,\mathrm{div}_{\mathcal{S}}\mathbf{u}_{\rm tan}.$$

5. Material surfaces

5.1. *Kinematical relations*

Assume that $\mathcal{S}(t)$ is a material surface so that, necessarily, *the fluid velocity is continuous across* $\mathcal{S}(t)$. Assume further that $\mathcal{A}(t)$ is a material subsurface of $\mathcal{S}(t)$, so that boundary curve $\partial\mathcal{A}(t)$ a material curve.† Then:

(i) The fluid velocity $\mathbf{u}$ is a velocity field for $\mathcal{S}$; hence the normal velocity of $\mathcal{S}$ and the normal fluid-velocity coincide,

$$V = \mathbf{u}\cdot\mathbf{n}. \tag{5.1}$$

(ii) The migrationally normal velocity-field for $\mathcal{S}$ coincides with the fluid velocity,

$$\mathbf{u} = V\mathbf{n} + \mathbf{u}_{\rm tan}. \tag{5.2}$$

(iii) The material time-derivative $\dot\varphi$ coincides with the time-derivative $\mathring{\varphi}$ following the surface as described by the migrationally normal velocity-field (4.6),

$$\dot\varphi = \mathring{\varphi}. \tag{5.3}$$

(iv) The normal migrational velocity $V^{\rm mig}_{\partial\mathcal{A}}$ vanishes.

Assertion (i) is immediate, as is the relation

$$V_{\partial\mathcal{A}} = \mathbf{u}\cdot\boldsymbol{\nu}, \tag{5.4}$$

which implies (iv). By (i),

$$\mathbf{v} = V\mathbf{n} + \mathbf{u}_{\rm tan} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \mathbf{u}_{\rm tan} = \mathbf{u},$$

which is (ii). Finally, by (ii) and (3.6), the trajectories used to compute (4.7) satisfy

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \mathbf{u}(\mathbf{z}(t),t)$$

and hence represent trajectories of material points. Thus (iii) is satisfied.

5.2. *Transport relations for material surfaces*

The following transport relations follow as consequences of $(4.10)_2$:

If $\mathcal{S}(t)$ is a material surface and $\mathcal{A}(t)$ a material subsurface of $\mathcal{S}(t)$, with boundary curve $\partial\mathcal{A}(t)$ a material curve, then given any scalar surface-field $\varphi(\mathbf{x},t)$,

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}}\varphi\,\mathrm{d}a &= \int_{\mathcal{A}}\Big\{\dot\varphi + \varphi\,\mathrm{div}_{\mathcal{S}}\mathbf{u}_{\rm tan} - \varphi\,(\mathbf{u}\cdot\mathbf{n})K\Big\}\,\mathrm{d}a,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}}\varphi\,\mathrm{d}a &= \int_{\mathcal{A}}\Big\{\dot\varphi + \varphi\,\mathrm{div}_{\mathcal{S}}\mathbf{u}\Big\}\,\mathrm{d}a.\end{aligned}\right\} \tag{5.5}$$

The first of (5.5) follows directly upon using (5.1) and (5.3) in $(4.10)_2$. To establish the second of (5.5), note that, by (2.3),

$$-(\mathbf{u}\cdot\mathbf{n})K = (\mathbf{u}\cdot\mathbf{n})\mathrm{div}_{\mathcal{S}}\mathbf{n} = \mathrm{div}_{\mathcal{S}}\big((\mathbf{u}\cdot\mathbf{n})\mathbf{n}\big) - \underbrace{\mathbf{n}\cdot\mathrm{grad}_{\mathcal{S}}(\mathbf{u}\cdot\mathbf{n})}_{=0},$$

so that

$$-(\mathbf{u}\cdot\mathbf{n})K + \mathrm{div}_{\mathcal{S}}\mathbf{u}_{\rm tan} = \mathrm{div}_{\mathcal{S}}\mathbf{u},$$

† Stated differently: $\mathcal{S}$, $\mathcal{A}$, and $\partial\mathcal{A}$ convect with the fluid.

and $(5.5)_1$ reduces to $(5.5)_2$.

REMARKS

(i) The relation (4.9) between the time-derivatives $\overset{\circ}{\varphi}$ and $\overset{\square}{\varphi}$ of a scalar surface-field is, in some respects, an analog of the relation

$$\dot{\Phi} = \frac{\partial \Phi}{\partial t} + \mathbf{u} \cdot \operatorname{grad} \Phi \tag{5.6}$$

between the material and spatial time-derivatives of a field $\Phi(\mathbf{x}, t)$ whose spatial variable $\mathbf{x}$, at each time, belongs to *an open region* in three-dimensional space: $\overset{\circ}{\varphi}$ is analogous to the material time-derivative $\dot{\Phi}$, $\overset{\square}{\varphi}$ to the spatial time-derivative $\partial\Phi/\partial t$ (cf. (3.14) and (5.3)).

(i) A consequence of the relation $(5.5)_1$ is that if

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{A}} \varphi \, \mathrm{d}a = 0$$

for all $\mathcal{A}$, then

$$\dot{\varphi} - \varphi(\mathbf{u} \cdot \mathbf{n})K + \varphi \operatorname{div}_{\mathcal{S}} \mathbf{u}_{\tan} = 0. \tag{5.7}$$

This relation is consistent with (6) of Stone (1990) provided $\partial\varphi/\partial t$ is computed via the normally constant extension of φ as defined in §3.4, so that

$$\frac{\partial \varphi}{\partial t} = \dot{\varphi} - \mathbf{u}_{\tan} \cdot \operatorname{grad}_{\mathcal{S}} \varphi. \tag{5.8}$$

The relation (5.8) is formally analogous to the relation between the material and spatial time-derivatives of a bulk field.

(ii) The version $(5.5)_2$ of the transport relation for a material surface is established, for example, by Slattery (1972).

(iii) For an evolving *flat* material surface, the equation $(5.5)_1$ represents the two-dimensional version of the Reynolds (1903) transport relation (cf. Gurtin (1981, p. 78)).

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