TOWARDS STUDYING OF THE HIGHER RANK THEORY OF STABLE PAIRS

BY

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DISSERTATION

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Abstract

This thesis is composed of two parts. In the first part we introduce a higher rank analog of the Pandharipande-Thomas theory of stable pairs [28] on a Calabi-Yau threefold X. More precisely, we develop a moduli theory for frozen triples given by the data $\mathcal{O}_X^{\oplus r}(-n) \xrightarrow{\phi} F$ where F is a sheaf of pure dimension 1. The moduli space of such objects does not naturally determine an enumerative theory: that is, it does not naturally possess a perfect symmetric obstruction theory. Instead, we build a zero-dimensional virtual fundamental class by hand, by truncating a deformation-obstruction theory coming from the moduli of objects in the derived category of X. This yields the first deformation-theoretic construction of a higher-rank enumerative theory for Calabi-Yau threefolds. We calculate this enumerative theory for local $\mathbb{P}^1$ using the Graber-Pandharipande [10] virtual localization technique.

In the second part of the thesis we compute the Donaldson-Thomas type invariants associated to frozen triples using the wall-crossing formula of Joyce-Song [18] and Kontsevich-Soibelman [22].

In memory of Jerrold E. Marsden (1942-2010)

to my parents Zhaleh and Majid

and to Maryam.

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Chapter 1

Introduction

The work of algebraic geometers to understand the rigorous mathematical structure of Gromov-Witten invariants has led to introduction of new theories such as Donaldson-Thomas [32] and Pandharipande-Thomas theories [28]. The numerical invariants computed in each theory are conjecturally related to each other but the complete understanding of the connection between these invariants and invariants in Gromov-Witten theory has not yet been achieved. During several past years there has been a growth of interest in computing invariants associated to higher rank analogue of these theories. Toda [33] and Nagao [23] have succeeded in computing a class of higher rank Donaldson-Thomas type invariants using the the wall-crossing technology developed by Kontsevich-Soibelman [22] and Joyce-Song [18]. In this thesis we introduce two different techniques in computing the higher rank Donaldson-Thomas type invariants. The first involves the deformation theoretic higher rank enumerative theory for Calabi-Yau threefolds. One of our main results is the construction of a zero-dimensional virtual fundamental class for objects given as higher rank analogue of stable pairs in [28]. We carry out calculations over toric Calabi-Yau threefolds such as local $\mathbb{P}^1$ to compute invariants associated to these objects using the method of virtual localization [10]. In what follows we explain some of the required background in more detail:

In [27] and [28] the authors introduce stable pairs given by a tuple (F, s) where $s \in \mathrm{H}^0(X, F)$ and F is a pure sheaf with fixed Hilbert polynomial and fixed second Chern character which has one dimensional support. It is shown that there exists a virtual fundamental class of degree zero over the moduli space of stable pairs and the invariants are defined by integration against this class. In this thesis we define a higher rank analogue of stable pairs:

Let X be a nonsingular Calabi-Yau 3-fold over $\mathbb{C}$ with $\mathrm{H}^1(\mathcal{O}_X) = 0$ and with a fixed polarization L.

A triple of type (P_1, P_2) over X is given by a tuple (E_1, E_2, ϕ) where E_1 and E_2 have fixed Hilbert polynomials P_1 and P_2 respectively, E_2 is a pure sheaf with one dimensional support over X and $\phi : E_1 \to E_2$ is a holomorphic morphism.

We will introduce the notion of frozen triples of type (P_2, r) which means that in a given triple (E_1, E_2, ϕ), $E_1 \cong \mathcal{O}_X^{\oplus r}(-n)$ and E_2 has fixed Hilbert polynomial P_2. In other words we "freeze" E_1 to be isomorphic to $\mathcal{O}_X^{\oplus r}(-n)$ but the choice of this isomorphism is not fixed. We will also work with closely related objects called highly frozen triples given as quadruples (E_1, E_2, ϕ, ψ) where E_1, E_2 and ϕ have the same definition as before but this time we have "highly" frozen the triple by fixing a choice of isomorphism $\psi : E_1 \cong \mathcal{O}_X^{\oplus r}(-n)$.

We study the frozen and the highly frozen triples in families. The key strategy in construction of the moduli space of triples is to view a triple (E_1, E_2, ϕ) as an oriented tree with two vertices $\bullet \xrightarrow{\phi} \bullet$ which is decorated with the data associated to E_1 and E_2 (such as their Hilbert polynomials). Schmitt in [30] has given a GIT construction of moduli space of oriented trees with n vertices composed of torsion free sheaves. In addition, a more general treatment for similar GIT constructions when the corresponding sheaves are pure is given by Malte wandel [35]. However, since the GIT constructions normally give rise to coarse (rather than fine) moduli spaces we switch gears and essentially give a stacky construction of our moduli spaces.

One obtains a numerical stability condition by considering stability of points in the underlying (coarse GIT) moduli space using the Hilbert-Mumford criterion. In special case, for moduli space of frozen triples, it is shown that via some rearrangements the stability condition for frozen triples is written as the stability used by Le Potier for coherent systems. Schmitt's and, more specifically, Le Potier's stability conditions depend on choice of a stability parameter q. In the first part of this thesis (where we compute the invariants using virtual localization tecchnique) we consider the large limit stability which is equivalent to choosing $q \to \infty$, hence our stability becomes compatible with the stability used in [28] by Pandharipande and Thomas. Later, in the second part of the thesis, we consider the $q \to 0$ limit stability and we compute similar higher rank invariants using the wallcrossing technique developed by Kontsevich-Soibelman [22] and Joyce-Song [18].

The computation of invariants associated to frozen or highly frozen triples depends on construction of the virtual fundamental class over their moduli spaces. The key ingredient in construction of the virtual class is a well-behaved deformation-obstruction theory. The description of the deformation obstruction theory differs from case to case depending on the geometric structure of the moduli space under consideration, hence it is important to study the geometry of the moduli spaces of frozen and highly frozen triples.

We show that, $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$, the moduli space of stable frozen triples of type (P_2,r) is an Artin stack. Moreover, $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$, the moduli space of stable highly frozen triples of type (P_2,r), is a Deligne-Mumford stack. It is crucial to find the relationship between $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. It is shown that $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a principal $\mathrm{GL}_r(\mathbb{C})$ bundle over $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and there exists a forgetful map $\pi^{\mathfrak{M}}_{\mathrm{FT}}:\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')\to\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.

It is important to note that throughout this thesis we work over the open substacks $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')\subset\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')\subset\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ as follows:

1. $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')=\{(E_1,E_2,\phi,\psi)\in\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')\mid \mathrm{H}^1(E_2(n))=0\}$.
2. $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')=\{(E_1,E_2,\phi)\in\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')\mid \mathrm{H}^1(E_2(n))=0\}$.

We construct a well-behaved deformation obstruction theory for DM stack of highly frozen triples $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and hence obtain a virtual fundamental class:

The first step is to understand the deformations of frozen and highly frozen triples. Take a parametrizing scheme S of finite type over $\mathbb{C}$. A family of frozen triples is given by a tuple $(\mathcal{E},\mathcal{F},\phi)$ where $\mathcal{E}\cong\mathcal{O}^{\oplus r}_{X\times S}(-n)$, $\mathcal{F}$ denotes a family of pure one dimensional sheaves flat over S and

$$\phi:\mathcal{E}\to\mathcal{F}.$$

Moreover, the fiber of this family over every point $s\in S$ is given by a stable frozen triple over X. Take a family of stable pairs (stable frozen triples of rank 1) over S. Let $I^{\bullet}_S$ be the family of complexes associated to this family. Consider a nilpotent thickening, S' of S. Define the deformation of $I^{\bullet}_S$ as a complex $I^{\bullet}_{S'}$ such that the derived restriction of $I^{\bullet}_{S'}$ to S is quasi-isomorphic to $I^{\bullet}_S$. In Theorem 2.7 [28] the authors show that for such nilpotent thickenings to all orders, the complex

$I^\bullet_{S'}$ is quasi-isomorphic to the complex

$$\mathcal{O}_{X\times S'}(-n) \to \mathcal{F}.$$

Here we follow a similar strategy. We show how the frozen triples of rank $r > 1$ deform in families. Take any point $p \in \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. Here p is given as a stable frozen triple represented by a complex

$$I^\bullet : E \to F.$$

We compute the tangent space of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ at p. We show that there exists a map of groups

$$\mathrm{End}(\mathcal{O}_X(-n)^{\oplus r}) \xrightarrow{g} \mathrm{Hom}(I^\bullet, F)$$

and the tangent space is obtained as the quotient of $\mathrm{Hom}(I^\bullet, F)$ by the image of $\mathrm{End}(\mathcal{O}_X(-n)^{\oplus r})$ under g. The following theorem describes this statement.

Theorem (5.15): *Use notation in Definition 2.10. Fix a map $f : S \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. Let S' be a square-zero extension of S with ideal $\mathcal{I}$. Let $\mathcal{D}ef_S(S', \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))$ denote the deformation space of the map f obtained by the set of possible deformations, $f' : S' \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. The following statement is true:*

$$\mathcal{D}ef_S(S', \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')) \cong \mathrm{Hom}(I^\bullet_S, F) \otimes \mathcal{I} / \mathrm{Im}\left((\mathrm{End}(\mathcal{M}_S) \to \mathrm{Hom}(I^\bullet_S, \mathcal{F})) \otimes \mathcal{I}\right) \tag{1.1}$$

We show that the tangent space at p is isomorphic to deformations of the complex $I^\bullet$ with fixed determinant that represents p. The deformation theory of this complex is obtained by $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ and $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ where sub-index 0 indicates the trace-free group:

Theorem (5.10): *Let $p \in \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ be a point represented by a τ'-limit-stable frozen triple $\{(\mathcal{O}_X(-n)^{\oplus r}, F, \phi)\}$. Let*

$$I^\bullet := \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F$$

be a complex with trivial determinant. The following is true:

$$T_p\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0. \tag{1.2}$$

More generally we prove that the deformation $I^\bullet_{S'}$ of $I^\bullet_S$ for nilpotent thickenings of S to all orders is quasi-isomorphic to the complex given by

$$\mathcal{O}^{\oplus r}_{X\times S'}(-n) \to \mathcal{F}.$$

As mentioned above, the moduli stack of highly frozen triples has the structure of a DM stack. In this case, the deformation obstruction theory is given by a morphism in the derived category:

$$ob : \mathbb{G}^\bullet \to \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')},$$

where $\mathbb{G}^\bullet$ is a perfect complex of amplitude $[-1,0]$ and moreover, $h^0(ob)$ is an isomorphism and $h^{-1}(ob)$ is an epimorphism. Here $\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')}$ denotes the 2 term truncated cotangent complex, in degrees -1 and 0, associated to the moduli stack.

For the case of Artin stack of stable frozen triples, the truncated cotangent complex contains a nonzero term in degree 1 and for deformation-obstruction theory, we require a morphism in the derived category:

$$ob : \mathbb{E}^\bullet \to \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')},$$

where $\mathbb{E}^\bullet$ is a perfect complex of amplitude $[-1,1]$ and moreover $h^1(ob)$ and $h^0(ob)$ are isomorphisms and $h^{-1}(ob)$ is an epimorphism [25].

We construct a deformation obstruction theory over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ which has all the nice cohomological properties however it is perfect of amplitude $[-2,1]$.

Theorem (6.7): *There exists a map in the derived category,*

$$R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0 \otimes \pi^*_X\omega_X\right)[2] \xrightarrow{ob} \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}.$$

After suitable truncations, there exists a 4 term complex $\mathbb{E}^{\bullet}$ *of locally free sheaves , such that* $\mathbb{E}^{\bullet\vee}$ *is self-symmetric of amplitude* $[-2,1]$ *and there exists a map in the derived category,*

$$\mathbb{E}^{\bullet\vee} \xrightarrow{ob^t} \mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \tag{1.3}$$

such that $h^{-1}(ob^t)$ *is surjective, and* $h^0(ob^t)$ *and* $h^1(ob^t)$ *are isomorphisms. .*

The computation of invariants over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ or $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ using the conventional methods is not possible because of the following main reasons:

Key obstacles: The truncation of $\mathbb{E}^{\bullet\vee}$ in Theorem 6.7 from 4 terms to 3 terms is not possible over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$, otherwise one may use Noseda's conjectural construction in [25] to directly define a virtual fundamental class for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. On the other hand constructing a well-behaved deformation obstruction theory over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ using the techniques discussed in [28] is not possible either, hence we propose the following strategy:

Strategy: Pullback $\mathbb{E}^{\bullet\vee}$ in Theorem 6.7 to $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ via the forgetful map $\pi^{\mathfrak{M}}_{\mathrm{FT}}$. Consider a Deligne-Mumford affine cover of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. Then show that locally one may apply a suitable truncation mechanism to the pulled-back complex so that it is ensured that the truncated complex satisfies the conditions for the perfect deformation-obstruction theory of perfect amplitude $[-1,0]$ over the DM stack of stable highly frozen triples:

Theorem (6.12) *Consider the 4-term deformation obstruction theory* $\mathbb{E}^{\bullet\vee}$ *of perfect amplitude* $[-2,1]$ *over* $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.

1. Locally in the étale topology over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ *there exists a perfect two-term deformation obstruction theory of perfect amplitude* $[-1,0]$ *which is obtained from the suitable local truncation of the pullback* $(\pi^{\mathfrak{M}}_{\mathrm{FT}})^*\mathbb{E}^{\bullet\vee}$.

2. This local theory defines a globally well-behaved virtual fundamental class over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

The construction mentioned above can be extended to the case where X is given as a toric Calabi

Yau threefold. There exists an action of $\mathbf{G} = \mathbf{T} \times \mathrm{T}_0$ on the DM stack of highly frozen triples. The action of $\mathbf{T}$ is induced by $(\mathbb{C}^*)^3$ action on X, and T_0 denotes the action of $(\mathbb{C}^*)^r$ such that over a point $p \in \mathfrak{H}^r_{\mathrm{s,HFT}}(\tau')$ represented by $\mathcal{O}_X(-n)^{\oplus r} \to F$, each factor of $\mathbb{C}^*$ in $(\mathbb{C}^*)^r$ scales each factor in the fibers of $\mathcal{O}_X(-n)^{\oplus r}$ independently. We show that when $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1)$ then $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') = \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') = \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. It is shown that the action of $\mathbf{G}$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ induces a weight decomposition on the $\mathbf{G}$-equivariant stable highly frozen triples due to which the triples decompose as:

$$I^{\bullet,\mathbf{G}} \cong \bigoplus_{i=1}^{r} (\mathcal{O}_X(-n) \to F_i)\,. \tag{1.4}$$

This decomposition, in particular, is due to the action of T_0 on the highly frozen triples. The consequence of identifying $\mathbf{G}$-equivariant stable highly frozen triples as multiple copies of stable pairs as in (1.4), is that the $\mathbf{G}$-fixed components of the moduli stack of highly frozen triples is obtained as an r-fold product of $\mathbf{T}$-fixed components of the moduli stack of stable pairs which are conjectured by Pandharipande and Thomas in [28] (Conjecture 2) to be nonsingular and compact. Let $\mathbf{Q}$ denote a $\mathbf{G}$-fixed component of $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$. Let $(G_{0,\mathbf{Q}})^{\mathbf{G}}$ and $(G_{1,\mathbf{Q}})^{\mathbf{G}}$ denote the $\mathbf{G}$-equivariant terms in degrees 0 and 1 of the restriction to $\mathbf{Q}$ of the dual of $\mathbb{G}^\bullet$ in Theorem 6.12. Given that $\mathbf{Q}$ is nonsingular and compact, by the virtual localization formula [10], the virtual fundamental class of $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ is obtained as:

$$\left[\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')\right]^{vir} = \sum_{\mathbf{Q} \subset \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')} \iota_{\mathbf{Q}*}\left(\frac{e(G_{1,\mathbf{Q}})}{e(G_{0,\mathbf{Q}})} \cdot e(T_{\mathbf{Q}}) \cap [\mathbf{Q}]\right). \tag{1.5}$$

The invariants associated to the highly frozen triples can be obtained by

$$\mathrm{HFT}(r,n,\beta) = \int_{\left[\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')\right]^{vir}} 1.$$

We generalize the method of box counting in [27] and compute the invariants associated to highly frozen triples for $r = 2$ over total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ (Example 11.2). We also present an algorithm for similar computations when X is given as total space of $\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2$

(Example 11.3).

In Chapter 12 we use a different approach to compute the invariants of objects with similar properties to frozen and highly frozen triples for when the stability parameter $q \to 0$. Here we use the wall-crossing techniques. Assuming that the sheaf F appearing in frozen triples is given as a quotient sheaf with zero-dimensional support, using our calculations one obtains compatible results with computations in [23]. Our strategy here is to use both ideas of Joyce-Song [18] and Kontsevich-Soibelman [22] in wall-crossings to compute the invariants, however our computations and notations mainly follow the work of Joyce and Song [18].

Chapter 2

Definition of triples

Definition 2.1. Let X be a nonsingular projective Calabi-Yau 3-fold over $\mathbb{C}$ (i.e $K_X \cong \mathcal{O}_X$ and $\pi_1(X) = 0$ which implies $\mathrm{H}^1(\mathcal{O}_X) = 0$) with a fixed polarization L . A holomorphic triple supported over X is given by (E_1, E_2, ϕ) consisting of a torsion free coherent sheaf E_1 and a pure sheaf with one dimensional support E_2, together with a holomorphic morphism $\phi : E_1 \to E_2$.

A homomorphism of triples from $(\acute{E}_1, \acute{E}_2, \acute{\phi})$ to (E_1, E_2, ϕ) is a commutative diagram:

$$\begin{array}{ccc} \acute{E}_1 & \xrightarrow{\acute{\phi}} & \acute{E}_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

Remark 2.2. A triple (E_1, E_2, ϕ) of type (P_1, P_2, β) is given by a triple such that $P(E_1(m)) = P_1$ and $P(E_2(m)) = P_2$ and $\beta = \mathrm{ch}_2(E_2)$ as defined in Definition 2.8. During the discussion, for simplicity, we omit β and write a triple of type (P_1, P_2).

Remark 2.3. Since by assumption the sheaf E_2 has one dimensional support, the Hilbert polynomial of E_2 in variable m satisfies:

$$P(E_2(m)) = \chi(E_2(m)) = m \int_\beta c_1(L) + d. \tag{2.1}$$

Here $c_1(L)$ is the first Chern class of the fixed polarization L over X and $d \in \mathbb{Z}$ and β as before is $\mathrm{ch}_2(E_2)$. Note that P_2 is a polynomial of degree$= \dim(\mathrm{Supp}(E_2)) = 1$ and by rank of E_2 (denoted by $rk(E_2)$) we mean the leading coefficient of P_2.

Definition 2.4. A *frozen triple* of rank r is a special case of a holomorphic triple where $E_1 \cong$

$\mathcal{O}_X(-n)^{\oplus r}$ for some $n \in \mathbb{Z}$.

Remark 2.5. By freezing the triple we mean fixing E_1 to be isomorphic with $\mathcal{O}_X(-n)^{\oplus r}$. We do not make a choice of such an isomorphism here. Later we fix an isomorphism $\psi : E_1 \xrightarrow{\cong} \mathcal{O}_X(-n)^{\oplus r}$and we call the triples *highly-frozen* triples.

Definition 2.6. Use the notation above. Let S be a $\mathbb{C}$ scheme of finite type and let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ be the corresponding projections. An *S-flat family* of triples over X is a triple $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ consisting of a morphism of $\mathcal{O}_{X\times S}$ modules $\mathcal{E}_1 \xrightarrow{\phi} \mathcal{E}_2$ such that $\mathcal{E}_1$ and $\mathcal{E}_2$ are flat over S and for every point $s \in S$ the fiber $(\mathcal{E}_1, \mathcal{E}_2, \phi) \mid_s$ is given by a holomorphic triple as in Definitions 2.1.

Two S-flat families of triples $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ and $(\mathcal{E}'_1, \mathcal{E}'_2, \phi')$ are isomorphic if there exists a commutative diagram of the form:

$$\begin{array}{ccc} \acute{\mathcal{E}}_1 & \xrightarrow{\acute{\phi}} & \acute{\mathcal{E}}_2 \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{E}_1 & \xrightarrow{\phi} & \mathcal{E}_2 \end{array}$$

Definition 2.7. An S-flat family of frozen-triples is a triple $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ consisting of a morphism of $\mathcal{O}_{X\times S}$ modules $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy the condition of Definition 2.6 and moreover $\mathcal{E}_1 \cong \pi_X^*\mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{M}_S$ where $\mathcal{M}_S$ is a vector bundle of rank r on S.

Two S-flat families of frozen-triples $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ and $(\mathcal{E}'_1, \mathcal{E}'_2, \phi')$ are isomorphic if there exists a commutative diagram:

$$\begin{array}{ccc} \acute{\mathcal{E}}_1 & \xrightarrow{\acute{\phi}} & \acute{\mathcal{E}}_2 \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{E}_1 & \xrightarrow{\phi} & \mathcal{E}_2 \end{array}$$

Definition 2.8. A frozen-triple of class β and of fixed Hilbert polynomial P_2 is a frozen-triple (E_1, E_2, ϕ) such that the Hilbert polynomial of E_2 is equal to P_2 and $\beta = \mathrm{ch}_2(E_2)$. Having fixed r in $E_1 \cong \mathcal{O}_X^{\oplus r}(-n)$, we denote these frozen triples as frozen triples of type (P_2, r).

Now we define highly frozen triples.

Definition 2.9. A *highly frozen triple* is a quadruple (E_1, E_2, ϕ, ψ) where (E_1, E_2, ϕ) is a frozen triple as in Definition 2.4 and $\psi : E_1 \xrightarrow{\cong} \mathcal{O}_X(-n)^{\oplus r}$ is a fixed choice of isomorphism. A morphism between highly frozen triples $(E'_1, E'_2, \phi', \psi')$ and (E_1, E_2, ϕ, ψ) is a morphism $E'_2 \xrightarrow{\rho} E_2$ such that the following diagram is commutative.

$$\begin{array}{ccccc} \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\psi'^{-1}} & E'_1 & \xrightarrow{\acute{\phi}} & E'_2 \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle \rho} \\ \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\psi^{-1}} & E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

Definition 2.10. An S-flat family of highly frozen-triples is a quadruple $(\mathcal{E}_1, \mathcal{E}_2, \phi, \psi)$ consisting of a morphism of $\mathcal{O}_{X\times S}$ modules $\mathcal{E}_1 \xrightarrow{\phi} \mathcal{E}_2$ such that $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy the condition of Definition 2.6 and moreover $\psi : \mathcal{E}_1 \xrightarrow{\cong} \pi_X^*\mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{O}_S^{\oplus r}$ is a fixed choice of isomorphism.

Two S-flat families of highly frozen-triples $(\mathcal{E}_1, \mathcal{E}_2, \phi, \psi)$ and $(\mathcal{E}'_1, \mathcal{E}'_2, \phi', \psi')$ are isomorphic if there exists a commutative diagram:

$$\begin{array}{ccccc} \pi_X^*\mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{O}_S^{\oplus r} & \xrightarrow{\psi'^{-1}} & \mathcal{E}'_1 & \xrightarrow{\acute{\phi}} & \mathcal{E}'_2 \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\ \pi_X^*\mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{O}_S^{\oplus r} & \xrightarrow{\psi^{-1}} & \mathcal{E}_1 & \xrightarrow{\phi} & \mathcal{E}_2 \end{array}$$

Definition 2.11. Let $q_1(m)$ and $q_2(m)$ be positive rational polynomials of degree at most 2. A triple $T = (E_1, E_2, \phi)$ of type (P_1, P_2) is called $\acute{\tau}$-semistable (respectively, stable) if for any subsheaves F_1 of E_1 and F_2 of E_2 such that $0 \neq F_1 \oplus F_2 \neq E_1 \oplus E_2$ and $\phi(F_1) \subset F_2$:

$$\begin{aligned} & q_2(m)\left(P_{F_1} - rk(F_1)\left(\frac{P_1}{rk(E_1)} - \frac{q_1(m)}{rk(E_1)}\right)\right) \\ & + q_1(m)\left(P_{F_2} - rk(F_2)\left(\frac{P_2}{rk(E_2)} + \frac{q_2(m)}{rk(E_2)}\right)\right) \leq 0/resp. < 0. \end{aligned} \tag{2.2}$$

The construction of the parametrizing scheme of stable frozen and highly frozen triples is a specialization of Schmitt's construction of moduli space of oriented trees with n vertices [30] to the case where $n = 2$. Although it is straightforward to specialize Schmitt's construction to our setup, in order to keep completeness, we have included this construction in the later sections.

2.1 Stability for frozen triples and higher rank PT pairs

Comparison between $\acute{\tau}$-stability and Le Potier's stability for coherent systems

Consider the special case in which the triples are given as coherent systems, i.e when $E_1 \cong \Gamma \otimes \mathcal{O}_X$ such that $\Gamma \subset \mathrm{H}^0(E_2)$. For simplicity we denote this coherent system by (Γ, E_2). Recall that by a sub-coherent system we mean a pair $(\acute{\Gamma}, \acute{E}_2) \subset (\Gamma, E_2)$ which is given by $\acute{\Gamma} \otimes \mathcal{O}_X \to \acute{E}_2$, i.e a sub-sheaf $0 \to \acute{E}_2 \xrightarrow{i} E_2$ and $\acute{\Gamma} \subset \mathrm{H}^0(\acute{E}_2)$ such that $i(\acute{\Gamma}) \subset \Gamma$.

Remark 2.12. We intend to work with one stability parameter. We use the rational function $q(m)$ instead of $q_1(m)$ and $q_2(m)$ by setting $q_2(m)/q_1(m) := q(m)$.
Assume $\Gamma \otimes \mathcal{O}_X \to E_2$ is $\acute{\tau}$-semistable, i.e for all $(\acute{\Gamma}, \acute{E}_2) \subset (\Gamma, E_2)$:

$$\begin{aligned} & q(m)\left(dim(\Gamma') \cdot P_{\mathcal{O}_X} - dim(\Gamma') \cdot \left(\frac{dim(\Gamma) \cdot P_{\mathcal{O}_X}}{dim(\Gamma)} - \frac{q_1(m)}{dim(\Gamma)}\right)\right) \\ & + \left(P_{\acute{E}_2} - rk(\acute{E}_2)\left(\frac{P_{E_2}}{rk(E_2)} + \frac{q_2(m)}{rk(E_2)}\right)\right) \leq 0, \end{aligned} \tag{2.3}$$

Hence one obtains:

$$\begin{aligned} & q(m)\left(\cancel{dim(\Gamma') \cdot P_{\mathcal{O}_X}} - \cancel{dim(\Gamma') \cdot P_{\mathcal{O}_X}} + \frac{dim(\Gamma') \cdot q_1(m)}{dim(\Gamma)}\right) \\ & + \left(P_{\acute{E}_2} - rk(\acute{E}_2)\left(\frac{P_{E_2}}{rk(E_2)} + \frac{q_2(m)}{rk(E_2)}\right)\right) \leq 0, \end{aligned} \tag{2.4}$$

By carefully rewriting, one obtains:

$$\frac{P_{E_2}}{rk(E_2)} + \frac{q_2(m)}{\dim(\Gamma)} \cdot \frac{\dim(\Gamma)}{rk(E_2)} \geq \frac{P_{\acute{E}_2}}{rk(\acute{E}_2)} + \frac{q_2(m)}{\dim(\Gamma)} \cdot \frac{\dim(\acute{\Gamma})}{rk(\acute{E}_2)} \tag{2.5}$$

Which is similar to Le Potier's criteria for stability of (Γ, E_2) if we require this inequality to hold for every choice of sub-coherent systems $(\acute{\Gamma}, \acute{E}_2)$. We explain this similarity in the remark below.

Remark 2.13. One may rescale the stability parameter with constant numbers. For example choosing the stability parameter to be $q(m) = \frac{q_2(m)}{q_1(m)\cdot\dim(\Gamma)}$ gives the stability condition for coherent systems. The coherent systems are naively a subset of frozen triples in Definition 2.8 and so far we have shown that every $\acute{\tau}$-semistable triple, given as a coherent system, is stable in the sense of Le Potier [29] and with more effort it can be shown that a stable coherent system (thought of as a frozen triple) would also be stable with respect to stability condition for frozen triples. We leave the proof of this fact to the interested reader.

2.1.1 Statement of $\acute{\tau}$-stability for frozen triples of type (P_2, r)

Use notation of Definition 2.8 and Remark 2.3. We study stability for frozen triples of type (P_2, r). Fix a frozen triple (E_1, E_2, ϕ) of type (P_2, r). The subtriples of this frozen triple are given by triples of the form (G_1, G_2, ψ) for which the following diagram commutes:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{\psi} & G_2 \\ \downarrow & & \downarrow \\ E_1 \cong \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & E_2 \end{array} \quad . \tag{2.6}$$

The stability assumption means that for (E_1, E_2, ϕ) the following condition is satisfied:

$\forall G_1 \subset E_1$ and $\forall G_2 \subset E_2$ such that $0 \neq G_1 \oplus G_2 \neq \mathcal{O}_X(-n)^{\oplus r} \oplus E_2 \neq 0$ and $\phi(G_1) \subset G_2$:

$$q_2(m)\left(P_{G_1} - rk(G_1)\left(\frac{P_{E_1}}{r} - \frac{q_1(m)}{r}\right)\right) +$$
$$q_1(m)\left(P_{G_2} - rk(G_2)\left(\frac{P_{E_2}}{rk(E_2)} + \frac{q_2(m)}{rk(E_2)}\right)\right) \leq 0. \tag{2.7}$$

Taking the sub-triple to be $\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\psi} G_2$ such that $G_2 \subset E_2$ then the stability condition is written as:

$$q_2(m)\left(\cancel{P_{\mathcal{O}_X(-n)^{\oplus r}}} - \cancel{r\left(\frac{P_{\mathcal{O}_X(-n)^{\oplus r}}}{r}\right)}\right) + q_2(m) \cdot q_1(m)$$
$$+ q_1(m)\left(P_{G_2} - rk(G_2)\left(\frac{P_{E_2}}{rk(E_2)} + \frac{q_2(m)}{rk(E_2)}\right)\right) < 0. \tag{2.8}$$

As in Remark 2.12, since we are interested in the ratio $\frac{q_2(m)}{q_1(m)}$, we assume $q_2(m) = q(m)$ and $q_1(m) = 1$, so dividing by $rk(G_2)$ we obtain:

$$\frac{P_{G_2}}{rk(G_2)} + \frac{q(m)}{rk(G_2)} \leq \frac{P_{E_2}}{rk(E_2)} + \frac{q(m)}{rk(E_2)}. \tag{2.9}$$

Which is again somewhat similar to Le Potier's condition for coherent systems [29]. We are now ready to give a complete $\acute{\tau}$-stability condition for frozen triples of type (P_2, r):

Definition 2.14. Let $q(m)$ be given by a polynomial with rational coefficients such that its leading coefficient is positive. A frozen triple (E_1, E_2, ϕ) of type (P_2, r) is $\acute{\tau}$-stable with respect to $q(m)$ if and only if:

1. for all proper nonzero subsheaves $G \subset E_2$ for which ϕ does not factor through G we have:

$$\frac{P_G}{rk(G)} < \frac{P_{E_2}}{rk(E_2)} + \frac{q(m)}{rk(E_2)}.$$

2. For all subsheaves, $G \subset E_2$ which the map ϕ factors through:

$$q(m) + \left(P_G - rk(G)\left(\frac{P_{E_2}}{rk(E_2)} + \frac{q(m)}{rk(E_2)}\right)\right) < 0. \tag{2.10}$$

Remark 2.15. It is trivially seen that equations (2.9) and (2.10) are exactly equivalent to each other.

2.1.2 $q(m) \to \infty$ limit stability for frozen triples of type (P_2, r)

As we discussed in the previous sections, whether we study a triple of type (P_1, P_2) or a frozen triple of type (P_2, r) the stability condition depends on the parameter $q(m)$. As a consequence, the moduli of $\acute{\tau}$-(semi)stable objects depends on this parameter too. We consider moduli of frozen triples when $q(m) \to \infty$. In [28] (Lemma 3.1) Pandharipande and Thomas study the stable pairs $\phi : \mathcal{O}_X \to F$ and show that asymptotically their notion of stability is equivalent to requiring the sheaf F to be given as a pure one dimensional sheaf and the map ϕ to be generically surjective. We show below that the statement of asymptotic stability condition for frozen and highly frozen triples is similar to stability of PT pairs [28].

Definition 2.16. Fix $q(m)$ to be given as a polynomial of degree at least 2 with rational coefficients such that its leading coefficient is positive. A frozen (respectively highly frozen) triple of type (P_2, r) is called to be $q(m) \to \infty$ τ'-limit-stable if it is stable in the sense of Definition 2.14 with respect to this fixed choice of $q(m)$.

Lemma 2.17. *Let $q(m)$ be a polynomial as in Definition 2.16. A frozen triple (E_1, E_2, ϕ) of type (P_2, r) is $\acute{\tau}$-limit-stable if and only if the map $E_1 \xrightarrow{\phi} E_2$ has zero dimensional cokernel.*

Proof. For simplicity, we use $\mathcal{O}_X^{\oplus r}(-n)$ instead of E_1. The exact sequence $0 \to K \to \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} E_2 \to Q \to 0$ induces a short exact sequence:

$$0 \to \mathrm{Im}(\phi) \to E_2 \to Q \to 0$$

Therefore one obtains the following commutative diagram of the triples:

$$\begin{array}{ccc} \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & \mathrm{Im}(\phi) \\ = \downarrow & & \downarrow \\ \mathcal{O}_X(-n)^{\oplus r} & \longrightarrow & E_2 \end{array}$$

Now we assume that $\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} E_2$ is a $q(m) \to \infty \acute{\tau}$-limit-stable triple :

$$q(m) + \left(P_{\mathrm{Im}(\phi)} - rk(\mathrm{Im}(\phi)) \cdot \left(\frac{P_{E_2}}{rk(E_2)} + \frac{q(m)}{rk(E_2)} \right) \right) < 0. \tag{2.11}$$

In other words by rearrangement:

$$q(m) \left(1 - \frac{rk(\mathrm{Im}(\phi))}{rk(E_2)} \right) < rk(\mathrm{Im}(\phi)) \frac{P_{E_2}}{rk(E_2)} - P_{\mathrm{Im}(\phi)}.$$

Consider the polynomials on both sides of inequality (2.11) with respect to the variable m. One sees that the right hand side of (2.11) is a polynomial in m of degree at most 1. However by the choice $q(m)$ as in Definition 2.16 one sees that the left hand side of the inequality is given by a polynomial of degree at least two with positive leading coefficient. Hence the left hand side becomes larger than the right hand side and the only way for the inequality to make sense is to have the left hand side to be equal to zero, i.e $rk(\mathrm{Im}(\phi)) = rk(E_2)$ and therefore Q must be a zero dimensional sheaf. For the other direction: Assume that Q is *not* a zero dimensional sheaf and the triple is $\acute{\tau}$-limit-stable. Now by similar argument, since degree of $q(m)$ is chosen to be sufficiently large enough, $rk(\mathrm{Im}(\phi)) = rk(E_2)$ which contradicts the assumption of Q not being zero dimensional sheaf and this finishes the proof. □

By Lemma 2.17 it is seen that $q(m) \to \infty$ τ'-limit stable pairs are given as the higher rank analog of PT stable pairs [28].

Remark 2.18. We proved in Lemma 2.17 that the notion of $q(m) \to \infty$ coincides with the notion of $q(m)$-stability for a suitable choice of $q(m)$ given in Definition 2.16. The important outcome of this conclusion is that for a suitable choice of $q(m)$ the notion of $q(m) \to \infty$ τ'-limit-stability comes from a GIT notion of stability.

Chapter 3

Construction of moduli stacks

To construct a well-behaved moduli space of stable triples or frozen-triples the first step is to make sure that the family of (semistable triples of a given type (P_1, P_2) is bounded.

3.1 Boundedness

In this section we quote the results obtained by Malte Wandel in [35] which helps us to get a well-behaved scheme that parametrizes the triples of a fixed given type. In [35] (Definition 1.1) Wandel studies the construction of the moduli space of objects $\phi : \mathcal{D} \to \mathcal{E}$ denoted as pairs. These objects are defined similar to triples in Definition 2.1. The author introduces the notion of Hilbert polynomial and reduced Hilbert polynomial for a pair [35] (Definition 1.3). Moreover, the author defines a semistability condition denoted as δ-semistability [35] (Definition 1.4) where δ is given as stability parameter. Replacing δ with $q(m)$, it is easily seen that the Wandel's notion of δ-semistability is completely compatible with our notion of τ'-semistability in Definition 2.14. In order to construct the underlying parameter scheme of triples one needs a boundedness criterion for the family of triples of type (P_1, P_2). Here we state some of the theorems in [35] without any proofs which ensure one to obtain the required boundedness conditions for the family of triples. The following statements can all be adapted to our case once one replaces the notion of pairs and δ-semistability in [35] with our notion of triples and τ'-smeistability respectively.

Proposition 3.1. *[35](Proposition 2.1) Given a pair $\phi : \mathcal{D} \to \mathcal{E}$, Let P and δ be polynomials. Then there is a constant C depending only on P and $\mathcal{D}$ such that for every $\mathcal{O}_X$-module $\mathcal{E}$ occurring in a δ-semistable pair we have $\mu_{\max}(\mathcal{E}) \leq C$. In particular, the family of pairs which are semistable* *with respect to any stability parameter δ having the fixed Hilbert polynomial P is bounded.*

Following this proposition it is shown in [35] (Proposition 2.4) that a family of δ-semistable pairs with given fixed numerical data (such as fixed Hilbert polynomial) satisfy n-regularity condition. Hence, it is shown that given a bounded family of δ-semistable pairs, the sheaves $\mathcal{D}$ and $\mathcal{E}$ appearing in the family satisfy the condition that for some large enough integer n the sheaves $\mathcal{D}(n)$ and $\mathcal{E}(n)$ are globally generated [35] (look following Definition 3.2).

3.2 Definition of moduli stacks as categories fibered in groupoids

Definition 3.2. Use Definition 2.9. Define $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ to be the fibered category $\mathfrak{p}:\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')\to Sch/\mathbb{C}$ such that:

1. For all $S\in Sch/\mathbb{C}$ the objects in $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ are S-flat families of τ'-stable highly frozen triples of type (P_2,r) as in Definition 2.10.

2. Given a morphism of $\mathbb{C}$-schemes $g:S\to K$ and two families of highly frozen triples $T_S:=(\mathcal{E}_1,\mathcal{E}_1,\phi,\psi)_S$ and $\acute{T}_K:=(\mathcal{E}'_1,\mathcal{E}'_1,\phi',\psi')_K$ as in Definition 2.10 (sub-index indicates the base parameter scheme over which the family is constructed), a morphism $T_S\to\acute{T}_K$ in $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is defined by an isomorphism:

$$\nu_S: T_S\xrightarrow{\cong}(g\times\mathbf{1}_X)^*\acute{T}_K.$$

Definition 3.3. Use Definition 2.7. Define $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ to be he fibered category $\mathfrak{p}:\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')\to Sch/\mathbb{C}$ such that:

1. For all $S\in Sch/\mathbb{C}$ the objects in $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ are S-flat families of frozen triples of type (P_2,r) as in Definition 2.7.

2. Given a morphism of $\mathbb{C}$-schemes $g:S\to K$ and two families of frozen triples $T_S:=(\mathcal{E}_1,\mathcal{E}_1,\phi)_S$ and $\acute{T}_K:=(\mathcal{E}'_1,\mathcal{E}'_1,\phi')_K$ as in Definition 2.7 (sub-index indicates the base parameter scheme over

which the family is construced), a morphism $T_S \to \acute{T}_K$ in $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ is defined by an isomorphism:

$$\nu_S : T_S \xrightarrow{\cong} (g \times \mathbf{1}_X)^* \acute{T}_K.$$

Proposition 3.4. *Use definitions 3.2 and 3.3. The fibered categories* $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ *and* $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ *are stacks.*

Proof. This is immediate from faithfully flat descent of coherent sheaves and homomorphisms of coherent sheaves [34] (Theorem 4.23). □

Remark 3.5. There exists a forgetful morphism $g' : \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \to \mathcal{B}\,\mathrm{GL}_r(\mathbb{C})$ which is given by taking a frozen triple $\{(E_1, E_2, \phi)\} \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ to $\{E_1\} \in \mathcal{B}\,\mathrm{GL}_r(\mathbb{C})$ by forgetting E_2 and ϕ.

Proposition 3.6. *The natural diagram:*

$$\begin{array}{ccc} \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') & \xrightarrow{\quad g \quad} & pt = \mathrm{Spec}(\mathbb{C}) \\ \Big\downarrow{\scriptstyle \pi^{\mathfrak{M}}_{\mathrm{FT}}} & & \Big\downarrow{\scriptstyle i} \\ \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') & \xrightarrow{\quad \acute{g} \quad} & \mathcal{B}\,\mathrm{GL}_r(\mathbb{C}) = \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right] \end{array}, \tag{3.1}$$

is a fibered diagram in the category of stacks. In particular $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ *is a* $\mathrm{GL}_r(\mathbb{C})$*-torsor over* $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$*. It is true that locally in the flat topology* $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \cong \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]$*. This isomorphism does not hold true globally unless* $r = 1$*.*

Proof. We show that there exists a forgetful map $\pi^{\mathfrak{M}}_{\mathrm{FT}} : \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \to \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ which induces a map from $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]$ to $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and show that this map has an inverse locally but not globally unless $r = 1$. First we prove the claim for $r = 1$.

By definition $\mathfrak{M}^{P_2,1}_{s,\mathrm{HFT}}(\tau')$ stands for the moduli stack of rank 1 τ'-stable highly frozen triples. Moreover $\mathfrak{M}^{(P_2,1)}_{s,\mathrm{FT}}(\tau')$ stands for the moduli stack of rank 1 τ'-stable frozen triples. For $r = 1$, $\mathrm{GL}_1(\mathbb{C}) = \mathbb{G}_m$. For a $\mathbb{C}$-scheme S, an S-point of $\mathfrak{M}^{(P_2,1)}_{s,\mathrm{HFT}}(\tau') \times [\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}]$ is identified with the data $(\mathcal{O}_{X\times S}(-n) \to E_2, \mathcal{L}_S)$ where $\mathcal{L}_S$ is a $\mathbb{G}_m$ line bundle over S. Let $\pi_S : X \times S \to S$ be the natural projection onto the second factor. There exists a map that sends this point to an S-point $a \in$

$(\mathfrak{M}^{(P_2,1)}_{s,\mathrm{FT}}(\tau'))(S)$ which is obtained by tensoring with $\mathcal{L}_S$, i.e $\mathcal{O}_X(-n)\boxtimes\mathcal{L}_S \xrightarrow{\phi^{\mathcal{L}}} E_2\boxtimes\mathcal{L}_S$. Note that tensoring $\mathcal{O}_{X\times S}(-n)$ with $\pi_S^*\mathcal{L}_S$ does not change the fact that $\mathcal{O}_{X\times S}(-n)\mid_{s\in S}\cong \mathcal{O}_X(-n)\boxtimes\mathcal{L}_S\mid_{s\in S}$ fiber by fiber.

Moreover there exists a section map $s:\mathfrak{M}^{(P_2,1)}_{s,\mathrm{FT}}(\tau')\to\mathfrak{M}^{(P_2,1)}_{s,\mathrm{HFT}}(\tau')\times[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}]$. Simply take an S-point $[\mathcal{O}_X(-n)\boxtimes\mathcal{L}_S\to E_2]\in(\mathfrak{M}^{(P_2,1)}_{s,\mathrm{FT}}(\tau'))(S)$ and send to an S-point in $(\mathfrak{M}^{(P_2,1)}_{s,\mathrm{HFT}}(\tau')\times[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}])(S)$ by the map

$$[\mathcal{O}_X(-n)\boxtimes\mathcal{L}_S\to E_2]\mapsto([\mathcal{O}_{X\times S}(-n)\to E_2\otimes\pi_S^*\mathcal{L}_S^{-1}],\mathcal{L}_S).$$

Note that since $\mathcal{L}_S$ is a line bundle over S then it is invertible and hence a section map is always well defined and $\mathfrak{M}^{(P_2,1)}_{s,\mathrm{FT}}(\tau')$ is a gerbe over $\mathfrak{M}^{(P_2,1)}_{s,\mathrm{HFT}}(\tau')$. To proceed further we state the following definition.

Definition 3.7. Consider a stack $(\mathfrak{Y},p_{\mathfrak{Y}}:\mathfrak{Y}\to Sch/\mathbb{C})$. Given Two morphism of stacks $p_1:\mathfrak{X}\to\mathfrak{Y}$ and $p_2:\mathfrak{X}'\to\mathfrak{Y}$, the fibered product of $\mathfrak{X}$ and $\mathfrak{X}'$ over $\mathfrak{Y}$ is defined by the category whose objects are defined by triples (x,x',α) where $x\in\mathfrak{X}$ and $x'\in\mathfrak{X}'$ respectively and $\alpha:p_1(x)\to p_2(x')$ is an arrow in $\mathfrak{Y}$ such that $p_{\mathfrak{Y}}(\alpha)=\mathrm{id}$. Moreover the morphisms $(x,x',\alpha)\to(y,y',\beta)$ are defined by the tuple $(\phi:x\to y,\psi:x'\to y')$ such that

$$p_2(\psi)\circ\alpha=\beta\circ p_1(\phi):p_1(x)\to p_2(y').$$

Now let $r>1$. There exists a forgetful map $\pi^{\mathfrak{M}}_{\mathrm{FT}}:\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')\to\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ which takes (E_1,E_2,ϕ,ψ) to (E_1,E_2,ϕ) by forgetting the choice of isomorphism, ψ. Moreover, there exists a map $g':\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')\to\mathcal{B}\,\mathrm{GL}_r(\mathbb{C})$ by Remark 3.5. Finally there exists the natural projection $i:\mathrm{Spec}(\mathbb{C})\to\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]=\mathcal{B}\,\mathrm{GL}_r(\mathbb{C})$. It follows directly from Definition 3.7 that the diagram:

$$\begin{array}{ccc}
\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') & \xrightarrow{\quad g\quad} & pt=\mathrm{Spec}(\mathbb{C})\\
\Big\downarrow{\scriptstyle \pi^{\mathfrak{M}}_{\mathrm{FT}}} & & \Big\downarrow{\scriptstyle i}\\
\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') & \xrightarrow{\quad \acute{g}\quad} & \mathcal{B}\,\mathrm{GL}_r(\mathbb{C})=\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]
\end{array}$$

is a fibered diagram and $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') = \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \times_{\mathcal{B}\,\mathrm{GL}_r(\mathbb{C})} pt$. Here one cannot use the same argument used for frozen triples of rank 1 to conclude that there exists a section map $s : \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \to \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]$, since as we showed, the S-point of $\mathcal{B}\,\mathrm{GL}_r(\mathbb{C})$ is a $\mathrm{GL}_r(\mathbb{C})$ bundle of rank r over S and this vector bundle is trivializable locally but not globally. Therefore locally in the flat topology one may think of $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ as isomorphic to $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \times [\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}]$ but not globally. □

3.3 Moduli stacks as algebraic stacks

3.3.1 The Parameter Scheme of $\acute{\tau}$-stable highly frozen triples of type (P_2, r)

Replacing the pairs and δ-semistability in [35] (Section 2) with triples and τ'-semistability and adapting the results of propositions 2.1 and 2.4 in [35] to our case one finds that there exists an integer n' such that for all coherent sheaves E_1 and E_2 appearing in a family of $\acute{\tau}$-(semi)stable triples (E_1, E_2, ϕ), $E_1(n')$ and (in particular) $E_2(n')$ are globally generated. Now use notation of Definition 2.6. One first constructs an S-flat family of coherent sheaves E_2 with fixed Hilbert polynomial P_2. By construction the family of coherent sheaves E_2 appearing in a τ'-stable triple is bounded and moreover the large enough twist $E_2(n')$ is globally generated. Fix such n' and let V_2 be a complex vector space of dimension $d_2 = P(n')$ given as $V_2 = \mathrm{H}^0(E_2 \otimes L^{n'})$. Let $\mathcal{Q}_2$ denote $\mathrm{Quot}_{P_2}(V_2 \otimes \mathcal{O}_X(-n'))$. Now we fix a large enough integer n (not necessarily equal to n'). We construct a scheme which parameterizes morphisms $\mathcal{O}_X^{\oplus r}(-n) \to E_2$: There exists a bundle $\mathcal{P}$ over $\mathcal{Q}_2$ whose fibers parametrize $\mathrm{H}^0(E_2(n))$. It is trivially seen that the fibers of the bundle $\mathcal{P}^{\oplus r}$ parametrize $\mathrm{H}^0(E_2(n))^{\oplus r}$. In other words the fibers of $\mathcal{P}^{\oplus r}$ parametrize the maps $E_1 \to E_2$ such that $E_1 = \mathcal{O}_X^{\oplus r}(-n)$. Now let

$$\mathfrak{S}^{(P_2,r,n)}_s(\tau') \subset \mathcal{P}^{\oplus r} \tag{3.2}$$

be given as an open subscheme of $\mathcal{P}^{\oplus r}$ whose fibers parametrize τ'-stable highly frozen triples $E_1 \to E_2$.

$$\begin{array}{ccc} & & V_2 \otimes \mathcal{O}_X(-n') \\ & & \downarrow \\ \mathcal{O}_X(-n) & \xrightarrow{\phi} & E_2 \end{array}$$

Remark 3.8. Given a $\mathbb{C}$-scheme, S, a quasi-projective $\mathbb{C}$-scheme $\mathcal{A}$ and a complex group G, the quotient stack $\left[\frac{\mathcal{A}}{G}\right]$ consists of pairs $(\mathbf{P}, \pi)$ such that P is a principal G-bundle over S and $\pi : \mathbf{P} \to \mathcal{A}$ is a G-equivariant morphism.

Theorem 3.9. *Let* $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$ *be the stable locus of the parametrizing scheme of highly frozen triple of type* (P_2, r) *as in* (3.2)*. Let* $\left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right]$ *be the stack-theoretic quotient of* $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$ *by* $\mathrm{GL}(V_2)$ *where* V_2 *is defined as in Section 3.3.1. There exists an isomorphism of groupoids*

$$\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau') \cong \left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right]. \tag{3.3}$$

Proof. Consider the scheme $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$. First one shows that there exists a functor $\mathfrak{q} : \left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right] \to \mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$. Then one shows that there exists a functor in the opposite direction and finally one proves that the composition of the two functors is a natural isomorphism of categories fibered in groupoids. Look at [8] for more general treatment. Diaconescu [6] uses a similar proof to construct the moduli stack of ADHM sheaves supported over a curve as a quotient stack. Fix a parametrizing scheme S over $\mathbb{C}$. The fiber of the quotient stack $\left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right]_S$ over S consists of pairs $(\mathbf{P}, \pi_{\mathfrak{S}})$ as in Remark 3.8 (where $\mathcal{A}$ in Remark 3.8 is replaced by $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$). Let $\mathbb{T} := \mathbb{E}_1 \xrightarrow{\phi} \mathbb{E}_2$ be the universal τ'-stable frozen triple of type (P_2, r) over $X \times \mathfrak{S}_s^{(P_2,r,n)}(\tau')$. Given:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\pi_{\mathfrak{S}}} & \mathfrak{S}_s^{(P_2,r,n)}(\tau') \\ p \downarrow & & \\ S & & \end{array}, \tag{3.4}$$

one obtains a diagram:

$$\begin{array}{ccc} \mathbf{P} \times X & \xrightarrow{(\pi_{\mathfrak{S}})_X} & \mathfrak{S}_s^{(P_2,r,n)}(\tau') \times X \\ {\scriptstyle (p \times \mathbf{1}_X)} \downarrow & & \\ S \times X & & \end{array}$$

Let K be a $\mathbb{C}$-scheme and let $g : S \to K$ be a morphism of $\mathbb{C}$-schemes. A morphism in $\left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right]$ between two objects:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\pi_{\mathfrak{S}}} & \mathfrak{S}_s^{(P_2,r,n)}(\tau') \\ {\scriptstyle p} \downarrow & & \\ S & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\pi'_{\mathfrak{S}}} & \mathfrak{S}_s^{(P_2,r,n)}(\tau') \\ {\scriptstyle p'} \downarrow & & \\ K & & \end{array}$$

is given by a commutative diagram:

$$\begin{array}{ccccc} & & & & \mathfrak{S}_s^{(P_2,r,n)}(\tau') \\ & & {\scriptstyle \pi_{\mathfrak{S}}} \nearrow & & \nwarrow {\scriptstyle g^*\acute{\pi}_{\mathfrak{S}}} \\ \mathbf{P} & \xrightarrow{\nu} & g^*\acute{\mathbf{P}} & & \\ {\scriptstyle p} \downarrow & & \downarrow {\scriptstyle g^*p'} & & \\ S & \xrightarrow{=} & S & & \end{array} \tag{3.5}$$

such that ν is an isomorphism of principal $\mathrm{GL}(V_2)$-bundles over S. Note that by the construction in Section 3.3.1 the objects parametrized by $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$ are given by a morphism $\mathcal{O}_X(-n)^{\oplus r} \to E_2$ such that E_2 (itself) is given as a flat quotient $V_2 \otimes \mathcal{O}_X(-n') \twoheadrightarrow E_2$. Now define a morphism

$$q' : \mathfrak{S}_s^{(P_2,r,n)}(\tau') \to \mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$$

by forgetting the surjection $V_2 \otimes \mathcal{O}_X(-n') \twoheadrightarrow E_2$. Note that by construction and since the map $\pi_{\mathfrak{S}}$ in diagram (3.4) is $\mathrm{GL}(V_2)$-equivariant then one obtains a map from S to $\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$ i.e one obtains an induced diagram:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\pi_{\mathfrak{S}}} & \mathfrak{S}_s^{(P_2,r,n)}(\tau') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q'} \\ S & \longrightarrow & \mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau') \end{array}. \tag{3.6}$$

Since $g^*\pi'_{\mathfrak{S}} \circ \nu = \pi_{\mathfrak{S}}$, it is guaranteed that $(\pi_{\mathfrak{S}} \times \mathbf{1}_X)^*\mathbb{T} \cong (g \times \mathbf{1}_X)^*((\acute{\pi_{\mathfrak{S}}}) \times \mathbf{1}_X)^*\mathbb{T}$ and this isomorphism descends to $T_S \xrightarrow{\cong} (g \times \mathbf{1}_X)^*\acute{T}_K$ where T_S and $\acute{T}_K$ are as in Definition 3.2. Hence the map q' in (3.6) factors through a map $\mathfrak{q} : \left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right] \to \mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$.

For the other direction one uses Lemma 4.3.1 in [14] and the methods described in [6]. Let $\pi_S : X \times S \to S$ and $\pi_X : X \times S \to X$. Let $\mathcal{E}_2$ denote the S-flat family of coherent shaves appearing in τ'-stable family of triples of type (P_2, r). By Definition 3.3 $\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')(S)$ is given by an S-flat family of frozen highly triples if type (P_2, r). Since the Hilbert polynomial of E_2 in the family is fixed then there exists a large enough integer n such that E_2 appearing in the family is n-regular and $E_2(n)$ is globally generated. This ensures that the direct image sheaves of $\mathcal{E}_2(n)$ via $\pi_S : X \times S \to S$ is globally generated and the higher direct image vanishes. i.e.:

$$\mathcal{B} := (\pi_S)_*(\mathcal{E}_2 \otimes \pi_X^*\mathcal{O}_X(n')) \tag{3.7}$$

is globally generated. Then there exists a surjective morphism:

$$\pi_S^*\mathcal{B} \otimes_{X \times_{\mathbb{C}} S} \pi_X^*(\mathcal{O}_X(-n')) \to \mathcal{E}_2 \to 0. \tag{3.8}$$

Let the principal $\mathrm{GL}(V_2)$-bundle $p : \mathbf{P} \to S$ be defined as:

$$\mathbf{P} = \mathbb{I}som(V_2 \otimes \mathcal{O}_S \to \mathcal{B}). \tag{3.9}$$

Since $\mathbf{P}$ is given by a frame bundle over S then one has:

$$p^*\mathcal{B} \cong V_2 \otimes \mathcal{O}_{\mathbf{P}}.$$

Now pull back by $(p \times \mathbf{1}_X) :\to S \times X$ and obtain:

$$(p \times \mathbf{1}_X)^*(\pi_S^*\mathcal{B} \otimes_{X \times_{\mathbb{C}} S} \pi_X^*(\mathcal{O}_X(-n'))) \to (p \times \mathbf{1}_X)^*\mathcal{E}_2 \to 0. \tag{3.10}$$

On the other hand:

$$(p \times \mathbf{1}_X)^*(\pi_S^*\mathcal{B} \otimes_{X \times_{\mathbb{C}} S} \pi_X^*(\mathcal{O}_X(-n'))) \cong V_2 \otimes (p \times \mathbf{1}_X)^*(\pi_X^*(\mathcal{O}_X(-n'))).$$

Let $\mathcal{E}_2^{\mathbf{P}} = (p \times 1_X)^*\mathcal{E}_2$. One obtains an isomorphism:

$$p^*\mathcal{B} \cong (\pi_{\mathbf{P}})_* \left[\mathcal{E}_2^{\mathbf{P}} \otimes_{X \times \mathbf{P}} (p \times \mathbf{1}_X)^*(\pi_X^*\mathcal{O}_X(n'))\right]. \tag{3.11}$$

Let $\{b\} \in \mathbf{P}$ be a closed point. By evaluation at $\{b\}$ one obtains:

$$p^*\mathcal{B} \mid_{X \times \{b\}} \cong V_2 \otimes O_{X \times \{b\}}$$

and

$$(\pi_{\mathbf{P}})_* \left[\mathcal{E}_2^{\mathbf{P}} \otimes_{X \times P} (p \times \mathbf{1}_X)^*(\pi_X^*\mathcal{O}_X(n'))\right] \mid_{X \times \{b\}} \cong \mathrm{H}^0 \left[\mathcal{E}_2^{\mathbf{P}}(n') \mid_{X \times \{b\}}\right]. \tag{3.12}$$

twisting the map:

$$V_2 \otimes O_{X \times \{b\}}(-n) \to \mathcal{E}_2^{\mathbf{P}} \mid_{X \times \{b\}}$$

by n and taking the zero cohomology one obtains an isomorphism:

$$\mathrm{H}^0\left[V_2 \otimes O_{X\times\{b\}}\right] \xrightarrow{\cong} \mathrm{H}^0\left[\mathcal{E}_2^{\mathbf{P}}(n') \mid_{X\times\{b\}}\right]. \tag{3.13}$$

Hence for every closed point $\{b\} \in \mathbf{P}$ one gets an isomorphism in the level of zero cohomologies. We use this fact to construct a family of stable highly frozen triples parametrized by $\mathbf{P}$. This family is obtained by applying $(\pi_{\mathbf{P}})_*$ to the following morphism over $\mathbf{P}$:

$$V_2 \otimes_{X\times\mathbf{P}} (p \times \mathbf{1}_X)^*(\pi_X^*\mathcal{O}_X(-n')) \to \mathcal{E}_2^{\mathbf{P}}. \tag{3.14}$$

This family is naturally $\mathrm{GL}(V_2)$-equivariant by construction and it gives rise to a classifying $\mathrm{GL}(V_2)$-equivariant morphism $\mathbf{P} \to \mathfrak{S}_s^{(P_2,r,n)}(\tau')$.

Now consider two objects in $\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$ and a morphism between them. This data by Definition 3.3 is a pair (g, ν_S) such that $g : S \to K$ and $\nu_S : T_S \to (g \times \mathbf{1}_X)^*\acute{T}_K$. However since these two families determine principal $\mathrm{GL}(V_2)$-bundles over S and K respectively, we obtain a morphism of principal $\mathrm{GL}(V_2)$-bundles:

$$\begin{array}{ccc} \mathbf{P} \cong (g\times\mathbf{1}_X)^*\acute{\mathbf{P}} & \xrightarrow{\nu} & \acute{\mathbf{P}} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \acute{p}} \\ S & \xrightarrow{g} & K \end{array} \tag{3.15}$$

Let $h : \mathbf{P} \xrightarrow{\cong} (g \times \mathbf{1}_X)^*\acute{\mathbf{P}}$, it is verified that the family $h^*\nu^*\acute{T}_K$ and T_S are isomorphic. Therefore there exists a functor $\mathfrak{j} : \mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau') \to \left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right]$ and also it is verified that $\mathfrak{q} \circ \mathfrak{j}$ and $\mathfrak{j} \circ \mathfrak{q}$ are natural isomorphisms. $\square$

One may use the above results (i.e the natural isomorphism in Theorem 3.9) in order to obtain an alternative definition of the moduli stack of τ'-stable highly frozen triples of type (P_2, r) as the quotient stack $\left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}(V_2)}\right]$.

Remark 3.10. By Definition 3.2 and construction of $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$ in Section 3.3.1 $\mathrm{GL}_r(\mathbb{C})$ acts compatibly on both sides of the isomorphism (3.3). The next corollary gives the algebraic structure of the moduli stack of frozen triples.

Corollary 3.11. *Use Proposition 3.6, Theorem 3.9 and Remark 3.10. Let* $\left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}_r(\mathbb{C})\times \mathrm{GL}(V_2)}\right]$ *be the stack-theoretic quotient of* $\mathfrak{S}_s^{(P_2,r,n)}(\tau')$ *by* $\mathrm{GL}_r(\mathbb{C})\times \mathrm{GL}(V_2)$ *where* V_2 *is defined as in Section 3.3.1. There exists an isomorphism of groupoids:*

$$\mathfrak{M}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau') \cong \left[\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{\mathrm{GL}_r(\mathbb{C})\times \mathrm{GL}(V_2)}\right].$$

Theorem 3.12. *Consider* $q(m)\to\infty$ τ'*-limit stability as in Lemma 2.17. The moduli stack* $\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$ *for such choice of stability parameter* $q(m)$ *is a Deligne-Mumford (DM) stack.*

Proof. It is enough to show that for every $\mathbb{C}$-point $p\in \mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')(\mathrm{Spec}(\mathbb{C}))$ its' stabilizer group $\mathrm{Stab}_{\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')}(p)$ is finite. Since the point p is represented by a τ'-stable highly frozen triple (E_1,E_2,ϕ,ψ), then $\mathrm{Stab}_{\mathfrak{M}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')}(p)$ is obtained by the automorphism group of (E_1,E_2,ϕ,ψ). Hence it is enough to show that the automorphism group of any such (E_1,E_2,ϕ,ψ) is a finite group. The following lemma shows that the automorphism group of a τ'-limit-stable highly frozen triple has one element which is the identity.

Lemma 3.13. *Given a* τ'*-limit-stable highly frozen triple* (E_1,E_2,ϕ,ψ) *as in Definition 2.9 and a commutative diagram*

$$\begin{array}{ccccc}
\mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\psi^{-1}} & E_1 & \xrightarrow{\phi} & E_2\\
{\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle\rho}\\
\mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\psi^{-1}} & E_1 & \xrightarrow{\phi} & E_2
\end{array},$$

the map ρ *is given by* id_{E_2}*.*

Proof. Since ψ is a choice of isomorphism, for simplicity replace E_1 by $\mathcal{O}_X(-n)^{\oplus r}$ and consider the diagram:

$$\begin{array}{ccc}
\mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & E_2\\
{\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\rho}\\
\mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & E_2
\end{array}, \tag{3.16}$$

The diagram (3.16) induces:

$$\begin{array}{ccccc} \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & \mathrm{Im}(\phi) & \hookrightarrow & E_1 \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \rho\,|_{\mathrm{Im}(\phi)}} & & \downarrow{\scriptstyle \rho} \\ \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & \mathrm{Im}(\phi) & \hookrightarrow & E_1 \end{array}.$$

By commutativity of (3.16) $\rho \circ \phi = \phi \circ \mathrm{id} = \phi$, then $\rho(\mathrm{Im}(\phi)) = \mathrm{Im}(\phi)$. Hence $\rho(\mathrm{Im}(\phi)) \subset \mathrm{Im}(\phi)$. It follows that $\rho\,|_{\mathrm{Im}(\phi)} = \mathrm{id}_{\mathrm{Im}(\phi)}$. Indeed if $s \in \mathrm{Im}(\phi)(\mathcal{U})$ where $\mathcal{U} \subset X$ is affine open with $\tilde{s} \in \mathcal{O}_X(-n)^{\oplus r}(\mathcal{U})$ satisfying $\phi(\tilde{s}) = s$, then $\rho(s) = \rho(\phi(\tilde{s})) = \phi(\mathrm{id}(\tilde{s})) = \phi(\tilde{s}) = s$. Now apply $\mathrm{Hom}(-, E_2)$ to the short exact sequence

$$0 \to \mathrm{Im}(\phi) \to E_2 \to Q \to 0$$

where Q denotes the corresponding cokernel. One obtains:

$$0 \to \mathrm{Hom}(Q, E_2) \to \mathrm{Hom}(E_2, E_2) \to \mathrm{Hom}(\mathrm{Im}(\phi), E_2).$$

Since (E_1, E_2, ϕ, ψ) is $q(m) \to \infty$ τ'-limit-stable then by Lemma 2.17 Q is a sheaf with 0-dimensional support. Hence by purity of E_2, $\mathrm{Hom}(Q, E_2) \cong 0$. Hence one obtains an injection

$$\mathrm{Hom}(E_2, E_2) \hookrightarrow \mathrm{Hom}(\mathrm{Im}(\phi), E_2).$$

Now

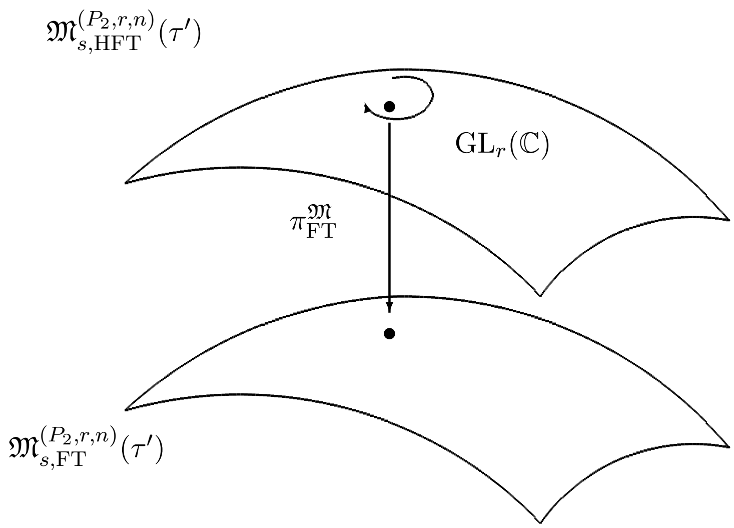
$$\rho\,|_{\mathrm{Im}(\phi)} = \mathrm{id}_{\mathrm{Im}(\phi)} = (\mathrm{id}_{E_2})\,|_{\mathrm{Im}(\phi)}\ .$$

So $\rho = \mathrm{id}_{E_2}$.

This finishes the proof of Lemma 3.13 as well as Theorem 3.12. □

Remark 3.14. It is seen from work of Malte Wandel [35] (Section 3) that $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ more than being a DM stack has the structure of a quasi-projective scheme. We will use this fact later in discussing the construction of deformation obstruction theory over $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

Now we compare the infinitesimal structure, i.e the deformations of objects in $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ with that of objects in $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. Note that throughout the rest of this study by τ'-stability we mean the $q(m) \to \infty$ τ'-limit-stability.



Chapter 4

Further discussions on stability

The material in this chapter is included for completeness. Although we constructed the moduli of stable frozen and highly frozen triples as stacks, one finds it interesting to study the GIT properties of these moduli spaces. The motivation behind this attempt is to show that our notion of τ'-stability comes from a GIT notion of stability. We state our observations in the most general setting. In other words we study triples of type (P_1, P_2). We consider the GIT stability for the coarse moduli space which is underlying the moduli stack of triples of type (P_1, P_2). To start, consider $\mathcal{Q}_i = \mathrm{Quot}(V_i \otimes \mathcal{O}_X(-n), P_i)$ as in Section 3.3.1 for $i = 1, 2$.

Lemma 4.1. *[14] (Lemma 4.3.2). Let* $\{a_i\} = [V_i \otimes \mathcal{O}_X(-n) \to E_i] \in \mathcal{Q}_i$ *be a closed point such that* $E_i(n)$ *are globally generated and* $\mathrm{H}^0(a_i(n)) : \mathrm{H}^0(V_i \otimes \mathcal{O}_X) \to \mathrm{H}^0(E_1(n))$ *is an isomorphism. Then there exists a natural injective homomorphism* $\mathrm{Aut}(E_i) \to \mathrm{GL}(V_i)$ *such that the image is the stabilizer of the point* $\{a_i\}$ *in* $\mathcal{Q}_i$*.*

Proof. Given an element $e_i \in \mathrm{Aut}(E_i)$ consider the map $\mathrm{Aut}(E_i) \to \mathrm{GL}(V_i)$ defined by

$$e_i \mapsto \mathrm{H}^0(a_i(n))^{-1} \circ \mathrm{H}^0(e_i(n)) \circ \mathrm{H}^0(a_i(n)) \tag{4.1}$$

Since n is chosen to be large enough, the above map automatically is injective. Now by definition of isomorphism of quotients, an element $g_i \in \mathrm{GL}(V_i)$ belongs to stabilizer of the point $\{a_i\}$ under the action of $\mathrm{GL}(V_i)$ *if and only if* there exists an automorphism of E_i, e_i, such that $a_i \circ g_i = e_i \circ a_i$. $\square$

There exists a left action of $\mathrm{GL}(V_i)$ which, on the closed points of $\mathcal{Q}_i$, is defined by:

$$\begin{aligned} & g_i \cdot [a_i : V_i \otimes \mathcal{O}_X(-n) \to E_i] = \\ & [V_i \otimes \mathcal{O}_X(-n) \xrightarrow{g_i^{-1} \otimes id|_{\mathcal{O}_X(-n)}} V_i \otimes \mathcal{O}_X(-n) \to E_i]. \end{aligned} \tag{4.2}$$

Since the center of the group $Z_i \subset \mathrm{GL}(V_i)$ lies in stabilizer of $\{a_i\}$, for all $\{a_i\} \in \mathcal{Q}_i$ one may take the group acting on $\mathcal{Q}_i$ to be the quotient of $\mathrm{GL}(V_i)$ by Z_i, i.e: $\mathrm{PGL}(V_i)$. On the other hand, since the composite map of groups $\mathrm{SL}(V_i) \to \mathrm{GL}(V_i) \to \mathrm{PGL}(V_i)$ is surjective, then by choosing the group to be $\mathrm{SL}(V_i)$ one will not loose any information. Hence essentially we take the group acting on $\mathcal{Q}_i$ to be given by $\mathrm{SL}(V_i)$ however for the moment we stick to $\mathrm{GL}(V_i)$. Consider $\mathfrak{S}$ as a subscheme of $\mathcal{A} = \mathcal{Q}_1 \times \mathcal{P} \times \mathcal{Q}_2$ where $\mathcal{P}$ is given by the projective Hom-bundle $\mathbf{P}(\mathrm{Hom}(V_1, V_2))$. One can show that there exists an action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ on $\mathfrak{S}$ that leaves $\mathfrak{S}$ invariant.

4.1 Linearization over $\mathfrak{S}$ and Hilbert-Mumford criterion

We state the main result of this section:

Theorem 4.2. *Let $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Then there exists a G-linearized ample line bundle $\mathcal{L}^{\mathrm{GL}(V_1)\times \mathrm{GL}(V_2)}$ over $\mathfrak{S}$ such that for every closed point $\{\mathfrak{u}\} \in \mathfrak{S}$, $\{\mathfrak{u}\}$ is (semi)stable with respect to the Hilbert-Mumford criterion if and only if it is $\acute{\tau}$-(semi)stable.*

Proof. Since $\mathfrak{S} \subset \mathcal{Q}_1 \times \mathcal{P} \times \mathcal{Q}_2$, to obtain a G-linearized line bundle over $\mathfrak{S}$ we start from the scheme $\mathcal{A} = \mathcal{Q}_1 \times \mathcal{P} \times \mathcal{Q}_2$. Here we explain how to choose a linearization over $\mathcal{A}$ which provides us with a numerical stability condition which is compatible with the criterion of Definition 2.11.

Consider the universal quotient over $\mathcal{Q}_i \times X$ for $i = 1, 2$. Let $\tilde{\pi}_{\mathcal{Q}_i} : \mathcal{Q}_i \times X \to \mathcal{Q}_i$ be the projection. Since the Hilbert polynomial of E_i is fixed and $n \gg 0$, $\tilde{\pi}_{\mathcal{Q}_i*}\tilde{\mathcal{E}}_i(n)$ is globally generated. There exists an embedding of $\mathcal{Q}_i$ into the Grassmannian which parametrizes the locally free quotients of $\tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X})$ with rank $P_i(n)$. To further embed the obtained Grassmannian into a projective scheme one uses the Plücker embedding of the Grassmannian. If $[\tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X}) \to \tilde{\pi}_{\mathcal{Q}_i*}\tilde{\mathcal{E}}_i(n)]$

is the tautological quotient on $\mathrm{Grass}(\tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X}), P_i(n))$, then the $P_i(n)$'th exterior power

$$h : \bigwedge^{P_i(n)} \tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X}) \to det(\tilde{\pi}_{\mathcal{Q}_i*}\tilde{\mathcal{E}}_i(n)),$$

induces a closed immersion

$$\mathrm{Grass}(\tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X}), P_i) \xhookrightarrow{k_2^i} \mathbb{P}(\bigwedge^{P_i(n)} \tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X})),$$

hence one obtains the composite inclusion:

$$\mathcal{Q}_i \xhookrightarrow{k_1^i} \mathrm{Grass}(\tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X}), P_i(n)) \xhookrightarrow{k_2^i} \mathbb{P}(\bigwedge^{P_i(n)} \tilde{\pi}_{\mathcal{Q}_i*}(V_i \otimes \mathcal{O}_{\mathcal{Q}_i \times X})) := \mathfrak{P}_i. \qquad (4.3)$$

Take the ample line bundle $\mathcal{O}_{\mathfrak{P}_i}(1)$, pull back via $\mathcal{Q}_i \xrightarrow{k_2^i \circ k_1^i} \mathfrak{P}_i$ and obtain the following isomorphism of coherent sheaves over $\mathcal{Q}_i$:

$$L_{\mathcal{Q}_i} := (k_2^i \circ k_1^i)^* \mathcal{O}_{\mathfrak{P}_i}(1) \cong \det(\tilde{\pi}_{\mathcal{Q}_i*}\tilde{\mathcal{E}}_i(n)). \qquad (4.4)$$

Note that $\det(\tilde{\pi}_{\mathcal{Q}_i*}\tilde{\mathcal{E}}_i(n))$ are line bundles over $\mathcal{Q}_i$ which are equivariant with respect to the action of $\mathrm{GL}(V_i)$ for $i = 1, 2$. The linearized line bundle over $\mathfrak{S}$ is given as

$$\mathcal{L}^{(\mathrm{GL}(V_1)\times \mathrm{GL}(V_2))} = L_{\mathcal{Q}_1}^{d_1} \boxtimes L_{\mathcal{Q}_2}^{d_2} \boxtimes \mathcal{O}_{\mathcal{P}}(1)^{d_3}, \qquad (4.5)$$

Remark 4.3. To define a linearized line bundle over $\mathfrak{S}$ which satisfies the condition of Theorem 4.2, one needs to assign particular values to d_1, d_2 and d_3 in (4.5) such that the stability (in the sense of Hilbert-mumford criterion) of a point in $\mathfrak{S}$ with respect to $\mathcal{L}$ results in the $\acute{\tau}$-stability of the triple represented by this point. We will see that what highly affects the stability of triples is the values assigned to the ratios $\frac{d_1}{d_2}$, $\frac{d_1}{d_3}$ and $\frac{d_2}{d_3}$ rather than d_i's themselves.

We state the following theorem without proof which is the special case of a result obtained by Schmitt in [30].

Theorem 4.4. *[30] (Theorem. 4.6). The categorical good quotient*

$$\mathfrak{S}_{ss}(\tau')//(\mathrm{GL}(V_1) \times \mathrm{GL}(V_2))$$

exists as a projective scheme and moreover the stable locus $\mathfrak{S}_s(\tau')//(\mathrm{GL}(V_1) \times \mathrm{GL}(V_2))$ *is a geometric quotient for* $\mathfrak{S}_s(\tau')$ *with respect to the action of* $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$.

By arguments in [14] (Lemma. 4.3.1) it is proved that the categorical quotient $\mathfrak{S}_s(\tau')//(\mathrm{GL}(V_1) \times \mathrm{GL}(V_2))$ co-represents the moduli functor which induces the moduli functor $\mathfrak{M}_{\mathrm{T}}^{(P_1,P_2)}$ of $\acute{\tau}$-(semi)stable triples of type (P_1, P_2). Next we briefly review the Hilbert-Mumford criterion for the linearized line bundle over $\mathfrak{S}$. We pick a particular linearized line bundle over $\mathfrak{S}$ by assigning suitable values to $\frac{d_1}{d_2}$, $\frac{d_1}{d_3}$ and $\frac{d_2}{d_3}$. Eventually we use the Hilbert-Mumford criterion to show that when closed points in $\mathfrak{S}$ are GIT-(semi)stable with respect to $\mathcal{L}^{(\mathrm{GL}(V_1)\times(\mathrm{GL}(V_2)))}$, then their corresponding triples are $\acute{\tau}$-(semi)stable and vice versa.

Let G be a reductive group acting on a scheme X equipped with a G-equivariant linearization $\mathcal{L}^G$. Let $\lambda(t) : \mathbb{G}_m \to G$, be a nontrivial one-parameter subgroup of G. Given a point $x \in X$ acted on by G, the Hilbert-Mumford character of the linearized line bundle $\mathcal{L}^G$ is defined as:

$$\mu^{\mathcal{L}^G}(x, \lambda) := -r, \tag{4.6}$$

where r is the weight of the action of $\mathbb{G}_m$ on the fibers of $\mathcal{L}^G$ over the fixed point x_0 of the action of $\mathbb{G}_m$ on X induced by λ. For more clarification, let $\sigma : G \to X$ be defined as it is shown in the diagram below:

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda(t)} & G \\ \downarrow & & \downarrow \sigma \\ \mathbb{A}^1 & \xrightarrow{j} & X \end{array} \tag{4.7}$$

where $\sigma : G \to X$ is defined as $\sigma(g) : g \to \sigma(x, \lambda(t))$ and $j : \mathbb{A}^1 \to X$ is the unique extension of action of $\sigma \circ \lambda(t)$ to action of $\mathbb{A}^1$ on X. Since the ample line bundle $\mathcal{L}^G$ is linearized with respect to the action of G, the restriction of the line bundle to any two points on X which are in the same orbit of G produces isomorphic fibers.

$$\mathcal{L}^G \mid_x \xrightarrow[\Phi]{\cong} \mathcal{L}^G \mid_{g\cdot x}, \tag{4.8}$$

for all $g \in G$. Now let $x_0 = \lim_{t\to 0} \sigma(x, \lambda(t))$ be a fixed point of this action. We conclude that $\Phi(x_0, \lambda(t)) = t^\alpha \cdot id_{\mathcal{L}^G|_{x_0}}$. Now define $\mu^{\mathcal{L}^G}(x, \lambda(t)) = \alpha$ for every one parameter subgroup λ of G and define the weight r of the action of G as

$$r = -\alpha \tag{4.9}$$

We state the Hilbert-Mumford theorem.

Theorem 4.5. *[14](Theorem 4.2.11). Let G be a reductive group acting on a scheme X equipped with a G-equivariant linearized line bundle $\mathcal{L}^G$. A point $\{x\} \in X$ is semistable if and only if for all nontrivial one-parameter subgroups $\lambda : \mathbb{G}_m \to G$, one has*

$$\mu^{\mathcal{L}^G}(x, \lambda) \geq 0,$$

and $\{x\}$ is stable if and only if strict inequality holds for all non-trivial λ.

Next we apply this criterion to our setup. We take the groups acting on $\mathcal{Q}_i$ to be given by $\mathrm{SL}(V_i)$. Consider points $\{a_i\} : [V_i \otimes \mathcal{O}_X(-n) \to E_i]$ over $\mathcal{Q}_i$. In order to determine the limit point for the action of any one-parameter subgroup λ of $\mathrm{GL}(V_i)$ we need to decompose V_i in to their weight spaces $V_i = \bigoplus_{j\in\mathbb{Z}} V_i^j$. Let E_i^j be defined as the image of V_i^j under a_i. It is seen that

$$lim_{t\to 0}\lambda(t) \cdot [a_i] = [V_i \otimes \mathcal{O}_X(-n) \to \bigoplus_j E_i^j]. \tag{4.10}$$

For proof look at [14] (Lemma 4.4.3). The weight of the action of one parameter subgroup of $\mathrm{SL}(V_i)$

via $\lambda(t)$ on the fiber of $L_{\mathcal{Q}_i}$ for $i = 1, 2$ is given by

$$\sum_{j\in\mathbb{Z}} j \cdot P(E_i^j, n).$$

Let $\lambda_1 : \mathbb{G}_m \to \mathrm{SL}(V_1)$ and $\lambda_2 : \mathbb{G}_m \to \mathrm{SL}(V_2)$ be the one parameter sub-groups of $\mathrm{SL}(V_1)$ and $\mathrm{SL}(V_2)$ respectively. The weight of the action of $\mathbb{G}_m \times \mathbb{G}_m$ via $\lambda_1(t) \times \lambda_2(s)$ on the fiber of $L_{\mathcal{Q}_1}^{d_1} \boxtimes L_{\mathcal{Q}_2}^{d_2}$ over a point $\{a_1\} \times \{a_2\} \in \mathcal{Q}_1 \times \mathcal{Q}_2$ is given as

$$P_{12} := d_1 \cdot \sum_{j\in\mathbb{Z}} j \cdot P(E_1^j, n) + d_2 \cdot \sum_{j\in\mathbb{Z}} j \cdot P(E_2^j, n), \tag{4.11}$$

Remark 4.6. V_i's decompose in to the weight spaces such that:

$$\sum_{j\in\mathbb{Z}} j \cdot \dim(V_i^j) = 0$$

Therefore we obtain the following identities for $i = 1, 2$:

$$\sum_{j\in\mathbb{Z}} j \cdot P(E_i^j, n) = \frac{1}{\dim(V_i)} \sum_{j\in\mathbb{Z}} j \cdot (\dim(V_i) P(E_i^j, n) - \dim(V_i^j) P(E_i, n)).$$

Over $\mathcal{Q}_1 \times \mathcal{Q}_2$ this identity is written as:

$$\begin{aligned} P_{12} = & \frac{d_1}{\dim(V_1)} \cdot \sum_{j\in\mathbb{Z}} j \cdot (\dim(V_1) P(E_1^j, n) - \dim(V_1^j) P(E_1, n)) \\ & + \frac{d_2}{\dim(V_2)} \cdot \sum_{j\in\mathbb{Z}} j \cdot (\dim(V_2) P(E_2^j, n) - \dim(V_2^j) P(E_2, n)). \end{aligned} \tag{4.12}$$

The character of the action of $\mathbb{G}_m \times \mathbb{G}_m$ on the fiber of $\mathcal{O}_{\mathcal{P}}(1)^{d_3}$ over a point $\phi \in \mathcal{P}$ is given by

$$P_{\mathcal{P}} = -\frac{d_3}{\dim(V_1)} \cdot \sum_{j\in\mathbb{Z}} j \cdot \dim(V_1^j) + \frac{d_3}{\dim(V_2)} \cdot \sum_{j\in\mathbb{Z}} j \cdot \dim(V_2^j).$$

Hence by adding the characters P_{12} and $P_{\mathcal{P}}$ one obtains the character of the action of $\mathbb{G}_m \times \mathbb{G}_m$ via $\lambda_1(t) \times \lambda_2(s)$ on the fiber of $\mathcal{L}^{\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)} = L_{\mathcal{Q}_1}^{d_1} \otimes L_{\mathcal{Q}_2}^{d_2} \otimes \mathcal{O}_{\mathcal{P}}(1)^{d_3}$ over a point $(\{a_1\}, \{a_2\}, \phi) \in$

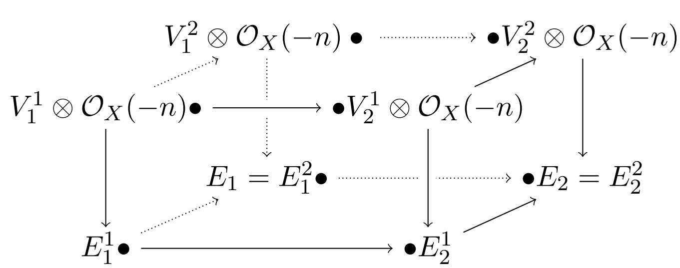
$\mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{P}$:

$$\begin{aligned} r &= P_{12} - \frac{d_3}{\dim(V_1)} \cdot \sum_{j \in \mathbb{Z}} j \cdot \dim(V_1^j) + \frac{d_3}{\dim(V_2)} \cdot \sum_{j \in \mathbb{Z}} j \cdot \dim(V_2^j) \\ &= \frac{d_1}{\dim(V_1)} \cdot \sum_{j \in \mathbb{Z}} j \cdot (\dim(V_1) P(E_1^j, n) - \dim(V_1^j) P(E_1, n)) \\ &+ \frac{d_2}{\dim(V_2)} \cdot \sum_{j \in \mathbb{Z}} j \cdot (\dim(V_2) P(E_2^j, n) - \dim(V_2^j) P(E_2, n)) \\ &- \frac{d_3}{\dim(V_1)} \cdot \sum_{j \in \mathbb{Z}} j \cdot \dim(V_1^j) + \frac{d_3}{\dim(V_2)} \cdot \sum_{j \in \mathbb{Z}} j \cdot \dim(V_2^j). \end{aligned} \tag{4.13}$$

By rearranging the terms in identity (4.13) we obtain:

$$\begin{aligned} r = &- \sum_{j \in \mathbb{Z}} j \cdot \dim(V_1^j) \left(\frac{d_1}{\dim(V_1)} \cdot P(E_1, n) + \frac{d_3}{\dim(V_1)} \right) \\ &- \sum_{j \in \mathbb{Z}} j \cdot \dim(V_1^j) \left(\frac{d_2}{\dim(V_2)} \cdot P(E_1, n) + \frac{d_3}{\dim(V_2)} \right) \\ &\frac{d_1}{\dim(V_1)} \cdot \sum_{j \in \mathbb{Z}} j \cdot \dim(V_1) P(E_1^j, n) + \frac{d_2}{\dim(V_2)} \cdot \sum_{j \in \mathbb{Z}} j \cdot \dim(V_2) P(E_2^j, n) \end{aligned} \tag{4.14}$$

Now assigning the right choice of weights to d_1, d_2, d_3 would enable one to obtain a stability condition compatible with the numerical stability condition for triples as in Definition 2.11. Set $\frac{d_3}{d_1} = q_1(n)$ and $\frac{d_3}{d_2} = q_2(n)$ where $q_1(n)$ and $q_2(n)$ are defined as in Definition 2.11. To show the compatibility of the two notions of stability the usual procedure is to pick a filtration with two terms $0 \subset V_i^1 \subset V_i^2 = V_i$ which induces $0 \subset E_i^1 \subset E_i^2 = E_i$ for $i = 1, 2$. Then putting $-r > 0$ provides the required numerical stability condition as an inequality. Given the commutative diagram:



(4.15)

The triple (E_1, E_2, ϕ) is stable in the sense of Hilbert-Mumford criterion if and only if:

$$\begin{aligned} q_2(n)\left(P_{E_1^1} - rk(E_1^1)\left(\frac{P_{E_1}}{rk(E_1)} - \frac{q_1(n)}{rk(E_1)}\right)\right) + \\ q_1(n)\left(P_{E_1^2} - rk(E_1^2)\left(\frac{P_{E_2}}{rk(E_2)} + \frac{q_2(n)}{rk(E_2)}\right)\right) < 0, \end{aligned} \tag{4.16}$$

For which we have used the description of r mentioned above, we have used the fact that $\dim(V_i) = \dim(\mathrm{H}^0(E_i \otimes \mathcal{O}_X(n)))$ and finally we have used some rearrangements. The inequality (4.16) is identical with the inequality in 2.11. This finishes the proof of Theorem 4.2. □

Chapter 5

Deformations of frozen triples

In this section, we describe the deformation theory of frozen and highly frozen triples.

5.1 Preliminaries

As we showed, the construction of the moduli stack of stable frozen triples depends on a choice of two fixed large enough integers $n \gg 0$ and $n' \gg 0$. The first integer appears in the description of a stable highly frozen triple $\mathcal{O}_X(-n)^{\oplus r} \to E_2$ and the second integer is the one for which $E_2(n')$ becomes globally generated and hence there exists a surjective map $V_2 \otimes \mathcal{O}_X(-n') \twoheadrightarrow E_2$. We also observed that according to Wandel [35] (Proposition 2.4) given a bounded family of stable triples $E_1 \to E_2$ there exists an integer n' such that for every tuple (E_1, E_2) appearing in the family $E_1(n')$ and $E_2(n')$ are globally generated over X. The fact that the sheaf $E_2(n')$ is globally generated for large enough values of n' does not a priori imply that $\mathrm{H}^i(E_2(n)) = 0$ for all $i > 0$ and our fixed choice of n. Hence we introduce the following definition:

Definition 5.1. Consider $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ in definitions 3.2 and 3.3 respectively. Define the open substacks $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \subset \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \subset \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ as follows:

1. $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') = \{(E_1, E_2, \phi, \psi) \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \mid \mathrm{H}^1(E_2(n)) = 0\}$.
2. $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') = \{(E_1, E_2, \phi) \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \mid \mathrm{H}^1(E_2(n)) = 0\}$.

Remark 5.2. From now on all our calculations are carried out over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and the results in chapters 5, 6 and 7 hold true for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ only. Also we assume that it is implicitly understood that in the following sections by the "*moduli stack of frozen or highly frozen triples*" we mean the open substack of the corresponding moduli stacks as defined in Definition 5.1.

Remark 5.3. As we will see in Chapter 8 there exist situations such as when X is given as a toric variety given by the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\to\mathbb{P}^1$ where it is seen that one obtains the vanishing of higher cohomologies of $E_2(n)$ assuming that $\mathcal{O}(-n)^{\oplus r}\to E_2$ is τ'-stable. In other words for such fixed choice of X one has $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')=\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')=\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. Hence later for such specific situations we will not distinguish between $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$, $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and their corresponding open substacks in Definition 5.1.

Lemma 5.4. *Let* $[\mathcal{O}_X(-n)^{\oplus r}\xrightarrow{\phi}E_2]$ *correspond to a point of* $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ *or* $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. *Then:*

$$\mathrm{Ext}^2(F,\mathcal{O}_X(-n))\cong 0\cong\mathrm{Ext}^1(F,\mathcal{O}_X(-n)). \tag{5.1}$$

Proof. Use Serre duality and obtain:

$$\begin{aligned}&\mathrm{Ext}^i(F,\mathcal{O}_X(-n))\cong(\mathrm{Ext}^{3-i}(\mathcal{O}_X(-n),F\otimes\omega_X)^\vee\\&\cong\mathrm{Ext}^{3-i}(\mathcal{O}_X(-n),F)^\vee\cong H^{3-i}(F(n))^\vee.\end{aligned} \tag{5.2}$$

The statement follows from the definitions of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. □

Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X. Let $I^\bullet$ be an object of the derived category given by the complex represented by a τ'-stable frozen triple

$$I^\bullet:=\mathcal{O}_X(-n)^{\oplus r}\xrightarrow{\phi}E_2$$

with $\mathcal{O}_X(-n)^{\oplus r}$ in degree 0 and E_2 in degree 1. Let $K:=\mathrm{Ker}(\phi)$ and $Q:=\mathrm{Coker}(\phi)$. There exist the following exact triangles in the derived category:

$$E_2[-1]\to I^\bullet\to\mathcal{O}_X(-n)^{\oplus r}\to E_2\to\cdots \tag{5.3}$$

$$K\to I^\bullet\to Q[-1]\to K[1]\to\cdots \tag{5.4}$$

Lemma 5.5. *Suppose that $\mathcal{O}(-n)^{\oplus r} \to E_2$ is τ'-stable. Then the following statement is true:*

$$\operatorname{Ext}^{\leq -1}(I^\bullet, I^\bullet) = 0. \tag{5.5}$$

Proof. Note that $\operatorname{Ext}^k(I^\bullet, I^\bullet) = 0$ for $k \leq -2$ by degree considerations. We now consider $k = -1$. Apply $\operatorname{Hom}(I^\bullet, \cdot)$ to (5.3) and obtain:

$$\begin{aligned} &\cdots \to \operatorname{Ext}^{-2}(I^\bullet, E_2) \to \operatorname{Ext}^{-1}(I^\bullet, I^\bullet) \\ &\to \operatorname{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) \to \operatorname{Ext}^{-1}(I^\bullet, E_2) \to \cdots \end{aligned} \tag{5.6}$$

Now apply $\operatorname{Hom}(\cdot, E_2)$ to (5.4) and obtain:

$$\begin{aligned} &\cdots \to \operatorname{Ext}^i(Q[-1], E_2) \to \operatorname{Ext}^i(I^\bullet, E_2) \to \\ &\operatorname{Ext}^i(K, E_2) \to \operatorname{Ext}^{i+1}(Q[-1], E_2) \cdots \end{aligned} \tag{5.7}$$

Combining the exact sequence (5.6) and exact sequences obtained from (5.7) for $i = -2$ and $i = -1$ we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\vdots & & & & \vdots & & \\
\downarrow & & & & \downarrow & & \\
\operatorname{Ext}^{-2}(Q[-1], E_2) & & & & \operatorname{Ext}^{-1}(Q[-1], E_2) & & \\
\downarrow & & & & \downarrow & & \\
\cdots \longrightarrow \operatorname{Ext}^{-2}(I^\bullet, E_2) & \longrightarrow & \operatorname{Ext}^{-1}(I^\bullet, I^\bullet) & \to & \operatorname{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) \longrightarrow \operatorname{Ext}^{-1}(I^\bullet, E_2) & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \\
\operatorname{Ext}^{-2}(K, E_2) & & & & \operatorname{Ext}^{-1}(K, E_2) & & \\
\downarrow & & & & \downarrow & & \\
\operatorname{Ext}^{-1}(Q[-1], E_2) & & & & \operatorname{Hom}(Q[-1], E_2) & & \\
\downarrow & & & & \downarrow & & \\
\vdots & & & & \vdots & &
\end{array} \tag{5.8}$$

It is easy to see that $\operatorname{Ext}^{-2}(Q[-1], E_2) \cong 0$, $\operatorname{Ext}^{-2}(K, E_2) \cong 0$ and $\operatorname{Ext}^{-1}(K, E_2) \cong 0$ for degree reasons. Moreover, $\mathrm{Ext}^{-1}(Q[-1], E_2) = \mathrm{Hom}(Q, E_2) \cong 0$ since Q is zero dimensional (by limit stability) and E_2 is of pure dimension one. Hence the above commutative diagram takes the following form:

$$\begin{array}{ccccccc}
 & \vdots & & & & \vdots & \\
 & \downarrow & & & & \downarrow & \\
 & 0 & & & & 0 & \\
 & \downarrow & & & & \downarrow & \\
\cdots \to & \mathrm{Ext}^{-2}(I^\bullet, E_2) & \to \mathrm{Ext}^{-1}(I^\bullet, I^\bullet) & \to & \mathrm{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) & \to \mathrm{Ext}^{-1}(I^\bullet, E_2) & \to \cdots \\
 & \downarrow & & & & \downarrow & \\
 & 0 & & & & 0 & \\
 & \downarrow & & & & \downarrow & \\
 & 0 & & & & 0 & \\
 & \downarrow & & & & \downarrow & \\
 & \vdots & & & & \vdots &
\end{array} \tag{5.9}$$

hence

$$\mathrm{Ext}^{-2}(I^\bullet, E_2) \cong 0 \ \text{ and } \ \mathrm{Ext}^{-1}(I^\bullet, E_2) \cong 0, \tag{5.10}$$

and therefore

$$\mathrm{Ext}^{-1}(I^\bullet, I^\bullet) \cong \mathrm{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)).$$

Now apply $\mathrm{Hom}(\cdot, \mathcal{O}_X^{\oplus r}(-n))$ to (5.3) and obtain:

$$\begin{aligned}
&\cdots \to \mathrm{Ext}^{-1}(E_2, \mathcal{O}_X^{\oplus r}(-n)) \to \mathrm{Ext}^{-1}(\mathcal{O}_X^{\oplus r}(-n), \mathcal{O}_X^{\oplus r}(-n)) \\
&\to \mathrm{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) \to \mathrm{Hom}(E_2, \mathcal{O}_X^{\oplus r}(-n)) \to \cdots .
\end{aligned} \tag{5.11}$$

Now $\mathrm{Ext}^{-1}(\mathcal{O}_X^{\oplus r}(-n), \mathcal{O}_X^{\oplus r}(-n)) \cong 0$ by degree reasons and $\mathrm{Hom}(E_2, \mathcal{O}_X^{\oplus r}(-n)) \cong 0$ by purity of $\mathcal{O}_X^{\oplus r}(-n)$. Hence $\mathrm{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) \cong 0$ and

$$\mathrm{Ext}^{-1}(I^\bullet, I^\bullet) \cong \mathrm{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) \cong 0.$$

□

From now on for simplicity we change our notation and replace E_2 with F.

5.1.1 Deformations of $[\mathcal{O}_X(-n) \to F]$

Throughout this section we consider a rank 1 τ'-stable frozen triple. We also emphasize that for the results of this section we do not require $\mathrm{H}^1(\mathcal{O}_X)$ to vanish.

Lemma 5.6. *The complex $I^\bullet$ that represents a $\acute{\tau}$-limit-stable frozen triple $\mathcal{O}_X(-n) \xrightarrow{\phi} F$ is simple as an object in the derived category i.e:*

$$\mathcal{H}om(I^\bullet, I^\bullet) = \mathcal{O}_X. \tag{5.12}$$

Proof. Look at the proof of Lemma 1.15 in [28]. The proof follows by replacing $\mathcal{O}_X$ appearing in stable pairs by $\mathcal{O}_X(-n)$. $\square$

Proposition 5.7. *Let $I^\bullet := \mathcal{O}_X(-n) \xrightarrow{\phi} F$. Given a point $p \in \mathfrak{H}^{(P_2,1,n)}_{s,\mathrm{FT}}(\tau')$ represented by the frozen triple $\{(\mathcal{O}_X(-n), F, \phi)\}$, the following is true:*

$$T_p\mathfrak{H}^{(P_2,1,n)}_{s,\mathrm{FT}}(\tau') \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0, \tag{5.13}$$

where $T_p\mathfrak{H}^{(P_2,1,n)}_{s,\mathrm{FT}}(\tau')$ denotes the tangent space at p and the sub-index 0 denotes the trace-free group.

Proof. It is known that the first order deformation of a stable pair (a stable rank 1 frozen triple) is governed by the group $\mathrm{Ext}^0(I^\bullet, F)$ [28]. Hence we know that $T_p\mathfrak{H}^{(P_2,1,n)}_{s,\mathrm{FT}}(\tau') = \mathrm{Ext}^0(I^\bullet, F)$. On the other hand by the work of Huybrechts and Thomas [15] the deformations of $I^\bullet$ with trivial determinant is obtained by $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$. Apply $\mathrm{Hom}(I^\bullet, \cdot)$ to the exact triangle $I^\bullet \to \mathcal{O}_X(-n) \to F$ and obtain the following exact sequence:

$$\begin{aligned} \cdots \to \mathrm{Hom}(I^\bullet, I^\bullet) \to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)) \to \mathrm{Hom}(I^\bullet, F) \\ \to \mathrm{Ext}^1(I^\bullet, I^\bullet) \to \mathrm{Ext}^1(I^\bullet, \mathcal{O}_X(-n)) \to \mathrm{Ext}^1(I^\bullet, F) \to \cdots \end{aligned} \tag{5.14}$$

Apply $\operatorname{Hom}(\cdot, \mathcal{O}_X(-n))$ to $I^\bullet \to \mathcal{O}_X(-n) \to F$ and obtain the following long exact sequence:

$$\begin{aligned}&\cdots \to \operatorname{Ext}^1(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) \to \operatorname{Ext}^1(I^\bullet, \mathcal{O}_X(-n))\\ &\to \operatorname{Ext}^2(F, \mathcal{O}_X(-n)) \to \cdots\end{aligned} \tag{5.15}$$

Now combine these two exact sequences and obtain:

$$\begin{array}{ccccccc} & & & & \vdots & & \\ & & & & \downarrow & & \\ & & & & \operatorname{Ext}^1(F, \mathcal{O}_X(-n)) & & \\ & & & & \downarrow & & \\ & & \mathrm{H}^1(\mathcal{O}_X) & \xrightarrow{\cong} & \operatorname{Ext}^1(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) & & \\ & & \downarrow & & \downarrow & & \\ \cdots \to \operatorname{Hom}(I^\bullet, F) & \to & \operatorname{Ext}^1(I^\bullet, I^\bullet) & \longrightarrow & \operatorname{Ext}^1(I^\bullet, \mathcal{O}_X(-n)) & \longrightarrow & \operatorname{Ext}^1(I^\bullet, F) \\ & & & & \downarrow & & \\ & & & & \operatorname{Ext}^2(F, \mathcal{O}_X(-n)) & & \\ & & & & \downarrow & & \\ & & & & \vdots & & \end{array} \tag{5.16}$$

Since $p \in \mathfrak{H}_{s,\mathrm{FT}}^{(P_2,1,n)}(\tau')$, we may apply Lemma 5.4 and the diagram 5.16 takes the form:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & \mathrm{H}^1(\mathcal{O}_X) & \xrightarrow{\cong} & \operatorname{Ext}^1(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) & & \\ & & \downarrow & & \downarrow \cong & & \\ \cdots \to \operatorname{Hom}(I^\bullet, F) & \to & \operatorname{Ext}^1(I^\bullet, I^\bullet) & \xrightarrow{\alpha} & \operatorname{Ext}^1(I^\bullet, \mathcal{O}_X(-n)) & \longrightarrow & \operatorname{Ext}^1(I^\bullet, F) \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array} \tag{5.17}$$

Because

$$\operatorname{Ext}^1(I^\bullet, I^\bullet) \cong \operatorname{Ext}^1(I^\bullet, I^\bullet)_0 \oplus \operatorname{Im}[\mathrm{H}^1(\mathcal{O}_X) \to \operatorname{Ext}^1(I^\bullet, I^\bullet)],$$

we conclude from the middle square of (5.17) that the canonical map $\operatorname{Ker}(\alpha) \to \operatorname{Ext}^1(I^\bullet, I^\bullet)_0$ is an isomorphism. Hence one obtains the exact sequence:

$$\begin{aligned}&\mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)) \to \mathrm{Hom}(I^\bullet, F) \to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \\ &\to 0 \to \mathrm{Ext}^1(I^\bullet, F) \to \mathrm{Ext}^2(I^\bullet, I^\bullet)_0.\end{aligned} \tag{5.18}$$

Remark 5.8. Note that in obtaining the exact sequence (5.18) one could have assumed that the condition $\mathrm{H}^1(\mathcal{O}_X) \cong 0$ holds true. Then one obtains

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & 0 \cong \mathrm{H}^1(\mathcal{O}_X) & \xrightarrow{\cong} & \mathrm{Ext}^1(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) \cong 0 & & \\ & & \downarrow & & \downarrow \cong & & \\ \cdots \longrightarrow \mathrm{Hom}(I^\bullet, F) & \longrightarrow & \mathrm{Ext}^1(I^\bullet, I^\bullet) & \xrightarrow{\alpha} & \mathrm{Ext}^1(I^\bullet, \mathcal{O}_X(-n)) & \longrightarrow & \mathrm{Ext}^1(I^\bullet, F) \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array} \tag{5.19}$$

Hence it is easy to see that in this case:

$$\mathrm{Ext}^1(I^\bullet, I^\bullet) \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \oplus \mathrm{Im}[\mathrm{H}^1(\mathcal{O}_X) \to \mathrm{Ext}^1(I^\bullet, I^\bullet)] \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0.$$

Now consider the exact sequence:

$$\begin{aligned}&\cdots \to \mathrm{Hom}(F, \mathcal{O}_X(-n)) \to \mathrm{Hom}(\mathcal{O}_X(-n), \mathcal{O}_X(-n)) \\ &\to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)) \to \mathrm{Ext}^1(F, \mathcal{O}_X(-n)) \to \cdots,\end{aligned} \tag{5.20}$$

where $\mathrm{Hom}(F, \mathcal{O}_X(-n))$ and $\mathrm{Ext}^1(F, \mathcal{O}_X(-n))$ vanish by Lemma 5.4. Hence one concludes that

$\mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}$. Therefore (5.18) is written as:

$$\begin{aligned} &0 \to \mathrm{Hom}(I^\bullet, I^\bullet) \to \mathbb{C} \to \mathrm{Hom}(I^\bullet, F) \\ &\to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \to 0 \to \mathrm{Ext}^1(I^\bullet, F) \to \mathrm{Ext}^2(I^\bullet, I^\bullet)_0. \end{aligned} \tag{5.21}$$

Now $\mathrm{Hom}(I^\bullet, I^\bullet) \cong \mathbb{C}$ by Lemma 5.6 and so (5.21) takes the form:

$$\begin{aligned} &0 \to \mathbb{C} \xrightarrow{\cong} \mathbb{C} \to \mathrm{Hom}(I^\bullet, F) \cong \\ &\mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \to 0 \to \mathrm{Ext}^1(I^\bullet, F) \to \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 \end{aligned} \tag{5.22}$$

and this finishes the proof of Proposition 5.7. □

5.2 Deformations of $\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F$

Now return to a frozen triple of rank r. Let X be a Calabi-Yau threefold with $\mathrm{H}^1(\mathcal{O}_X) \cong 0$.

Lemma 5.9. *Let $\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F$ be a $\acute{\tau}$-limit-stable frozen triple represented by the complex $I^\bullet$. Then there exists an injective map:*

$$\mathrm{Hom}(I^\bullet, I^\bullet) \hookrightarrow \mathrm{End}(\mathcal{O}_X(-n)^{\oplus r}). \tag{5.23}$$

Proof. Apply $\mathrm{Hom}(I^\bullet, \cdot)$ to $F[-1] \to I^\bullet \to \mathcal{O}_X(-n)^{\oplus r}$ and obtain the following exact sequence:

$$\mathrm{Ext}^{-1}(I^\bullet, F) \to \mathrm{Hom}(I^\bullet, I^\bullet) \to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \mathrm{Hom}(I^\bullet, F) \to \cdots \tag{5.24}$$

Now apply $\mathrm{Hom}(\cdot, \mathcal{O}_X(-n)^{\oplus r})$ to the same exact triangle and obtain:

$$\begin{aligned}&\mathrm{Hom}(F, \mathcal{O}_X(-n)^{\oplus r}) \to \mathrm{End}(\mathcal{O}_X(-n)^{\oplus r})\\ &\to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \mathrm{Ext}^1(F, \mathcal{O}_X(-n)^{\oplus r}) \to \cdots\end{aligned} \tag{5.25}$$

Using Lemma 5.4 one gets the following isomorphism:

$$\mathrm{End}(\mathcal{O}_X(-n)^{\oplus r}) \cong \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}). \tag{5.26}$$

Now apply (5.10) to conclude via (5.24) that $\mathrm{Hom}(I^\bullet, I^\bullet) \to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r})$ is injective. □

Now we state the main theorem in this section.

Theorem 5.10. *Let $p \in \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ be a point represented by a τ'-limit-stable frozen triple $\{(\mathcal{O}_X(-n)^{\oplus r}, F, \phi)\}$. Let*

$$I^\bullet := \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F$$

be a complex with trivial determinant. The following is true:

$$T_p\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0. \tag{5.27}$$

Proof. We repeat the same argument in Proposition 5.7. and obtain the diagram:

$$\begin{array}{ccccccc}
 & & & & \vdots & & \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Ext}^1(F, \mathcal{O}_X^{\oplus r}(-n)) & & \\
 & & & & \downarrow & & \\
 & & \mathrm{H}^1(\mathcal{O}_X) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathfrak{gl}_r(\mathbb{C}) & & \\
 & & \downarrow & & \downarrow & & \\
\cdots \to \mathrm{Hom}(I^\bullet, F) & \to & \mathrm{Ext}^1(I^\bullet, I^\bullet) & \longrightarrow & \mathrm{Ext}^1(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) & \longrightarrow & \cdots \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Ext}^2(F, \mathcal{O}_X(-n)^{\oplus r}) & & \\
 & & & & \downarrow & & \\
 & & & & \vdots & &
\end{array} \tag{5.28}$$

By Lemma 5.4 we get:

$$\begin{array}{ccccc} & & & & \vdots \\ & & & & \downarrow \\ & & & & 0 \\ & & & & \downarrow \\ & & \mathrm{H}^1(\mathcal{O}_X) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathfrak{gl}_r(\mathbb{C}) \\ & & \downarrow & & \downarrow \\ \cdots \to \mathrm{Hom}(I^\bullet, F) & \to & \mathrm{Ext}^1(I^\bullet, I^\bullet) & \longrightarrow & \mathrm{Ext}^1(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \longrightarrow \cdots \\ & & & & \downarrow \\ & & & & 0 \\ & & & & \downarrow \\ & & & & \vdots \end{array} \tag{5.29}$$

Now recall that $\mathrm{H}^1(\mathcal{O}_X) \cong 0$ by assumption. As in Remark 5.8 we obtain:

$$\begin{aligned} &0 \to \mathrm{Hom}(I^\bullet, I^\bullet) \to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \mathrm{Hom}(I^\bullet, F) \\ &\to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \to 0 \to \mathrm{Ext}^1(I^\bullet, F), \end{aligned} \tag{5.30}$$

apply the functor $\mathrm{Hom}(\cdot, \mathcal{O}_X(-n)^{\oplus r})$ to $F[-1] \to I^\bullet \to \mathcal{O}_X(-n)^{\oplus r}$ and obtain:

$$\begin{aligned} &\mathrm{Hom}(F, \mathcal{O}_X(-n)^{\oplus r}) \to \mathfrak{gl}_r(\mathbb{C}) \\ &\to \mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \mathrm{Ext}^1(F, \mathcal{O}_X(-n)^{\oplus r}) \to \cdots . \end{aligned} \tag{5.31}$$

By purity of $\mathcal{O}_X(-n)^{\oplus r}$ and Lemma 5.4, we have $\mathrm{Hom}(F, \mathcal{O}_X(-n)^{\oplus r}) \cong 0 \cong \mathrm{Ext}^1(F, \mathcal{O}_X(-n)^{\oplus r})$ hence

$$\mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \cong \mathfrak{gl}_r(\mathbb{C}),$$

and one obtains the following exact sequence:

$$\begin{aligned}&0 \to \mathrm{Hom}(I^\bullet, I^\bullet) \to \mathfrak{gl}_r(\mathbb{C}) \to \mathrm{Hom}(I^\bullet, F)\\ &\xrightarrow{k} \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \to 0 \to \mathrm{Ext}^1(I^\bullet, F).\end{aligned} \tag{5.32}$$

In order to continue the proof of Theorem 5.10 one needs to study the deformation space of frozen triples. In order to carry this out first one obtains the deformation space of a highly frozen triple, then one uses the result of Proposition 3.6 to obtain a comparison between deformation spaces of frozen triples and highly frozen triples:

Proposition 5.11. *Given a τ'-stable highly frozen triple (E_1, F, ϕ, ψ) represented by the complex $I^\bullet : [\mathcal{O}_X(-n)^{\oplus r} \to F]$ its space of infinitesimal deformations is given by* $\mathrm{Hom}(I^\bullet, F)$.

Proof. A square zero embedding $S \hookrightarrow S'$ is a closed immersion whose defining ideal $\mathcal{I}$ satisfies $\mathcal{I}^2 = 0$. Given a square zero embedding and a family of highly frozen triples over S, a flat deformation of this family over $\acute{S}$ is a completion of the following commutative diagram with the missing arrows (and exact rows).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X\times\acute{S}}(-n)^{\oplus r} \otimes \mathcal{I} & \longrightarrow & \mathcal{O}_{X\times\acute{S}}(-n)^{\oplus r} & \longrightarrow & \mathcal{O}_{X\times S}(-n)^{\oplus r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}\otimes\mathcal{I} & \dashrightarrow & \acute{\mathcal{F}} & \dashrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array} \tag{5.33}$$

Following a method described by Illusie in [16] (Chapter IV) for deformation of graded modules and graded morphisms of graded modules, one needs to think of $\mathcal{O}_{X\times_{\mathbb{C}}S}$ as a graded algebra in degree zero. Therefore one obtains from $\mathcal{F}$ the graded $\mathcal{O}_{X\times S}$-algebra, $\mathcal{F}_{gr} := \mathcal{O}_{X\times S} \oplus \mathcal{F}$ such that $\mathcal{O}_{X\times S}$ sits in degree zero and the second summand sits in degree one. We similarly define $\mathcal{F}'_{gr}$.

Hence one obtains a commutative diagram of graded $\mathcal{O}_{X\times S}$-algebras.

$$\begin{array}{ccccccccc} 0 \to & \left(\mathcal{O}_{X\times S} \oplus \mathcal{O}_{X\times \acute{S}}(-n)^{\oplus r}\right) \otimes \mathcal{I} & \to & \mathcal{O}_{X\times \acute{S}} \oplus \mathcal{O}_{X\times \acute{S}}(-n)^{\oplus r} & \to & \mathcal{O}_{X\times S} \oplus \mathcal{O}_{X\times S}(-n)^{\oplus r} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{F}\otimes\mathcal{I} & \dashrightarrow & \acute{\mathcal{F}}_{gr} & \dashrightarrow & \mathcal{O}_{X\times S}\oplus\mathcal{F} & \longrightarrow 0 \end{array}, \tag{5.34}$$

here the bottom row in degree zero is given by ([16], 3.1):

$$0 \to 0 \to \mathcal{O}_{X\times S} \to \mathcal{O}_{X\times S} \to 0 \tag{5.35}$$

and in degree one it is given by the bottom row in (5.33). We know that the obstruction to complete this diagram is given by composition of morphisms:

$$\begin{aligned} & L_{\mathcal{O}_{X\times S}\oplus\mathcal{F}/\mathcal{O}_{X\times S}\oplus\mathcal{O}_{X\times S}(-n)^{\oplus r}} \to L_{\mathcal{O}_{X\times S}\oplus\mathcal{O}_{X\times S}(-n)^{\oplus r}/\mathcal{O}_X} \otimes \mathcal{F}[1] \\ & \to \mathcal{I}\otimes(\mathcal{O}_{X\times S}(-n)\oplus\mathcal{O}_{X\times S}(-n)^{\oplus r})\otimes(\mathcal{O}_{X\times S}\oplus\mathcal{F}) \to \mathcal{I}\otimes\mathcal{F}[2] \end{aligned} \tag{5.36}$$

where L is the cotangent complex. Let $k^1(-)$ of a graded module denote the degree one component of that module. Now we state Illusie's result in [16] (Chapter IV 3.2.12):

Theorem 5.12. *Given* $I^\bullet := [\mathcal{O}_{X\times S}(-n)^{\oplus r} \xrightarrow{\phi} \mathcal{F}]$*, there exists an element*

$$ob \in \operatorname{Ext}^2_{\mathcal{D}^b(X\times S)}(\operatorname{Cone}(\phi), \mathcal{I}\otimes\mathcal{F})$$

whose vanishing is necessary and sufficient to complete Diagram (5.34)*. If* $ob = 0$ *then the set of isomorphism classes of completions forms a torsor under* $\operatorname{Ext}^1_{\mathcal{D}^b(X\times S)}(\operatorname{Cone}(\phi), \mathcal{I}\otimes\mathcal{F})$*.*

Here, $\operatorname{Cone}(\phi) = I^\bullet_S[1]$. Moreover, the obstructions $ob : \operatorname{Cone}(\phi) \to \mathcal{I}\otimes\mathcal{F}$ are given by the composite morphism [16] (3.2.14.3):

$$\begin{aligned}&\mathrm{Cone}(\phi) \to k^1\left(L_{\mathcal{O}_{X\times S}\oplus\mathcal{O}_{X\times S}(-n)^{\oplus r}/\mathcal{O}_X}\otimes\mathcal{F}[1]\right)\\ &\to k^1\left(\mathcal{I}\otimes(\mathcal{O}_{X\times S}(-n)\oplus\mathcal{O}_{X\times S}(-n)^{\oplus r})\otimes(\mathcal{O}_{X\times S}\oplus\mathcal{F})\right)\to\mathcal{I}\otimes\mathcal{F}[2].\end{aligned} \tag{5.37}$$

Another way of stating this theorem is to say that the obstructions are given by:

$$\mathrm{Cone}(\phi) \to L_{\mathcal{O}_{X\times S}/\mathcal{O}_X}\otimes\mathcal{F}[1]\to\mathcal{I}\otimes\mathcal{F}[2], \tag{5.38}$$

the set of such composite homomorphisms is given by $\mathrm{Hom}(I^{\bullet}_S[1],\mathcal{I}\otimes\mathcal{F}[2])\cong\mathrm{Ext}^1(I^{\bullet}_S,\mathcal{I}\otimes\mathcal{F})\cong\mathrm{Ext}^1(I^{\bullet}_S,\mathcal{F})\otimes\mathcal{I}$, similarly if $ob=0$, then the set of isomorphism classes of deformations of highly frozen triples makes a torsor under $\mathrm{Ext}^1(I^{\bullet}_S[1],\mathcal{I}\otimes\mathcal{F})\cong\mathrm{Hom}(I^{\bullet}_S,\mathcal{I}\otimes\mathcal{F})\cong\mathrm{Hom}(I^{\bullet}_S,\mathcal{F})\otimes\mathcal{I}$ and this finishes the proof of Proposition 5.11 □

Now we use the result of Proposition 5.11 and Proposition 3.6 to study the space of infinitesimal deformations of a frozen triple.

Proposition 5.13. *The tangent space of the moduli stack of $\acute{\tau}$-limit-stable frozen triples at a point $\{p\}:(E,F,\phi)$ represented by a complex $I^{\bullet}:=[E\to F]$ (where $E\cong\mathcal{O}_X^{\oplus r}(-n)$) is given by:*

$$\mathrm{T}_{\{p\}}\,\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')\cong\mathrm{Hom}(I^{\bullet},F)/\,\mathrm{Im}(\mathfrak{gl}_r(\mathbb{C})\to\mathrm{Hom}(I^{\bullet},F)). \tag{5.39}$$

Equivalently

$$\mathrm{T}_{\{p\}}\,\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')\cong\mathrm{Coker}\left[\mathrm{Hom}(I^{\bullet},\mathcal{O}_X(-n)^{\oplus r})\to\mathrm{Hom}(I^{\bullet},F)\right]. \tag{5.40}$$

Proof. Since our analysis is over a point in the moduli stack, we assume that $S=Spec(\mathbb{C})$ and S' is a square-zero extension over S. Therefore via $S\leftrightarrows S'$ one writes $\mathcal{O}_{S'}\cong\mathcal{O}_S\oplus\mathcal{O}_{S'}\otimes\mathcal{I}$ as an $\mathcal{O}_S$-module. Now use the result of Proposition 5.11. The tangent space of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ at a stable frozen triple of type (P_2,r) is given by the space of infinitesimal deformations of that triple. Use the notation in Definition 2.10. Suppose that $\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_{S'}\xrightarrow{\phi'}\acute{\mathcal{F}}$ is a flat deformation of the family of frozen triples $\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_S\xrightarrow{\phi}\mathcal{F}$ over $\acute{S}$. Similar to 5.33 to obtain the set of such flat deformations one needs to consider the commutative diagram below:

$$\begin{array}{ccccccccc} 0 \to & (\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_{S'})\otimes\mathcal{I} & \to & \mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_{S'} & \to & \mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_S & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{F}\otimes\mathcal{I} & \longrightarrow & \acute{\mathcal{F}} & \longrightarrow & \mathcal{F} & \longrightarrow 0 \end{array} \tag{5.41}$$

The tangent space $\mathrm{T}_{\{p\}}\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$, i.e the set of extensions in (5.41) is given by:

$$\mathrm{Ext}^1\left(\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_{S'}\xrightarrow{\phi'}\acute{\mathcal{F}},(\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_{S'})\otimes\mathcal{I}\xrightarrow{\phi'}\mathcal{F}\otimes\mathcal{I}\right), \tag{5.42}$$

we use the isomorphisms $\left(\mathcal{O}_{X\times\acute{S}}(-n)^{\oplus r}\right)\otimes\mathcal{I}\cong(\mathcal{O}_{X\times S}(-n)^{\oplus r})\otimes\mathcal{I}$ and the notation introduced earlier. Now fix a trivialization $\psi_{\mathcal{M}}:\mathcal{M}_S\xrightarrow{\cong}\mathcal{O}_S^{\oplus r}$. This induces a fixed choice of isomorphism $\psi:\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_S\xrightarrow{\cong}\mathcal{O}^{\oplus r}_{X\times S}(-n)$. Now use the fact that S is a point hence S' is split over S. Therefore one obtains the following splitting of $\mathcal{O}_{X\times S}$-modules:

$$\mathcal{O}_{X\times\acute{S}}(-n)^{\oplus r}\cong\mathcal{O}_{X\times S}(-n)^{\oplus r}\oplus\left(\mathcal{O}_{X\times\acute{S}}(-n)^{\oplus r}\right)\otimes\mathcal{I}, \tag{5.43}$$

Now replace $\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{M}_S$ with the fixed choice of $\mathcal{O}^{\oplus r}_{X\times S}(-n)$ in the top row of (5.41). Moreover use the splitting property in (5.43). The commutative diagram in 5.41 induces:

$$\begin{array}{ccccccccc} 0 \longrightarrow & 0 & \longrightarrow & \mathcal{O}_{X\times S}(-n)^{\oplus r} & \xrightarrow{\cong} & \mathcal{O}_{X\times S}(-n)^{\oplus r} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{F}\otimes\mathcal{I} & \longrightarrow & \acute{\mathcal{F}} & \longrightarrow & \mathcal{F} & \longrightarrow 0 \end{array}, \tag{5.44}$$

where the $\mathcal{O}_{X\times S}(-n)^{\oplus r}$ appearing in the upper row are given as a choice of trivialization of E appearing in the frozen triple (E,F,ϕ). The set of extensions in (5.44) is given by:

$$\mathrm{Ext}^1(\mathcal{O}^{\oplus r}_{X\times S}(-n)\xrightarrow{\phi}\mathcal{F},(\mathcal{F}\otimes\mathcal{I})[-1])\cong\mathrm{Ext}^0(I^\bullet_S,\mathcal{F}\otimes\mathcal{I})\cong\mathrm{Hom}(I^\bullet_S,\mathcal{F})\otimes\mathcal{I} \tag{5.45}$$

where by Proposition 5.11 $\mathrm{Hom}(I_S^\bullet, \mathcal{F})$ is the space of infinitesimal deformations of the highly frozen triple represented by the complex $I_S^\bullet := O_{X\times S}(-n)^{\oplus r} \to \mathcal{F}$.

Hence it is seen that when $S = \mathrm{Spec}(\mathbb{C})$ one obtains the deformation space of a $\mathbb{C}$-point in $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ from the deformation space of a $\mathbb{C}$-point in $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ by making a choice of isomorphism $\psi : \mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{M}_S \xrightarrow{\cong} \mathcal{O}^{\oplus r}_{X\times S}(-n)$. It is also seen that there exists a map $\mathrm{Hom}(I_S^\bullet, \mathcal{F}) \to \mathrm{T}_{\{p\}}\,\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and the kernel of this map corresponds to the choices of trivialization of $\mathcal{O}_X(-n)\otimes \pi_S^*\mathcal{M}_S$ which were not fixed in obtaining the diagram in (5.44), i.e $\mathfrak{gl}_r(\mathbb{C})$. In other words over a $\mathbb{C}$-point in the moduli stack one obtains a short exact sequence of $\mathbb{C}$-vector spaces

$$\mathrm{Hom}(I^\bullet, I^\bullet) \to \mathfrak{gl}_r(\mathbb{C}) \to \mathrm{Hom}(I_S^\bullet, \mathcal{F}) \to \mathrm{T}_{\{p\}}\,\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \to 0. \tag{5.46}$$

Note that when $S = \mathrm{Spec}(\mathbb{C})$ then $I_S^\bullet \cong I^\bullet$ canonically. Also it is true that for large n one has $\mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \cong \mathrm{End}(\mathcal{O}_X(-n)^{\oplus r}) \cong \mathfrak{gl}_r(\mathbb{C})$. Now replace $\mathfrak{gl}_r(\mathbb{C})$ with $\mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r})$ and conclude that the space of infinitesimal deformations of a frozen triple in $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$, i.e the tangent space of the moduli stack at a $\mathbb{C}$-point, is obtained as

$$\mathrm{T}_{\{p\}}\,\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \cong \mathrm{Coker}\left[\mathrm{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \mathrm{Hom}(I^\bullet, F)\right] \tag{5.47}$$

and this finishes the proof of Proposition 5.13 as well as Theorem 5.10 □

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Remark 5.14. Another way of observing the result obtained in 5.13 is to compare the tangent spaces of the moduli stacks of τ'-stable highly frozen triples and frozen triples. Since $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a $\mathrm{GL}_r(\mathbb{C})$ torsor over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$, therefore at every point $\{p\}$ one obtains the following exact sequence of the corresponding tangent spaces:

$$\mathfrak{gl}_r(\mathbb{C}) \to T_{\{p\}}\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \to \mathrm{T}_{\{p\}}\,\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \to 0, \tag{5.48}$$

hence it is immediately seen that $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \cong \mathrm{Coker}[\mathfrak{gl}_r(\mathbb{C}) \to T_{\{p\}}\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')]$. But $T_{\{p\}}\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \cong \mathrm{Hom}(I^\bullet, \mathcal{F})$ by Proposition 5.11 and this proves the result obtained in 5.13.

Now we analyze the infinitesimal deformations of frozen triples in more generality, i.e we do not assume that S is a point or S' is an S-scheme. We assume that S is an affine scheme of finite type over $\mathbb{C}$ and $S \hookrightarrow S'$ is a square-zero embedding of $\mathbb{C}$-schemes.

Theorem 5.15. *Use notation in Definition 2.10. Fix a map* $f : S \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. *Let* S' *be a square-zero extension of* S *with ideal* $\mathcal{I}$. *Let* $\mathcal{D}ef_S(S', \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))$ *denote the deformation space of the map* f *obtained by the set of possible deformations,* $f' : S' \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. *The following statement is true:*

$$\mathcal{D}ef_S(S', \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')) \cong \mathrm{Hom}(I^\bullet_S, F) \otimes \mathcal{I} / \,\mathrm{Im}\left((\mathrm{End}(\mathcal{M}_S) \to \mathrm{Hom}(I^\bullet_S, \mathcal{F})) \otimes \mathcal{I} \right) \tag{5.49}$$

Proof. Let $g : S \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ denote the map of $\mathbb{C}$-stacks. Given the square-zero extension S' one may ask if the map g is extendable to a map $g' : S' \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. If g is extendable, then by Proposition 5.11 we know that the set of such extensions is given by $\mathrm{Hom}(I^\bullet_S, \mathcal{F}) \otimes \mathcal{I}$. Let $\pi^{\mathfrak{M}}_{\mathrm{FT}} : \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ denote the forgetful map in Proposition 3.6. Via composition, one obtains a map $\pi^{\mathfrak{M}}_{\mathrm{FT}} \circ g : S \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. One may ask further if the map $\pi^{\mathfrak{M}}_{\mathrm{FT}} \circ g$ can be extended to a map $\pi^{\mathfrak{M}}_{\mathrm{FT}} \circ g' : S' \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') & \xleftarrow{\quad g \quad} & S \\ \downarrow{\scriptstyle \pi^{\mathfrak{M}}_{\mathrm{FT}}} & \swarrow{\scriptstyle \pi^{\mathfrak{M}}_{\mathrm{FT}} \circ g} & \downarrow \\ \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') & \longleftarrow\cdots\cdots & S' \end{array}, \tag{5.50}$$

by Theorem 5.10 we have shown that the following exact sequence exists over $X \times S$.

$$\mathfrak{g}l_r(\mathbb{C}) \otimes \mathcal{O}_S \to \mathrm{Hom}(I^\bullet_S, \mathcal{F}) \to \mathrm{Ext}^1(I^\bullet_S, I^\bullet_S) \to 0.$$

Let $\mathcal{D}ef_S(S', \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau'))$ denote the deformation space of the map $\pi^{\mathfrak{M}}_{\mathrm{FT}} \circ g$ obtained by set of possible extensions $\pi^{\mathfrak{M}}_{\mathrm{FT}} \circ g'$. By Proposition 5.11 we have shown that

$$\mathcal{D}ef_S(S', \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')) \cong \mathrm{Hom}(I^\bullet_S, \mathcal{F}) \otimes \mathcal{I}.$$

Moreover, by Definition 2.10 and via evaluating the moduli functor associated to the moduli stack of frozen triples on S', one obtains a family of frozen triples represented by the complex $\mathcal{O}_X^{\oplus r}(-n)\boxtimes\mathcal{M}_S\to\mathcal{F}$ over $X\times S$. By assumption both S and S' are chosen to be affine schemes therefore it is not hard to see that the flat deformation of the locally free sheaf $\mathcal{M}_S$ over S' is trivial:

Lemma 5.16. *Let $\mathcal{M}_S$ be a vector bundle of rank r over S such that $\mathcal{M}_S\cong\mathcal{O}_S^{\oplus r}$. Given a square-zero extension $S\hookrightarrow S'$ of affine $\mathbb{C}$-schemes, the flat deformations of $\mathcal{M}_S$ over S' is trivial, i.e the flat extension of $\mathcal{M}_S$ over S' is given by $\mathcal{M}'_S$ a vector bundle of rank r over S' such that $\mathcal{M}'_S\cong\mathcal{O}_{S'}^{\oplus r}$*

Proof. Replace $\mathcal{M}_S$ with $\mathcal{O}_S^{\oplus r}$. There exists an exact sequence

$$0\to\mathcal{O}_S^{\oplus r}\otimes\mathcal{I}\to\mathcal{M}'_S\to\mathcal{O}_S^{\oplus r}\to 0.$$

Since S' is affine, we get an exact sequence:

$$0\to\mathrm{H}^0(\mathcal{O}_S^{\oplus r}\otimes\mathcal{I})\to\mathrm{H}^0(\mathcal{M}'_S)\to\mathrm{H}^0(\mathcal{O}_S^{\oplus r})\to 0.$$

Let $e_1,e_2,\cdots,e_r$ be the canonical generators of $\mathcal{O}_S^{\oplus r}$. Choose lifts $e'_1,e'_2,\cdots,e'_r\in\mathrm{H}^0(\mathcal{M}'_S)$. These sections define a homomorphism $\phi:\mathcal{O}_{S'}^{\oplus r}\xrightarrow{e'_1,e'_2,\cdots,e'_r}\mathcal{M}'_S$. Moreover the homomorphism ϕ becomes an isomorphism upon restriction to S. Since $S\subset S'$ is a nilpotent thickening, by Nakyama's lemma, this implies that ϕ is an isomorphism. $\square$

Hence there exists a surjective map

$$\mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau'))\to\mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))\to 0.$$

Moreover by construction, there exists a natural map $\mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))\to\mathrm{Ext}^1(I^\bullet_S,I^\bullet_S)\otimes\mathcal{I}$, therefore one obtains the following commutative diagram:

$$\begin{array}{ccccccc}
\mathfrak{gl}_r(\mathcal{O}_S)\otimes\mathcal{I} & \xrightarrow{e} & \mathrm{Hom}(I^\bullet_S,\mathcal{F})\otimes\mathcal{I} & \xrightarrow{k} & \mathrm{Ext}^1(I^\bullet_S,I^\bullet_S)\otimes\mathcal{I} & \longrightarrow & 0\\
\uparrow\cong & & \uparrow\cong & & \uparrow & & \\
\mathrm{End}(\mathcal{M}_S)\otimes\mathcal{I} & \xrightarrow{e'} & \mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')) & \xrightarrow{k'} & \mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')) & \longrightarrow & 0
\end{array}, \quad (5.51)$$

by commutativity of the above diagram and surjectivity of the maps k and k', one concludes the following isomorphisms:

$$\begin{aligned}
&\mathrm{Ext}^1(I^\bullet_S, I^\bullet_S)\otimes\mathcal{I}\cong \mathrm{Hom}(I^\bullet_S,\mathcal{F})\otimes\mathcal{I}/\operatorname{Im}(e)\\
&\cong \mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau'))/\operatorname{Im}(e')\cong \mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')),
\end{aligned} \tag{5.52}$$

therefore

$$\mathcal{D}ef_S(S',\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))\cong \mathrm{Hom}(I^\bullet_S,\mathcal{F})\otimes\mathcal{I}/\operatorname{Im}\Big((\mathrm{End}(\mathcal{M}'_S)\to\mathrm{Hom}(I^\bullet_S,\mathcal{F}))\otimes\mathcal{I}\Big).$$

□

Remark 5.17. There exists several ways to obtain the deformation space of a family of highly frozen triples in Proposition 5.11 with respect to the complex $I^\bullet_S$ that represents this family. For example in some special cases such as when F is a locally free sheaf, one may deform the highly frozen triple in Proposition 5.11 using a different method discussed in [3] and [9]. The S-flat family $\mathcal{F}$ in this case parametrizes the locally free, S-flat sheaves over S. Given a family of highly frozen triples represented by the complex $\mathcal{O}_{X\times S}(-n)^{\oplus r}\xrightarrow{\phi}\mathcal{F}$ one may consider the complex:

$$\begin{aligned}
&C^\bullet: 0\to\mathcal{E}nd(\mathcal{O}_{X\times S}(-n)^{\oplus r}\oplus\mathcal{E}nd(\mathcal{F}))\\
&\xrightarrow{\Delta}\mathcal{H}om(\mathcal{O}_{X\times S}(-n)^{\oplus r},\mathcal{F})\to 0,
\end{aligned} \tag{5.53}$$

where $\Delta(\psi_1,\psi_2)=\phi\circ\psi_1-\psi_2\circ\phi$. By the results in [3], there exists a long exact sequence of

Hyper-cohomologies:

$$\begin{aligned}
&0 \to \mathbb{H}^0(C^\bullet) \to \mathrm{H}^0(X \times S, \mathcal{E}nd(\mathcal{O}_{X\times S}(-n)^{\oplus r}) \oplus \mathcal{E}nd(\mathcal{F}))) \\
&\to \mathrm{H}^0(X \times S, \mathcal{H}om(\mathcal{O}_{X\times S}(-n)^{\oplus r}, \mathcal{F})) \to \mathbb{H}^1(C^\bullet) \\
&\to \mathrm{H}^1(X \times S, \mathcal{E}nd(\mathcal{O}_{X\times S}(-n)^{\oplus r}) \oplus \mathcal{E}nd(\mathcal{F}))) \\
&\to \mathrm{H}^1(X \times S, \mathcal{H}om(\mathcal{O}_{X\times S}(-n)^{\oplus r}, \mathcal{F})) \to \mathbb{H}^2(C^\bullet) \to \cdot
\end{aligned} \tag{5.54}$$

the space of infinitesimal deformations of the highly frozen triple is given by $\mathbb{H}^1(C^\bullet)$ and the obstructions are given by $\mathbb{H}^2(C^\bullet)$, [3]. One proves that when $\mathrm{H}^1(\mathcal{O}_X) = 0$, $\mathbb{H}^1(C^\bullet) \cong \mathrm{Hom}(I^\bullet_S, F)$: Using the local to global spectral sequence, the fact that $\mathrm{H}^1(\mathcal{O}_X) = 0$, Serre duality and $n \gg 0$ one simplifies (5.54) as follows:

$$\begin{aligned}
&0 \to \mathbb{H}^0(C^\bullet) \to \mathfrak{gl}_r(\mathcal{O}_S) \oplus \mathrm{End}(\mathcal{F}) \to \mathrm{H}^0(\mathcal{F}(n))^{\oplus r} \to \mathbb{H}^1(C^\bullet) \\
&\to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to 0 \to \mathbb{H}^2(C^\bullet) \to \cdot
\end{aligned} \tag{5.55}$$

recall that by applying $\mathrm{Hom}(\cdot, \mathcal{F})$ to the exact triangle $I^\bullet_S \to \mathcal{O}_{X\times S}(-n)^{\oplus r} \to \mathcal{F}$, one obtains

$$0 \to \mathrm{End}(F) \to \mathrm{H}^0(\mathcal{F}(n))^{\oplus r} \to \mathrm{Hom}(I^\bullet_S, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to 0. \tag{5.56}$$

Now compare these two exact sequences and obtain a commutative diagram:

$$\begin{array}{ccccccccc}
\mathfrak{gl}_r(\mathcal{O}_S) \oplus \mathrm{End}(F) & \longrightarrow & \mathrm{H}^0(\mathcal{F}(n))^{\oplus r} & \longrightarrow & \mathbb{H}^1(C^\bullet) & \longrightarrow & \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) & \longrightarrow & 0 \\
\downarrow & & \cong\downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\
\mathrm{End}(\mathcal{F}) & \longrightarrow & \mathrm{H}^0(\mathcal{F}(n))^{\oplus r} & \longrightarrow & \mathrm{Hom}(I^\bullet_S, \mathcal{F}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) & \longrightarrow & 0
\end{array} \tag{5.57}$$

The first and fifth vertical maps in (5.57) are surjective and injective respectively and the second and fourth are isomorphisms, therefore by 5-Lemma the map $\mathbb{H}^1(C^\bullet) \to \mathrm{Hom}(I^\bullet_S, \mathcal{F})$ is an isomorphism as we expected. Before finishing this section, for completeness, we state two theorems which can be proved as a corollary of Theorem 5.10.

Corollary 5.18. *Let $I^\bullet_S$ be defined as in Theorem 5.10. The first order deformation $I^\bullet_{S'}$ over S' of $I^\bullet_S$ with trivial determinant is quasi-isomorphic to a complex:*

$$\{\mathcal{O}_{X\times_{\mathbb{C}}S'}(-n)^{\oplus r} \xrightarrow{\phi} \mathcal{F}'\}$$

Proof. This is a direct consequence of Theorem 5.10. □

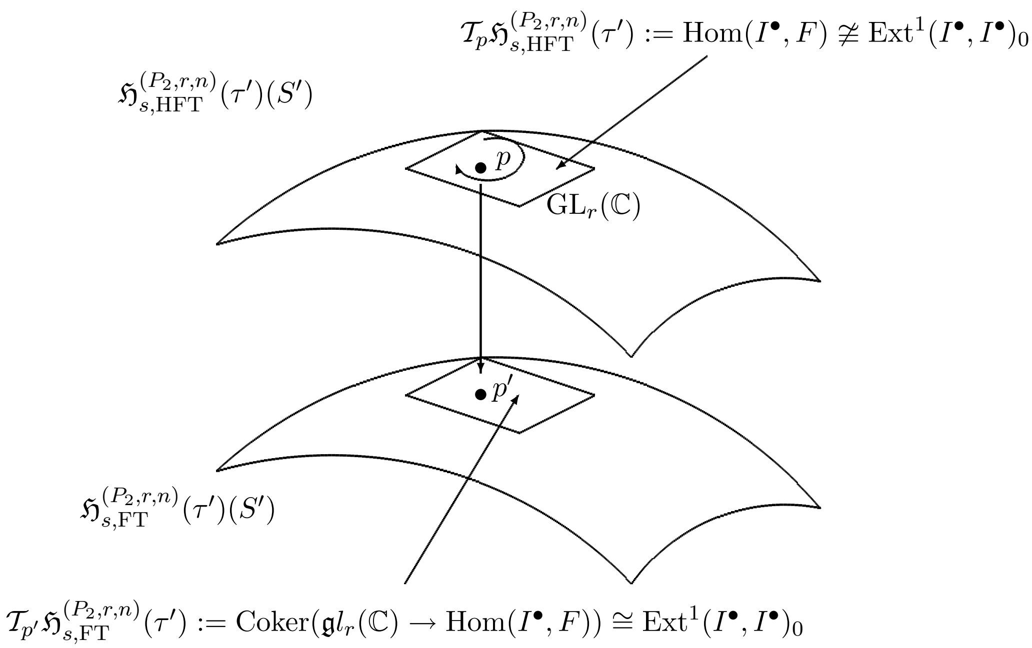
Now we consider higher order deformations:

Corollary 5.19. *Let $I^\bullet_S$ be defined as in Theorem 5.10. The higher order deformation $I^\bullet_{S'}$ over $\acute{S}$ of $I^\bullet_S$ with trivial determinant is quasi-isomorphic to a complex:*

$$\{\mathcal{O}_{X\times_{\mathbb{C}}\acute{S}}(-n)^{\oplus r} \xrightarrow{\phi'} \acute{\mathcal{F}}\}$$

Proof. This is a direct consequence of Theorem 5.10. □

Remark 5.20. In conclusion, by applying Illusie's method to the deformation theory of highly frozen triples, we concluded that the highly frozen triples do not deform in general as objects in the derived category (which are associated to them), but the frozen triples satisfy this property. The schematic picture below explains this conclusion pictorially.


$\mathcal{T}_p\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') := \mathrm{Hom}(I^\bullet, F) \ncong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$
$\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')(S')$
p
$\mathrm{GL}_r(\mathbb{C})$
p'
$\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')(S')$
$\mathcal{T}_{p'}\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') := \mathrm{Coker}(\mathfrak{gl}_r(\mathbb{C}) \to \mathrm{Hom}(I^\bullet, F)) \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$

Chapter 6

Deformation-obstruction theories

By Theorem 3.9 the moduli stack of stable frozen triples $\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$ is an Artin stack.

Notation 1: By a perfect complex $\mathbb{E}^\bullet$ in $\mathcal{D}(X)$ of perfect amplitude $[a,b]$ we mean a complex satisfying the condition that for every point $p \in X$ there exists an open neighborhood $\mathcal{U}_p$ over which there exists a complex of vector bundles $\mathbb{R}^\bullet$ whose terms R^i vanish for $i < a$ and $i > b$ and $\mathbb{E}^\bullet \mid_{\mathcal{U}_p}$ is quasi-isomorphic to $\mathbb{R}^\bullet$.

Notation 2: By a perfect complex of strongly perfect amplitude $[a,b]$ we mean a complex $\mathbb{E}^\bullet$ in $\mathcal{D}(X)$ satisfying the condition that there exists globally a complex of vector bundles $\mathbb{R}^\bullet$ such that $R^i = 0$ for $i < a$ or $i > b$ and such that $\mathbb{E}^\bullet \cong \mathbb{R}^\bullet$ in $\mathcal{D}(X)$.

Definition 6.1. Following [21] and [26]) by definition a perfect deformation-obstruction theory for $\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$ is given by a perfect 3-term complex $\mathbb{E}^{\bullet\vee}$ of strongly perfect amplitude $[-1,1]$ and a map in the derived category:

$$\mathbb{E}^{\bullet\vee} \xrightarrow{\phi} \mathbb{L}^\bullet_{\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')}$$

such that $h^1(\phi)$ and $h^0(\phi)$ are isomorphisms and $h^{-1}(\phi)$ is an epimorphism.

Remark 6.2. Notation: The reason for having superscript $\vee$ in $\mathbb{E}^{\bullet\vee}$ appearing in statement of Definition 6.1 will be justified through our construction later.

Remark 6.3. Here $\mathbb{L}^\bullet_{\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')}$ is the truncated cotangent complex of the Artin moduli stack of τ'-stable frozen triples concentrated in degrees -1, 0 and 1 whose pullback via the projection map $\pi_{\mathrm{FT}}^{\mathfrak{M}} : \mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau') \to \mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$ has the form:

$$(\pi_{\mathrm{FT}}^{\mathfrak{M}})^*\mathbb{L}^\bullet_{\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')} : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathfrak{A}} \mid_{\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')} \to \mathfrak{K}^\vee \otimes \mathcal{O}_{\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')},$$

where $\mathfrak{K}^\vee$, by construction, is the dual of the Lie algebra of $\mathrm{GL}_r(\mathbb{C})$. Here, $\mathfrak{A}$ denotes the smooth Artin stack that one needs to embed $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ into, in order to obtain the truncated cotangent complex, $\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}$. Finally $\mathcal{I}$ is the ideal corresponding to this embedding.

We will show later that there exists a complex $\mathbb{E}^{\bullet\vee} \in \mathcal{D}^b(\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))$ which satisfies the cohomological properties of a perfect deformation-obstruction theory for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. However as we will see, this complex is perfect of wrong amplitude. In other words it is perfect of amplitude $[-2,1]$ (instead of being perfect of amplitude $[-1,1]$ as in Definition 6.1).

(1). By Theorem 3.12 $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a DM stack. In this situation the truncated cotangent complex takes the form:

$$\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathfrak{A}} \mid_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} .$$

Here as Behrend and Fantechi define in [2], a perfect deformation-obstruction theory is given by a perfect 2 term complex $\mathbb{G}^\bullet$ of strongly perfect amplitude $[-1,0]$ and a map in the derived category:

$$\mathbb{G}^\bullet \xrightarrow{\phi} \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')},$$

such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is an epimorphism. Unfortunately, using usual direct methods, the construction of such $\mathbb{G}^\bullet$ for our setup fails in general.

(2). To solve the issue in **(1)**, first we show that there exists a 4-term perfect deformation obstruction theory of strongly perfect amplitude $[-2,1]$ over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. Then we pullback this complex via the projection map $\pi^{\mathfrak{M}}_{\mathrm{FT}} : \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and we apply a suitable local truncation to the pulled-back complex and define a perfect deformation obstruction theory of perfect amplitude $[-1,0]$ over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. Finally we show that locally we can construct virtual fundamental cycles which (under Assumption 7.15) glue to each other to give rise to a globally well-defined virtual fundamental class over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

Remark 6.4. From now on by a perfect complex of perfect amplitude $[a,b]$ we mean a perfect complex of strongly perfect amplitude $[a,b]$.

6.1 Discussion on perfect obstruction theory over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$

Given X a smooth projective Calabi-Yau threefold over $\mathbb{C}$ and S a parametrizing scheme of finite type over $\mathbb{C}$, by Theorem 5.10, we showed that the tangent space at every point of the moduli stack of $\acute{\tau}$-limit-stable frozen triples is isomorphic to the space of deformations of the complex with fixed determinant which represents the stable frozen triple. In this section we use this result to construct a deformation obstruction complex for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.To save space let us temporarily introduce the following notation:

1. $\mathfrak{H} := \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$
2. $A := \mathcal{O}_{X\times\mathfrak{H}} \oplus M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n)$ where M is a vector bundle of rank r.
3. $\pi_{\mathfrak{H}} : X \times \mathfrak{H} \to \mathfrak{H}$

Consider the universal exact triangle determined by the universal complex representing a universal stable frozen triple over $X \times \mathfrak{H}$:

$$\mathbb{I}^{\bullet} \to M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n) \to \mathbb{F} \tag{6.1}$$

Now consider the following commutative diagram:

$$\begin{array}{ccc} X \times \mathfrak{H} & \xrightarrow{\pi_{\mathfrak{H}}} & \mathfrak{H} \\ \pi_X \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{C} \end{array} \tag{6.2}$$

Apply $R\mathcal{H}om(\cdot, \mathbb{I}^{\bullet}) \otimes \pi_X^*\omega_X[2]$ to this triangle and obtain the composition of morphisms:

$$\begin{aligned} &R\mathcal{H}om(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet}) \otimes \omega_{\pi_{\mathfrak{H}}}[2] \to R\mathcal{H}om(\mathbb{F}, \mathbb{I}^{\bullet}) \otimes \omega_{\pi_{\mathfrak{H}}}[3] \to \\ &R\mathcal{H}om(M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n), \mathbb{I}^{\bullet}) \otimes \omega_{\pi_{\mathfrak{H}}}[3] \end{aligned} \tag{6.3}$$

There exists a map from the trace-free $R\mathcal{H}om$ to $R\mathcal{H}om$ so we get the following composition of morphisms in $\mathcal{D}^b(X \times \mathfrak{H})$:

$$\begin{aligned} &R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_{\pi_{\mathfrak{H}}}[2] \to R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[2] \\ &\to R\mathcal{H}om(\mathbb{F}, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[3] \to R\mathcal{H}om(M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n), \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[3] \end{aligned} \tag{6.4}$$

Theorem 6.5. *There exists a map in $\mathcal{D}^b(\mathfrak{H})$:*

$$R\pi_{\mathfrak{H}*}(R\mathcal{H}om(\mathbb{F}, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[3]) \xrightarrow{\psi} \mathbb{L}^\bullet_{\mathfrak{H}} \tag{6.5}$$

Proof. One needs to apply the result of Illusie [16] (Section 4.2) in Theorem 5.12 to the universal complex $\mathbb{I}^\bullet : M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n) \xrightarrow{\tilde{\phi}} \mathbb{F}$. Since we will not eventually use $R\pi_{\mathfrak{H}*}(R\mathcal{H}om(\mathbb{F}, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[3])$ as a suitable candidate for the deformation obstruction theory of $\mathfrak{H} := \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and since the proof follows directly from the proof of Joyce and Song in [18] (Theorem 14.7) applied to $\mathbb{I}^\bullet : M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n) \xrightarrow{\tilde{\phi}} \mathbb{F}$ we omit providing a detailed proof here and leave this to the reader. □

Remark 6.6. Note that the complex $R\mathcal{H}om(\mathbb{F}, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[3]$ is neither perfect of amplitude $[-1, 1]$ nor it defines a deformation theory for moduli stack of frozen triples. However by (6.4) one obtains:

$$\begin{aligned} &R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_{\pi_{\mathfrak{H}}}[2] \to R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{h}}}[2] \\ &\to R\pi_{\mathfrak{H}} * (R\mathcal{H}om(\mathbb{F}, \mathbb{I}^\bullet) \otimes \omega_{\pi_{\mathfrak{H}}}[3]) \to \mathbb{L}^\bullet_{\mathfrak{H}} \end{aligned} \tag{6.6}$$

Now consider the composite morphism in the derived category:

$$R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_{\pi_{\mathfrak{H}}}[2] \to \mathbb{L}^\bullet_{\mathfrak{H}}. \tag{6.7}$$

Note that for every point $\{p\} \in \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ represented by a complex $I^\bullet$ the fiber of $R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes$ $\omega_{\pi_{\mathfrak{H}}}[2]$ over $I^\bullet$ is a complex which has 4 non-vanishing cohomologies (by taking cohomologies in degrees $-2, \cdots, 1$) equal to $Ext^{2+i}(I^\bullet, I^\bullet)_0$.

6.2 Non-perfect deformation-obstruction theory of amplitude $[-2, 1]$ over $\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$

Theorem 6.7. *There exists a map in the derived category,*

$$R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2] \xrightarrow{ob} \mathbb{L}^\bullet_{\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')}.$$

After suitable truncations, there exists a 4 term complex $\mathbb{E}^\bullet$ of locally free sheaves , such that $\mathbb{E}^{\bullet\vee}$ is self-symmetric of amplitude $[-2, 1]$ and there exists a map in the derived category,

$$\mathbb{E}^{\bullet\vee} \xrightarrow{ob^t} \mathbb{L}^\bullet_{\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')} \tag{6.8}$$

such that $h^{-1}(ob^t)$ is surjective, and $h^0(ob^t)$ and $h^1(ob^t)$ are isomorphisms.

Proof. In what follows we use the notation $\mathfrak{H} := \mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$.

Consider the universal complex:

$$\mathbb{I}^\bullet = [M \otimes \mathcal{O}_{X\times\mathfrak{H}}(-n) \to \mathbb{F}] \in \mathcal{D}^b(X \times \mathfrak{H}).$$

Since the composition of the maps $id : \mathcal{O}_{X\times\mathfrak{H}} \to \mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$ and $tr : \mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \to \mathcal{O}_{X\times\mathfrak{H}}$ is multiplication by $rk(\mathbb{I}^\bullet)$, one obtains a splitting $\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \cong \mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \oplus \mathcal{O}_{X\times\mathfrak{H}}$. Recall that by discussions in Section 3.3.1 $\mathfrak{H} = [\frac{\mathfrak{S}_s^{(P_2,r,n)}(\tau')}{G}]$ where $G = \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}(V_2)$.
For simplicity denote $\mathfrak{S} := \mathfrak{S}_s^{(P_2,r,n)}(\tau')$. Let $\mathbb{I}^\bullet_{\mathfrak{S}}$ denote the pullback of $\mathbb{I}^\bullet$ to $X \times \mathfrak{S}$. We write $L^\bullet$ to mean the full, untruncated cotangent complex, and write $\mathbb{L}^\bullet = \tau^{\geq -1}L^\bullet$ for the truncated cotangent complex. Consider the *Atiyah class* $\mathbb{I}^\bullet_{\mathfrak{S}} \to L^\bullet_{X\times\mathfrak{S}} \otimes \mathbb{I}^\bullet_{\mathfrak{S}}[1]$ defined by Illusie [16] (Section

IV2.3.6). The Atiyah class can be identified with a class in $\mathrm{Ext}^1(\mathbb{I}_{\mathfrak{S}}, L^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}})$. The composite

$$\mathbb{I}^\bullet_{\mathfrak{S}} \to L^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1] \to \tau^{\geq -1}L^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1] = \mathbb{L}^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1]$$

is the truncated Atiyah class of [15], Section 2.2.

By [14] (Proposition 2.1.10) the complex $\mathbb{I}^\bullet$ is perfect. It then follows from Corollaire IV.2.3.7.4 of [16] that the composite $\mathbb{I}^\bullet_{\mathfrak{S}} \to \mathbb{L}^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1] \to \Omega^1_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1]$, when identified with a 1-extension, agrees with the canonical 1-extension

$$0 \to \Omega^1_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}} \to \mathcal{P}^1_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}} \to \mathbb{I}^\bullet_{\mathfrak{S}} \to 0 \tag{6.9}$$

defined by tensoring with the first-order principal parts $\mathcal{P}^1_{X\times\mathfrak{S}}$.

We want to show that the Atiyah class descends to $X\times\mathfrak{H} = X\times[\frac{\mathfrak{S}}{G}]$ where $G = \mathrm{GL}_r(\mathbb{C})\times\mathrm{GL}(V_2)$ (where this identification comes from discussions in Section 3.3). More precisely, this means the following. Let $q_{\mathfrak{H}} : X\times\mathfrak{S} \to X\times\mathfrak{H}$ denote the projection. Then we want a morphism $\mathbb{I}^\bullet \to L^\bullet_{X\times\mathfrak{H}}\otimes\mathbb{I}^\bullet[1]$ on $\mathfrak{H}$, such that the natural composite $q^*_{\mathfrak{H}}\mathbb{I}^\bullet \to q^*_{\mathfrak{H}}L^\bullet_{X\times\mathfrak{H}}\otimes q^*_{\mathfrak{H}}\mathbb{I}^\bullet[1] \to L^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1]$ agrees with the Atiyah class of Illusie. The complex $\mathbb{I}^\bullet_{\mathfrak{S}}$ is G-equivariant by construction (it comes via pullback from $X\times\mathfrak{H}$), and the construction of the Atiyah class shows that it too is naturally G-equivariant. The pulled back cotangent complex $q^*_{\mathfrak{H}}L^\bullet_{X\times\mathfrak{H}}$ has the following description. There is a natural composite map $L^\bullet_{X\times\mathfrak{S}} \to \Omega^1_{X\times\mathfrak{S}} \to \mathfrak{g}^\vee\otimes\mathcal{O}_{X\times\mathfrak{S}}$, where the second map is dual to the infinitesimal $\mathfrak{g}$-action (and $\mathfrak{g} = \mathrm{Lie}(G)$). Then $q^*_{\mathfrak{H}}L^\bullet_{X\times\mathfrak{H}} \simeq \mathrm{Cone}[L^\bullet_{X\times\mathfrak{S}} \to \mathfrak{g}^\vee\otimes\mathcal{O}_{X\times\mathfrak{S}}][-1]$. Thus, to prove that the Atiyah class descends to $X\times\mathfrak{H}$ in the sense explained above, it suffices to show that the composite $\mathbb{I}^\bullet_{\mathfrak{S}} \to \mathbb{L}^\bullet_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1] \to \Omega^1_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{X\times\mathfrak{S}}[1] \to \mathfrak{g}^\vee\otimes\mathbb{I}^\bullet_{\mathfrak{S}}[1]$ represents an equivariantly split extension. By the above discussion, this extension is obtained by pushing out the principal parts extension (6.9) along the natural map $\Omega^1_{X\times\mathfrak{S}}\otimes\mathbb{I}^\bullet_{\mathfrak{S}} \to \mathfrak{g}^\vee\otimes\mathbb{I}^\bullet_{\mathfrak{S}}$. Just as a splitting of the principal parts extension corresponds to a choice of connection, however, a splitting of its pushout corresponds to a choice of an L-connection [5] (Section 4) where $L = \mathfrak{g}\otimes\mathcal{O}_{X\times\mathfrak{S}}$ is the action Lie algebroid associated to the infinitesimal G-action. Since $\mathbb{I}^\bullet$ is G-equivariant, it comes equipped with a $\mathfrak{g}\otimes\mathcal{O}_{X\times\mathfrak{G}}$-connection, hence a G-equivariant splitting of the required 1-extension. It follows that the Atiyah class descends to $X\times\mathfrak{H}$.

We now have the truncated Atiyah class of the universal complex, given by a class in

$$\begin{aligned}&\mathrm{Ext}^1_{X\times\mathfrak{H}}(\mathbb{I}^\bullet,\mathbb{I}^\bullet\otimes\mathbb{L}^\bullet_{X\times\mathfrak{H}})\cong\mathrm{Ext}^1_{X\times\mathfrak{H}}(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet),\mathbb{L}^\bullet_{X\times\mathfrak{H}})\\&\cong\mathrm{Ext}^1_{X\times\mathfrak{H}}(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\oplus\mathcal{O}_{X\times\mathfrak{H}},\mathbb{L}^\bullet_{X\times\mathfrak{H}}),\end{aligned}\tag{6.10}$$

where $\mathbb{L}^\bullet_{X\times\mathfrak{H}}$ denotes the truncated cotangent complex of $X\times\mathfrak{H}$. Note that over $X\times\mathfrak{H}$, $\mathbb{L}^\bullet_{X\times\mathfrak{H}}=\mathcal{L}\pi_X^*\mathbb{L}^\bullet_X\oplus\mathcal{L}\pi^*_{\mathfrak{H}}\mathbb{L}^\bullet_{\mathfrak{H}}$. Since the projection maps are flat the derived pullbacks are the same as the usual pullbacks. One obtains the following map between the Ext groups:

$$\mathrm{Ext}^1_{X\times\mathfrak{H}}(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\oplus\mathcal{O}_{X\times\mathfrak{H}},\mathbb{L}^\bullet_{X\times\mathfrak{H}})\to\mathrm{Ext}^1_{X\times\mathfrak{H}}(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0,\pi^*_{\mathfrak{H}}\mathbb{L}^\bullet_{\mathfrak{H}}).\tag{6.11}$$

On the other hand:

$$\begin{aligned}&\mathrm{Ext}^1_{X\times\mathfrak{H}}\left(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0,\pi^*_{\mathfrak{H}}\mathbb{L}^\bullet_{\mathfrak{H}}\right)\\&\cong\mathrm{Ext}^{\dim(X)+\dim(\mathfrak{H})-1}_{X\times\mathfrak{H}}\left(\mathcal{L}\pi^*_{\mathfrak{H}}\mathbb{L}^\bullet_{\mathfrak{H}},R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_{X\times\mathfrak{H}}\right)^\vee\\&\cong\mathrm{Ext}^{\dim(X)+\dim(\mathfrak{H})-1}_{\mathfrak{H}}\left(\mathbb{L}^\bullet_{\mathfrak{H}},R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_{X\times\mathfrak{H}}\right)\right)^\vee\\&\cong\mathrm{Ext}^{\dim(\mathfrak{H})-[\dim(X)+\dim(\mathfrak{H})-1]}_{\mathcal{M}^P_{\mathrm{FT}}}\left(R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_{X\times\mathfrak{H}}\right),\mathbb{L}^\bullet_{\mathfrak{H}}\otimes\omega_{\mathfrak{H}}\right)\\&\cong\mathrm{Ext}^{-\dim(X)+1}_{\mathfrak{H}}\left(R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_{X\times\mathfrak{H}}\right)\otimes R\pi_{\mathfrak{H}*}\mathcal{L}\pi^*_{\mathfrak{H}}\omega^{-1}_{\mathfrak{H}},\mathbb{L}^\bullet_{\mathfrak{H}}\right),\end{aligned}\tag{6.12}$$

where the first isomorphism is obtained by Serre duality, the second isomorphism is induced by the adjointness property of the left derived pullback and the right derived pushforward and the third isomorphism is obtained by Serre duality. By projection formula and the definition of the relative dualizing sheaf

$$\omega_{\pi_{\mathfrak{H}}}=\omega_{X\times\mathfrak{H}}\otimes\omega^{-1}_{\mathfrak{H}}=\pi^*_X\omega_X$$

the last term in (6.12) is rewritten as:

$$\mathrm{Ext}_{\mathfrak{H}}^{-\dim(X)+1}\left(R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^{\bullet},\mathbb{I}^{\bullet})_0\otimes\pi_X^*\omega_X\right),\mathbb{L}^{\bullet}_{\mathfrak{H}}\right). \tag{6.13}$$

Since X is a three-fold, (6.13) is rewritten as:

$$R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^{\bullet},\mathbb{I}^{\bullet})_0\otimes\pi_X^*\omega_X\right)[2]\to\mathbb{L}^{\bullet}_{\mathfrak{H}}. \tag{6.14}$$

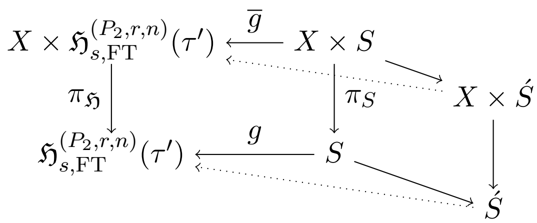
Therefore, it is seen that the truncated Atiyah class of the universal complex over the moduli stack of τ'-stable frozen triples, induces a well defined map in the derived category as in (6.7). Next we show that this morphism in the derived category defines a relative deformation-obstruction theory for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. □

Proposition 6.8. *The morphism given by* (6.14) *defines a relative deformation-obstruction theory for* $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.

Proof. We follow the same strategy as in [15] , [28]. Given a morphism of $\mathbb{C}$-stacks $S\xrightarrow{g}\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and a square zero embedding $S\hookrightarrow\acute{S}$, by the theory of cotangent complexes there exists a morphism in $\mathcal{D}^b(S)$: $\mathbb{L}^{\bullet}_S\to\mathbb{L}^{\bullet}_{S/S'}\cong[\mathcal{I}_{S\subset S'}\to\Omega_{S'}\mid_S]$.There exists a morphism: $[\mathcal{I}_{S\subset S'}\to\Omega_{S'}\mid_S]\to\mathcal{I}_{S\subset S'}[1]$, hence we obtain a morphism in $\mathcal{D}^b(S)$:

$$e: g^*\mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}\to\mathbb{L}^{\bullet}_S\to\mathcal{I}_{S\subset\acute{S}}[1] \tag{6.15}$$

where $\mathcal{I}_{S\subset\acute{S}}$ is the ideal of $S\subset\acute{S}$ and $e\in\mathrm{Ext}^1(g^*\mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')},\mathcal{I}_{S\subset\acute{S}})$. Now e is equal to zero if and only if there exists a lift $\acute{g}:\acute{S}\to\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and moreover if such g' exists then the set of isomorphism extensions forms a torsor under $\mathrm{Hom}(g^*\mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')},\mathcal{I}_{S\subset\acute{S}})$. Consider the following commutative diagram:



. (6.16)

Pullback the morphism in (6.14) by g and obtain:

$$g^* R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2] \to g^*\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \tag{6.17}$$

This induces a natural composite morphism in $\mathcal{D}^b(S)$:

$$o : g^* R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2] \to g^*\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \to \mathbb{L}^\bullet_S \to \mathcal{I}_{S\subset\acute{S}}[1], \tag{6.18}$$

where $o \in \mathrm{Ext}^{-1}(g^* R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right), \mathcal{I}_{S\subset\acute{S}})$. One shows that there exists an extension of g to $\acute{g}$ if and only if o vanishes and moreover the set of such extensions forms a torsor under

$$\mathrm{Ext}^{-2}(g^* R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right), \mathcal{I}_{S\subset\acute{S}}).$$

By (6.16) and the flatness of π_S one obtains the following isomorphism:

$$g^* R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2] \cong R\pi_{S*}\left(R\mathcal{H}om(\overline{g}^*\mathbb{I}^\bullet, \overline{g}^*\mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2]. \tag{6.19}$$

Hence one obtains:

$$R\pi_{S*}\left(R\mathcal{H}om(\overline{g}^*\mathbb{I}^\bullet, \overline{g}^*\mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2] \to g^*\mathbb{L}^\bullet_{\mathcal{M}^{P,s}_{\mathrm{FT}}} \to \mathbb{L}^\bullet_S \to \mathcal{I}_{S\subset\acute{S}}[1] \tag{6.20}$$

therefore:

$$o \in \operatorname{Ext}^{-1}(R\pi_{S*}\left(R\mathcal{H}om(\overline{g}^*\mathbb{I}^\bullet, \overline{g}^*\mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right), \mathcal{I}_{S\subset \acute{S}}) \cong \operatorname{Ext}^2_{X\times S}(\overline{g}^*\mathbb{I}^\bullet, \overline{g}^*\mathbb{I}^\bullet \otimes \pi_S^*\mathcal{I}_{S\subset \acute{S}})_0 \quad (6.21)$$

by a similar argument to (6.12). By results of Thomas in [32] the trace of the obstruction class is the obstruction to deform $det(\overline{g}^*\mathbb{I}^\bullet)$. So this is enough to conclude that $o = 0$ if and only if there exist deformations of $\overline{g}^*\mathbb{I}^\bullet$ from $X \times S$ to $X \times \acute{S}$. Moreover the set of such deformations forms a torsor under $\operatorname{Ext}^1_{X\times S}(\overline{g}^*\mathbb{I}^\bullet, \overline{g}^*\mathbb{I}^\bullet \otimes \pi_S^*\mathcal{I}_{S\subset \acute{S}})_0$. By definition of relative moduli stack, the deformations of $\overline{g}^*\mathbb{I}^\bullet$ are in one to one correspondence with deformations of g to $\acute{g}$ and this finishes the proof of Proposition 6.8. □

Now we show that the deformation-obstruction theory in Proposition 6.8 is globally quai-isomorphic to a 4 term complex of vector bundles.

6.3 Truncation to a perfect 4 term complex of vector bundles

Lemma 6.9. *Given S a smooth scheme of finite type over $\mathbb{C}$ and $X \to S$ a smooth projective morphism of relative dimension n, If F is a flat family of coherent sheaves on the fibers of $f : X \to \mathbb{C}$ then there exists a locally free resolution*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to F$$

Such that $R^n f_* F_m$ is locally free for $m = 0, \cdots, n$, $R^i f_* F_m = 0$ for $i \neq n$ and $m = 0, \cdots, n$.

Proof. Look at [14] (Proposition 2.1.10). □

Proposition 6.10. *The complex $R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2]$ in (6.14) is quasi-isomorphic to a 4 term complex of locally free sheaves.*

Proof. Consider the universal complex $\mathbb{I}^\bullet$ over $X \times \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. By Lemma 6.9 there exists a finite locally free resolution $A^\bullet$ of $\mathbb{I}^\bullet$. There exists an isomorphism $(A^\bullet)^\vee \otimes A^\bullet \cong \mathcal{O}_{X\times\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \oplus ((A^\bullet)^\vee \otimes A^\bullet)_0$. Now define the quasi-isomorphism class of trace-free Homs by $R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \cong ((A^\bullet)^\vee \otimes A^\bullet)_0$. Each term in the complex $((A^\bullet)^\vee \otimes A^\bullet)_0$ is a coherent locally free sheaf over $X \times \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ flat over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. Since the projection map $\pi_{\mathfrak{H}} : X \times \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ has relative dimension 3, by Lemma 6.9 there exists a locally free resolution of length 4 associated to each term in $((A^\bullet)^\vee \otimes A^\bullet)_0$. From this point the proof follows Lemma 2.10 [28] . Let the complex $B^\bullet$ be a sufficiently negative locally free resolution of $((A^\bullet)^\vee \otimes A^\bullet)_0$ trimmed to start at least 4 places earlier than $((A^\bullet)^\vee \otimes A^\bullet)_0$, then $R^{\leq 2}\pi_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')*}B^m = 0$ for all m and $R^3\pi_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')*}B^m$ is locally free. Let $\mathbb{E}^{\bullet\vee}$ be defined as the complex with

$$\mathbb{E}^j \cong R^3\pi_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')*}B^{j+3}. \tag{6.22}$$

The complex $\mathbb{E}^\bullet$ is a complex of locally free sheaves, and quasi-isomorphic to $R\pi_{\mathfrak{H}*}R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$. Restricting this complex to a point $\{b\} \in \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ (i.e base change) one obtains a complex whose cohomologies compute $\mathrm{Ext}^i(I^\bullet, I^\bullet)_0$. By the property of $I^\bullet$ shown earlier, the negative Ext groups vanish. Hence, one obtains a complex whose nonvanishing cohomologies are given by $\mathrm{Ext}^0(I^\bullet, I^\bullet)_0, \cdots \mathrm{Ext}^4(I^\bullet, I^\bullet)_0, \cdots$. However since X is Calabi-Yau, by Serre duality $\mathrm{Ext}^i(I^\bullet, I^\bullet) \cong \mathrm{Ext}^{3-i}(I^\bullet, I^\bullet)$. Hence for $i > 3$, $\mathrm{Ext}^i(I^\bullet, I^\bullet)_0 \cong 0$ and the only non-vanishing cohomologies are $\mathrm{Ext}^0(I^\bullet, I^\bullet)_0 \cdots \mathrm{Ext}^3(I^\bullet, I^\bullet)_0$. Note that, by Serre duality, $\mathrm{Hom}(I^\bullet, I^\bullet)_0 \cong Ext^3(I^\bullet, I^\bullet)_0^\vee$. Now apply Lemma 2.10 in [28] to $\mathbb{E}^{\bullet\vee}$ in (6.22). The complex $\mathbb{E}^{\bullet\vee}$ is quasi-isomorphic to a 4 term complex of locally free sheaves.

The truncated cotangent complex of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ is concentrated in degrees -1,0 and 1:

$$\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}} \mid_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \to (\mathfrak{gl}_r(\mathbb{C}))^\vee \otimes \mathcal{O}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}.$$

By Theorem 6.7 and Proposition 6.8 one obtains a morphism in the derived category

$$\mathbb{E}^{\bullet\vee} \xrightarrow{ob^t} \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}.$$

Restrict to a point $b \in \mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$ and obtain:

$$h^{-1}(ob^t\mid_b) : \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 \to \mathrm{Ext}^1(I^\bullet, F)$$

which is surjective by the construction and the exact sequence in (5.22). Moreover, by Propositions 5.13 and 5.15, the cohomology map in degree zero

$$h^0(ob^t\mid_b) : \mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \to \mathrm{Coker}(\mathfrak{gl}_r(\mathbb{C}) \to \mathrm{Hom}(I^\bullet, F))$$

is an isomorphism. Finally, $h^1(ob^t\mid_b)$ is an isomorphism mapping the automorphisms of the object in the derived category to the automorphisms of the associated frozen triple. This finishes the proof of theorem 6.7. □

To obtain a relative deformation obstruction theory over $\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$ one needs to truncate the complex $\mathbb{E}^{\bullet\vee}$ so that it does not have any cohomology in degree -2. The cohomological truncation of $\mathbb{E}^{\bullet\vee}$ on degree -2 over $\mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$ can not solve this issue since the truncated complex may not be perfect of amplitude $[-1,1]$. We will show in the next section that the pull back of the complex $\mathbb{E}^{\bullet\vee}$ via the map

$$\pi_{\mathrm{FT}}^{\mathfrak{M}} : \mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau') \to \mathfrak{H}_{s,\mathrm{FT}}^{(P_2,r,n)}(\tau')$$

and a suitable truncation over $\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$ provide a candidate for deformation obstruction theory over $\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$.

6.4 A perfect Deformation-obstruction theory of amplitude $[-1,0]$ over $\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$

In this section we propose a strategy to find a suitable deformation-obstruction theory over the smooth components of the moduli stack of highly frozen triples. First we prove a statement about the self duality of the complex $\mathbb{E}^\bullet$ obtained in Proposition 6.10.

Lemma 6.11. *The complex* $\mathbb{E}^\bullet$ *in Proposition 6.10 is self-dual in the sense of [1]. In other words there exists a quasi-isomorphism of complexes:*

$$\mathbb{E}^\bullet \xrightarrow{\cong} \mathbb{E}^{\bullet\vee}[1]$$

Proof. Use the notation in Section 6.1. The derived dual of $\mathbb{E}^\bullet$ over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is given by

$$\mathbb{E}^{\bullet\vee} := R\mathcal{H}om(\mathbb{E}^\bullet, \mathcal{O}_{\mathfrak{H}}).$$

By Proposition 6.10 $\mathbb{E}^\bullet$ is quasi-isomorphic to $R\pi_{\mathfrak{H}*}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2]$. Now use Grothendieck duality and obtain the following isomorphisms:

$$\begin{aligned}
&R\mathcal{H}om(\mathbb{E}^\bullet, \mathcal{O}_{\mathfrak{H}}) \cong R\pi_{\mathfrak{H}*}(R\mathcal{H}om_{X\times\mathfrak{H}}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2], \pi^!\mathcal{O}_{\mathfrak{H}}))\\
&\cong R\pi_{\mathfrak{H}*}(R\mathcal{H}om_{X\times\mathfrak{H}}\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_X^*\omega_X\right)[2], \pi_X^*\omega_X[3])\\
&\cong R\pi_{\mathfrak{H}*}R\mathcal{H}om(\mathcal{O}_{X\times\mathfrak{H}}, R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]) \cong \mathbb{E}^\bullet[-1].
\end{aligned} \tag{6.23}$$

Hence we conclude that $\mathbb{E}^{\bullet\vee}[1] \cong \mathbb{E}^\bullet$. Note that the second isomorphism in (6.23) is obtained using the fact that X is a Calabi-Yau threefold and hence $\omega_X \cong \mathcal{O}_X$. □

An alternative obstruction bundle for HFT

Consider the forgetful map $\pi^{\mathfrak{M}}_{\mathrm{FT}} : \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. We pullback the four-term deformation obstruction theory of perfect amplitude $[-2,1]$ over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ via $\pi^{\mathfrak{M}}_{\mathrm{FT}}$. After suitably truncating the pulled-back complex we define a perfect two-term deformation obstruction theory of amplitude $[-1,0]$ over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

Theorem 6.12. *Consider the 4-term deformation obstruction theory* $\mathbb{E}^{\bullet\vee}$ *of perfect amplitude* $[-2,1]$ *over* $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.

1. Locally in the Zariski topology over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ *there exists a perfect two-term deformation obstruction theory of perfect amplitude* $[-1,0]$ *which is obtained from the suitable local truncation of the pullback* $(\pi^{\mathfrak{M}}_{\mathrm{FT}})^*\mathbb{E}^{\bullet\vee}$.

2. This local theory (under Assumption 7.15) defines a globally well-behaved virtual fundamental class over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

Proof: Here we prove the first part of the theorem. For notational simplicity, denote $\pi = \pi^{\mathfrak{M}}_{\mathrm{FT}}$. As we showed before, since $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ is an Artin stack, the cotangent complex of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ has a term in degree 1. By the canonical exact triangle of relative cotangent complexes in the derived category, we have:

$$\pi^*\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \to \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} \to \Omega_\pi \to \pi^*\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}[1], \tag{6.24}$$

By Theorem 6.7, $\mathbb{E}^{\bullet\vee} \xrightarrow{ob} \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}$ is a perfect deformation obstruction theory of amplitude $[-2,1]$ for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$, such that $h^0(ob), h^1(ob)$ are isomorphisms and $h^{-1}(ob)$ is an epimorphism.

Proposition 6.13. *Let $\mathcal{U} = \coprod_i \mathcal{U}_i$ be an atlas of affine schemes for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. Fix one the maps $q : \mathcal{U}_i \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. The following isomorphism holds true in $\mathcal{D}^b(\mathcal{U}_i)$:*

$$\mathrm{Hom}(q^*\Omega_\pi, q^*(\pi^*\mathbb{E}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}[1])) \cong \mathrm{Hom}(q^*\Omega_\pi, q^*(\pi^*\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}[1])). \tag{6.25}$$

Proof. Consider the exact triangle

$$q^*(\pi^*\mathbb{E}^{\bullet\vee}) \xrightarrow{ob_i^t} q^*(\pi^*\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}) \to \mathrm{Cone}(ob_i^t) \tag{6.26}$$

induced by the pulling back (via $\pi \circ q : \mathcal{U}_i \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$) the deformation obstruction theory in Theorem 6.7. By Proposition 3.6, and the exact triangle in (6.24):

$$\Omega_\pi \cong \mathfrak{K}^\vee \otimes \mathcal{O}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')},$$

where $\mathfrak{K}^\vee$ by construction is the dual of the Lie algebra of $\mathrm{GL}_r(\mathbb{C})$. Hence $q^*\Omega_\pi \cong \mathfrak{K}^\vee \otimes \mathcal{O}_{\mathcal{U}_i}$. Now apply $\mathrm{Hom}^0(q^*\Omega_\pi,)$ to the exact triangle (6.26) and obtain

$$\begin{aligned}&\mathrm{Hom}^0(q^*\Omega_\pi, \mathrm{Cone}(ob_i^t)) \to \mathrm{Hom}^0(q^*\Omega_\pi, q^*(\pi^*\mathbb{E}^{\bullet\vee})[1])\\ &\to \mathrm{Hom}^0(q^*\Omega_\pi, q^*(\pi^*\mathbb{L}^\bullet)[1]) \to \mathrm{Hom}^0(q^*\Omega_\pi, \mathrm{Cone}(ob_i^t)[1]).\end{aligned} \tag{6.27}$$

We prove the statement of the theorem by showing that

$$\mathrm{Hom}^0(q^*\Omega_\pi, \mathrm{Cone}(ob_i^t)) \cong 0 \cong \mathrm{Hom}^0(q^*\Omega_\pi, \mathrm{Cone}(ob_i^t)[1]). \tag{6.28}$$

Now consider the long exact sequence of cohomology induced by the exact triangle in 6.26:

$$\begin{aligned}&0 \to \cancel{\mathrm{h}^{-3}(q^*(\pi^*\mathbb{E}^{\bullet\vee}))} \to \cancel{\mathrm{h}^{-3}(q^*(\pi^*\mathbb{L}^\bullet))} \to \mathrm{h}^{-3}(\mathrm{Cone}(ob_i^t)) \xrightarrow{\cong} \mathrm{h}^{-2}(q^*(\pi^*\mathbb{E}^{\bullet\vee}))\\ &\to \cancel{\mathrm{h}^{-2}(q^*(\pi^*\mathbb{L}^\bullet))} \to \mathrm{h}^{-2}(\mathrm{Cone}(ob_i^t)) \to \mathrm{h}^{-1}(q^*(\pi^*\mathbb{E}^{\bullet\vee})) \twoheadrightarrow \mathrm{h}^{-1}(q^*(\pi^*\mathbb{L}^\bullet))\\ &\to \cancel{\mathrm{h}^{-1}(\mathrm{Cone}(ob_i^t))} \to \mathrm{h}^{0}(q^*(\pi^*\mathbb{E}^{\bullet\vee})) \xrightarrow{\cong} \mathrm{h}^{0}(q^*(\pi^*\mathbb{L}^\bullet)) \to \cancel{\mathrm{h}^{0}(\mathrm{Cone}(ob_i^t))}\\ &\to \mathrm{h}^{1}(q^*(\pi^*\mathbb{E}^{\bullet\vee})) \xrightarrow{\cong} \mathrm{h}^{1}(q^*(\pi^*\mathbb{L}^\bullet)) \to \cancel{\mathrm{h}^{1}(\mathrm{Cone}(ob_i^t))}\\ &\to \cancel{\mathrm{h}^{2}(q^*(\pi^*\mathbb{E}^{\bullet\vee}))} \to \cancel{\mathrm{h}^{2}(q^*(\pi^*\mathbb{L}^\bullet))} \to 0\end{aligned} \tag{6.29}$$

where we have used the fact that $q^*(\pi^*\mathbb{L}^\bullet)$ and $q^*(\pi^*\mathbb{E}^{\bullet\vee})$ are perfect complexes of amplitudes $[-1, 1]$ and $[-2, 1]$ respectively and $\mathrm{h}^i(ob_i^t)$ is an isomorphism for $i = 0, 1$ and a surjection for $i = -1$. Hence we conclude that $\mathrm{Cone}(ob_i^t))$ has cohomologies on degrees -2 and -3 only. Now use the fact that one can replace the complex $\mathrm{Cone}(ob_i^t))$ with a representative complex $\mathbf{A}^\bullet$ such that $\mathbf{A}^k = 0$ for $k \geq -1$. Now we use the following lemma.

Lemma 6.14. *If $\mathcal{U}$ is an affine scheme and $\boldsymbol{A}^\bullet$ is a complex with $\boldsymbol{A}^k = 0$ for $k \geq -1$, then* $\mathrm{Hom}^0(\mathcal{O}_\mathcal{U}, \boldsymbol{A}^\bullet[l]) \cong 0$ *for all $l \geq 0$.*

Proof. We use the general fact that given complexes $\mathbf{G}$ and $\mathbf{F}$, in order to compute the Grothendieck hypercohomolgy $\mathrm{Hom}^i(\mathbf{G}, \mathbf{F})$, one replaces $\mathbf{F}$ with its injective resolution $\mathbf{F} \to \mathbf{K}^\bullet$. Moreover replacing $\mathbf{F}$ with $\mathbf{K}^\bullet$ is equivalent with replacing $\mathbf{G}$ with $\mathbf{P}^\bullet$ such that $\mathbf{P}^\bullet \to \mathbf{G}$ is a projective resolution. Now use the fact that locally over $\mathcal{U}$, $\mathcal{O}_\mathcal{U}$ is given as a free and in particular projective module hence its projective resolution consists of one term and one can use $\mathcal{O}_{\mathcal{U}}$ itself instead of the complex $\mathbf{P}^\bullet$ in order to compute $\mathrm{Hom}^0(\mathcal{O}_{\mathcal{U}}, \mathbf{A}^\bullet[l])$ which is isomorphic to zero since $\mathcal{O}_{\mathcal{U}}$ is a flasque sheaf sitting in degree zero. □

Now use the fact that by construction $q^*\Omega_\pi \cong \mathfrak{K}^\vee \otimes \mathcal{O}_{\mathcal{U}}$ and apply the result of Lemma 6.14 by replacing $\mathcal{O}_{\mathcal{U}}$ with $q^*\Omega_\pi$ and obtain the vanishings in (6.28). This finishes the proof of Proposition 6.13. □

Lemma 6.15. *Let $q : \mathcal{U} \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $q' : \mathcal{U}' \to \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ be given as fixed affine charts over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ such that the isomorphism in Proposition 6.13 holds true over $\mathcal{U}$ and $\mathcal{U}'$. Let $p_1 : \mathcal{U} \times_{q\times q'} \mathcal{U}' \to \mathcal{U}$ and $p_2 : \mathcal{U} \times_{q\times q'} \mathcal{U}' \to \mathcal{U}'$ be the corresponding projections. Then*

$$\mathrm{Hom}^0(p_2^*(q^*\Omega_\pi, q^*(\pi^*\mathbb{E}^{\bullet\vee})[1])) \cong \mathrm{Hom}^0(p_2^*(q^*\Omega_\pi, q^*(\pi^*\mathbb{L}^{\bullet\vee})[1]))$$

Proof. Because $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a quasi-projective scheme (Remark 3.14) then an intersection of affine subschemes of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is affine. Now apply Proposition 6.13 to $\mathcal{U} \times_{q\times q'} \mathcal{U}'$. □

In what follows in order to save space we denote by $\mathfrak{H}_{\mathrm{FT}} := \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and $\mathfrak{H}_{\mathrm{HFT}} := \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. Now fix $\mathcal{U}_i$, by the local existence of the map g_i in Proposition 6.13 there exists a commutative diagram over $\mathcal{U}_i$:

$$\begin{array}{ccccccccc}
\pi^*\mathbb{E}^{\bullet\vee}\mid_{\mathcal{U}_i} & \longrightarrow & \mathrm{Cone}(g_i)[-1] & \longrightarrow & \Omega_\pi\mid_{\mathcal{U}_i} & \xrightarrow{g_i} & \pi^*\mathbb{E}^{\bullet\vee}[1]\mid_{\mathcal{U}_i} & \longrightarrow & \mathrm{Cone}(g_i) \\
\downarrow{\scriptstyle \pi^*(ob)\mid_{\mathcal{U}_i}} & & \downarrow{\scriptstyle ob'} & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi^* ob[1]\mid_{\mathcal{U}_i}} & & \downarrow \\
\pi^*\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}}\mid_{\mathcal{U}_i} & \longrightarrow & \mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}\mid_{\mathcal{U}_i} & \longrightarrow & \Omega_\pi\mid_{\mathcal{U}_i} & \longrightarrow & \pi^*\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}}[1]\mid_{\mathcal{U}_i} & \longrightarrow & \mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}[1]\mid_{\mathcal{U}_i}
\end{array} \tag{6.30}$$

Lemma 6.16. *The map $ob' : \mathrm{Cone}(g_i)[-1] \to \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')}\mid_{\mathcal{U}_i}$ defines a perfect 3-term deformation obstruction theory of amplitude $[-2,0]$ for $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ over $\mathcal{U}_i$.*

Proof. : We show that $\mathrm{Cone}(g)[-1]$ is concentrated in degrees -2, -1 and 0, moreover $h^0(ob')$ is an isomorphism and $h^{-1}(ob')$ is an epimorphism. The proof uses the long exact sequence of cohomologies. For $h^{-1}(ob')$ one obtains:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & h^{-1}(\pi^*\mathbb{E}^{\bullet\vee}\mid_{\mathcal{U}_i}) \cong \pi^* h^{-1}(\mathbb{E}^{\bullet\vee}\mid_{\mathcal{U}_i}) & \xrightarrow{\cong} & h^{-1}(\mathrm{Cone}(g_i)[-1]) & \longrightarrow & 0 \\
\downarrow & & \downarrow \pi^*(h^{-1}(ob)) & & \downarrow h^{-1}(ob') & & \downarrow \\
0 & \longrightarrow & h^{-1}(\pi^*\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}}\mid_{\mathcal{U}_i}) \cong \pi^* h^{-1}(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}}\mid_{\mathcal{U}_i}) & \xrightarrow{\cong} & h^{-1}(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}\mid_{\mathcal{U}_i}) & \longrightarrow & 0
\end{array}, \tag{6.31}
$$

the top horizontal isomorphism is due to the fact that

$$
\mathrm{Cone}(g_i)[-1] : E^{-2} \to E^{-1} \to E^0 \oplus \Omega_\pi \to E^1,
$$

where E^i correspond to the terms of the complex $\mathbb{E}^{\bullet\vee}\mid_{\mathcal{U}_i}$. The vanishings on the left and right of the top and bottom rows of (6.31) are due to the fact that Ω_π is a sheaf concentrated in degree zero. By Theorem 6.7, the second vertical map (from left) is a surjection and by commutativity of the diagram (6.31) the map $h^{-1}(ob')$ is surjective. In degrees 0 and 1 one obtains:

$$
\begin{array}{ccccccccccc}
0 \to & \pi^*\,\mathrm{h}^0(\mathbb{E}^{\bullet\vee}\mid_{\mathcal{U}_i}) & \to & \mathrm{h}^0(\mathrm{Cone}(g_i)[-1]) & \to & \Omega_\pi\mid_{\mathcal{U}_i} & \to & \pi^*\,\mathrm{h}^1(\mathbb{E}^{\bullet\vee}\mid_{\mathcal{U}_i}) & \to & \mathrm{h}^1(\mathrm{Cone}(g)[-1]) & \to 0 \\
\downarrow & \downarrow \pi^*\,\mathrm{h}^0(ob)\mid_{\mathcal{U}_i} & & \downarrow \mathrm{h}^0(ob') & & \downarrow \mathrm{id} & & \downarrow \pi^*\,\mathrm{h}^1(ob)\mid_{\mathcal{U}_i} & & \downarrow \mathrm{h}^1(ob') & \downarrow \\
0 \to & \pi^*\,\mathrm{h}^0(\mathbb{L}^\bullet_{\mathfrak{H}_{s,\mathrm{FT}}}\mid_{\mathcal{U}_i}) & \to & \mathrm{h}^0(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}\mid_{\mathcal{U}_i}) & \to & \Omega_\pi\mid_{\mathcal{U}_i} & \to & \pi^*\,\mathrm{h}^1(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}}\mid_{\mathcal{U}_i}) & \to & \mathrm{h}^1(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}\mid_{\mathcal{U}_i}) & \to 0
\end{array}. \tag{6.32}
$$

In this diagram, $h^1(\mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')}\mid_{\mathcal{U}_i}) \cong 0$ since over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ the truncated cotangent complex does not have cohomology in degree 1. Moreover $\pi^* h^1(ob)\mid_{\mathcal{U}_i}$ is an isomorphism by Theorem 6.7. Hence $h^1(ob') \cong 0$. Moreover by Theorem 6.7, $\pi^* h^0(ob)$ is an isomorphism, hence by the commutativity of the diagram (6.32), $h^0(ob')$ is an isomorphism. This finishes the proof of Lemma 6.16. □

In order to obtain a perfect deformation obstruction theory of amplitude $[-1, 0]$, one needs to truncate the complex $\mathrm{Cone}(g_i)[-1]$ so that it does not have any cohomology in degree -2.

The self-duality of $\mathbb{E}^\bullet$ gives a diagram of morphisms in the derived category:

$$\begin{array}{ccccc} \mathbb{E}^\bullet & \xrightarrow[\text{q-isom}]{\cong} & \mathbb{E}^{\bullet\vee}[1] & \longrightarrow & \mathrm{Cone}(g_i) \\ {\scriptstyle g_i^\vee}\downarrow & & & & \\ \mathrm{T}_\pi\,|_{\mathcal{U}_i}\,[1] & & & & \end{array} \tag{6.33}$$

Lemma 6.17. *The natural map*

$$\mathrm{Hom}^0_{\mathcal{D}(\mathcal{U}_i)}(\mathbb{E}^\bullet, \mathrm{T}_\pi\,|_{\mathcal{U}_i}\,[1]) \leftarrow \mathrm{Hom}^0_{\mathcal{D}(\mathcal{U}_i)}(\mathrm{Cone}(g_i), \mathrm{T}_\pi\,|_{\mathcal{U}_i}\,[1]) \tag{6.34}$$

is an isomorphism.

Proof. Note that $\mathcal{U}_i$ is affine and $\mathrm{T}_\pi\,|_{\mathcal{U}_i}\,[1] \cong \mathcal{O}_{\mathcal{U}_i}^{\dim(\mathfrak{K})}[1]$, so the statement reduces to knowing that $\mathrm{H}^1(\mathbb{E}^{\bullet\vee}) \to \mathrm{H}^1(\mathrm{Cone}(g_i)^\vee)$ is an isomorphism. This follows since $\mathbb{E}^{\bullet\vee}[1] \to \mathrm{Cone}(g_i)$ is an isomorphism on H^{-1} as shown in diagram (6.31). □

By Lemma 6.17 $g_i^\vee$ factors through a map

$$\mathrm{Cone}(g_i) \to \mathrm{T}_\pi\,|_{\mathcal{U}_i}\,[1]$$

which is unique up to homotopy. We make such a choice of lift and denote it again by $g_i^\vee$. Now consider the exact triangle

$$\mathrm{Cone}(g_i^\vee)[-1] \to \mathrm{Cone}(g_i)[-1] \xrightarrow{g^\vee} T_\pi\,|_{\mathcal{U}_i} \to \mathrm{Cone}(g_i^\vee). \tag{6.35}$$

Denote $\mathbb{G}^\bullet\,|_{\mathcal{U}_i} := \mathrm{Cone}(g_i^\vee)[-1]$. In order to finish the proof of Theorem 6.12, we need one more lemma.

Lemma 6.18. *The complex $\mathbb{G}^\bullet\,|_{\mathcal{U}_i}$ defines a perfect deformation obstruction theory of amplitude $[-1,0]$ for $\mathcal{U}_i$.*

Proof. : By construction

$$\mathbb{G}^{\bullet} \mid_{\mathcal{U}_i} := \pi^* E^{-2} \to \pi^* E^{-1} \oplus T_{\pi} \mid_{\mathcal{U}_i} \to \pi^* E^{0} \oplus \Omega_{\pi} \mid_{\mathcal{U}_i} \to \pi^* E^{1}.$$

This complex has no cohomology in degree 1 and -2, i.e in the following commutative diagram, the top row is quasi-isomorphic to the bottom row:

$$\begin{array}{ccccccc} \pi^* E^{-2} & \xrightarrow{d'} & \pi^* E^{-1} \oplus T_{\pi} \mid_{\mathcal{U}_i} & \longrightarrow & \pi^* E^{0} \oplus \Omega_{\pi} \mid_{\mathcal{U}_i} & \xrightarrow{d} & \pi^* E^{1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Coker}(d') & \longrightarrow & \mathrm{Ker}(d) & \longrightarrow & 0 \end{array} , \quad (6.36)$$

moreover there exists a morphism

$$\mathbb{G}^{\bullet} \mid_{\mathcal{U}_i} \to \mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} \mid_{\mathcal{U}_i}$$

which is given by the composition of

$$\mathbb{G}^{\bullet} \mid_{\mathcal{U}_i} \to \mathrm{Cone}(g)[-1]$$

and

$$\mathrm{Cone}(g)[-1] \to \mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} \mid_{\mathcal{U}_i} .$$

By Lemma 6.16, this map satisfies the condition of being a deformation obstruction theory. This finishes the first part of the proof of Theorem 6.12. □

To prove that the local deformation obstruction theory in Theorem 6.12 under Assumption 7.15 defines a globally well-behaved virtual fundamental class, we need some preparation in the next section.

Remark 6.19. The difference between the construction in Theorem 6.12 and the construction in [28] (2.3) for rank 1 triples (stable pairs) is that, for theory of stable pairs, the terms $\mathrm{Hom}(I^{\bullet}, I^{\bullet})_0$ and $\mathrm{Ext}^3(I^{\bullet}, I^{\bullet})_0$ are equal to zero by stability, however in higher rank, the stability condition does not ensure that the stable objects, $I^\bullet$, in the derived category are simple objects. Hence applying Theorem 6.12 is necessary to obtain a perfect deformation obstruction theory of amplitude $[-1, 0]$ for $\mathcal{U}_i$.

Chapter 7

Virtual fundamental class for HFT

In Proposition 6.13 we proved the local existence of a map in the derived category

$$\Omega_\pi \mid_{\mathcal{U}_i} \xrightarrow{g} \pi^*\mathbb{E}^{\bullet\vee}[1] \mid_{\mathcal{U}_i},$$

where $\mathcal{U}_i$ were open subsets given as elements of a smooth cover $\mathcal{U} = \coprod_i \mathcal{U}_i$ of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ (Noseda in [25] refers to charts in Proposition 6.13 as charts with lifting property). We locally constructed the perfect deformation obstruction theory in the first part of Theorem 6.12. To prove the second part of Theorem 6.12 we need to check that the virtual fundamental cycles obtained from this construction are independent of choice of local charts. In other words, we need to prove that the local virtual cycles obtained from Theorem 6.12 glue to each other and define a well-behaved global virtual fundamental cycle. We also emphasize here that in gluing the local virtual fundamental cycles we will require an addition assumption (Assumption 7.15) which we will discuss later. First, we need some background.

7.1 Background

Definition 7.1. [2](Section 1). Let $\mathfrak{M}$ be a DM stack and $S = \bigoplus_i S^i$ be a quasi- coherent sheaf of $\mathcal{O}_{\mathfrak{M}}$-algebras such that $S^0 = \mathcal{O}_{\mathfrak{M}}$, S^1 is coherent and S is locally generated by S^1. Then the affine $\mathfrak{M}$-scheme, $C = \mathrm{Spec}(S)$ is called the cone over $\mathfrak{M}$ if the following additional conditions are satisfied:

There exists a zero section:

$$\mathfrak{M} \xrightarrow{0} C$$

and a multiplication map

$$\mathbb{A}^1 \times C \to C$$

induced by $(\lambda, c) \mapsto \lambda \cdot c$. Moreover the multiplication map satisfies the axiom that for any $\lambda, \mu \in \mathbb{A}^1$ and $c \in C$, $\lambda \cdot (\mu \cdot c) = (\lambda\mu) \cdot c$, $1 \cdot c = c$ and $0 \cdot c = 0$.

Definition 7.2. [2](Section 1). Given a coherent sheaf F over $\mathfrak{M}$ we get an associated cone (a linear space):

$$C(F) : \mathrm{Spec}(\mathrm{Sym}^{\bullet}(F)) \to \mathfrak{M}$$

such that for any $\mathfrak{M}$-stack T we get

$$C(F)(T) = \mathrm{Hom}(F_T, \mathcal{O}_T).$$

Any cone $C = \mathrm{Spec}(\bigoplus_i S^i)$ is a closed subcone of the associated cone $A(C) = \mathrm{Spce}(\mathrm{Sym}^{\bullet}(S^1))$, called the Abelian Hull of C.

Definition 7.3. [2](Section 1). If E is a vector bundle and $d : E \to C$ is a morphism of cones, we say that C is an E-cone if C is invariant under the action of E on $A(C)$:

$$E \times C \to C,$$

induced by $(\nu, \gamma) \mapsto d\nu + \gamma$.

Here we include some statements proved by Seibert in [31] without proof.

Lemma 7.4. *[31](Lemma 2.1). Let $\Phi_{\bullet} : E_{\bullet} \to F_{\bullet}$ be a commutative square between the complexes of linear spaces $E_{\bullet} : E_0 \to E_1$ and $F_{\bullet} : F_0 \to F_1$ and let $C \hookrightarrow E_1$ be an E_0-cone, then $\Phi_1^{-1}(C) \hookrightarrow F_1$ is an F_0-cone.*

Definition 7.5. [31](Definition 2.2). Let $\Phi_{\bullet} : E_{\bullet} \to F_{\bullet}$ be a commutative square in $\mathrm{Lin}(\mathfrak{M})$ and $C \subset E_1$ an E_0-cone. Then the F_0-cone

$$\Phi_{\bullet}^{!} C = \Phi_1^{-1}(C)$$

in F_1 is called the pullback of C under $\Phi_\bullet$

Lemma 7.6. *[31](Lemma 2.5). Given $\Phi_\bullet : E_\bullet \to F_\bullet$, a commutative square in $\mathrm{Lin}(\mathfrak{M})$ and F_0 a vector bundle such that $\Phi_\bullet$ induces an isomorphism on h^0 and a closed embedding of linear spaces on h^1, then*

$$0 \to F_0 \xrightarrow{(\Phi_0,-d')} E_0 \oplus F_1 \xrightarrow{q} E_1$$

is exact. In this case we say that going down is applicable to $\Phi_\bullet$.

Definition 7.7. Let $\Phi_\bullet : E_\bullet \to F_\bullet$ be a commutative square in $\mathrm{Lin}(\mathfrak{M})$ such that going down is applicable to $\Phi_\bullet$. Let $C \subset F_1$ be an F_0-cone. The unique cone $\overline{C} \subset \mathrm{Im}(q) \subset E_1$ with $q^{-1}(\overline{C}) = E_0 \oplus C$ is called pushforward of C by $\Phi_\bullet$, denoted by

$$(\Phi_\bullet)_!(C).$$

Lemma 7.8. *[25](Lemma 24). Given a square diagram of linear spaces over $\mathfrak{M}$*

$$\begin{array}{ccc} A_\bullet & \xrightarrow{\alpha} & B_\bullet \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ C_\bullet & \xrightarrow{\gamma} & D_\bullet \end{array} \quad . \tag{7.1}$$

which commutes up to homotopy. Assume that going down is applicable to δ and β and $h^1(C_\bullet) \to h^1(D_\bullet)$ is injective. Then

$$\delta_! \alpha^! = \gamma^! \beta_!$$

7.2 Gluing the local cone stacks

Followed by constructions in [2], we choose a local embedding over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ over which we construct the normal cone. Then we prove that the normal cone is independent of this local embedding, i.e it remains invariant under base change. Then we glue the local normal cones constructed over each local embedding and obtain a global cone stack. Eeventually a global virtual fundamental class is constructible from a global normal cone stack [2].

Definition 7.9. [2](Definition 3.7). A local embedding of $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is given a by a tuple $(\mathcal{U}, M)$ fitting in a diagram:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & M \\ {\scriptstyle \pi_{\mathcal{U}}}\downarrow & & \\ \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') & & \end{array} , \tag{7.2}$$

where

1. $\mathcal{U}$ is an affine $\mathbb{C}$-scheme of finite type.

2. M is an affine $\mathbb{C}$-scheme of finite type.

3. $\pi_{\mathcal{U}}$ is an étale morphism.

4. f is a local immersion.

Given a local embedding $(\mathcal{U}, M)$, let $\mathcal{I}$ be the ideal corresponding to embedding of $\mathcal{U}$ in M. By restricting the cotangent complex to $\mathcal{U}$ we obtain a map

$$\phi : f^*\mathbb{L}^{\bullet}_M \to \left[\mathcal{I}/\mathcal{I}^2 \to \Omega_M \mid_{\mathcal{U}}\right],$$

such that ϕ induces an isomorphism in h^{-1} and h^0. Moreover, by [2] (Proposition 2.6), we obtain a morphism of cone stacks

$$\phi^{\vee} : \left[N_{\mathcal{U}/M}/T_{M|_{\mathcal{U}}}\right] \to \pi^*_{\mathcal{U}}\mathfrak{N}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')},$$

where $N_{\mathcal{U}/M} = \mathrm{Spec}(\mathrm{Sym}^{\bullet}(\mathcal{I}/\mathcal{I}^2))$ is the normal sheaf of $\mathcal{U}$ in M. The normal cone of $\mathcal{U}$ in M is obtained by $C(\mathcal{U}_\alpha, M) = \mathrm{Spec}(\bigoplus_i \mathcal{I}^i/\mathcal{I}^{i+1})$ which is a $T_{M|_{\mathcal{U}}}$-cone inside $N_{\mathcal{U}/M}$. In what will follow $\mathfrak{N}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')}$ denotes the intrinsic normal sheaf over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.
Fix the open smooth chart $\mathcal{U} \xrightarrow{\pi_{\mathcal{U}}} \mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$, a local embedding with the lifting property [25] as

in Theorem 6.12. Consider the perfect deformation obstruction theory

$$\mathbb{G}^\bullet \mid_{\mathcal{U}} \to \pi_{\mathcal{U}}^* \mathbb{L}^\bullet_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')}$$

in Lemma 6.18. To continue we need some background about semi-perfect obstruction theories from [12]. First we state the definition of the numerical equivalence . Let $\mathcal{M}$ be an artin stack and $X \to \mathcal{M}$ a representable morphism of a DM stack to an Artin stack. Now let $\mathcal{U} = \coprod_\alpha \mathcal{U}_{\alpha \in \Lambda}$ be a DM cover of X by affine schemes. Consider $\mathcal{U}_\alpha \to \mathcal{M}$ for some $\alpha \in \Lambda$.

Definition 7.10. [12] (Definition 2.5) Let $\iota : T \to T'$ be a closed subscheme with T' local Artinian. Let $\mathcal{I}$ be the ideal sheaf of T in T' and let $\mathfrak{m}$ be the ideal sheaf of the closed point of T'. We call ι a small extension if $\mathcal{I} \cdot \mathfrak{m} = 0$. Given a small extension $(T, T', \mathcal{I}, \mathfrak{m})$ that fits into a commutative square

$$\begin{array}{ccc} T & \xrightarrow{g} & \mathcal{U}_\alpha \\ \iota \downarrow & & \downarrow \\ T' & \longrightarrow & \mathcal{M} \end{array} , \tag{7.3}$$

so that the image of g contains a closed point $p \in \mathcal{U}_\alpha$, finding a morphism $g' : T' \to \mathcal{U}_\alpha$ that commutes with the arrows in (7.3) is called "infinitesimal lifting problem of $\mathcal{U}_\alpha/\mathcal{M}$ at p".

Given a perfect relative deformation obstruction theory $\mathbb{G}^\bullet \to \mathbb{L}^\bullet_{X/\mathcal{M}}$ denote by $\mathbb{G}^\bullet \mid_{\mathcal{U}_\alpha}$ the restriction of $\mathbb{G}^\bullet$ to $\mathcal{U}_\alpha$. Let $\phi : \mathbb{G}^\bullet \mid_{\mathcal{U}_\alpha} \to \mathbb{L}^\bullet_{\mathcal{U}_\alpha/\mathcal{M}}$ and $\phi' : \mathbb{G}'^\bullet \mid_{\mathcal{U}_\alpha} \to \mathbb{L}^\bullet_{\mathcal{U}_\alpha/\mathcal{M}}$ be given as two perfect relative deformation obstruction theories over $\mathcal{U}_\alpha \to \mathcal{M}$.

Definition 7.11. Given a $\mathcal{U}_\alpha \to \mathcal{M}$ let $\phi : \mathbb{G}^\bullet \to \mathbb{L}^\bullet_{\mathcal{U}_\alpha/\mathcal{M}}$ be a perfect obstruction theory. For the infinitesimal lifting problem in Definition 7.10 we call the image

$$ob(\phi, g, T, T') := \mathrm{H}^1(\phi^\vee)(\omega(g, T, T')) \in \mathrm{Ext}^1(g^* \mathbb{G}^\bullet, \mathcal{I}) = \mathrm{Ob}(\phi, p) \otimes \mathcal{I} \tag{7.4}$$

the obstruction class (of ϕ) to the lifting problem

Definition 7.12. [12] (Definition 2.9) We call ϕ and ϕ' ν-equivalent if there exists an isomorphism of sheaves:

$$\psi : \mathrm{H}^1(\mathbb{G}^\bullet) \to \mathrm{H}^1(\mathbb{G}'^\bullet) \tag{7.5}$$

so that for every closed point $p \in \mathcal{U}_\alpha$ and any infinitesimal lifting problem of $\mathcal{U}_\alpha/\mathcal{M}$ at p as in Definition 7.10 we have

$$\psi\mid_p (ob(\phi, g, T, T')) = ob(\phi', g, T, T') \in \mathrm{Ob}(\phi', p) \otimes_{\mathbf{k}} I.$$

Now let $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ be given as two charts with the lifting property as in Theorem 6.12. Moreover let $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Let $\mathcal{U}_{\alpha\beta} \xrightarrow{f_\alpha} \mathcal{U}_\alpha$ and $\mathcal{U}_{\alpha\beta} \xrightarrow{f_\beta} \mathcal{U}_\beta$ be the corresponding maps. By our construction, one locally obtains perfect deformation obstruction theories of amplitude$[-1, 0]$ given by $\phi_\alpha : \mathbb{G}^\bullet \mid_{\mathcal{U}_\alpha} \to \mathbb{L}^\bullet_{\mathcal{U}_\alpha/\mathcal{M}}$ and $\phi_\beta : \mathbb{G}^\bullet \mid_{\mathcal{U}_\beta} \to \mathbb{L}^\bullet_{\mathcal{U}_\beta/\mathcal{M}}$. For the notational convenience denote $\mathbb{G}^\bullet \mid_{\mathcal{U}_\alpha} \to \mathbb{L}^\bullet_{\mathcal{U}_\alpha/\mathcal{M}}$ and $\mathbb{G}^\bullet \mid_{\mathcal{U}_\beta} \to \mathbb{L}^\bullet_{\mathcal{U}_\beta/\mathcal{M}}$ by $\mathbb{G}^\bullet_\alpha \to \mathbb{L}^\bullet_\alpha$ and $\mathbb{G}^\bullet_\beta \to \mathbb{L}^\bullet_\beta$ respectively.

Proposition 7.13. *Let $f^*_\alpha \phi_\alpha$ and $f^*_\beta \phi_\beta$ denote the pullback of ϕ_α and ϕ_β to $\mathcal{U}_{\alpha\beta}$. Then $f^*_\alpha \phi_\alpha$ and $f^*_\beta \phi_\beta$ are ν-equivalent over $\mathcal{U}_{\alpha\beta}$.*

Proof. We have to show that given a diagram

$$\begin{array}{ccc} T & \xrightarrow{g_{\alpha\beta}} & \mathcal{U}_{\alpha\beta} \\ \iota\downarrow & & \downarrow \\ T' & \longrightarrow & \mathcal{M} \end{array}, \tag{7.6}$$

there exists a map $\psi : \mathrm{H}^1(f^*_\alpha \mathbb{G}^\bullet_\alpha)^\vee \xrightarrow{\cong} \mathrm{H}^1(f^*_\beta \mathbb{G}^\bullet_\beta)^\vee$ such that given a class $ob(f^*_\alpha \phi_\alpha, g_{\alpha\beta}, T, T') \in \mathrm{H}^1(f^*_\alpha(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}} \mid_{\mathcal{U}_\alpha})^\vee)$ (Look at diagram (7.6)) and for every point $p \in \mathcal{U}_{\alpha\beta}$ we have

$$\psi\mid_p ob(f^*_\alpha \phi_\alpha, g_{\alpha\beta}, T, T') = ob(f^*_\beta \phi_\beta, g_{\alpha\beta}, T, T').$$

Apply the result of Proposition 6.13 to $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ and obtain unique isomorphisms as in (6.25) over $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$. Now use the fact that $\mathcal{U}_{\alpha\beta}$ is affine and pull back the obtained isomorphisms via

f_α and f_β to $\mathcal{U}_{\alpha\beta}$ and obtain a unique isomorphism

$$\mathrm{Hom}(\Omega_\pi \mid_{\mathcal{U}_{\alpha\beta}}, \mathbb{E}^\bullet_{\alpha\beta}) \cong \mathrm{Hom}(\Omega_\pi \mid_{\mathcal{U}_{\alpha\beta}}, \mathbb{L}^\bullet_{\alpha\beta}).$$

Now apply Lemma 6.18 and conclude that by the uniqueness property there exists an isomorphism in $\mathcal{D}^b(\mathcal{U}_{\alpha\beta})$ given by

$$\kappa_{\alpha\beta} : f^*_\alpha \mathbb{E}^\bullet_\alpha \to f^*_\beta \mathbb{E}^\bullet_\beta.$$

By assumption $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are given as charts with lifting property (Theorem 6.12), hence there exists lifts $\mathrm{Hom}(\Omega_\pi \mid_{\mathcal{U}_\alpha}, \mathbb{E}^\bullet_\alpha[1])$ and $\mathrm{Hom}(\Omega_\pi \mid_{\mathcal{U}_\beta}, \mathbb{E}^\bullet_\beta[1])$ given by $g_\alpha : \Omega_\pi \mid_{\mathcal{U}_\alpha} \to \mathbb{E}^\bullet_\alpha[1]$ and $g_\beta : \Omega_\pi \mid_{\mathcal{U}_\beta} \to \mathbb{E}^\bullet_\beta[1]$ over $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ respectively. Now consider the pullbacks $f^*_\alpha \Omega_\pi \mid_{\mathcal{U}_\alpha} [-1] \to f^*_\alpha \mathbb{E}^\bullet_\alpha$ and $f^*_\alpha \Omega_\pi \mid_{\mathcal{U}_\alpha} [-1] \to f^*_\alpha \mathbb{E}^\bullet_\alpha$ and note that by Proposition 6.13 $f^*_\alpha g_\alpha$ and $f^*_\beta g_\beta$ are homotopic to each other over $\mathcal{U}_{\alpha\beta}$ and satisfy the equation:

$$f^*_\alpha g_\alpha - f^*_\beta g_\beta = d \circ h_{\alpha\beta} + h_{\alpha\beta} \circ d$$

where $h_{\alpha\beta}$ is given as a choice of homotopy. Now take the cone of $f^*_\alpha g_\alpha$ and $f^*_\beta g_\beta$ and obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Cone}(f^*_\alpha g_\alpha)[-1] & \longrightarrow & f^*_\alpha \Omega_\pi \mid_{\mathcal{U}_\alpha} & \xrightarrow{f^*_\alpha g_\alpha} & f^*_\alpha \mathbb{E}^\bullet_\alpha[1] & \longrightarrow & \mathrm{Cone}(f^*_\alpha g_\alpha) \\
\downarrow J_{\alpha\beta}[-1] & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow J_{\alpha\beta} \\
\mathrm{Cone}(f^*_\beta g_\beta)[-1] & \longrightarrow & f^*_\beta \Omega_\pi \mid_{\mathcal{U}_\beta} & \xrightarrow{f^*_\beta g_\beta} & f^*_\beta \mathbb{E}^\bullet_\beta[1] & \longrightarrow & \mathrm{Cone}(f^*_\beta g_\beta),
\end{array} \tag{7.7}$$

where $J_{\alpha\beta} := \begin{pmatrix} \mathrm{id} & h_{\alpha\beta} \\ 0 & \mathrm{id} \end{pmatrix}$. Since the first and the second rows in diagram (7.7) are given by exact triangles one computes the long exact sequence of cohomologies and obtains the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow \mathrm{H}^i(f_\alpha^*\Omega_\pi\mid_{\mathcal{U}_\alpha}[-1]) & \longrightarrow & \mathrm{H}^i(f_\alpha^*\mathbb{E}_\alpha^\bullet) & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\alpha^*g_\alpha))[-1] & \longrightarrow \cdots \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{H}^i(J_{\alpha\beta})[-1] & \\
\cdots \longrightarrow \mathrm{H}^i(f_\beta^*\Omega_\pi\mid_{\mathcal{U}_\beta}[-1]) & \longrightarrow & \mathrm{H}^i(f_\beta^*\mathbb{E}_\beta^\bullet) & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\beta^*g_\beta))[-1] & \longrightarrow \cdots,
\end{array} \tag{7.8}$$

Now use [7] (Proposition 4.10) and conclude that the left, middle and right squares in (7.8) are commutative square diagrams for all i. By computing the cohomologies in the level of $i=-1$ one obtains:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^{-1}(f_\alpha^*\mathbb{E}_\alpha^\bullet) & \xrightarrow[\rho_1]{\cong} & \mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^*g_\alpha)[-1]) & \longrightarrow 0 \\
& \downarrow \mathrm{id} & & \downarrow \mathrm{H}^{-1}(J_{\alpha\beta}[-1]) & \\
0 \longrightarrow & \mathrm{H}^{-1}(f_\beta^*\mathbb{E}_\beta^\bullet) & \xrightarrow[\rho_2]{\cong} & \mathrm{H}^{-1}(\mathrm{Cone}(f_\beta^*g_\beta)) & \longrightarrow 0,
\end{array} \tag{7.9}$$

where the vanishings on the ends are due to the fact that $\mathrm{H}^i(f_\alpha^*\Omega_\pi\mid_{\mathcal{U}_\alpha}[-1])\cong 0$ and $\mathrm{H}^i(f_\beta^*\Omega_\pi\mid_{\mathcal{U}_{beta}}[-1])\cong 0$ for $i=-1,0$. Hence we conclude that by commutativity of the middle square $\mathrm{H}^{-1}(J_{\alpha\beta}[-1])$ is an isomorphism of cohomologies and moreover, given any $\nu\in\mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^*g_\alpha)[-1])$:

$$\mathrm{id}\circ\rho_1^{-1}(\nu)=\rho_2^{-1}\circ\mathrm{H}^{-1}(J_{\alpha\beta}[-1])(\nu). \tag{7.10}$$

Note that given a choice of homotopy $h_{\alpha\beta}^\vee$ satisfying

$$f_\alpha^*g_\alpha^\vee - f_\beta^*g_\beta^\vee = d\circ h_{\alpha\beta}^\vee + h_{\alpha\beta}^\vee\circ d$$

and via restriction of the exact triangle in (6.35) to $\mathcal{U}_{\alpha\beta}$ and similar to the above procedure we obtain a commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Cone}(f_\alpha^* g_\alpha^\vee)[-1] & \longrightarrow & \mathrm{Cone}(f_\alpha^* g_\alpha)[-1] & \xrightarrow{f_\alpha^* g_\alpha^\vee} & f_\alpha^* \mathrm{T}_\pi \mid_{\mathcal{U}_\alpha} & \longrightarrow & \mathrm{Cone}(f_\alpha^* g_\alpha^\vee) \\
\downarrow {\scriptstyle J_{\alpha\beta}^\vee[-1]} & & \downarrow {\scriptstyle J_{\alpha\beta}[-1]} & & \downarrow {\scriptstyle \mathrm{id}} & & \downarrow {\scriptstyle J_{\alpha\beta}^\vee} \\
\mathrm{Cone}(f_\beta^* g_\beta^\vee)[-1] & \longrightarrow & \mathrm{Cone}(f_\beta^* g_\beta)[-1] & \xrightarrow{f_\beta^* g_\beta^\vee} & f_\beta^* \mathrm{T}_\pi \mid_{\mathcal{U}_\beta} & \longrightarrow & \mathrm{Cone}(f_\beta^* g_\beta^\vee).
\end{array} \tag{7.11}$$

Similarly obtain a commutative diagram induced by the long exact sequences of cohomologies:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\alpha^* g_\alpha^\vee)[-1]) & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\alpha^* g_\alpha)[-1]) & \longrightarrow & \mathrm{H}^i(f_\alpha^* \mathrm{T}_\pi \mid_{\mathcal{U}_\alpha}) & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\alpha^* g_\alpha^\vee) & \longrightarrow & \cdots \\
& & \downarrow {\scriptstyle \mathrm{H}^i(J_{\alpha\beta}^\vee[-1])} & & \downarrow {\scriptstyle \mathrm{H}^i(J_{\alpha\beta}[-1])} & & \downarrow {\scriptstyle \mathrm{id}} & & \downarrow {\scriptstyle \mathrm{H}^i(J_{\alpha\beta}^\vee)} & & \\
\cdots & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\beta^* g_\beta^\vee)[-1]) & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\beta^* g_\beta)[-1]) & \longrightarrow & \mathrm{H}^i(f_\beta^* \mathrm{T}_\pi \mid_{\mathcal{U}_\beta}) & \longrightarrow & \mathrm{H}^i(\mathrm{Cone}(f_\beta^* g_\beta^\vee) & \longrightarrow & \cdots,
\end{array} \tag{7.12}$$

Now use [7] (Proposition 4.10) and conclude that the left, middle and right squares in (7.8) are commutative square diagrams for all i and in particular for $i = -1$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^* g_\alpha^\vee)[-1]) & \xrightarrow{q_1} & \mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^* g_\alpha)[-1]) & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^* g_\alpha^\vee) & \longrightarrow & \cdots \\
& & \downarrow {\scriptstyle \mathrm{H}^{-1}(J_{\alpha\beta}^\vee[-1])} & & \downarrow {\scriptstyle \mathrm{H}^{-1}(J_{\alpha\beta}[-1])} & & \downarrow {\scriptstyle \mathrm{id}} & & \downarrow {\scriptstyle \mathrm{H}^{-1}(J_{\alpha\beta}^\vee)} & & \\
0 & \longrightarrow & \mathrm{H}^{-1}(\mathrm{Cone}(f_\beta^* g_\beta^\vee)[-1]) & \xrightarrow{q_2} & \mathrm{H}^i(\mathrm{Cone}(f_\beta^* g_\beta)[-1]) & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^{-1}(\mathrm{Cone}(f_\beta^* g_\beta^\vee) & \longrightarrow & \cdots.
\end{array} \tag{7.13}$$

Hence by commutativity of the left square and the fact that $\mathrm{H}^{-1}(J_{\alpha\beta}[-1])$ is an isomorphism, then $\mathrm{H}^{-1}(J_{\alpha\beta}^\vee[-1])$ is an isomorphism and moreover, for any $\mu \in \mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^* g_\alpha^\vee)[-1])$ we have:

$$\mathrm{H}^{-1}(J_{\alpha\beta}[-1]) \circ q_1(\mu) = q_2 \circ \mathrm{H}^{-1}(J_{\alpha\beta}^\vee[-1])(\mu) \tag{7.14}$$

Now take an element $\mu \in \mathrm{H}^{-1}(\mathrm{Cone}(f_\alpha^* g_\alpha^\vee)[-1])$ and note that by (7.10) and (7.14) we have:

$$\mathrm{id} \circ \rho_1^{-1} \circ \mathrm{H}^{-1}(J_{\alpha\beta}[-1]) \circ q_1(\mu) = \rho_2^{-1} \circ \mathrm{H}^{-1}(J_{\alpha\beta}[-1]) \circ q_2 \circ \mathrm{H}^{-1}(J_{\alpha\beta}^\vee[-1])(\mu) \tag{7.15}$$

Moreover $\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}$ and $\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}}$ satisfy the condition that $\mathrm{H}^{-1}(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}}) \cong \mathrm{H}^{-1}(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{FT}}})$ hence one can easily see that there exist maps $\lambda_1 : \mathrm{H}^{-1}(f^*_\alpha \mathbb{E}^\bullet_\alpha) \to \mathrm{H}^{-1}(f^*_\alpha(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}} |_{\mathcal{U}_\alpha}))$ and $\lambda_2 : \mathrm{H}^{-1}(f^*_\beta \mathbb{E}^\bullet_\beta) \to \mathrm{H}^{-1}(f^*_\beta(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}} |_{\mathcal{U}_\beta}))$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^{-1}(f^*_\alpha \mathbb{E}^\bullet_\alpha) & \xrightarrow{\lambda_1} & \mathrm{H}^{-1}(f^*_\alpha(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}} |_{\mathcal{U}_\alpha})) \\ \mathrm{id}\downarrow & & \mathrm{id}\downarrow \\ \mathrm{H}^{-1}(f^*_\beta \mathbb{E}^\bullet_\beta) & \xrightarrow{\lambda_2} & \mathrm{H}^{-1}(f^*_\beta(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}} |_{\mathcal{U}_\beta})). \end{array} \tag{7.16}$$

Now by (7.16) and (7.15) it is seen that given $\mu \in \mathrm{H}^{-1}(\mathrm{Cone}(f^*_\alpha g^\vee_\alpha)[-1])$ we obtain an identity

$$\mathrm{id} \circ \lambda_1 \circ \mathrm{id} \circ \rho_1^{-1} \circ \mathrm{H}^{-1}(J_{\alpha\beta}[-1]) \circ q_1(\mu) = \lambda_2 \circ \mathrm{id} \circ \rho_2^{-1} \circ \mathrm{H}^{-1}(J_{\alpha\beta}[-1]) \circ q_2 \circ \mathrm{H}^{-1}(J^\vee_{\alpha\beta}[-1])(\mu). \tag{7.17}$$

Let $\psi^\vee := \mathrm{id} \circ \rho_2^{-1} \circ \mathrm{H}^{-1}(J_{\alpha\beta}[-1]) \circ q_2 \circ \mathrm{H}^{-1}(J^\vee_{\alpha\beta}[-1])$. So far we have seen that in the level of H^{-1} cohomology there exists a map $\psi^\vee : \mathrm{H}^{-1}(\mathrm{Cone}(f^*_\alpha g^\vee_\alpha)[-1]) \xrightarrow{\cong} \mathrm{H}^{-1}(\mathrm{Cone}(f^*_\beta g^\vee_\alpha)[-1])$ such that $\lambda_2 \circ \mathrm{Im}(\psi^\vee) = \mathrm{Im}(\lambda_1)$. Recall that by our notation $\mathbb{G}^\bullet |_{\mathcal{U}_i} := \mathrm{Cone}(g^\vee)[-1]$. Now dualize the construction and conclude that there exists a map $\psi : \mathrm{H}^1(f^*_\alpha \mathbb{G}^\bullet_\alpha)^\vee \xrightarrow{\cong} \mathrm{H}^1(f^*_\beta \mathbb{G}^\bullet_\beta)^\vee$ such that given a class $ob(f^*_\alpha \phi_\alpha, g_{\alpha\beta}, T, T') \in \mathrm{H}^1(f^*_\alpha(\mathbb{L}^\bullet_{\mathfrak{H}_{\mathrm{HFT}}} |_{\mathcal{U}_\alpha})^\vee)$ (Look at diagram (7.6)) and for every point $p \in \mathcal{U}_{\alpha\beta}$ we have

$$\psi |_p \, ob(f^*_\alpha \phi_\alpha, g_{\alpha\beta}, T, T') = ob(f^*_\beta \phi_\beta, g_{\alpha\beta}, T, T').$$

This finishes the proof of Proposition 7.13. □

Definition 7.14. [12](Definition 3.1). A semi perfect obstruction theory over $X \to \mathcal{M}$ consists of an étale covering $\mathcal{U} = \coprod_{\alpha \in \Lambda} \mathcal{U}_\alpha$ of X by schemes, and a truncated perfect relative obstruction theory

$$\phi_\alpha : \mathbb{G}^\bullet_\alpha \to \mathbb{L}^\bullet_{\mathcal{U}_\alpha/\mathcal{M}}$$

for each $\alpha \in \Lambda$ such that

1. for each α, β in Λ there is an isomorphism

$$\psi_{\alpha\beta} : \mathrm{H}^1(\mathbb{G}^\bullet_\alpha \mid_{\mathcal{U}_{\alpha\beta}}) \xrightarrow{\cong} \mathrm{H}^1(\mathbb{G}^\bullet_\beta \mid_{\mathcal{U}_{\alpha\beta}})$$

so that the collection $(\mathrm{H}^1(\mathbb{G}^\bullet_\alpha), \psi_{\alpha\beta})$ forms a descent datum of sheaves.

2. For any pair $\alpha, \beta \in \Lambda$ the obstruction theories $\phi_\alpha \mid_{\mathcal{U}_{\alpha\beta}}$ and $\phi_\beta \mid_{\mathcal{U}_{\alpha\beta}}$ are ν-equivalent.

The condition (1) above, that the ν-equivalences we have constructed induce a descent datum of sheaves on H^1, seems hard to guarantee in our setting. Namely, it requires that we carefully choose homotopies $h_{\alpha\beta}$ and $h^\vee_{\alpha\beta}$ on $\mathcal{U}_{\alpha\beta}$ so that the induced composite quasi-isomorphisms $\psi_{\gamma\alpha} \circ \psi_{\beta\gamma} \circ \psi_{\alpha\beta}$ induce the identity maps on H^1. For now, we do not see how to make such choices. Thus, we assume:

Assumption 7.15. The homotopies $h_{\alpha\beta}$ and $h^\vee_{\alpha\beta}$ can be chosen so that the collection $(\mathrm{H}^1(\mathbb{G}^\bullet_\alpha), \psi_{\alpha\beta})$ forms a descent datum of sheaves.

Theorem 7.16. *Suppose that Assumption 7.15 holds. Then the local deformation obstruction theory obtained in Proposition 6.18 defines a semi-perfect obstruction theory over* $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

Proof. For part (2) apply Proposition 7.13 and conclude that $\phi_\alpha \mid_{\mathcal{U}_{\alpha\beta}} = f^*_\alpha \phi_\alpha$ and $\phi_\beta \mid_{\mathcal{U}_{\alpha\beta}} = f^*_\beta \phi_\beta$ are ν-equivalent. Now we prove part (1). First apply Proposition 7.13 and obtain the map

$$\psi_{\alpha\beta} : \mathrm{H}^1(\mathbb{G}^{\bullet,\vee}_\alpha \mid_{\mathcal{U}_{\alpha\beta}}) \xrightarrow{\cong} \mathrm{H}^1(\mathbb{G}^{\bullet,\vee}_\beta \mid_{\mathcal{U}_{\alpha\beta}}).$$

By Assumption 7.15, $(\mathrm{H}^1(\mathbb{G}^{\bullet,\vee}_\alpha), \psi_{\alpha\beta})$ forms a descent datum. This completes the proof. □

Remark 7.17. The assumption that the descent condition holds should, morally speaking, be unnecessary. The local models $\mathbb{G}^\bullet_\alpha$ can always be glued up to higher homotopies, and thus should always give an ∞-stack in which the virtual normal cone lives. We expect that in the future a good intersection theory for ∞-stacks would allow one to construct a virtual cycle using this ∞-stack. Such a construction is beyond the scope of the present thesis, however.

Since we proved the existence of a semi-perfect obstruction theory over $\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ now we can apply the result of [12] (Theorem 3.3) and obtain a virtual fundamental class $[\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')]^{vir}$ over $\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$:

Given

$$\phi_\alpha : \mathbb{G}_\alpha^\bullet \to \mathbb{L}_{\mathcal{U}_\alpha/\mathcal{M}}^\bullet$$

apply Spec(Sym$^\bullet$) to both sides of this morphism and obtain a map between the complexes of linear spaces over $\mathcal{U}_\alpha$, i.e:

$$L_\bullet^\alpha := \mathrm{Spec}(\mathrm{Sym}^\bullet)(\mathbb{L}_{\mathcal{U}_\alpha/\mathcal{M}}^\bullet) \to \mathrm{Spec}(\mathrm{Sym}^\bullet)(\mathbb{G}_\alpha^\bullet) := \mathbb{G}_\bullet^\alpha.$$

which induces an isomorphism in h^0 and a closed embedding in h^1. Let $\mathbb{G}_\bullet^\alpha : G_0 \to G_1$. It is easily seen that $C(\mathcal{U}_\alpha, M)$ is a G_0-cone inside G_1.

Now choose another local embedding $(\mathcal{U}_\beta, M_\beta)$ with the lifting property and obtain a commutative diagram:

$$\begin{array}{ccccc} & & \mathcal{U}_\beta & \xrightarrow{f'} & M_\beta \\ & \swarrow^{\pi_\alpha} & \downarrow p & & \downarrow q \\ \mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau') & \xleftarrow{\pi_\beta} & \mathcal{U}_\alpha & \xrightarrow{f} & M_\alpha. \end{array} \tag{7.18}$$

Consider the following notation:

$$\begin{aligned} L_\bullet^\alpha &= \left[T_{M|_{\mathcal{U}_\alpha}} \to N_{\mathcal{U}_\alpha/M}\right] \\ L_\bullet^\beta &= \left[T_{M_\beta|_{\mathcal{U}_\beta}} \to N_{\mathcal{U}_\beta/M_\beta}\right] \\ \phi^\alpha &: L_\bullet^\alpha \to \mathbb{G}_\bullet^\alpha \\ \phi^\beta &: L_\bullet^\beta \to \mathbb{G}_\bullet^\beta. \end{aligned} \tag{7.19}$$

By construction we have seen that there exists a commutative square diagram in the derived category of complexes of linear spaces over $\mathcal{U}_\beta$:

$$\begin{array}{ccc} L^\beta_\bullet & \xrightarrow{a} & p^*L^\alpha_\bullet \\ \phi^\beta \big\downarrow & & \big\downarrow p^*\phi^\alpha \\ \mathbb{G}^\beta_\bullet & \xrightarrow{b} & p^*\mathbb{G}^\alpha_\bullet, \end{array} \tag{7.20}$$

Note that by the definition of perfect deformation obstruction theory, the vertical maps in (7.20) induce isomorphisms in H^0 and closed embedding in H^1.

Lemma 7.18. *Given the local normal cone $C(\mathcal{U}_\alpha, M) \subset N_{\mathcal{U}_\alpha/M_\alpha}$ as a $\pi_\alpha^* G_0$-cone inside $\pi_\alpha^* G_1$. Consider the diagram in* (7.20)*. Then*

$$\phi^\beta_! a^!(p^*C(\mathcal{U}_\alpha, M_\beta)) = b^!(p^*\phi^\alpha)_!(C(\mathcal{U}_\alpha, M_\alpha)).$$

Proof. By Proposition 7.13 $\mathrm{H}^1(b)$ is given by $\psi_{\beta\alpha}$ which is isomorphism and in particular injective. Moreover by Lemma 6.18 the two vertical maps in (7.20) induce isomorphisms in H^0 and closed embeddings in H^1, i.e going down is applicable to both vertical maps in (7.20). Now apply Lemma 7.8. □

Lemma 7.19. *Consider the commutative diagram* (7.20)*. It is true that*

$$a^!(p^*C(\mathcal{U}_\alpha, M_\alpha)) = C(\mathcal{U}_\beta, M_\beta),$$

where $C(\mathcal{U}_\beta, M_\beta)$ is a $\pi_\beta^* G_0$-cone inside $\pi_\beta^* G_1$.

Proof. Apply Lemma 46 in [25] to charts $(\mathcal{U}_\alpha, M_\alpha)$ and $(\mathcal{U}_\beta, M_\beta)$. □

Proposition 7.20. *Given the commutative diagram in* (7.20)*, it is true that*

$$\phi^\beta_!(C(\mathcal{U}_\beta, M_\beta)) = b^!(p^*\phi^\alpha)_!(C(\mathcal{U}_\alpha, M)).$$

Proof. By Lemma 7.18,

$$\phi_!^{\beta} a^!(p^* C(\mathcal{U}_\alpha, M)) = b^!(p^*\phi^\alpha)_!(C(\mathcal{U}_\alpha, M)).$$

On the other hand, by Lemma 7.19,

$$a^!(p^* C(\mathcal{U}_\alpha, M)) = C(\mathcal{U}_\beta, M_\beta),$$

hence one obtains

$$\phi_!^{\alpha}(C(\mathcal{U}_\beta, M_\beta)) = \phi_!^{\beta}(a^!(p^* C(\mathcal{U}_\alpha, M))) = b^!(p^*\phi^\alpha)_!(C(\mathcal{U}_\alpha, M)).$$

Now use the compatibility obtained in Proposition 7.20 and obtain a compatibility statement for the virtual fundamental classes as in Proposition 3.4 in [12] and conclude that the local virtual fundamental classes obtained from local virtual normal cones over $\mathcal{U}_\alpha$ glue to each other and give rise to a global virtual moduli cycle over $\mathfrak{H}_{s,\mathrm{HFT}}^{(P_2,r,n)}(\tau')$. This finishes the proof of the second part of Theorem 6.12. □

Chapter 8

Classification of torus-fixed HFT

In this section we classify the torus equivariant stable highly frozen triples. We assume that the base threefold X is given by the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\to\mathbb{P}^1$ (local $\mathbb{P}^1$) or $\mathcal{O}_{\mathbb{P}^2}(-3)\to\mathbb{P}^2$ (local $\mathbb{P}^2$). For now, to explain our strategy, we stick to local $\mathbb{P}^1$, the constructions and results can all be extended to local $\mathbb{P}^2$. In order to have a well-defined definition of stability for triples over a quasi-projective variety X, as we explain below, we use the geometric analogue of τ'-limit stability to define the stable frozen triples.

8.0.1 Torus actions on the moduli stack of highly frozen triples supported on local $\mathbb{P}^1$

Let X be given as total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\to\mathbb{P}^1$. Consider the ample line bundle over X given by $\mathcal{O}_X(1)$. By our earlier notation, $\mathfrak{M}^{(P_2,r,n)}_{\mathrm{s,HFT}}(\tau')$ denotes the moduli stack of stable rank r highly frozen triples $\mathcal{O}_X(-n)^{\oplus r}\to F$ in which F has Hilbert polynomial P_2. However in the setting of this section X is given as a toric non-compact variety and the Hilbert polynomial of F is not well-defined. Therefore in order to define stability, we use the geometric stability criteria for triples which is equivalent to τ'-limit stability.

Definition 8.1. Given X as total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\to\mathbb{P}^1$, the highly frozen triples (E,F,ϕ,ψ) and frozen triples (E,F,ϕ) are called stable if $\mathrm{Coker}(\phi)$ has zero-dimensional support.

Let $\mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau')$ denote the stack of τ'-stable highly frozen triples over X. The reason to change our notation is that from now on we use a geometric criterion for stability of triples and we omit the superscript P_2 in the notation. Since X is a toric threefold, there exists a canonical $\mathbf{T}= (\mathbb{C}^*)^3$ action on X. Now fix an equivariant structure for $\mathcal{O}_X(-n)^{\oplus r}$. We will show that having fixed this equivariant structure the action of $\mathbf{T}$ on X induces an action of $\mathbf{T}$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ by pullback. Moreover there exists an extra $\mathrm{T}_0 = (\mathbb{C}^*)^r$ action on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ for which the points in T_0-fixed loci corresponding to this action take the form of direct sums of Pandharipande-Thomas stable pairs. In the following section we describe each induced action more carefully.

8.1 The geometric action of $\mathbf{T} = (\mathbb{C}^*)^3$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ over local $\mathbb{P}^1$

Background (the $\mathbf{T}$ action on X)

We study the natural induced action of $\mathbf{T} = (\mathbb{C}^*)^3$ on the moduli stack of highly frozen triples supported over a local Calabi-Yau threefold X given by the total space of $N = \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{C}$ where $\mathcal{C} \cong \mathbb{P}^1$:

$$X = \mathbf{Spec}(\mathrm{Sym}^\bullet(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))).$$

Let L_1 and L_2 denote the first and second copy of $\mathcal{O}_{\mathbb{P}^1}(-1)$ in N. Let $\mathcal{U}_\alpha$ be given as the local patch around $0 \in \mathcal{C}$. By fixing equivariant structures on L_1 and L_2 we can see that there exists an action of $(\mathbb{C}^*)^3$ on X which is given locally over $\mathcal{U}_\alpha$ by

$$(\lambda_1, \lambda_2, \lambda_3) * (l_1, l_2, s) = (\lambda_1 l_1, \lambda_2 l_2, \lambda_3 s),$$

where l_1 and l_2 denote any local non-vanishing sections of L_1 and L_2 respectively and s denotes the local coordinate along $\mathcal{C}$. Later in Example 11.2 we carry out computations with a two dimensional sub-torus of $\mathbf{T}$ which fixes the Calabi-Yau form of X, however for now we stick to this notation.

Definition 8.2. Define the divisor $D_1 \subset X$ as the fiber of X over $0 \in \mathcal{C}$. Moreover, define $D_2 \subset X$ and $D_3 \subset X$ as $\mathrm{Tot}(L_1 \to \mathcal{C})$ and $\mathrm{Tot}(L_2 \to \mathcal{C})$ respectively.

Let $l_1^\vee$ and $l_2^\vee$ denote any local sections of $L_1^\vee$ and $L_2^\vee$ in $\mathcal{U}_\alpha$. Here we give a local description of the modules associated to the structure sheaves of D_1, D_2, D_3 in $\mathcal{U}_\alpha$. It follows by the usual arguments that there exists an equivalence of categories

$$\mathcal{F} : \mathrm{Coh}(X) \xrightarrow{\cong} \mathrm{Mod}_{\mathrm{Sym}^\bullet(N^\vee)} .$$

Locally over $\mathcal{U}_\alpha$ the module over $\mathbb{C}[l_1^\vee, l_2^\vee, s]$ associated to the structure sheaf of X is given by the polynomial ring itself:

$$\mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}) = \mathbb{C}[l_1^\vee, l_2^\vee, s].$$

Let t_1 and t_2 and t_3 denote the weights of the action of $(\lambda_1, \lambda_2, \lambda_3)$ on $(0, l_1^\vee, 0)$, $(0, 0, l_2^\vee)$ and $(s, 0, 0)$ respectively. One observes that the action of $\mathbf{T}$ on the localized structure module obtained above gives it a decomposition into torus weight spaces, i.e locally:

$$\mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}) = \bigoplus_{(m,n,l)} \mathbb{C}[l_1^\vee, l_2^\vee, s](m, n, l). \tag{8.1}$$

Remark 8.3. Locally over $\mathcal{U}_\alpha$ the divisor D_1 is understood by the vanishing locus of s, therefore inorder to obtain the module associated to $\mathcal{O}_{\mathcal{U}_\alpha}(kD_1)$ we consider the $\mathbb{C}[l_1^\vee, l_2^\vee, s]$-module generated by $\frac{1}{s^k}$:

$$\mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}(kD_1)) = (\frac{1}{s})^k \mathbb{C}[s, l_1^\vee, l_2^\vee] \tag{8.2}$$

In this case this module is generated by $(\frac{1}{s})^k$ which lies in $\frac{1}{t_3^k}$ weight space. Similarly one may consider divisors D_2 and D_3 in X. For completeness we describe the module structure associated to $\mathcal{O}_{\mathcal{U}_\alpha}(kD_2)$ and $\mathcal{O}_{\mathcal{U}_\alpha}(kD_3)$:

$$\begin{aligned} \mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}(kD_2)) &= (\frac{1}{l_1^\vee})^k \mathbb{C}[s, l_1^\vee, l_2^\vee], \\ \mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}(kD_3)) &= (\frac{1}{l_2^\vee})^k \mathbb{C}[s, l_1^\vee, l_2^\vee], \end{aligned} \tag{8.3}$$

8.1.1 The induced action of T on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$

We show that the $\mathbf{T}$-action on X induces an action on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$. Given a τ'-stable highly frozen triple (E,F,ϕ,ψ) represented by the complex $\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F$ and $t\in\mathbf{T}$ we obtain a new highly frozen triple as follows: Let $\mathcal{U}\subset X$ be an open subset. Given $t\in\mathbf{T}$. Identify $\mathcal{O}_X(-1)$ with $\mathcal{O}_X(-D_1)$ where D_1 is defined in Definition 8.2. Hence locally over $\mathcal{U}$ sections of $O_X(-n)$ are identified with sections of $\mathcal{O}_X(-nD_1)$. Now given a section $s\in\Gamma(\mathcal{O}_X^{\oplus r}(-n)\mid_{t^{-1}\mathcal{U}})$, the composition $s\circ t^{-1}$ defines a map:

$$t^*\mathcal{O}_X^{\oplus r}(-n)\mid_{\mathcal{U}}\xrightarrow{\psi}\mathcal{O}_X^{\oplus r}(-n)\mid_{\mathcal{U}},$$

which is an isomorphism. In other words we have chosen an equivariant structure for $\mathcal{O}_X^{\oplus r}(-n)$. Therefore the induced inverse isomorphism ψ^{-1} defines a map $\mathcal{O}_X^{\oplus r}(-n)\xrightarrow{\psi^{-1}} t^*\mathcal{O}_X^{\oplus r}(-n)$. Now compose with sections of F and obtain a highly frozen triple:

$$\begin{array}{ccc}\mathcal{O}_X^{\oplus r}(-n) & & \\ \Big\downarrow{\scriptstyle \psi^{-1}} & & \\ t^*\mathcal{O}_X^{\oplus r}(-n) & \xrightarrow{t^*\phi} & t^*F,\end{array} \tag{8.4}$$

Hence we are able to obtain a new τ'-stable highly frozen triple (E,t^*F,ϕ',ψ) represented by the complex $\mathcal{O}_X^{\oplus r}(-n)\xrightarrow{\phi'} t^*F$ such that $\phi'=t^*\phi\circ\psi^{-1}$ in (8.4). One needs to show that the composite morphism in (8.4) induces a well-behaved pointwise action of $\mathbf{T}$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$. We prove this fact in several steps. First we show in more generality that there exists an action of $\mathbf{T}$ on the moduli stack of triples of type (P_1,P_2) (i.e $\mathfrak{M}^{(P_1,P_2),s}(\tau')$). Then we specialize to frozen triples and show that there exists a well-behaved action of $\mathbf{T}$ on $\mathfrak{M}^r_{\mathrm{s,FT}}(\tau')$ induced by the pullback. Then by Definition 8.4 and since the pointwise action of $\mathbf{T}$ on highly frozen triples is induced by the composition of the isomorphism ψ^{-1} and pulling back by the torus $(t^*\phi)$, the existence of a well-behaved action of $\mathbf{T}$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ follows as a corollary.

Proposition 8.4. *Let X be given as the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$. Let $\boldsymbol{T}$ be the $(\mathbb{C}^*)^3$ action on X. Having fixed an equivariant structure on $\mathcal{O}_X(1)$, there exists an induced action of $\boldsymbol{T}$*

on moduli stack of stable highly frozen triples given by:

$$\overline{m}^{\boldsymbol{T}} : \boldsymbol{T} \times \mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau') \to \mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau')$$
$$\overline{m}^{\boldsymbol{T}}(t, (E, F, \phi, \psi)) = \overline{m}_{t\,*}(E, F, \phi, \psi) \tag{8.5}$$

where $\overline{m}_{t\,*}(E, F, \phi, \psi)$ is defined by the composite morphism in 8.4.

Proof. The proof of this statement is completely borrowed from arguments in [20] (Proposition 4.1). Since $\mathrm{Pic}(\mathbf{T}) = 0$ any line bundle on X acquires an equivariant structure. Moreover, a fixed equivariant structure on $\mathcal{O}_X(1)$ induces an equivariant structure on any twist of $\mathcal{O}_X(1)$. Let n_3 be a suitable integer as in Section 3.3.1. We know that for any $n \geq n_3$ the sheaves E and F appearing in the family of triples are globally generated. Consider the Quot schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$ that parametrize the flat quotients E_1 and E_2 of sheaves $V_1 \otimes \mathcal{O}_X(-n)$ and $V_2 \otimes \mathcal{O}_X(-n)$ respectively with Hilbert polynomial P_1 and P_2.

The moduli stack of triples $E_1 \to E_2$ is obtained as the quotient stack of a scheme $\mathfrak{S}$ by the action of the group $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Note that $\mathfrak{S}$ is defined as a closed subscheme of $\mathcal{A} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{P}$ where $\mathcal{P}$ is the projective **Hom** bundle given as $\mathcal{P} = \mathbf{P}(\mathrm{Hom}(V_1, V_2)^{\vee})$. Let $\mathcal{H}_1 = V_1 \otimes \mathcal{O}_X(-n)$ and $\mathcal{H}_2 = V_2 \otimes \mathcal{O}_X(-n)$. Let $\sigma : \mathbf{T} \times X \to X$ denote the action of $\mathbf{T}$ on X and $p_2 : T \times X \to X$ denote the projection onto the second factor. The $\mathbf{T}$-equivariant structure on X induces the isomorphisms $\phi_{\mathcal{H}_i} : \sigma^*\mathcal{H}_i \xrightarrow{\cong} p_2^*\mathcal{H}_i$ for $i = 1, 2$ (where $\mathbf{T}$ acts trivially on V_1 and V_2). It is easy to see that the action of $\mathbf{T}$ on X lifts to $\mathcal{Q}_1 \times \mathcal{Q}_2$:

Let $\underline{\mathrm{Quot}}(\mathcal{H}_1, P_1) \times \underline{\mathrm{Quot}}(\mathcal{H}_1, P_1)$ be the functor which is representable by the product of Quot schemes $\mathcal{Q}_1 \times \mathcal{Q}_2$. In other words there exists an isomorphism of functors:

$$\Theta : \underline{\mathrm{Quot}}(\mathcal{H}_1, P_1) \times \underline{\mathrm{Quot}}(\mathcal{H}_1, P_1) \xrightarrow{\cong} \underline{\mathcal{Q}_1 \times \mathcal{Q}_2}, \tag{8.6}$$

where $\underline{\mathcal{Q}_1 \times \mathcal{Q}_2} = \mathrm{Hom}(-, \mathcal{Q}_1 \times \mathcal{Q}_2)$. Our goal is to define a regular action of $\mathbf{T}$ on $\mathcal{Q}_1 \times \mathcal{Q}_2$ given by the map $\overline{m}^{\mathbf{T}} : \mathbf{T} \times \mathcal{Q}_1 \times \mathcal{Q}_2 \to \mathcal{Q}_1 \times \mathcal{Q}_2$. Let $p_{12}^{\mathcal{Q}_1 \times \mathcal{Q}_2} : \mathbf{T} \times X \times \mathcal{Q}_1 \times \mathcal{Q}_2 \to \mathbf{T} \times X$ be the projection onto the first two factors and $\overline{m}^{\mathcal{Q}_1\times\mathcal{Q}_2} = \sigma \times \mathbf{1}_{\mathcal{Q}_1\times\mathcal{Q}_1}$ be the lift of the action of $\mathbf{T}$ on $X \times \mathcal{Q}_1 \times \mathcal{Q}_2$. Let $(\mathcal{H}_{\mathcal{Q}_1} \xrightarrow{u_1} \mathcal{E}_1, \mathcal{H}_{\mathcal{Q}_2} \xrightarrow{u_2} \mathcal{E}_2)$ be the universal family over $\mathcal{Q}_1 \times \mathcal{Q}_2$. It is seen that pre-composing $(\overline{m}^{\mathcal{Q}_1\times\mathcal{Q}_2})^*(u_1, u_2)$ with $(p_{12}^{\mathcal{Q}_1\times\mathcal{Q}_2})^*(\phi_{\mathcal{H}_1}^{-1}, \phi_{\mathcal{H}_2}^{-1})$ gives an element of

$$\underline{\mathrm{Quot}}(\mathcal{H}_1, P_1) \times \underline{\mathrm{Quot}}(\mathcal{H}_1, P_1)(\mathbf{T} \times \mathcal{Q}_1 \times \mathcal{Q}_2) \cong \mathrm{Hom}(\mathbf{T} \times \mathcal{Q}_1 \times \mathcal{Q}_2, \mathcal{Q}_1 \times \mathcal{Q}_2),$$

which defines the regular action of $\mathbf{T}$ on $\mathcal{Q}_1 \times \mathcal{Q}_2$. Let $t = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{T}$ be a closed point. Let $i_t : X \hookrightarrow \mathbf{T} \times X$ denote the injection. Let

$$\phi_{\mathcal{H}_1\times\mathcal{H}_2,t} = i_t^*(\phi_{\mathcal{H}_1}, \phi_{\mathcal{H}_2}) = (i_t^*\phi_{\mathcal{H}_1}, i_t^*\phi_{\mathcal{H}_2}) : t^*(\mathcal{H}_1, \mathcal{H}_2) \xrightarrow{\cong} (\mathcal{H}_1, \mathcal{H}_2).$$

Let $q \in \mathcal{Q}_1 \times \mathcal{Q}_2 : ([\mathcal{H}_1 \xrightarrow{u_1} E_1], [\mathcal{H}_2 \xrightarrow{u_2} E_2])$ be a closed point. It is easy to see that there exists a lift of the action of $t \in \mathbf{T}$ on $\mathcal{Q}_1 \times \mathcal{Q}_2$ which is obtained by $\overline{m}^{\mathbf{T}}(t, q) = q \cdot t$ and it corresponds to

$$([\mathcal{H}_1 \xrightarrow{(i_t^*\phi_{\mathcal{H}_1})^{-1}} t^*\mathcal{H}_1 \xrightarrow{t^*u_1} t^*E_1], [\mathcal{H}_2 \xrightarrow{(i_t^*\phi_{\mathcal{H}_2})^{-1}} t^*\mathcal{H}_2 \xrightarrow{t^*u_2} t^*E_2]). \tag{8.7}$$

The composite morphisms in (8.7) define the lifted action of $\mathbf{T}$ on $\mathcal{Q}_1 \times \mathcal{Q}_2$. Since the action of $\mathbf{T}$ on points of $\mathcal{P}$ is trivial one lifts the action of $\mathbf{T}$ to $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{P}$ where $\mathbf{T}$ acts on $\mathcal{Q}_1 \times \mathcal{Q}_2$ as described above and it acts trivially on the points $p = \mathrm{Hom}(V_1, V_2)^\vee \in \mathcal{P}$. Let $\mathfrak{S}$ be the scheme parametrizing triples of type (P_1, P_2). Let $\mathfrak{U} \subset \mathfrak{S}$ be the open subscheme of τ'-limit stable triples of type (P_1, P_2). The regular action of $\mathbf{T}$ on $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{P}$ restricts to the action of $\mathbf{T}$ on $\mathfrak{U}$:

$$\begin{array}{ccc} \mathbf{T} \times \mathfrak{U} & \xrightarrow{\quad \overline{m}^{\mathbf{T}} \quad} & \mathfrak{U} \\ {\scriptstyle id_{\mathbf{T}} \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ \mathbf{T} \times \mathfrak{M}_s^{(P_1,P_2)}(\tau') & & \mathfrak{M}_s^{(P_1,P_2)}(\tau'). \end{array} \tag{8.8}$$

Note that the action of $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ is trivial on $\mathbf{T}$ and the maps $\overline{m}^{\mathbf{T}}$, π and $\pi \circ \overline{m}^{\mathbf{T}}$ are G-equivariant. By the property of quotient stacks, this induces a G-equivariant map $\mathbf{T} \times \mathfrak{M}_s^{(P_1,P_2)}(\tau') \to \mathfrak{M}_s^{(P_1,P_2)}(\tau')$ which defines the induced action of $\mathbf{T}$ on $\mathfrak{M}_s^{(P_1,P_2)}(\tau')$. This proof restricts easily to the case where $\mathfrak{M}^{(P_1,P_2)}_{s}(\tau')$ is replaced by $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ and one obtains the action of $\mathbf{T}$ in Proposition 8.4 over moduli stack of stable frozen triples. Now we use the fact that given any $t \in \mathbf{T}$ the action of $\mathbf{T}$ on highly frozen triples is obtained by pre-composition of the the map t^{*} and the map defined by fixed choice of ψ^{-1} which we denoted by the choice of equivariant structure on $\mathcal{O}_X(-n)^{\oplus r}$. This by construction will automatically define an action of $\mathbf{T}$ on $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ (and hence on $\mathfrak{M}^{r}_{s,\mathrm{HFT}}(\tau')$). □

For more detailed discussion look at [20] (Proposition 4.1).

Proposition 8.5. *Let S be a parametrizing scheme of finite type over $\mathbb{C}$. Let $(\mathcal{E},\mathcal{F},\phi,\psi)_S$ denote a family of stable highly frozen triples over S. Suppose that for all $t = (\lambda_1,\lambda_2,\lambda_3) \in \boldsymbol{T}$:*

$$t^{*}((\mathcal{E},\mathcal{F},\phi,\psi)_S) \cong (\mathcal{E},\mathcal{F},\phi,\psi)_S. \tag{8.9}$$

then $(\mathcal{E},\mathcal{F},\phi,\psi)_S$ admits a $\boldsymbol{T}$-equivariant structure.

Proof. We give an adaptation of the proof given in [24] (Lemma 3.3) to our case. By assumption for any $t \in \mathbf{T}$ one has

$$t^{*}((\mathcal{E},\mathcal{F},\phi,\psi)_S \cong (\mathcal{E},\mathcal{F},\phi,\psi)_S.$$

Let $\sigma : \mathbf{T} \times X \to X$ denote the torus action on X and $p_2 : \mathbf{T} \times X \to X$ be the projection onto the second factor. Let $q : X \times S \to S$ be the projection onto S. One needs to show that there exists a map:

$$\rho : \mathrm{Ext}^0_{id_{\mathbf{T}}\times q}((p_2 \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S, (\sigma \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S) \to \mathcal{O}_{\mathbf{T}\times S}, \tag{8.10}$$

which is an isomorphism of line bundles over $\mathbf{T} \times S$. Here

$$\begin{aligned}&\mathrm{Ext}^0_{id_{\mathbf{T}}\times q}((p_2 \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S, (\sigma \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S)\\ &:= R^0(q \times \mathrm{id}_{\mathbf{T}})_{*}(\mathcal{H}om((p_2 \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S, (\sigma \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S)).\end{aligned} \tag{8.11}$$

By definition of $\mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau')$, choosing a family of stable highly frozen triples over S is equivalent to choosing a unique map $S \to \mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau')$. Since $(\sigma \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S$ and $(p_2 \times \mathrm{id}_S)^{*}(\mathcal{E},\mathcal{F},\phi,\psi)_S$

are two families over $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$, they both define maps $f : \mathbf{T} \times S \to \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ and $g : \mathbf{T} \times S \to \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ respectively. By the uniqueness property, both maps are uniquely isomorphic to each other. On the other hand by Lemma 3.13 the complexes representing τ'-stable highly frozen triples are simple objects hence:

$$\begin{aligned}
&\mathrm{Ext}^0_{id_{\mathbf{T}} \times q}((p_2 \times \mathrm{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \mathrm{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \\
&\cong \mathrm{Ext}^0_{id_{\mathbf{T}} \times q}((\mathcal{E}, \mathcal{F}, \phi, \psi)_{\mathbf{T} \times S}, (\mathcal{E}, \mathcal{F}, \phi, \psi)_{\mathbf{T} \times S}) \cong \mathcal{O}_{\mathbf{T} \times S}.
\end{aligned} \tag{8.12}$$

Now the inverse image of $1 \in \mathcal{O}_{\mathbf{T} \times S}$ via the map ρ in (8.10) gives a section of

$$\mathrm{Ext}^0_{id_{\mathbf{T}} \times q}((\mathcal{E}, \mathcal{F}, \phi, \psi)_{\mathbf{T} \times S}, (\mathcal{E}, \mathcal{F}, \phi, \psi)_{\mathbf{T} \times S})$$

which induces a section of

$$\mathrm{Ext}^0_{id_{\mathbf{T}} \times q}((p_2 \times \mathrm{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \mathrm{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S)$$

which induces a morphism $(p_2 \times \mathrm{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S \to (\sigma \times \mathrm{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$. Moreover, it can be checked that this morphism is an isomorphism (pointwise) for every point in the moduli stack of stable highly frozen triples hence it is an isomorphism and this finishes the proof. □

8.2 The non-geometric action of $\mathrm{T}_0 = (\mathbb{C}^*)^r$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ and the splitting property of stable highly frozen triples

Let $p \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ be represented by a highly frozen triple (E, F, ϕ, ψ). We introduce a non-geometric action of $\mathrm{T}_0 = (\mathbb{C}^*)^r$ on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$. By definition $\phi = (s_1, s_2, \cdots, s_r)$ where $s_i \in \mathrm{H}^0(F(n))$.

Definition 8.6. Define the action σ_0 of $\mathrm{T}_0 = (\mathbb{C}^*)^r$ on a point $p \in \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ given by a stable highly frozen triple (E, F, ϕ, ψ) via rescaling each copy of $\mathcal{O}_X(-n)$ independently by an element of

$\mathbb{C}^*$, i.e multiplication of each section of $F(n)$ independently by an element of $\mathbb{C}^*$:

$$\sigma_0((z_1, z_2, \cdots, z_r), [\mathcal{O}_{X\times S}(-n)^{\oplus r} \xrightarrow{\phi} F]) =$$
$$[\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\nu = \begin{pmatrix} z_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_r^{-1} \end{pmatrix}} \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F], \tag{8.13}$$

Equivalently the action σ_0 is obtained via the morphism ν from a τ'-stable highly frozen triple (E, F, ϕ, ψ) to $(E, F, \phi\circ\nu, \psi)$ as given in (8.13).

Lemma 8.7. *The action of* T_0 *satisfies the axioms of an action of* T_0 *on* $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.

Proof. One needs to prove that:

1. There exists an identity element $e \in \mathrm{T}_0$ such that $e * p = p$ for all $p \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$.
2. For all $g, h \in \mathrm{T}_0$ one has $(gh) * p = g * (h * p)$ for all $p \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$

Let the identity element of T_0 be given by $e = (1, 1, \cdots, 1)$. It is easily seen that $e * p = p$ for all $p \in \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$. Now let $g = (z_1, \cdots, z_r)$ and $h = (z'_1, \cdots, z'_r)$ be two elements of T_0. Note that $g * h = (z_1z'_1, \cdots, z_rz'_r)$. Therefore $g * (h * p)$ is obtained by the following composite map

$$\begin{aligned} &\sigma_0(h, \sigma_0(g, [\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F])) = \\ &[\mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\nu} \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\nu'} \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F], \end{aligned} \tag{8.14}$$

where $\nu'\circ\nu = \begin{pmatrix} z_1^{-1}z_1'^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_r^{-1}z_r'^{-1} \end{pmatrix}$. Therefore, T_0 obviously satisfies the second axiom. □

Proposition 8.8. *Let S be a parametrizing scheme of finite type over $\mathbb{C}$. Let $(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ denote a family of stable highly frozen triples over S. Suppose that for all $t_0 = (z_1, \cdots, z_r) \in \mathrm{T}_0$:*

$$\sigma_0(t_0, (\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \cong (\mathcal{E}, \mathcal{F}, \phi, \psi)_S \tag{8.15}$$

then $(\mathcal{E},\mathcal{F},\phi,\psi)_S$ *admits a* T_0*-equivariant structure:*

$$\sigma_0^*(\mathcal{E},\mathcal{F},\phi,\psi)_S \cong \tilde{p}_2^*(\mathcal{E},\mathcal{F},\phi,\psi)_S,$$

where $\tilde{p}_2 : \mathrm{T}_0 \times \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau') \to \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ *is the projection onto the second factor.*

Proof. The action of T_0 is directly defined over $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ hence one may directly apply the proof of Proposition 8.5 to T_0 instead of $\mathbf{T}$ and the universal family $(\mathbb{E},\mathbb{F},\phi,\psi)$ instead of $(\mathcal{E},\mathcal{F},\phi,\psi)_S$ and use the simpleness property of τ'-limit stable highly frozen triples. □

Remark 8.9. Since the stable highly frozen triples are T_0-equivariant, by Proposition 8.8 it is easily seen that the action of T_0 on a point $p \in \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ (represented by a stable highly frozen triple (E,F,ϕ,ψ)) given by the morphism ν in (8.13) induces a T_0-weight decomposition on $E \cong \mathcal{O}_X^{\oplus r}(-n)$. Let $(w_1,\cdots,w_r)$ denote the r-tuple of weights. In fact the only nontrivial weights due to action of T_0 are

$$(1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,\cdots,0,1).$$

Now consider the module M associated to the sheaf $\mathcal{O}_X^{\oplus r}(-n)$ and denote by M^0 the module associated to the sheaf $\mathcal{O}_X(-n)$. The graded piece of M^{T_0} which sits in $(1,0,\cdots,0)$ wight space is given by the module $M^0 \oplus 0 \oplus \cdots \oplus 0$ which we denote by M^0_1 and so on. On the other hand the $\mathbf{T}$-weight decomposition of $(M)^{\mathbf{T}}$ is given by (8.1). Therefore the $\mathbf{T}\times\mathrm{T}_0$-weight decomposition of $(M)^{\mathbf{T}\times\mathrm{T}_0}$ is given by

$$\begin{aligned}(M)^{\mathbf{T}\times\mathrm{T}_0} &\cong \left(\bigoplus_{(m_1,m_2,m_3)} M^0(m_1,m_2,m_3)\right)(1,0,\cdots,0)\oplus\\ &\cdots\oplus\left(\bigoplus_{(m_1,m_2,m_3)} M^0(m_1,m_2,m_3\right)(0,\cdots,0,1),\\ &\cong \bigoplus_{i=1}^r\left(\bigoplus_{(m_1,m_2,m_3)} M^0_i(m_1,m_2,m_3)\right)\end{aligned} \tag{8.16}$$

According to Propositions 8.5 and 8.8 the $\mathbf{T}\times \mathrm{T}_0$-fixed points of $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ are represented by highly frozen triples which admit $\mathbf{T}\times \mathrm{T}_0$-equivariant structure. Now let N denote the module associated to F. Given a $\mathbf{T}\times \mathrm{T}_0$-equivariant highly frozen triple $(M)^{\mathbf{T}\times \mathrm{T}_0}\to N^{\mathbf{T}\times \mathrm{T}_0}$, by the property of morphism between two graded sheaves of modules, the sheaf $N^{\mathbf{T}\times \mathrm{T}_0}$ admits a weight decomposition compatible to that of $M^{\mathbf{T}\times \mathrm{T}_0}$. Hence it is seen that a torus equivariant highly frozen triple admits a $\mathbf{T}\times \mathrm{T}_0$-weight decomposition of the following form:

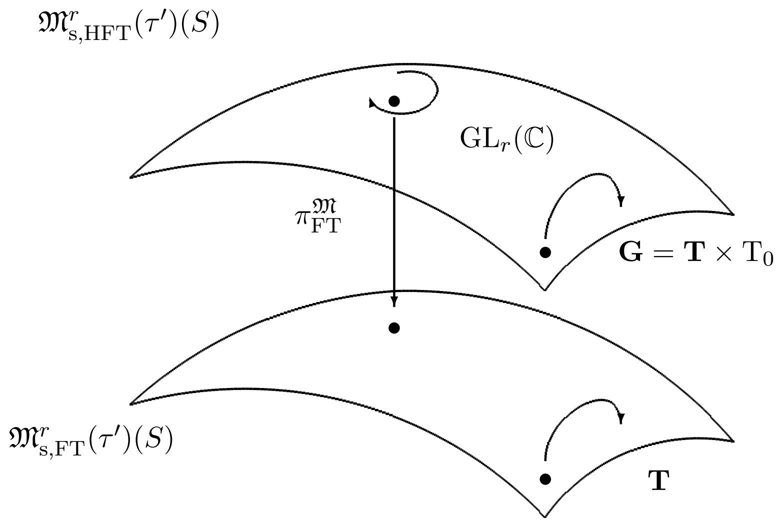
$$\begin{aligned}&[M^{\mathbf{T}\times \mathrm{T}_0}\to N^{\mathbf{T}\times \mathrm{T}_0}]\cong\\ &\cong\bigoplus_{i=1}^{r}\left[\bigoplus_{(m_1,m_2,m_3)}M^0_i(m_1,m_2,m_3)\to\bigoplus_{(m'_1,m'_2,m'_3)}N_i(m'_1,m'_2,m'_3)\right]\end{aligned} \tag{8.17}$$

Remark 8.10. The weight decomposition in (8.17) clarifies the fact that a $\mathbf{T}\times \mathrm{T}_0$-equivariant τ'-limit stable highly frozen triple is decomposable into r copies of $\mathbf{T}$-equivariant τ'-limit stable highly frozen triples of the form $\mathcal{O}_X(-n)\to F_i$:

$$[\mathcal{O}_X^{\oplus r}(-n)\to F]^{\mathbf{T}\times \mathrm{T}_0}\cong\bigoplus_{i=1}^{r}[\mathcal{O}_X(-n)\to F_i]^{\mathbf{T}}. \tag{8.18}$$

Hence the $\mathbf{T}\times \mathrm{T}_0$-equivariant highly frozen triples are given as a direct sum of r copies of $\mathbf{T}$-equivariant stable pairs in [28]. The important point to note is that what makes it possible to think of stable highly frozen triples in this setup as r copies of stable pairs is that our notion of τ'-limit-stability is compatible with the notion of stability in [28].

The following picture schematically shows the corresponding tori acting on $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ and $\mathfrak{M}^r_{\mathrm{s,FT}}(\tau')$.



$\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')(S)$
$\mathrm{GL}_r(\mathbb{C})$
$\pi^{\mathfrak{M}}_{\mathrm{FT}}$
$\mathbf{G} = \mathbf{T} \times \mathrm{T}_0$
$\mathfrak{M}^r_{\mathrm{s,FT}}(\tau')(S)$
$\mathbf{T}$

Chapter 9

Equivariant obstruction theory on HFT

The discussions in this section, except for some computational parts, are similar in nature to discussions in [27] (4.2). As we have showed in the last chapter there exists an action of $\mathbf{T}$ on $\mathfrak{M}^r_{\mathrm{s,FT}}(\tau')$ while $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ is acted on by $\mathbf{G} = \mathbf{T} \times \mathrm{T}_0$. Using the deformation obstruction theory obtained by Theorem 6.12, we compute a $\mathbf{G}$-invariant obstruction theory for the $\mathbf{G}$-fixed locus of $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$.

Remark 9.1. Note that since the construction of the virtual fundamental class in the second part of Theorem 6.12 depends on Assumption 7.15, we emphasize that all our virtual localization computations in this chapter and the following two chapters similarly hold true if Assumption 7.15 holds true.

Strategy

We assume that the $\mathbf{G}$-fixed components of $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ are compact and nonsingular: a consequence of identifying the highly frozen triple as multiple copies of PT pairs (Remark 8.10) is that the $\mathbf{G}$-fixed components of the moduli stack of highly frozen triples are obtained as r-fold products of $\mathbf{T}$-fixed components of the moduli stack of stable pairs which are conjectured by Pandharipande and Thomas in [28] (Conjecture 2) to be nonsingular and compact.

Let $\mathbf{Q}$ denote the $\mathbf{G}$-fixed locus of $\mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$. We assume that $\mathbf{Q}$ is nonsingular, connected and compact. Let $\iota_{\mathbf{Q}} : \mathbf{Q} \hookrightarrow \mathfrak{M}^r_{\mathrm{s,HFT}}(\tau')$ denote the natural embedding.

Let $\mathbb{G}^{\bullet}_{\mathbf{Q}} := (\iota_{\mathbf{Q}})^* \mathbb{G}^{\bullet}$ where $\mathbb{G}^{\bullet}$ is the deformation obstruction theory obtained in Theorem 6.12. Let $\mathbb{G}^{\bullet,\mathbf{G}}_{\mathbf{Q}}$ and $\mathbb{G}^{\bullet,\mathbf{m}}_{\mathbf{Q}}$ denote the sub-bundles of $\mathbb{G}^{\bullet}_{\mathbf{Q}}$ with zero and nonzero characters respectively. By Theorem 6.12 and the $\mathbf{G}$-fixed deformation obstruction theory restricted to $\mathbf{Q}$ is given by a map of perfect complexes:

$$\mathbb{G}^{\bullet,\mathbf{G}}_{\mathbf{Q}} \xrightarrow{\phi} \mathbb{L}^{\bullet}_{\mathbf{Q}}. \tag{9.1}$$

Here $\mathbb{G}^{\bullet,\mathbf{G}}_{\mathbf{Q}}$ is represented by a two term complex of vector bundles $G^{-1,\mathbf{G}}_{\mathbf{Q}} \to G^{0,\mathbf{G}}_{\mathbf{Q}}$. By the virtual localization formula:

$$\Big[\mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau')\Big]^{vir} = \sum_{\mathbf{Q}\subset\mathfrak{M}^{r}_{\mathrm{s,HFT}}(\tau')} \iota_{\mathbf{Q}*}\bigg(\frac{e(G^{\mathbf{m}}_{1,\mathbf{Q}})}{e(G^{\mathbf{m}}_{0,\mathbf{Q}})}\cap[\mathbf{Q}]^{vir}\bigg). \tag{9.2}$$

Where $G^{\mathbf{m}}_{0,\mathbf{Q}}$ and $G^{\mathbf{m}}_{1,\mathbf{Q}}$ denote the dual of $G^{0,\mathbf{m}}_{\mathbf{Q}}$ and $G^{-1,\mathbf{m}}_{\mathbf{Q}}$ respectively. Now we rewrite (9.2) with respect to the Euler classes $e(G_{1,\mathbf{Q}})$ and $e(G_{0,\mathbf{Q}})$ where $G_{0,\mathbf{Q}}$ and $G_{1,\mathbf{Q}}$ denote the dual of $G^{0}_{\mathbf{Q}}$ and $G^{-1}_{\mathbf{Q}}$ respectively. In order to do this ones needs to have the description of the virtual tangent space with respect to the $\mathbf{G}$-fixed deformation obstruction theory. If $\mathbf{Q}$ is assumed to be nonsingular, then

$$\mathbb{L}^{\bullet}_{\mathbf{Q}} := 0 \to \Omega_{\mathbf{Q}}.$$

The $\mathbf{G}$-fixed deformation obstruction theory (9.1) induces a composite morphism

$$G^{-1,\mathbf{G}}_{\mathbf{Q}} \to G^{0,\mathbf{G}}_{\mathbf{Q}} \xrightarrow{\phi} \Omega_{\mathbf{Q}}.$$

The kernel of this composite morphism is the obstruction bundle $\mathbf{K}$ and by definition:

$$[\mathbf{Q}]^{vir} = e(\mathbf{K}^{\vee}) \cap [\mathbf{Q}].$$

One computes the $\mathcal{K}$-theory class of $\mathbf{K}^{\vee}$ as follows:

$$[\mathbf{K}^{\vee}] = [G^{\mathbf{G}}_{1,\mathbf{Q}}] - [G^{\mathbf{G}}_{0,\mathbf{Q}}] + [T_{\mathbf{Q}}], \tag{9.3}$$

where $G^{\mathbf{G}}_{0,\mathbf{Q}}$ and $G^{\mathbf{G}}_{1,\mathbf{Q}}$ denote the dual of $G^{0,\mathbf{G}}_{\mathbf{Q}}$ and $G^{-1,\mathbf{G}}_{\mathbf{Q}}$ respectively. Therefore one has:

$$e(\mathbf{K}^{\vee}) = \frac{e(G^{\mathbf{G}}_{1,\mathbf{Q}})}{e(G^{\mathbf{G}}_{0,\mathbf{Q}})} \cdot e(T_{\mathbf{Q}}). \tag{9.4}$$

By (9.2) and (9.4) the virtual fundamental class of $\mathfrak{M}^r_{\text{s,HFT}}(\tau')$ is obtained as

$$\left[\mathfrak{M}^r_{\text{s,HFT}}(\tau')\right]^{vir} = \sum_{\mathbf{Q}\subset\mathfrak{M}^r_{\text{s,HFT}}(\tau')} \iota_{\mathbf{Q}*}\left(\frac{e(G_{1,\mathbf{Q}})}{e(G_{0,\mathbf{Q}})}\cdot e(T_{\mathbf{Q}})\cap[\mathbf{Q}]\right). \tag{9.5}$$

Now we compute the difference $[G_{0,\mathbf{Q}}]-[G_{1,\mathbf{Q}}]$ in the $\mathbf{G}$-equivariant $\mathcal{K}$-theory of $\mathbf{Q}$. Now consider a point $p\in\mathbf{Q}$ represented by the complex

$$I^{\bullet\mathbf{G}} := [\mathcal{O}_X^{\oplus r}(-n)\to F]^{\mathbf{G}}.$$

The difference $[G_{0,\mathbf{Q}}]-[G_{1,\mathbf{Q}}]$ over this point is the virtual tangent space at this point. We use the quasi isomorphism in diagram (6.36) to compute the virtual tangent space:

$$\begin{aligned}&\mathcal{T}^{\mathbf{Q}}_{I^\bullet} = [\text{Coker}(d')]-[\text{Ker}(d)] = \\ &\left([\pi^*E^1]-[\pi^*E^0]+[\pi^*E^{-1}]-[\pi^*E^{-2}]\right)+\left(\cancel{[T_\pi]}-\cancel{[\Omega_\pi]}\right),\end{aligned} \tag{9.6}$$

where E^i for $i=-1,\cdots,2$ are the corresponding terms of $\mathbb{E}^{\bullet\vee}$ in Lemma 6.10 and the cancellation in the second row is due to isomorphism of Ω_π and T_π which is seen from their triviality.

By the construction of $\mathbb{E}^{\bullet\vee}$ in Proposition 6.10 and since the point $p\in\mathbf{Q}$ is represented by $I^{\bullet,\mathbf{G}}$ the following identities hold true:

$$\begin{aligned}&[\pi^*E^1]-[\pi^*E^0]+[\pi^*E^{-1}]-[\pi^*E^{-2}] = \\ &= -[\text{Hom}(I^\bullet,I^\bullet)_0]+[\text{Ext}^1(I^\bullet,I^\bullet)_0]-[\text{Ext}^2(I^\bullet,I^\bullet)_0]+[\text{Ext}^3(I^\bullet,I^\bullet)_0].\end{aligned} \tag{9.7}$$

Hence the virtual tangent space in (9.6) is written as:

$$\mathcal{T}^{\mathbf{Q}}_{I^\bullet} = -[\mathrm{Hom}(I^\bullet, I^\bullet)_0] + [\mathrm{Ext}^1(I^\bullet, I^\bullet)_0] - [\mathrm{Ext}^2(I^\bullet, I^\bullet)_0]$$
$$+ [\mathrm{Ext}^3(I^\bullet, I^\bullet)_0] = [\chi(\mathcal{O}_X, \mathcal{O}_X)] - [\chi(I^\bullet, I^\bullet)]. \tag{9.8}$$

9.1 Computation of $\chi(\mathcal{O}_X, \mathcal{O}_X) - \chi(I^\bullet, I^\bullet)$

By definition

$$\chi(I^\bullet, I^\bullet) = \sum_{i,j=0}^{3} (-1)^{i+j} H^i(\mathcal{E}xt^j(I^\bullet, I^\bullet))$$

and

$$\chi(\mathcal{O}_X, \mathcal{O}_X) = \sum_{i,j=0}^{3} (-1)^{i+j} H^i(\mathcal{E}xt^j(\mathcal{O}_X, \mathcal{O}_X)),$$

here one may replace the cohomology terms with the Čech complex obtained with respect to an affine open cover $\bigcup_\alpha \mathcal{U}_\alpha$:

$$\chi(I^\bullet, I^\bullet) = \sum_{i,j=0}^{3} (-1)^{i+j} \mathfrak{C}^i(\mathcal{E}xt^j(I^\bullet, I^\bullet))$$

and

$$\chi(\mathcal{O}_X, \mathcal{O}_X) = \sum_{i,j=0}^{3} (-1)^{i+j} \mathfrak{C}^i(\mathcal{E}xt^j(\mathcal{O}_X, \mathcal{O}_X)).$$

By definition the sheaf F appearing in the stable highly frozen triples is pure of dimension 1 therefore the restriction of F over the triple and quadruple intersections of $\mathcal{U}_\alpha$'s vanishes and over such intersections $I^\bullet \cong \mathcal{O}_X^{\oplus r}(-n)$.

Definition 9.2. Define:

$$\begin{aligned}
\mathcal{T}^1_{[I^\bullet]} &= \bigoplus_{\alpha} \left(\mathrm{H}^0(\mathcal{U}_\alpha, \mathcal{O}_X) - \sum_j (-1)^j \, \mathrm{H}^0(\mathcal{U}_\alpha, \mathcal{E}xt^j(I^\bullet, I^\bullet))) \right) \\
\mathcal{T}^2_{[I^\bullet]} &= \bigoplus_{\alpha,\beta} \left(\mathrm{H}^0(\mathcal{U}_{\alpha\beta}, \mathcal{O}_X) - \sum_j (-1)^j \, \mathrm{H}^0(\mathcal{U}_{\alpha\beta}, \mathcal{E}xt^j(I^\bullet, I^\bullet))) \right) \\
\mathcal{T}^3_{[I^\bullet]} &= \bigoplus_{\alpha,\beta,\gamma} \big((1 - r^2) \, \mathrm{H}^0(\mathcal{U}_{\alpha\beta\gamma}, \mathcal{O}_X) \big) \\
\mathcal{T}^4_{[I^\bullet]} &= \bigoplus_{\alpha,\beta,\gamma,\delta} (1 - r^2) \, \mathrm{H}^0(\mathcal{U}_{\alpha\beta\gamma\delta}, \mathcal{O}_X).
\end{aligned} \tag{9.9}$$

By Definition 9.9 and (9.8) the virtual tangent space is obtained as:

$$\mathcal{T}_{[I^\bullet]} = \mathcal{T}^1_{[I^\bullet]} - \mathcal{T}^2_{[I^\bullet]} + \mathcal{T}^3_{[I^\bullet]} - \mathcal{T}^4_{[I^\bullet]} \tag{9.10}$$

Chapter 10

Virtual localization-vertex and edge calculation

Let $\mathbf{T}$ and T_0 be defined as before and $\mathbf{G} = \mathbf{T} \times \mathrm{T}_0$. Let (t_1, t_2, t_3) be defined as the weights of $\mathbf{T}$. Moreover let $(w_1, \cdots, w_r)$ be defined as weight of the action of T_0. Here w_i is given by tuples $(0, \cdots, 1, \cdots, 0)$ where 1 is positioned in the i'th position in the tuple. In this section we compute the $\mathbf{G}$-character of $\mathcal{T}^i_{[I^\bullet]}$ for $i = 1, \cdots 4$ in (9.10). We compute the vertex for $\mathbf{G}$-equivariant stable highly frozen triples which are identified with

$$\left[\bigoplus_{i=1}^{r} \left(\mathcal{O}_X^{\mathbf{T}}(-n) \to F_i^{\mathbf{T}}\right)\right]. \tag{10.1}$$

Recall that super-index $\mathbf{T}$ indicates equivariance with respect to the action of $\mathbf{T}$.
Choose a Čech cover $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ of X. The restriction of each copy of $O_X^{\mathbf{T}}(-n) \to F_i^{\mathbf{T}}$ in (10.1) to the underlying supporting curve $\mathcal{C}_\alpha$ of $F_i^{\mathbf{T}}$ induces an exact sequence of the form:

$$0 \to \mathcal{O}^{\mathbf{T}}_{\mathcal{C}_\alpha}(-n) \to (F_i^{\mathbf{T}})_\alpha \to (Q_i^{\mathbf{T}})_\alpha \to 0, \tag{10.2}$$

By τ'-stability the sheaf $(F_i^{\mathbf{T}})_\alpha$ may be zero and if it is nonzero then the cokernel $(Q_i^{\mathbf{T}})_\alpha$ has to be zero dimensional. Moreover by the splitting property of $\mathbf{G}$-equivariant highly frozen triples it is easily seen that

$$Q_\alpha^{\mathbf{G}} := \bigoplus_{i=1}^{r} (Q_i^{\mathbf{T}})_\alpha,$$

such that each $(Q_i^{\mathbf{T}})_\alpha$ has zero dimensional support: one has $\mathrm{Supp}(Q_\alpha^{\mathbf{G}}) := \bigcup_{i=1}^{r} \mathrm{Supp}(Q_i^{\mathbf{T}})_\alpha$ and if there exists $(Q_i^{\mathbf{T}})_\alpha$ for some i with one dimensional support then it contradicts with stability of the original highly frozen triple. Given $F_\alpha^{\mathbf{G}} = \bigoplus_{i=1}^{r} (F_i^{\mathbf{T}})_\alpha$, we use the procedure similar to [27]

(Section 4.4) and [4] (Section 4.7) to compute the $\mathbf{T}$ character of each summand, $(F_i^{\mathbf{T}})_\alpha$. Let $\mathrm{Ch}(F_i^{\mathbf{T}})_\alpha$ denote the $\mathbf{T}$-character of each summand. Let $(\mathbf{P}_i)_\alpha(t_1,t_2,t_3)$ denote the associated Poincaré polynomial of $(\mathbb{I}_i^\bullet)_\alpha := \left(\mathcal{O}_X^{\mathbf{T}}(-n)\to F_i^{\mathbf{T}}\right)|_\alpha$. The Poincaré polynomial of $(\mathbb{I}_i^\bullet)_\alpha$ is related to $\mathbf{T}$ character of F_i as:

$$\mathrm{Ch}(F_i^{\mathbf{T}})_\alpha = \frac{B_\alpha^n + (\mathbf{P}_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)}, \tag{10.3}$$

where the correction term B_α^n is the $\mathbf{T}$-character of $\mathcal{O}_X(-n)$ with the chosen equivariant structure. Now the $\mathbf{G}$-character of F_i is given by:

$$\mathrm{Ch}(F_i^{\mathbf{G}})_\alpha = w_i\cdot \mathrm{Ch}(F_i^{\mathbf{T}})_\alpha = \frac{B_\alpha^n\cdot w_i + w_i\cdot(\mathbf{P}_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)}, \tag{10.4}$$

where w_i is the weight corresponding to the action of T_0 on the i'th copy of $\mathcal{O}_X(-n)$ and on $F_i^{\mathbf{T}}$. The description of B_α^n depends on one's choice of equivariant structure. The $\mathbf{T}$-character of each $tr_\chi((\mathbb{I}_i^\bullet)_\alpha,(\mathbb{I}_i^\bullet)_\alpha)$ as computed in [4] (Section 4.7) is given as follows:

$$tr_\chi((\mathbb{I}_i^\bullet)_\alpha,(\mathbb{I}_i^\bullet)_\alpha) = \frac{w_i\cdot w_i^{-1}\cdot(\mathbf{P}_i)_\alpha\overline{(\mathbf{P}_i)}_\alpha}{(1-t_1)(1-t_2)(1-t_3)} = \frac{(\mathbf{P}_i)_\alpha\overline{(\mathbf{P}_i)}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}. \tag{10.5}$$

The dual bar operation is negation on $\mathcal{K}(\mathbf{Q}\,|_{\mathcal{U}_\alpha})$ and

$$t_i \to \frac{1}{t_i}$$

on the equivariant variables t_i. Since $\mathbb{I}_\alpha^{\bullet,\mathbf{G}} := \bigoplus_{i=1}^r(\mathbb{I}_i^{\bullet,\mathbf{T}})_\alpha$ the $\mathbf{G}$-character of $\chi(\mathbb{I}_\alpha^{\bullet,\mathbf{G}},\mathbb{I}_\alpha^{\bullet,\mathbf{G}})$ is obtained as:

$$tr_\chi(\mathbb{I}_\alpha^{\bullet,\mathbf{G}},\mathbb{I}_\alpha^{\bullet,\mathbf{G}}) = \sum_{\substack{1\le i\le r\\ 1\le j\le r}} \frac{w_iw_j^{-1}\cdot(\mathbf{P}_i)_\alpha\overline{(\mathbf{P}_j)}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}. \tag{10.6}$$

Moreover the $\mathbf{G}$-character of F_α appearing in $\mathbb{I}_\alpha^{\bullet,\mathbf{G}}$ is given by :

$$\mathrm{Ch}(F^{\mathbf{G}})_\alpha = \frac{\sum_{i=1}^r w_i\cdot B_\alpha^n + \sum_{i=1}^r w_i\cdot(\mathbf{P}_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)}, \tag{10.7}$$

since the $\mathbf{G}$-character of the α-summand of $\mathcal{T}^1_{[I\bullet]}$ in (9.9) is given by:

$$\frac{1-\sum_{\substack{1\leq i\leq r\\1\leq j\leq r}} w_iw_j^{-1}\cdot(\mathbf{P}_i)_\alpha\overline{(\mathbf{P}_j)}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}, \tag{10.8}$$

one computes the of the α-summand of the $\mathbf{G}$-character of $\mathcal{T}^1_{[I\bullet]}$ as a function of $\mathrm{Ch}(F^{\mathbf{G}}_\alpha)$:

$$\begin{aligned} tr_{R-_{\chi((\mathbb{I}^\bullet,\mathbf{G})_\alpha,(\mathbb{I}^\bullet,\mathbf{G})_\alpha)}} &= \mathrm{Ch}(F^{\mathbf{G}}_\alpha)\cdot(\sum_{j=1}^r w_j^{-1})\cdot\overline{B^n_\alpha}-\frac{\overline{\mathrm{Ch}(F^{\mathbf{G}}_\alpha)}\cdot(\sum_{i=1}^r w_i)\cdot B^n_\alpha}{t_1t_2t_3}\\ &+\mathrm{Ch}(F^{\mathbf{G}}_\alpha)\overline{\mathrm{Ch}(F^{\mathbf{G}}_\alpha)}\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}+\frac{1-(\sum_{i,j=1}^r w_iw_j^{-1})\cdot B^n_\alpha\overline{B^n_\alpha}}{(1-t_1)(1-t_2)(1-t_3)} \end{aligned} \tag{10.9}$$

10.0.1 Edge calculation

In this section we compute the $\mathbf{G}$-character of $\mathcal{T}^2_{[I\bullet]}$, $\mathcal{T}^3_{[I\bullet]}$ and $\mathcal{T}^4_{[I\bullet]}$. Assume that $\mathcal{U}_{\alpha\beta}$ is the affine patch over which the equivariant parameter t_1 is invertible. Given $F=\bigoplus_{i=1}^r F_i$, Let $(F_i)_{\alpha\beta}$ denote the restriction of F_i to $\mathcal{U}_{\alpha\beta}$. Let

$$\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})_i=\sum_{k_2,k_3\in\mu_{\alpha\beta}} t_2^{k_2}t_3^{k_3},$$

denote the $\mathbf{T}$-character associated to this restriction (Look at [4] (4.10)). The $\mathbf{G}$-character of $F_{\alpha\beta}$ is obtained as

$$\mathrm{Ch}(F^{\mathbf{G}}_{\alpha\beta})=\sum_{i=1}^r\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})_i\cdot w_i.$$

By the same argument as above and similar to computations in [4] (4.10) one relates the **G**-character of $\alpha\beta$'th summand of the virtual tangent space $\mathcal{T}^2_{[\boldsymbol{I}^\bullet]}$ in (9.9) to $\mathrm{Ch}(F^{\mathbf{G}}_{\alpha\beta})$:

$$tr_{R-\chi((\mathbb{I}^\bullet)_{\alpha\beta},(\mathbb{I}^\bullet)_{\alpha\beta})} = \Bigg[\mathrm{Ch}(F^{\mathbf{G}}_{\alpha\beta})(\sum_{j=1}^{r} w_j^{-1})\cdot\overline{B^n_{\alpha\beta}} - \frac{\overline{\mathrm{Ch}(F^{\mathbf{G}}_{\alpha\beta})}\cdot(\sum_{i=1}^r w_i)\cdot B^n_{\alpha\beta}}{t_2t_3} + \mathrm{Ch}(F^{\mathbf{G}}_{\alpha\beta})\overline{\mathrm{Ch}(F^{\mathbf{G}}_{\alpha\beta})}\frac{(1-t_2)(1-t_3)}{t_2t_3} + \frac{1-(\sum_{i,j=1}^r w_iw_j^{-1})\cdot B^n_{\alpha\beta}\overline{B^n_{\alpha\beta}}}{(1-t_2)(1-t_3)}\Bigg]\cdot\delta(t_1), \tag{10.10}$$

here $B^n_{\alpha\beta}$ is a function of n and the correction term that needs to be inserted into Poincaré polynomial of $\mathcal{O}_X\mid_{\mathcal{U}_{\alpha\beta}}$ in order to obtain the Poincaré polynomial of $\mathcal{O}_X(-n)\mid_{\mathcal{U}_{\alpha\beta}}$ also we have used the notation $\delta(t_1)=\sum_{k\in\mathbb{Z}} t_i^k$. Now assume $\mathcal{U}_{\alpha\beta\gamma}$ is the affine patch over which the equivariant parameters t_1 and t_2 are invertible. The α,β,γ'th summand of $\mathcal{T}^3_{[I^\bullet]}$ in (9.9) is obtained as follows:

$$tr_{R-\chi((\mathbb{I}^\bullet)_{\alpha\beta\gamma},(\mathbb{I}^\bullet)_{\alpha\beta\gamma})} = \frac{(1-\sum_{i,j=1}^r w_iw_j^{-1})}{(1-t_3)}\delta(t_1)\delta(t_2). \tag{10.11}$$

Finally the **T**-character of $\mathcal{T}^4_{[I^\bullet]}$ in (9.9) is obtained as:

$$tr_{R-\chi((\mathbb{I}^\bullet)_{\alpha\beta\gamma\delta},(\mathbb{I}^\bullet)_{\alpha\beta\gamma\delta})} = (1-\sum_{i,j=1}^r w_iw_j^{-1})\delta(t_1)\delta(t_2)\delta(t_3). \tag{10.12}$$

Based on above discussion the **G**-character of the virtual tangent space over a point is obtained as follows:

$$tr_{R-\chi(,\mathbb{I}^\bullet)} = \sum_{\alpha} tr_{R-\chi((\mathbb{I}^\bullet)_\alpha,(\mathbb{I}^\bullet)_\alpha)} - \sum_{\alpha,\beta} tr_{R-\chi((\mathbb{I}^\bullet)_{\alpha\beta},(\mathbb{I}^\bullet)_{\alpha\beta})} + \sum_{\alpha,\beta,\gamma} tr_{R-\chi((\mathbb{I}^\bullet)_{\alpha\beta\gamma},(\mathbb{I}^\bullet)_{\alpha\beta\gamma})} - \sum_{\alpha,\beta,\gamma,\delta} tr_{R-\chi((\mathbb{I}^\bullet)_{\alpha\beta\gamma\delta},(\mathbb{I}^\bullet)_{\alpha\beta\gamma\delta})} \tag{10.13}$$

10.1 PT Redistribution

As It is seen the **G**-character of the virtual tangent space in (10.13) is equal to the addition of vertex contributions (the first summand on right hand side of (10.13)) and the remaining edge contributions. Similar to discussions in [27] (Section 4.6) one may redistribute the terms in (10.9), (10.10), (10.11) and (10.12) so that they become Laurent polynomials in the variables t_i:
Define

$$\mathrm{G}_{\alpha\beta} = \mathrm{Ch}(F_{\alpha}^{\mathbf{G}})(\sum_{j=1}^{r} w_j^{-1}) \cdot \overline{B_{\alpha\beta}^{n}} - \frac{\overline{\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{G}})} \cdot (\sum_{i=1}^{r} w_i) \cdot B_{\alpha\beta}^{n}}{t_2 t_3} + \mathrm{Ch}(F_{\alpha\beta}^{\mathbf{G}})\overline{\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{G}})}\frac{(1-t_2)(1-t_3)}{t_2 t_3} + \frac{1-(\sum_{i,j=1}^{r} w_i w_j^{-1}) \cdot B_{\alpha\beta}^{n}\overline{B_{\alpha\beta}^{n}}}{(1-t_2)(1-t_3)}. \quad (10.14)$$

In that case one can rewrite the edge character (10.10) similar to [27] (Equation 4.11). Similarly define

$$\mathrm{G}_{\alpha\beta\gamma} = \frac{(1-\sum_{i,j=1}^{r} w_i w_j^{-1})}{(1-t_3)}. \quad (10.15)$$

Hence (10.11) is rewritten as

$$\left(\frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1} + t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1^{-1}}\right)\frac{1}{1-t_2} + t_2^{-1}\left(\frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1} + t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1^{-1}}\right)\frac{1}{1-t_2^{-1}}. \quad (10.16)$$

Note that here we expand the first term of the edge character in $\left(\frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1} + t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1^{-1}}\right)$ in ascending powers of t_1 and the second term in descending powers of t_1. We follow the same rule and expand the first term in (10.16) in ascending powers of t_2 and the second term in descending powers of t_2. Finally define

$$\mathrm{G}_{\alpha\beta\gamma\delta} = (1-\sum_{i,j=1}^{r} w_i w_j^{-1}). \quad (10.17)$$

Hence (10.12) is rewritten as

$$\left(\left(\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1} + t_1^{-1} \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} \right) \frac{1}{1-t_2} + t_2^{-1} \left(\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1} + t_1^{-1} \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} \right) \frac{1}{1-t_2^{-1}} \right) \frac{1}{1-t_3}$$
$$+ t_3^{-1} \left(\left(\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1} + t_1^{-1} \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} \right) \frac{1}{1-t_2} + t_2^{-1} \left(\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1} + t_1^{-1} \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} \right) \frac{1}{1-t_2^{-1}} \right) \frac{1}{1-t_3^{-1}}, \quad (10.18)$$

where we expand the first term in (10.18) in ascending powers of t_3 and the second term in descending powers of t_3. Now for each $\mathcal{U}_\alpha$ define a new vertex character similar to [27] (Equation 4.12):

$$V_\alpha = tr_{R-_{\chi((\mathbb{I}^\bullet, \mathbf{G})_\alpha, (\mathbb{I}^\bullet, \mathbf{G})_\alpha)}} + \sum_{i=1}^{3} \frac{\mathrm{G}_{\alpha\beta_i}(t_{i'}, t_{i"})}{1-t_i} \quad (10.19)$$

where $\beta_1, \beta_2, \beta_3$ are the three neighboring vertices and

$$(t_i, t_{i'}, t_{i"}) = (t_1, t_2, t_3).$$

Moreover redefine the edge character $\mathrm{E}_{\alpha\beta}$ as in [27] (Section 4.6):

$$\mathrm{E}_{\alpha\beta} = t_1^{-1} \frac{\mathrm{G}_{\alpha\beta}(t_2, t_3)}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta}(t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m'_{\alpha\beta}})}{1-t_1^{-1}} \quad (10.20)$$

Here the integers $m_{\alpha\beta}$ and $m'_{\alpha\beta}$ are determined by the normal bundle $\mathcal{N}_{\mathcal{C}_{\alpha\beta}/X}$ to the supporting curve $\mathcal{C}_{\alpha\beta} := \mathrm{Supp}(F_{\alpha\beta})$:

$$\mathcal{N}_{\mathcal{C}_{\alpha\beta}/X} = \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}).$$

Similarly redefine $\mathrm{E}_{\alpha\beta\gamma}$ and $\mathrm{E}_{\alpha\beta\gamma\delta}$ respectively as:

$$\mathrm{E}_{\alpha\beta\gamma} =$$
$$t_2^{-1} \left(t_1^{-1} \frac{\mathrm{G}_{\alpha\beta\gamma}(t_3)}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta\gamma}(t_3 t_1^{m'_{\alpha\beta}})}{1-t_1^{-1}} \right) \frac{1}{1-t_2^{-1}} - \left(t_1^{-1} \frac{\mathrm{G}_{\alpha\beta}(t_3)}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta\gamma}(t_3 t_1^{-m'_{\alpha\beta}})}{1-t_1^{-1}} \right) \frac{1}{1-t_2^{-1}} \quad (10.21)$$

and

$$
\begin{aligned}
&\mathrm{E}_{\alpha\beta\gamma\delta} = \\
&t_3^{-1}\left(t_2^{-1}\left(t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}}\right)\frac{1}{1-t_2^{-1}} - \left(t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}}\right)\frac{1}{1-t_2^{-1}}\right)\frac{1}{1-t_3^{-1}} \\
&-\left(t_2^{-1}\left(t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}}\right)\frac{1}{1-t_2^{-1}} - \left(t_1^{-1}\frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}} - \frac{\mathrm{G}_{\alpha\beta\gamma\delta}}{1-t_1^{-1}}\right)\frac{1}{1-t_2^{-1}}\right)\frac{1}{1-t_3^{-1}}
\end{aligned}
\tag{10.22}
$$

According to the above redistributions the **G**-character of the virtual tangent space in (10.13) can be rewritten as:

$$
tr_{R-\chi(,\mathbb{I}^\bullet)} = \sum_{\alpha} V_\alpha + \sum_{\alpha\beta} \mathrm{E}_{\alpha\beta} + \sum_{\alpha\beta\gamma} \mathrm{E}_{\alpha\beta\gamma\delta} + \sum_{\alpha\beta\gamma\delta} \mathrm{E}_{\alpha\beta\gamma\delta} \tag{10.23}
$$

Remark 10.1. Given a torus fixed component $\mathbf{Q}$ of the moduli stack of highly frozen triples denote $V_{\mathbf{Q}} = \sum_\alpha V_\alpha$ where V_α are defined as in (10.19). By discussions in [27] (Section 4.7) on can define the integral of the evaluation of the contribution of (10.9) on $\mathbf{Q}$, i.e:

$$
w(\mathbf{Q}) = \int_{\mathbf{Q}} e(\mathrm{T}_{\mathbf{Q}})e(-V_{\mathbf{Q}}). \tag{10.24}
$$

Hence by substituting $w(\mathbf{Q})$ in (10.24) in Equation 4.14 of [27] one obtains a definition for the equivariant vertex of the moduli stack of highly frozen triples.

Chapter 11

Calculation of examples

In this section for computational purposes we restrict to the case where $r = 2$.

Proposition 11.1. *Use result obtained in Lemma 2.17. and Remark 8.9. Given a $\acute{\tau}$-limit stable $\boldsymbol{G}$-equivariant highly frozen triple $\mathcal{O}_X(-n)^{\oplus 2,\boldsymbol{G}} \xrightarrow{\phi^{\boldsymbol{G}}} F^{\boldsymbol{G}}$ of type $(P_2, 2)$ with supporting curve $\mathcal{C}$ for F consider the finite length $\boldsymbol{G}$-equivariant cokernel $Q^{\boldsymbol{G}}$ given by* $\mathrm{Coker}(\phi)^{\boldsymbol{G}}$. *Then $Q^{\boldsymbol{G}} \cong Q_1^{\boldsymbol{T}} \oplus Q_2^{\boldsymbol{T}}$ such that each $Q_i^{\boldsymbol{T}}$ for $i = 1, 2$ is given as a subsheaf of*

$$\mathcal{H} = \varinjlim_r \left(\mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}})/\mathcal{O}_{\mathcal{C}}\right). \tag{11.1}$$

In other words a τ'-limit stable $\boldsymbol{G}$-equivariant highly frozen triple of rank 2 with support $\mathcal{C}$ is equivalent to a subsheaf of $\mathcal{H}$ in (11.1) *for $r \gg 0$. Look at similar statement for rank 1 highly frozen triples in [28] (Proposition 1.8).*

Proof. Since

$$\mathcal{O}_X(-n)^{\oplus 2,\mathbf{G}} \to F^{\mathbf{G}} := \bigoplus_{i=1}^{2} (\mathcal{O}_X^{\mathbf{T}}(-n) \to F_i^{\mathbf{T}}),$$

each $\mathcal{O}_X^{\mathbf{T}}(-n) \to F_i^{\mathbf{T}}$ restricted to the supporting curve of F_i, is identified with $Q_i^{\mathbf{T}}$ appearing in

$$0 \to \mathcal{O}_C^{\mathbf{T}}(-n) \to F_i^{\mathbf{T}} \to Q_i^{\mathbf{T}} \to 0,$$

and by Proposition 1.8 of [28] Q_i is identified with a subsheaf of the quasi-coherent sheaf

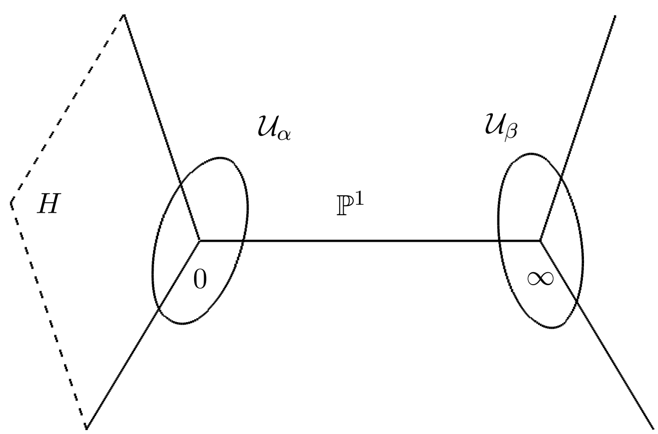
$$\varinjlim_r \mathcal{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}})/\mathcal{O}_{\mathcal{C}}.$$

It is easily seen that the cokernel of the original **G**-equivariant highly frozen triple , restricted to $\mathcal{C}$ and identified with $\bigoplus_{i=1}^{2} Q_i^{\mathbf{T}}$, is a subsheaf of direct sum of two copies of the same quasi-coherent sheaf. □

The above proposition enables one to use the method of melting crystals as described in Section 2.2 in [27]. Here we omit discussion about monomial ideals since they are discussed in the literature and instead we directly apply the method of PT in [27] to our setup. We solve two examples over two specific partitions. First we do a calculation for X given by total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$. We will also give a computational recipe for the case where X is given by total space of $\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2$.

11.0.1 Examples

Example 11.2. (**Local** $\mathbb{P}^1$). Assume that X is given as the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$. There exists two affine patches $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ covering X. The partitions associated to the Newton polyhedron of X on each patch are given as three dimensional partitions with $\mu_1 = (1), \mu_2 = (0), \mu_3 = (0)$ [27] (Example 4.9). We compute the vertex associated to the moduli stack of highly frozen triples of rank=2. The following picture describes the fibers of X over 0 and ∞ on the base $\mathbb{P}^1$. The hyperplane H is given as a fixed choice of equivariant structure.



Let $\mathcal{U}_\alpha$,$\mathcal{U}_\beta$ denote affine open patches over the divisors $0, \infty$ on the base $\mathbb{P}^1$ respectively. In order to obtain the Poincaré polynomial of F one needs a fixed choice of equivariant structure. Let $\mathbb{C}^*$

act on $\mathbb{C}^4$ by

$$t(x_0, x_1, x_2, x_3) = (tx_0, tx_1, t^{-1}x_2, t^{-1}x_3).$$

We identify X as a quotient $X \cong (\mathbb{C}^4 \backslash Z)/\mathbb{C}^*$ where $Z \subset \mathbb{C}^4$ is obtained by setting $x_0 = x_1 = 0$. Let $([x_0 : x_1], x_2, x_3)$ denote the coordiantes in X where $[x_0 : x_1]$ denote the homogeneous coordinates along the base $\mathbb{P}^1$ and x_2, x_3 denote the fiber coordinates. Locally in the $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ patches the defining coordinates are given as $(\frac{x_1}{x_0}, x_2x_0, x_3x_0)$ and $(\frac{x_0}{x_1}, x_2x_1, x_3x_1)$ respectively. Consider the $\mathcal{U}_\alpha$ patch. Let us denote the local coordinates in this patch by $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ where $\tilde{x}_1 = \frac{x_1}{x_0}, \tilde{x}_2 = x_2x_0, \tilde{x}_3 = x_3x_0$. Let $H \subset X$ denote the hyperplane obtained as the fiber of X over $0 \in \mathbb{P}^1$, i.e locally in $\mathcal{U}_\alpha$ by setting $\tilde{x}_1 = 0$. Now consider the action of $\mathbf{T} = \mathbb{C}^3$ on X where locally over $\mathcal{U}_\alpha$ is given by

$$(\lambda_1, \lambda_2, \lambda_3) \cdot \tilde{x}_i = \lambda_i \cdot \tilde{x}_i.$$

We identify an action of $(\mathbb{C}^*)^2$ on X which preserves the Calabi-Yau form by considering a subtorus $\mathrm{T}' \subset \mathbf{T}$ such that

$$\mathrm{T}' = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{T} \mid \lambda_1\lambda_2\lambda_3 = 1\}. \tag{11.2}$$

Let $\tilde{t}_1, \cdots, \tilde{t}_3$ denote the characters corresponding to the action of λ_i. Identify $\mathcal{O}_X(-1) \cong \mathcal{O}_X(-H)$. Locally over $\mathcal{U}_\alpha$ the Poincaré polynomial of $\mathcal{O}_X(-n) \mid_{\mathcal{U}_\alpha}$ is obtained as,

$$\frac{\tilde{t}_1^n}{(1-\tilde{t}_2)(1-\tilde{t}_2)(1-\tilde{t}_3)}.$$

Restriction to the affine open patch β is equivalent to the change of local variables,

$$\begin{aligned} \tilde{t}_1 &\mapsto \tilde{t}_1^{-1} \\ \tilde{t}_2 &\mapsto \tilde{t}_2\tilde{t}_1 \\ \tilde{t}_3 &\mapsto \tilde{t}_3\tilde{t}_1, \end{aligned} \tag{11.3}$$

hence the Poincaré polynomial of $\mathcal{O}_X(-n)\mid_{\mathcal{U}_\beta}$ is obtained by:

$$\frac{1}{(1-\tilde{t}_1^{-1})(1-(\tilde{t}_2\tilde{t}_1))(1-(\tilde{t}_3\tilde{t}_1))}.$$

Note that in this case the terms B^n_α and B^n_β in (10.9) are $\tilde{t}_1^n$ and 1 respectively. Finally the **T**-character of the Poincaré polynomial of $\mathcal{O}_X(-n)\mid_{\mathcal{U}_{\alpha\beta}}$ is obtained as,

$$\left(\frac{1}{(1-\tilde{t}_2)(1-\tilde{t}_3)}\right)\delta(\tilde{t}_1),$$

here the term $B^n_{\alpha\beta}$ in (10.10) is equal to 1. The vertex in (10.9) over the two patches α and β is obtained as follows:

$$\begin{aligned}
tr_{R_{-\chi((\mathbb{I}^\bullet)_\alpha,(\mathbb{I}^\bullet)_\alpha)}} &= \mathrm{Ch}(F^{\mathbf{T}}_\alpha)\cdot\frac{(w_1^{-1}+w_2^{-1})}{\tilde{t}_1^n} - \frac{\overline{\mathrm{Ch}(F^{\mathbf{T}}_\alpha)}\cdot(w_1+w_2)\cdot\tilde{t}_1^n}{\tilde{t}_1\tilde{t}_2\tilde{t}_3}\\
&+\mathrm{Ch}(F^{\mathbf{T}}_\alpha)\overline{\mathrm{Ch}(F^{\mathbf{T}}_\alpha)}\frac{(1-\tilde{t}_1)(1-\tilde{t}_2)(1-\tilde{t}_3)}{\tilde{t}_1\tilde{t}_2\tilde{t}_3}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_1)(1-\tilde{t}_2)(1-\tilde{t}_3)}\\
tr_{R_{-\chi((\mathbb{I}^\bullet)_\beta,(\mathbb{I}^\bullet)_\beta)}} &= \mathrm{Ch}(F^{\mathbf{T}}_\beta)\cdot(w_1^{-1}+w_2^{-1}) - \frac{\overline{\mathrm{Ch}(F^{\mathbf{T}}_\beta)}\cdot(w_1+w_2)}{\tilde{t}_1^{-1}(\tilde{t}_2\tilde{t}_1)(\tilde{t}_3\tilde{t}_1)}\\
&+\mathrm{Ch}(F^{\mathbf{T}}_\beta)\overline{\mathrm{Ch}(F^{\mathbf{T}}_\beta)}\frac{(1-\tilde{t}_1^{-1})(1-(\tilde{t}_2\tilde{t}_1))(1-(\tilde{t}_3\tilde{t}_1))}{\tilde{t}_1^{-1}(\tilde{t}_2\tilde{t}_1)(\tilde{t}_3\tilde{t}_1)}\\
&+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_1^{-1})(1-(\tilde{t}_2\tilde{t}_1))(1-(\tilde{t}_3\tilde{t}_1))}.
\end{aligned}\tag{11.4}$$

Similarly the edge character in (10.10) is obtained as follows:

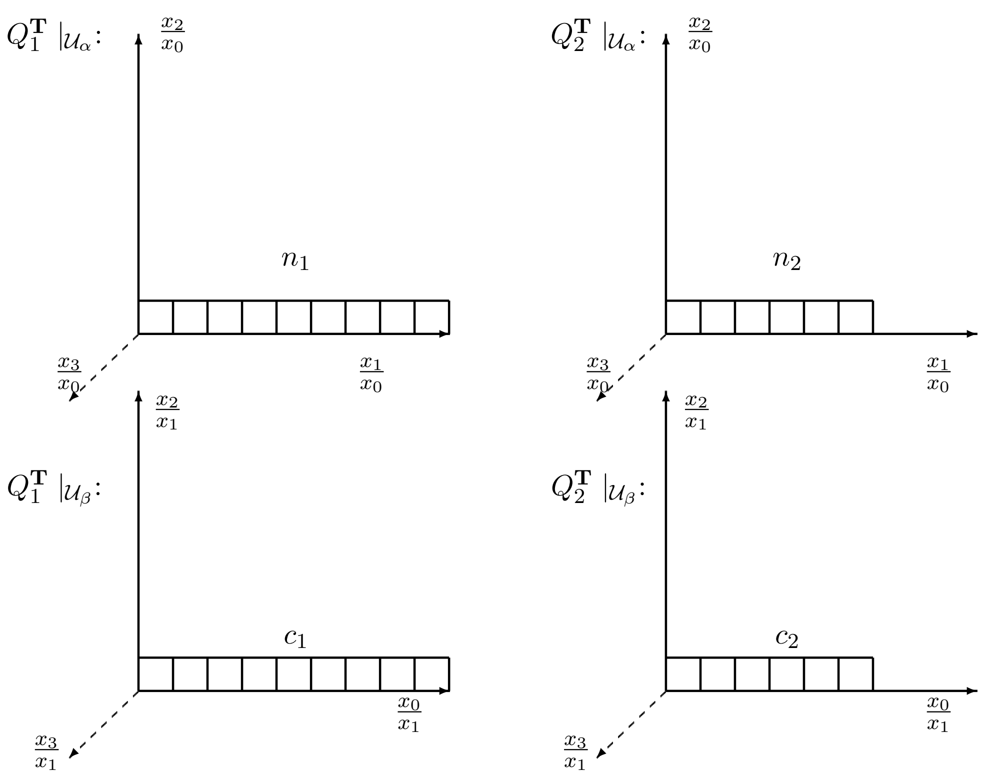
$$\begin{aligned}
tr_{R_{-\chi((\mathbb{I}^\bullet)_{\alpha\beta},(\mathbb{I}^\bullet)_{\alpha\beta})}} &= \Bigg(\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})\cdot(w_1^{-1}+w_2^{-1}) - \frac{\overline{\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})}\cdot(w_1+w_2)}{\tilde{t}_2\tilde{t}_3}+\\
&\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})\overline{\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})}\frac{((1-\tilde{t}_2)(1-\tilde{t}_3)}{\tilde{t}_2\tilde{t}_3}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_2)(1-\tilde{t}_3)}\Bigg)\delta(\tilde{t}_1).
\end{aligned}\tag{11.5}$$

Now we compute $\mathrm{Ch}(F^{\mathbf{T}}_\alpha),\mathrm{Ch}(F^{\mathbf{T}}_\beta),\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\beta})$. Let us for the moment assume that the **G**-fixed locus of the moduli stack is composed of only one component. Later we extend our computation to the more general case. Let $Q^{\mathbf{G}}$ denote the cokernel $\mathrm{Coker}(\phi^{\mathbf{G}})$ associated to $\mathcal{O}_X(-n)^{\oplus 2,\mathbf{G}} \xrightarrow{\phi^{\mathbf{G}}} F^{\mathbf{G}}$. Let $L(Q^{\mathbf{G}}) = k$ be the length of $Q^{\mathbf{G}}$. Suppose $L(Q^{\mathbf{G}} \mid_{\mathcal{U}_\alpha}) = k_1$ and $L(Q^{\mathbf{G}} \mid_{\mathcal{U}_\beta}) = k_2$ hence $k_1 + k_2 = k$. Now use the fact that by construction $Q^{\mathbf{G}} \cong Q_1^{\mathbf{T}} \oplus Q_2^{\mathbf{T}}$. Let $L(Q_1^{\mathbf{T}} \mid_{\mathcal{U}_\alpha}) = n_1$ and $L(Q_2^{\mathbf{T}} \mid_{\mathcal{U}_\alpha}) = n_2$. Moreover assume $L(Q_1^{\mathbf{T}} \mid_{\mathcal{U}_\beta}) = c_1$ and $L(Q_2^{\mathbf{T}} \mid_{\mathcal{U}_\beta}) = c_2$. So this means that we have the constraint that

$$n_1 + n_2 = k_1 \quad \text{and} \quad c_1 + c_2 = k_2.$$

It is seen that the box contribution associated to $Q^{\mathbf{G}}$ is obtained by considering the box contributions associated to $Q_1^{\mathbf{T}}$ and $Q_2^{\mathbf{T}}$ as follows:

$$\begin{aligned} \mathrm{Ch}(F_\alpha^{\mathbf{G}}) &= \frac{\tilde{t}_1^{-n_1}}{(1-\tilde{t}_1)} + \frac{\tilde{t}_1^{-n_2}}{(1-\tilde{t}_1)} \\ \mathrm{Ch}(F_\beta^{\mathbf{G}}) &= \frac{\tilde{t}_1^{c_1}}{(1-\tilde{t}_1^{-1})} + \frac{\tilde{t}_1^{c_2}}{(1-\tilde{t}_1^{-1})} \end{aligned} \tag{11.6}$$



Now we consider the case that the **G**-fixed locus of the moduli stack contains more than one component. In this case to compute the contribution of box configurations one needs to consider all possible tuples of six integers $(k_1, k_2, n_1, n_2, c_1, c_2)$ such that for a fixed value of k the following three relations are satisfied:

$$n_1 + n_2 = k_1, \quad c_1 + c_2 = k_2 \quad \text{and} \quad k_1 + k_2 = k.$$

Hence we obtain the following identities:

$$\begin{aligned}
\mathrm{Ch}(F_{\alpha}^{\mathbf{G}}) &= \sum_{n_1+n_2=k_1} \left(\frac{\tilde{t}_1^{-n_1}}{(1-\tilde{t}_1)} + \frac{\tilde{t}_1^{-n_2}}{(1-\tilde{t}_1)} \right) \\
\mathrm{Ch}(F_{\beta}^{\mathbf{G}}) &= \sum_{c_1+c_2=k_2} \left(\frac{\tilde{t}_1^{c_1}}{(1-\tilde{t}_1^{-1})} + \frac{\tilde{t}_1^{c_2}}{(1-\tilde{t}_1^{-1})} \right) \\
&\text{for all } \mathrm{k}_1, \mathrm{k}_2 \text{ such that } \mathrm{k}_1 + \mathrm{k}_2 = \mathrm{k}\,.
\end{aligned} \tag{11.7}$$

Moreover over the **G**-character of $F^{\mathbf{G}}$ restricted to $\mathcal{U}_{\alpha\beta}$ is given by $\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{G}}) = 2 \cdot \delta(t_1)$. Hence the vertex character over $\mathcal{U}_{\alpha}$ is obtained as:

$$\begin{aligned}
tr_{R-\chi(\mathbb{I}_{\alpha}^{\bullet},\mathbb{I}_{\alpha}^{\bullet})} &= (w_1^{-1} + w_2^{-1}) \cdot \sum_{n_1+n_2=k_1} \left(\frac{\tilde{t}_1^{-n_1-n}}{(1-\tilde{t}_1)} + \frac{\tilde{t}_1^{-n_2-n}}{(1-\tilde{t}_1)} \right) \\
&- (w_1 + w_2) \cdot \frac{1}{\tilde{t}_1\tilde{t}_2\tilde{t}_3} \sum_{n_1+n_2=k_1} \left(\frac{\tilde{t}_1^{n+n_1}}{(1-\tilde{t}_1^{-1})} + \frac{\tilde{t}_1^{n+n_2}}{(1-\tilde{t}_1^{-1})} \right) \\
&- \sum_{\substack{n_1+n_2=k_1 \\ m_1+m_2=k_1}} \left[\frac{\tilde{t}_1^{m_1-n_1+1}}{(1-\tilde{t}_1)^2} + \frac{\tilde{t}_1^{m_2-n_1+1}}{(1-\tilde{t}_1)^2} + \frac{\tilde{t}_1^{m_1-n_2+1}}{(1-\tilde{t}_1)^2} + \frac{\tilde{t}_1^{m_2-n_2+1}}{(1-\tilde{t}_1)^2} \right] \\
&\cdot \frac{(1-\tilde{t}_1)(1-\tilde{t}_2)(1-\tilde{t}_3)}{\tilde{t}_1\tilde{t}_2\tilde{t}_3} + \frac{1 - \frac{(w_1+w_2)^2}{w_1 w_2}}{(1-\tilde{t}_1)(1-\tilde{t}_2)(1-\tilde{t}_3)}.
\end{aligned} \tag{11.8}$$

The vertex character over $\mathcal{U}_\beta$ is obtained as:

$$
\begin{aligned}
& tr_{R-\chi(\mathbb{I}^\bullet_\beta,\mathbb{I}^\bullet_\beta)} = (w_1^{-1}+w_2^{-1})\cdot\sum_{c_1+c_2=k_2}\left(\frac{\tilde{t}_1^{c_1}}{(1-\tilde{t}_1^{-1})}+\frac{\tilde{t}_1^{c_2}}{(1-\tilde{t}_1^{-1})}\right)\\
& -(w_1+w_2)\cdot\frac{1}{\tilde{t}_1^{-1}(\tilde{t}_2\tilde{t}_1)(\tilde{t}_3\tilde{t}_1)}\sum_{c_1+c_2=k_2}\left(\frac{\tilde{t}_1^{-c_1}}{(1-\tilde{t}_1)}+\frac{\tilde{t}_1^{-c_2}}{(1-\tilde{t}_1)}\right)\\
& -\sum_{\substack{c_1+c_2=k_2\\ m_1+m_2=k_2}}\left[\frac{\tilde{t}_1^{c_1-m_1+1}}{(1-\tilde{t}_1)^2}+\frac{\tilde{t}_1^{c_1-m_2+1}}{(1-\tilde{t}_1)^2}+\frac{\tilde{t}_1^{c_2-m_1+1}}{(1-\tilde{t}_1)^2}+\frac{\tilde{t}_1^{c_2-m_2+1}}{(1-\tilde{t}_1)^2}\right]\\
& \cdot\frac{(1-\tilde{t}_1^{-1})(1-(\tilde{t}_2\tilde{t}_1))(1-(\tilde{t}_3\tilde{t}_1))}{\tilde{t}_1^{-1}(\tilde{t}_2\tilde{t}_1)(\tilde{t}_3\tilde{t}_1)}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_1^{-1})(1-(\tilde{t}_2\tilde{t}_1))(1-(\tilde{t}_3\tilde{t}_1))}.
\end{aligned}
\tag{11.9}
$$

The edge character over $\mathcal{U}_{\alpha\beta}$ is obtained as:

$$
tr_{R-\chi(\mathbb{I}^\bullet_{\alpha\beta},\mathbb{I}^\bullet_{\alpha\beta})} = \left(2(w_1^{-1}+w_2^{-1})-\frac{2\cdot(w_1+w_2)}{\tilde{t}_2\tilde{t}_3}+2\cdot\frac{(1-\tilde{t}_2)(1-\tilde{t}_3)}{\tilde{t}_2\tilde{t}_3}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_2)(1-\tilde{t}_3)}\right)\delta(\tilde{t}_1).
\tag{11.10}
$$

The $\mathbf{G}$-character of the virtual tangent space in (10.13) is obtained by the following equation:

$$
tr_{R-\chi(I^\bullet,I^\bullet)} = tr_{R-\chi(\mathbb{I}^\bullet_\alpha,\mathbb{I}^\bullet_\alpha)}+tr_{R-\chi(\mathbb{I}^\bullet_\beta,\mathbb{I}^\bullet_\beta)}-tr_{R-\chi(\mathbb{I}^\bullet_{\alpha\beta},\mathbb{I}^\bullet_{\alpha\beta})}
\tag{11.11}
$$

The computation of the right hand side of (11.11) would immediately become complicated for large values of k, hence we only do the calculations for $k=1$. Let $\mathbf{Q}^1$ denote the $\mathbf{G}$-fixed component of the moduli stack of rank 2 highly frozen triples over which the highly frozen triples $\mathcal{O}_X(-n)^{\oplus 2,\mathbf{G}} \xrightarrow{\phi} F^{\mathbf{G}}$ satisfy the condition that $L(\mathrm{Coker}(\phi)^{\mathbf{G}})=1$. By definition of the equivariant vertex in Remark 10.1 the coefficient of the degree 1 term in the vertex is obtained by the integral of the evaluation of the contribution of $V_{\mathbf{Q}^1}$ on $\mathbf{Q}^1$, i.e:

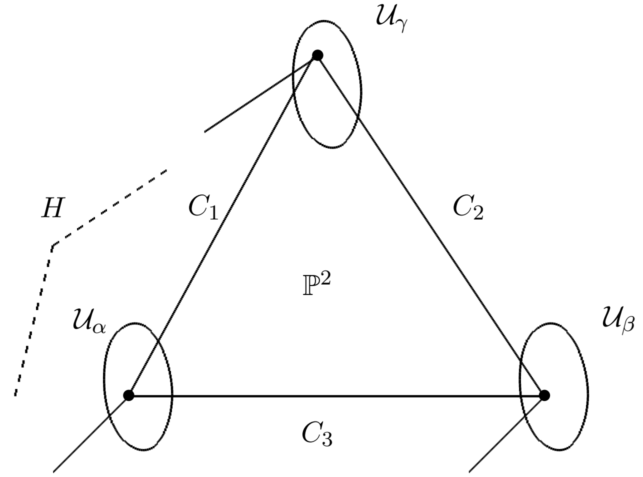
$$
w(\mathbf{Q}^1)=\int_{\mathbf{Q}^1}e(\mathrm{T}_{\mathbf{Q}^1})e(-V_{\mathbf{Q}^1})=\frac{(\mathrm{s}_1)(-n\mathrm{s}_1+\mathrm{s}_2+\mathrm{s}_3)(\mathrm{s}_1+n\mathrm{s}_1)}{(\mathrm{s}_2+\mathrm{s}_3)^3},
\tag{11.12}
$$

where s_i denote the equivariant characters corresponding to $\tilde{t}_i$, for similar discussions look at [27]

(Section 4.7) as well as the calculation in [27] (Lemma 5). Here we omit the calculations required for obtaining the right hand side of (11.12) however, we point out that the right hand side of (11.12) contains no contribution of w_i characters firstly because of so many cancelations involving the terms whose coefficients are given by polynomials in characters w_i and secondly because some of such terms are coupled with non-homogeneous polynomials of degree 1 in t_i (for example $w_1w_2+1+t_i$ for $i = 1, 2, 3$) whose **G**-equivariant Euler class vanishes since $e(w_1w_2+1+t_i) = (d_1+d_2)\cdot e(1)\cdot e(t_i) = 0$ (here $d_i = c_1(L_{w_i}) \in A^*_{\mathbf{G}}$ where L_{w_i} are the line bundles associated to the characters w_i). By the definition of the Calabi-Yau torus T' in (11.2), s_i satisfy the property that $\mathrm{s}_1 + \mathrm{s}_2 + \mathrm{s}_3 = 0$. Hence:

$$w(\mathbf{Q}^1) = (n+1)^2. \tag{11.13}$$

Example 11.3. (**Local** $\mathbb{P}^2$). This means that X is given as the total space of $\mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathbb{P}^2$. In this case there exists 3 torus fixed curves in the base $\mathbb{P}^2$ each of which are isomorphic to $\mathbb{P}^1$ and they intersect over the 3 torus fixed points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. Let $\mathcal{U}_\alpha$,$\mathcal{U}_\beta$,$\mathcal{U}_\gamma$, denote the affine open patches respectively. The outgoing partitions associated to the Newton polyhedron of X on each patch are given as three dimensional partitions with $\mu_1 = (1), \mu_2 = (1), \mu_3 = (0)$, look at [27] (Section 2.3) for more detail about partitions. To avoid unnecessary repetitions, we only provide the formula to compute the vertex, the interested reader can pursue the computations on his/her own.



Let $\mathbb{C}^*$ acts on $\mathbb{C}^4$ by

$$t(x_0, x_1, x_2, x_3) = (tx_0, tx_1, tx_2, t^{-3}x_3).$$

We identify X as a quotient $X \cong (\mathbb{C}^4 \backslash Z)/\mathbb{C}^*$ where $Z \subset \mathbb{C}^4$ is defined by setting $x_0 = x_1 = x_2 = 0$. Let $([x_0 : x_1 : x_2], x_3)$ denote the coordiantes in X where $[x_0 : x_1 : x_2]$ denote the homogeneous coordinates over the base $\mathbb{P}^2$ and x_3 denotes the fiber coordinate. Locally in the $\mathcal{U}_\alpha$, $\mathcal{U}_\beta$ and $\mathcal{U}_\gamma$ patches the defining coordinates are given as $(\frac{x_1}{x_0}, \frac{x_2}{x_0}, x_3x_0^3)$ and $(\frac{x_0}{x_1}, \frac{x_2}{x_1}, x_3x_1^3)$ and $(\frac{x_0}{x_2}, \frac{x_1}{x_2}, x_3x_2^3)$ respectively. Consider the $\mathcal{U}_\alpha$ patch. Let us denote the local coordinates in this patch by $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ where $\tilde{x}_1 = \frac{x_1}{x_0}, \tilde{x}_2 = \frac{x_2}{x_0}, \tilde{x}_3 = x_3x_0^3$. Let $H \subset X$ denote the hyperplane obtained by $\tilde{x}_1 = 0$. Now consider the action of $\mathbf{T} = \mathbb{C}^3$ on X where locally over $\mathcal{U}_\alpha$ is given by

$$(\lambda_1, \lambda_2, \lambda_3) \cdot \tilde{x}_i = \lambda_i \cdot \tilde{x}_i.$$

Here again we identify an action of $(\mathbb{C}^*)^2$ on X which preserves the Calabi-Yau form as in (11.2). Let $\tilde{t}_1, \cdots, \tilde{t}_3$ denote the characters corresponding to the action of λ_i. Identify $\mathcal{O}_X(-1) \cong \mathcal{O}_X(-H)$. Locally over $\mathcal{U}_\alpha$ the Poincaré polynomial of $\mathcal{O}_X(-n) \mid_{\mathcal{U}_\alpha}$ is obtained as,

$$\frac{\tilde{t}_1^n}{(1-\tilde{t}_2)(1-\tilde{t}_2)(1-\tilde{t}_3)}.$$

Restriction to the affine open patch β is equivalent to change of local variables,

$$\begin{aligned}\tilde{t}_1 &\mapsto \tilde{t}_1^{-1}\\ \tilde{t}_2 &\mapsto \frac{\tilde{t}_2}{\tilde{t}_1}\\ \tilde{t}_3 &\mapsto \tilde{t}_3\tilde{t}_1^3,\end{aligned} \tag{11.14}$$

hence the Poincaré polynomial of $\mathcal{O}_X(-n) \mid_{\mathcal{U}_\beta}$ is obtained by:

$$\frac{1}{(1-\tilde{t}_1^{-1})(1-(\frac{\tilde{t}_2}{\tilde{t}_1}))(1-(\tilde{t}_3\tilde{t}_1^3))}.$$

Restriction over the affine patch $\mathcal{U}_\gamma$ is equivalent to the change of local variables:

$$\tilde{t}_1 \mapsto \frac{\tilde{t}_1}{\tilde{t}_2}$$
$$\tilde{t}_2 \mapsto \tilde{t}_2^{-1}$$
$$\tilde{t}_3 \mapsto \tilde{t}_3\tilde{t}_2^3. \tag{11.15}$$

Note that unlike the case for local $\mathbb{P}^1$, $\mathcal{O}_X(-H)$ has nontrivial module structure over $\mathcal{U}_\alpha$ and $\mathcal{U}_\gamma$, hence the Poincaré polynomial of $\mathcal{O}_X(-n) \mid_{\mathcal{U}_\gamma}$ is obtained by:

$$\frac{(\frac{\tilde{t}_1}{\tilde{t}_2})^n}{(1-(\frac{\tilde{t}_1}{\tilde{t}_2}))(1-\tilde{t}_2^{-1})(1-(\tilde{t}_3\tilde{t}_2^3))}.$$

The terms B^n_α, B^n_β and B^n_γ in (10.9) are $\tilde{t}_1^n$, 1 and $\frac{\tilde{t}_1^n}{\tilde{t}_2^n}$ respectively. The **T**-character of the Poincaré polynomial of $\mathcal{O}_X(-n) \mid_{\mathcal{U}_{\alpha\beta}}$, $\mathcal{O}_X(-n) \mid_{\mathcal{U}_{\alpha\gamma}}$ and $\mathcal{O}_X(-n) \mid_{\mathcal{U}_{\beta\gamma}}$ are obtained respectively as:

$$\left(\frac{1}{(1-\tilde{t}_2)(1-\tilde{t}_3)}\right)\delta(\tilde{t}_1)$$
$$\left(\frac{\tilde{t}_3^n}{(1-\tilde{t}_1)(1-\tilde{t}_3)}\right)\delta(\tilde{t}_2)$$
$$\left(\frac{1}{(1-\tilde{t}_1^{-1})(1-\tilde{t}_3\tilde{t}_1^3)}\right)\delta(\frac{\tilde{t}_2}{\tilde{t}_1}). \tag{11.16}$$

The terms $B^n_{\alpha\beta}$, $B^n_{\alpha\gamma}$, $B^n_{\beta\gamma}$ in (10.10) are equal to 1, $\tilde{t}_3^n$ and 1 respectively.

The vertex in (10.9) over the two patches α and β is obtained as follows:

$$
\begin{aligned}
tr_{R-_{\chi((\mathbb{I}^\bullet)_\alpha,(\mathbb{I}^\bullet)_\alpha)}} &= \mathrm{Ch}(F_\alpha^{\mathbf{T}})\cdot\frac{(w_1^{-1}+w_2^{-1})}{\tilde{t}_1^n}-\frac{\overline{\mathrm{Ch}(F_\alpha^{\mathbf{T}})}\cdot(w_1+w_2)\tilde{t}_1^n}{\tilde{t}_1\tilde{t}_2\tilde{t}_3}\\
&+\mathrm{Ch}(F_\alpha^{\mathbf{T}})\overline{\mathrm{Ch}(F_\alpha^{\mathbf{T}})}\cdot\frac{(1-\tilde{t}_1)(1-\tilde{t}_2)(1-\tilde{t}_3)}{\tilde{t}_1\tilde{t}_2\tilde{t}_3}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_1)(1-\tilde{t}_2)(1-\tilde{t}_3)}\\
tr_{R-_{\chi((\mathbb{I}^\bullet)_\beta,(\mathbb{I}^\bullet)_\beta)}} &= \mathrm{Ch}(F_\beta^{\mathbf{T}})\cdot(w_1^{-1}+w_2^{-1})-\frac{\overline{\mathrm{Ch}(F_\beta^{\mathbf{T}})}\cdot(w_1+w_2)}{\tilde{t}_1^{-1}(\frac{\tilde{t}_2}{\tilde{t}_1})(\tilde{t}_3\tilde{t}_1^3)}\\
&+\mathrm{Ch}(F_\beta^{\mathbf{T}})\overline{\mathrm{Ch}(F_\beta^{\mathbf{T}})}\cdot\frac{(1-\tilde{t}_1^{-1})(1-(\frac{\tilde{t}_2}{\tilde{t}_1}))(1-(\tilde{t}_3\tilde{t}_1^3))}{\tilde{t}_1^{-1}(\frac{\tilde{t}_2}{\tilde{t}_1})(\tilde{t}_3\tilde{t}_1^3)}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_1^{-1})(1-(\frac{\tilde{t}_2}{\tilde{t}_1}))(1-(\tilde{t}_3\tilde{t}_1^3))}.
\end{aligned}
\tag{11.17}
$$

Moreover:

$$
\begin{aligned}
tr_{R-_{\chi((\mathbb{I}^\bullet)_\gamma,(\mathbb{I}^\bullet)_\gamma)}} &= \mathrm{Ch}(F_\alpha^{\mathbf{T}})\cdot\frac{(w_1^{-1}+w_2^{-1})\cdot\tilde{t}_2^n}{\tilde{t}_1^n}-\frac{\overline{\mathrm{Ch}(F_\gamma^{\mathbf{T}})}\cdot(w_1+w_2)\tilde{t}_1^{n-1}}{\tilde{t}_2^{n+1}\tilde{t}_3}\\
&+\mathrm{Ch}(F_\gamma^{\mathbf{T}})\overline{\mathrm{Ch}(F_\gamma^{\mathbf{T}})}\cdot\frac{(1-\frac{\tilde{t}_1}{\tilde{t}_2})(1-\tilde{t}_2^{-1})(1-\tilde{t}_3\tilde{t}_2^3)}{\tilde{t}_1\tilde{t}_2\tilde{t}_3}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_1\tilde{t}_2^{-1})(1-\tilde{t}_2^{-1})(1-\tilde{t}_3\tilde{t}_2^3)}.
\end{aligned}
\tag{11.18}
$$

Similarly the edge characters over $\mathcal{U}_{\alpha\beta},\mathcal{U}_{\alpha\gamma},\mathcal{U}_{\beta\gamma}$ are obtained as follows:

$$
\begin{aligned}
tr_{R-_{\chi((\mathbb{I}^\bullet)_{\alpha\beta},(\mathbb{I}^\bullet)_{\alpha\beta})}} &= \Bigg(\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{T}})\cdot(w_1^{-1}+w_2^{-1})-\frac{\overline{\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{T}})}\cdot(w_1+w_2)}{\tilde{t}_2\tilde{t}_3}+\\
&\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{T}})\overline{\mathrm{Ch}(F_{\alpha\beta}^{\mathbf{T}})}\cdot\frac{(1-\tilde{t}_2)(1-\tilde{t}_3)}{\tilde{t}_2\tilde{t}_3}+\frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_2)(1-\tilde{t}_3)}\Bigg)\delta(\tilde{t}_1).
\end{aligned}
\tag{11.19}
$$

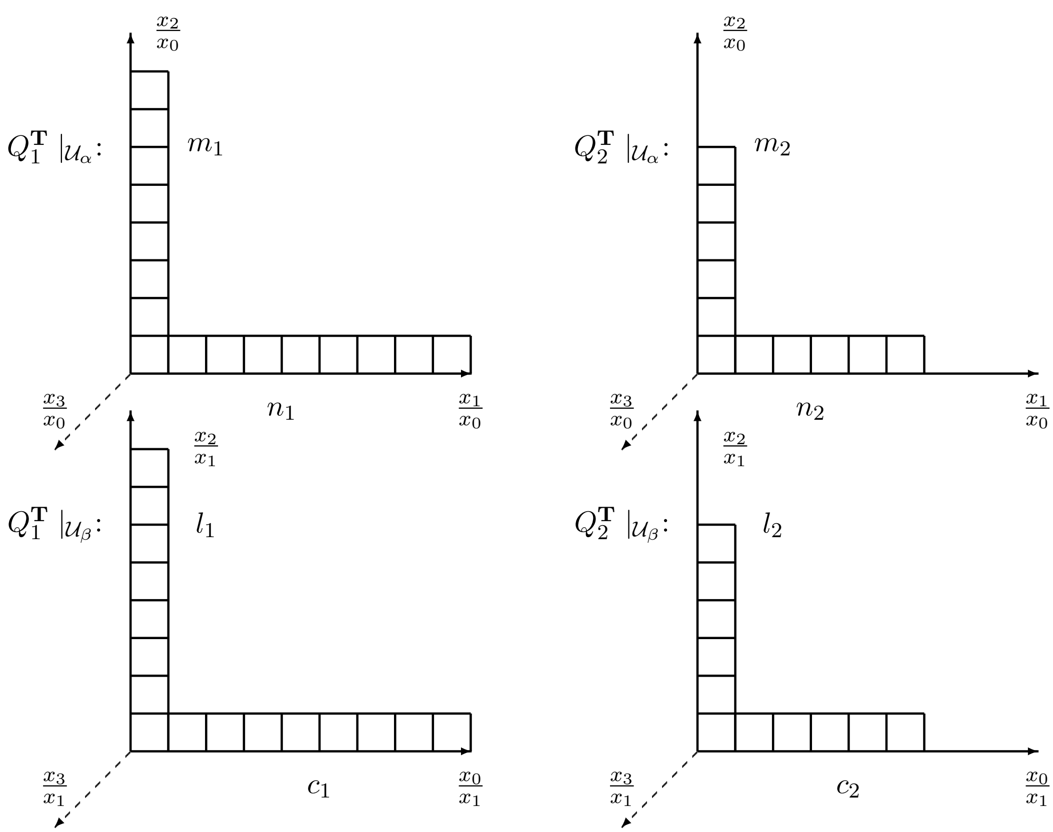
$$tr_{R_{-\chi((\mathbb{I}^\bullet)_{\alpha\gamma},(\mathbb{I}^\bullet)_{\alpha\gamma})}} = \Bigg(\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\gamma}) \cdot (w_1^{-1} + w_2^{-1}) \cdot \tilde{t}_3^{-n} - \frac{\overline{\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\gamma})} \cdot (w_1 + w_2) \cdot \tilde{t}_3^{n}}{\tilde{t}_1 \tilde{t}_3} +$$

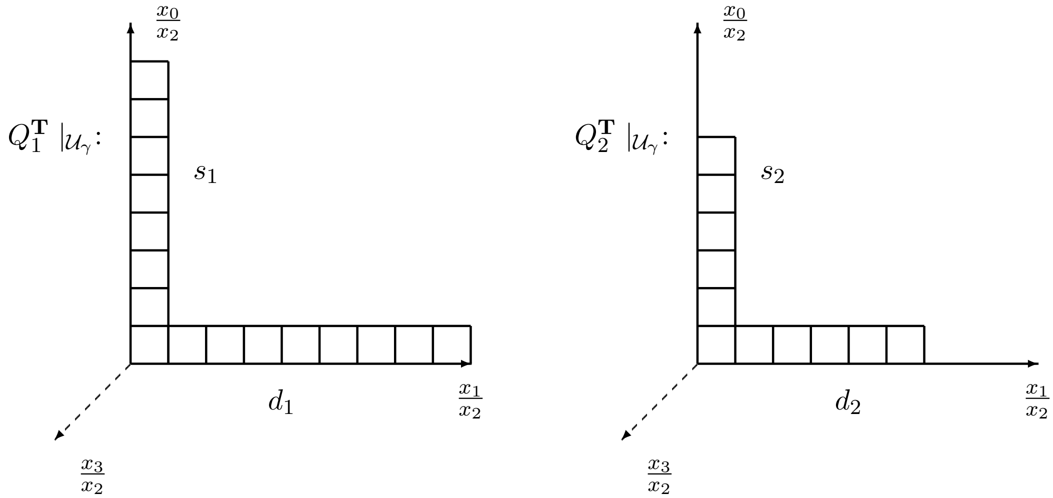
$$\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\gamma})\overline{\mathrm{Ch}(F^{\mathbf{T}}_{\alpha\gamma})} \cdot \frac{(1-\tilde{t}_1)(1-\tilde{t}_3)}{\tilde{t}_1\tilde{t}_3} + \frac{1 - \frac{(w_1+w_2)^2}{w_1 w_2}}{(1-\tilde{t}_1)(1-\tilde{t}_3)} \Bigg) \delta(\tilde{t}_2). \tag{11.20}$$

$$tr_{R_{-\chi((\mathbb{I}^\bullet)_{\beta\gamma},(\mathbb{I}^\bullet)_{\beta\gamma})}} = \Bigg(\mathrm{Ch}(F^{\mathbf{T}}_{\beta\gamma}) \cdot (w_1^{-1} + w_2^{-1}) - \frac{\overline{\mathrm{Ch}(F^{\mathbf{T}}_{\beta\gamma})} \cdot (w_1 + w_2)}{\tilde{t}_1^2 \tilde{t}_3} +$$

$$\mathrm{Ch}(F^{\mathbf{T}}_{\beta\gamma})\overline{\mathrm{Ch}(F^{\mathbf{T}}_{\beta\gamma})} \cdot \frac{(1-\tilde{t}_1^{-1})(1-\tilde{t}_3\tilde{t}_1^3)}{\tilde{t}_1^2\tilde{t}_3} + \frac{1 - \frac{(w_1+w_2)^2}{w_1 w_2}}{(1-\tilde{t}_1^{-1})(1-\tilde{t}_3\tilde{t}_1^3)} \Bigg) \delta(\frac{\tilde{t}_2}{\tilde{t}_1}). \tag{11.21}$$

Given the characters corresponding to $F^{\mathbf{T}}$ over each patch, by substitution in equation (10.13), one computes the vertex for arbitrary length k contributions. Note the difference in the box configurations in this case, in each patch the box configuration is given by the Young diagrams below.





Here the condition for $L(Q) = k$ (for fixed k) is given by:

$$m_1+m_2+n_1+n_2-2 = k_1, \quad l_1+l_2+c_1+c_2-2 = k_2 \quad , s_1+s_2+d_1+d_2-2 = k_3 \quad \text{and} \quad k_1+k_2+k_3 = k.$$

Chapter 12

Background on Wallcrossing

Joyce and Song in [18] compute the invariants of rank 1 frozen triples using the method of wall crossing. The general philosophy is to exploit the existence of an auxiliary category $\mathcal{B}_p$ whose objects consist of frozen triples (E, F, ϕ) but this time F, instead of only being pure and one dimensional, needs to be semistable with fixed reduced Hilbert polynomial equal to p. In this analysis, the authors classify the objects in $\mathcal{B}_p$ based on their numerical class (β, r), where β denotes the Chern character of F and r denotes the rank of E.

Strategy:

The key strategy is to define two suitable weak stability conditions $\tau^\bullet$ and $\tilde{\tau}$ for the objects of the category $\mathcal{B}_p$. The $\tilde{\tau}$-stable objects in $\mathcal{B}_p$ are given by objects closely related to the stable frozen triples and naively, (on the other side of the wall), the $\tau^\bullet$-stable objects in $\mathcal{B}_p$ are given by simpler objects such as Giseker semistable sheaves. Changing the weak stability condition, from $\tau^\bullet$ to $\tilde{\tau}$ and using the machinery of the Ringel-Hall algebra of stack functions discussed in [18], provides one with a wall-crossing identity in $\mathcal{B}_p$. Eventually one relates the weighted Euler characteristic of the moduli stack of $\tilde{\tau}$-(semi)stable objects to the weighted Euler characteristic of the moduli stack of $\tau^\bullet$-stable objects, which contains the Giseker (semi)stable sheaves. In the remaining chapters we discuss the computation of invariants of $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ with numerical class $(\beta, 2)$, following the Kontsevich-Soibelman [22] and Joyce-Song [18] wallcrossing machinery.

12.1 Preliminary Definitions

Definition 12.1. (Joyce and Song) [18] (Definition 13.1). Let X be a Calabi-Yau threefold equipped with ample line bundle $\mathcal{O}_X(1)$. Let τ denote the Giseker stability on the abelian category of coherent sheaves supported over X. Define $\mathcal{A}_p$ to be the sub-category of coherent sheaves whose objects are zero sheaves and non-zero τ-semistable sheaves with reduced Hilbert polynomial p.

Definition 12.2. (Joyce and Song) [18]. Define category $\mathcal{B}_p$ to be the category whose objects are triples (F, V, ϕ), where $F \in Obj(\mathcal{A}_p)$, V is a finite dimensional $\mathbb{C}$-vector space, and $\phi : V \to \text{Hom}(\mathcal{O}_X(-n), F)$ is a $\mathbb{C}$-linear map. Given (F, V, ϕ) and $(\acute{F}, \acute{V}, \acute{\phi})$ in $\mathcal{B}_p$ define morphisms $(F, V, \phi) \to (\acute{F}, \acute{V}, \acute{\phi})$ in $\mathcal{B}_p$ to be pairs of morphisms (f, g) where $f : F \to \acute{F}$ is a morphism in $\mathcal{A}_p$ and $g : V \to \acute{V}$ is a $\mathbb{C}$-linear map, such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & \text{Hom}(\mathcal{O}_X(-n), F) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \acute{V} & \xrightarrow{\acute{\phi}} & \text{Hom}(\mathcal{O}_X(-n), \acute{F}) \end{array} \tag{12.1}$$

Now we define the numerical class of objects in $\mathcal{B}_p$:

Definition 12.3. Let $\mathcal{A}$ denote any abelian category. Let $\mathcal{K}_0(\mathcal{A})$ denote the Grothendieck group of $\mathcal{A}$ generated by isomorphism classes $[E]$ of objects E in $\mathcal{A}$ which satisfy the relation $[E] = [F] + [G]$ if there exists a short exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$. The Euler form $\bar{\chi} : \mathcal{K}_0(\mathcal{A}) \times \mathcal{K}_0(\mathcal{A}) \to \mathbb{Z}$ is defined as:

$$\bar{\chi}([E], [F]) = \sum_{i \geq 0} (-1)^i \dim Ext^i(E, F).$$

Definition 12.4. Let $I = \{\alpha \in \mathcal{K}_0(\mathcal{A}) \mid \bar{\chi}(\alpha, \beta) = \bar{\chi}(\beta, \alpha) = 0, \forall \beta \in \mathcal{K}_0(\mathcal{A})\}$. Define the numerical Grothendieck group of $\mathcal{A}$ to be the quotient of $\mathcal{K}_0(\mathcal{A})$ by the two sided kernel of $\bar{\chi}$, i.e

$\mathcal{K}^{num}(\mathcal{A}) = \mathcal{K}_0(\mathcal{A})/I$. Define the positive cone of $\mathcal{A}$, $C(\mathcal{A}) \subset \mathcal{K}(\mathcal{A}) = \mathcal{K}^{num}(\mathcal{A})$ to be

$$C(\mathcal{A}) = \{[E] \in K(\mathcal{A}) \mid E \ncong 0, E \in \mathcal{A}\}. \tag{12.2}$$

The above definitions extend to the case where the abelian category is $\mathcal{A}_p$:

Definition 12.5. Define $\mathcal{K}(\mathcal{B}_p) = \mathcal{K}(\mathcal{A}_p) \oplus \mathbb{Z}$ where for $(F, V, \phi) \in \mathcal{B}_p$, $[(F, V, \phi)] = ([F], \dim(V))$.

We state following results by Joyce and Song without proof.

Lemma 12.6. *(Joyce and Song) [18] (Lemma 13.2). The category $\mathcal{B}_p$ is abelian and $\mathcal{B}_p$ satisfies the condition that for the underlying sheaves F the following is true:*

If $[F] = 0 \in \mathcal{K}^{num}(\mathcal{A}_p)$ then $F \cong 0$, moreover $\mathcal{B}_p$ is noetherian and artinian and the moduli stacks $\mathfrak{M}^{(\beta,d)}_{\mathcal{B}_p}$ are of finite type $\forall(\beta, d) \in C(\mathcal{B}_p)$.

Remark 12.7. The category $\mathcal{A}_p$ embeds as a full and faithful sub-category in $\mathcal{B}_p$ by $F \to (F, 0, 0)$, moreover it is shown by Joyce and Song in [18] that every object (F, V, ϕ) sits in a short exact sequence.

$$0 \to (F, 0, 0) \to (F, V, \phi) \to (0, V, 0) \to 0 \tag{12.3}$$

Definition 12.8. (Joyce and Song) [18]. Define $\bar{\chi}^{\mathcal{B}_p} : \mathcal{K}^{num}(\mathcal{B}_p) \times \mathcal{K}^{num}(\mathcal{B}_p) \to \mathbb{Z}$ by:

$$\begin{aligned} &\bar{\chi}^{\mathcal{B}_p}((\beta, d), (\gamma, e)) = \bar{\chi}(\beta - d[\mathcal{O}_X(-n)], \gamma - e[\mathcal{O}_X(-n)]) \\ &= \bar{\chi}(\beta, \gamma) - d\bar{\chi}([\mathcal{O}_X(-n)], \gamma) + e\bar{\chi}([\mathcal{O}_X(-n)], \beta) \end{aligned} \tag{12.4}$$

Definition 12.9. (Joyce and Song) [18] (Definition. 13.5). Define the positive cone of $\mathcal{B}_p$ by:

$$\mathcal{C}(\mathcal{B}_p) = \{(\beta, d) \mid \beta \in \mathcal{C}(\mathcal{A}_p) \text{ and } d \geq 0 \text{ or } \beta = 0 \text{ and } d > 0\}.$$

Next we recall the definition of weak (semi)stability from [18] for a general abelian category $\mathcal{A}$.

Definition 12.10. (Joyce and Song)[18](Definition. 3.5). Let $\mathcal{A}$ be an abelian category, $\mathcal{K}(\mathcal{A})$ be the quotient of $\mathcal{K}_0(\mathcal{A})$ by some fixed subgroup, and $C(\mathcal{A})$ the positive cone of $\mathcal{A}$. Suppose $(T, \leq)$ is a totally ordered set and $\tau : C(\mathcal{A}) \to T$ a map. We call $(\tau, T, \leq)$ a stability condition on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$ or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$ or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$. We call $(\tau, T, \leq)$ a weak stability condition on $\mathcal{A}$ if if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) \leq \tau(\beta) \leq \tau(\gamma)$ or $\tau(\alpha) \geq \tau(\beta) \geq \tau(\gamma)$.

For such $(\tau, T, \leq)$, we say that a nonzero object E in $\mathcal{A}$ is

1. τ-semistable if $\forall S \subset E$ where $S \ncong 0$, we have $\tau([S]) \leq \tau([E/S])$
2. τ-stable if $\forall S \subset E$ where $S \ncong 0$, we have $\tau([S]) < \tau([E/S])$
3. τ-unstable if it is not semistable.

In our analysis we apply the Definition 12.10 to objects in $\mathcal{A}_p$.

Remark 12.11. Note that the crucial point in understanding the weak stability of an object in an abelian category is that the criterion for stability is given by a comparison between sub-objects of this given object and its quotients which is different from the usual notion of stability.

Next we define the notion of permissible stability condition from [18] for the moduli stack of objects in a general abelian category $\mathcal{A}$. Here we assume that the moduli stack of τ-(semi)stable objects in $\mathcal{A}$ exists.

Definition 12.12. (Joyce and Song) [18] (Definition. 3.7). Let $(\tau, T, \leq)$ be a weak stability condition on $\mathcal{A}$. For $\alpha \in \mathcal{K}(\mathcal{A})$ let $\mathfrak{M}^{\alpha}_{ss}$ and $\mathfrak{M}^{\alpha}_{st}$ denote the moduli stacks of τ-(semi)stable objects $E \in \mathcal{A}$ with class $[E] = \alpha$ in $\mathcal{K}(\mathcal{A})$. We call $(\tau, T, \leq)$ permissible if

1. $\mathcal{A}$ is τ-Artinian (There exists no infinite chains of sub-objects with reducing slope of the subsequent quotients).
2. $\mathfrak{M}^{\alpha}_{ss}(\tau)$ is a finite type sub-stack of $\mathfrak{M}^{\alpha}_{\mathcal{A}}$ $\forall \alpha \in C(\mathcal{A})$.

One example of an abelian category for which there exists a moduli stack of (semi)stable objects which satisfy the condition in Definition 12.12 is the category of coherent sheaves, $\text{Coh}(X)$. In that case τ is the Giseker stability condition and from usual arguments, it is clear that there exist finite type open substacks $\mathfrak{M}^{\alpha}_{s}(\tau)$ and $\mathfrak{M}^{\alpha}_{ss}(\tau)$ of τ stable and τ-semistable sheaves $F \in \mathcal{A}$ with numerical class $\alpha \in \mathcal{K}^{num}(\text{Coh}(X))$.

Definition 12.13. (Joyce and Song) [18] (Definition. 13.5). Define the weak stability conditions $\tau^{\bullet}$, $\tilde{\tau}$ and τ^{n} in $\mathcal{B}_p$ by:

1. $\tau^\bullet(\beta, d) = 0$ if $d = 0$ and $\tau^\bullet(\beta, d) = -1$ if $d > 0$.
2. $\tilde{\tau}(\beta, d) = 0$ if $d = 0$ and $\tilde{\tau}(\beta, d) = 1$ if $d > 0$.
3. $\tau^n(\beta, d) = 0 \ \forall(\beta, d)$.

Chapter 13

Moduli stack of objects in $\mathcal{B}_p$

In this chapter we describe the moduli stack of semistable objects in $\mathcal{B}_p$. We construct this moduli stack of for the $\tilde{\tau}$-(weak)semistability condition. The constructions are similar for the case of the $\tau^\bullet$-(weak)semistability condition. In order to construct the moduli stack we give the definition of a new set of objects called the *rigidified* objects in $\mathcal{B}_p$.

Remark 13.1. By [18] (Page 185) there exists a natural embedding functor $\mathfrak{F} : \mathcal{B}_p \to D(X)$ which takes $(F, V, \phi_V) \in \mathcal{B}_p$ to an object in the derived category given by $\cdots \to 0 \to V \otimes \mathcal{O}_X(-n) \to F \to 0 \to \cdots$ where $V \otimes \mathcal{O}_X(-n)$ and F sit in degree -1 and 0. Assume that $\dim(V) = r$. In that case $V \otimes \mathcal{O}_X(-n) \cong \mathcal{O}_X(-n)^{\oplus r}$. Hence one may view an object $(F, V, \phi_V) \in \mathcal{B}_p$ as a triple (E, F, ϕ) represented by a complex $\phi : E \to F$ such that $E \cong \mathcal{O}_X(-n)^{\oplus r}$ (note the similarity between the objects in $\mathcal{B}_p$ and frozen triples in Definition 2.4).

Definition 13.2. Fix a parametrizing scheme of finite type S. Let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ denote the natural projections. Use the natural embedding functor $\mathfrak{F} : \mathcal{B}_p \to D(X)$ [18] (Page 185). Define the S-flat family of objects in $\mathcal{B}_p$ of type (β, r) as a complex

$$\pi_S^* M \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$$

sitting in degree -1 and 0 such that $\mathcal{F}$ is given by an S-flat family of semistable sheaves with fixed reduced Hilbert polynomial p with $\mathrm{Ch}(F) = \beta$ and M is a vector bundle of rank r over S. A morphism between two such S-flat families is given by a morphism between the complexes $\pi_S^* M \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^* M' \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{array}{ccc}
\pi_S^*M \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi_S} & \mathcal{F} \\
\downarrow & & \downarrow \\
\pi_S^*M' \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi'_S} & \mathcal{F}'.
\end{array}$$

Moreover an isomorphism between two such S-flat families in $\mathcal{B}_p$ is given by an isomorphism between the associated complexes $\pi_S^*M \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^*M' \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{array}{ccc}
\pi_S^*M \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi_S} & \mathcal{F} \\
\cong\downarrow & & \downarrow\cong \\
\pi_S^*M' \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi'_S} & \mathcal{F}'.
\end{array}$$

Note the similarity between definition of isomorphism between S-flat families of objects of type (β, r) in $\mathcal{B}_p$ and the isomorphism between two S-flat families of frozen triples of type (P_2, r) in Definition 2.7.

From now on whenever we mention objects in $\mathcal{B}_p$ we mean the objects which lie in the image of the natural embedding functor $\mathfrak{F} : \mathcal{B}_p \to D(X)$ [18] (Page 185). Moreover by the S-flat family of objects in $\mathcal{B}_p$, their morphisms (or isomorphisms) we mean the corresponding definitions as in Definition 13.2. Now we define the *rigidified* objects in $\mathcal{B}_p$. We give the category of these objects a new name $\mathcal{B}_p^{\mathbf{R}}$. However, we emphasize that it is implicitly understood that for us the category $\mathcal{B}_p^{\mathbf{R}}$ is the same as $\mathcal{B}_p$ together with an additional structure:

Definition 13.3. Define the category $\mathcal{B}_p^{\mathbf{R}}$ to be the category of rigidified objects in $\mathcal{B}_p$ whose objects are defined by tuples $(F, \mathbb{C}^{\oplus r}, \rho)$ where F is a coherent sheaf with reduced Hilbert polynomial p and $\mathrm{Ch}(F) = \beta$ and $\rho : \mathbb{C}^r \to \mathrm{Hom}(\mathcal{O}_X(-n), F)$ (for some r). Given two rigidified objects of fixed given type (β, r) as $(F, \mathbb{C}^{\oplus r}, \rho)$ and $(F', \mathbb{C}^{\oplus r}, \rho')$ in $\mathcal{B}_p^{\mathbf{R}}$ define morphisms $(F, \mathbb{C}^{\oplus r}, \rho) \to (F', \mathbb{C}^{\oplus r}, \rho')$

to be given by a morphism $f : F \to F'$ in $\mathcal{A}_p$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^{\oplus r} & \xrightarrow{\rho} & \mathrm{Hom}(\mathcal{O}_X(-n), F) \\ \mathrm{id}\downarrow & & \downarrow f \\ \mathbb{C}^{\oplus r} & \xrightarrow{\rho'} & \mathrm{Hom}(\mathcal{O}_X(-n), \acute{F}). \end{array} \tag{13.1}$$

Remark 13.4. There exists a natural embedding functor $\mathfrak{F}^{\mathbf{R}} : \mathcal{B}_p^{\mathbf{R}} \to D(X)$ which takes $(F, \mathbb{C}^{\oplus r}, \rho) \in \mathcal{B}_p^{\mathbf{R}}$ to an object in the derived category given by $\cdots \to 0 \to \mathbb{C}^{\oplus r} \otimes \mathcal{O}_X(-n) \to F \to 0 \to \cdots$ where $\mathbb{C}^{\oplus r} \otimes \mathcal{O}_X(-n)$ sits in degree -1 and F sits in degree 0. One may view an object in $\mathcal{B}_p^{\mathbf{R}}$ as a quadruple (E, F, ϕ, ψ) represented by a complex $\phi : E \to F$ such that $\psi : E \cong \mathcal{O}_X(-n)^{\oplus r}$ is a fixed choice of isomorphism (note the similarity between the objects in $\mathcal{B}_p^{\mathbf{R}}$ and highly frozen triples in Definition 2.9).

Definition 13.5. Fix a parametrizing scheme of finite type S. Let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ denote the natural projections. Use the natural embedding functor $\mathfrak{F}^{\mathbf{R}} : \mathcal{B}_p^{\mathbf{R}} \to D(X)$ in Remark 13.4. An S-flat family of objects of type (β, r) in $\mathcal{B}_p^{\mathbf{R}}$ is given by a complex

$$\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$$

sitting in degree -1 and 0 such that $\mathcal{F}$ is given by an S-flat family of semistable sheaves with fixed reduced Hilbert polynomial p with $\mathrm{Ch}(F) = \beta$. A morphism between two such S-flat families in $\mathcal{B}_p^{\mathbf{R}}$ is given by a morphism between the complexes $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{array}{ccc} \pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi_S} & \mathcal{F} \\ \mathrm{id}_{\mathcal{O}_{X\times S}}\downarrow & & \downarrow \\ \pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi'_S} & \mathcal{F}'. \end{array}$$

Moreover an isomorphism between two such S-flat families in $\mathcal{B}_p^{\mathbf{R}}$ is given by an isomorphism between the associated complexes $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{array}{ccc} \pi_S^*\mathcal{O}_S^{\oplus r}\otimes\pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi_S} & \mathcal{F} \\ \Big\downarrow \mathrm{id}_{\mathcal{O}_{X\times S}} & & \Big\downarrow \cong \\ \pi_S^*\mathcal{O}_S^{\oplus r}\otimes\pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\psi'_S} & \mathcal{F}'. \end{array}$$

Note the similarity between definition of isomorphism between S-flat families of objects of type (β, r) in $\mathcal{B}_p^{\mathbf{R}}$ and the isomorphism between two S-flat families of highly frozen triples of type (P_2, r) in definitions 2.9 and 2.10.

Similar to the way that we treated objects in $\mathcal{B}_p$ from now on whenever we mention objects in $\mathcal{B}_p^{\mathbf{R}}$ we mean the objects which lie in the image of the natural embedding functor $\mathfrak{F}^{\mathbf{R}} : \mathcal{B}_p^{\mathbf{R}} \to D(X)$ in Remark 13.4. Moreover by the S-flat family of objects in $\mathcal{B}_p^{\mathbf{R}}$, their morphisms (or isomorphisms) we mean the corresponding definitions as in Definition 13.5.

One associates the notion of weak $\tilde{\tau}$-semistability (or $\tau^{\bullet}$-stability) to objects in $\mathcal{B}_p$ and $\mathcal{B}_p^{\mathbf{R}}$. Now we show that there exists a moduli functor $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau}) : Sch/\mathbb{C} \to \text{Sets}$ which sends a $\mathbb{C}$-scheme S to an S-flat family of $\tilde{\tau}$-semsistable objects of type (β, r) in $\mathcal{B}_p$. Moreover we show that this moduli functor (as a functor with groupoid sections) is equivalent to a quotient stack. Finaly we show that the moduli stack $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_{p,ss}}(\tilde{\tau})$ is given by a stacky quotient of $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p^{\mathbf{R}},ss}(\tilde{\tau})$ (which itself is defined as the moduli stack of $\tilde{\tau}$-semistable objects of type (β, r) in $\mathcal{B}_p^{\mathbf{R}}$).

According to Definition 12.2 an object in the category $\mathcal{A}_p$ consists of semistable sheaves with fixed Hilbert polynomial p. As discussed in [13] (Theorem 3.37), the family of τ-semistable (i.e Giseker semistable) sheaves F on X such that F has a fixed Hilbert polynomial is bounded. Hence the family of τ-semistable sheaves F on X with Hilbert polynomial $P = \frac{k}{d!}\cdot p(t)$ for any $k = 0, 1, \cdots, N$ is also bounded. We consider a rigidified object $[\mathcal{O}_X^{\oplus r}(-n) \to F]$ of type (β, r). Moreover, we use the fact that by the Grothendieck-Riemann-Roch theorem fixing the Chern character (and hence the second Chern character) of a pure sheaf with one dimensional support is equivalent to fixing its Hilbert polynomial. Hence we start our construction with the assumption that the sheaf F appearing in the corresponding rigidified objects has a fixed Hilbert polynomial.

13.1 The underlying parameter scheme

Given a bounded family of τ-semistable sheaves F with fixed Hilbert polynomial P there exists an integer m such that for every sheaf F in the family, $F(m)$ is globally generated. First we construct an S-flat family of coherent sheaves F with fixed Hilbert polynomial P. To avoid confusion, here we slightly change our notation. We denote by $\mathcal{F}$ the family as a coherent $\mathcal{O}_{X\times S}$-module and by F we mean the fiber of this family over a geometric point of S. By construction, the family of coherent sheaves F appearing in a $\tilde{\tau}$-semistable rigidified object is bounded and moreover $F(n)$ is globally generated for all $n \geq m$.

Fix such n and let V be a complex vector space of dimension $d = P(n)$ given as $V = \mathrm{H}^0(F \otimes L^n)$. The line bundle L, as defined before, is the fixed polarization over X. Twisting the sheaf F by the fixed large enough n would ensure one to get a surjective morphism of coherent sheaves $V \otimes \mathcal{O}_X(-n) \to F$. One can construct a scheme parametrizing the flat quotients of $V \otimes \mathcal{O}_X(-n)$ with fixed given Hilbert polynomial. This by usual arguments provides us with Grothendieck's Quot-scheme. Here to shorten the notation we use $\mathcal{Q}$ to denote $\mathrm{Quot}_P(V \otimes \mathcal{O}_X(-n))$.

Now consider a sub-locus $\mathcal{Q}^{ss} \subset \mathcal{Q}$ which parametrizes the Giseker semistable sheaves F with fixed Hilbert polynomial P.

Definition 13.6. Define $\mathcal{P}$ over $\mathcal{Q}^{ss}$ to be the bundle whose fibers parametrize $\mathrm{H}^0(F(n))$. The fibers of the bundle $\mathcal{P}^{\oplus r}$ parametrize $\mathrm{H}^0(F(n))^{\oplus r}$.

In other words the fibers of $\mathcal{P}^{\oplus r}$ parametrize the maps $\mathcal{O}_X^{\oplus r}(-n) \to F$ (which define the complexes representing the objects in $\mathcal{B}_p^{\mathbf{R}}$). Now let $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau}) \subset \mathcal{P}^{\oplus r}$ be given as an open subscheme of $\mathcal{P}^{\oplus r}$ whose fibers parametrize $\tilde{\tau}$-semistable objects in $\mathcal{B}_p^{\mathbf{R}}$.

13.2 Stacky structure of $\mathfrak{M}_{\mathcal{B}_{p,ss}}^{(\beta,r)}(\tilde{\tau})$ and $\mathfrak{M}_{\mathcal{B}_{p,ss}^{\mathbf{R}}}^{(\beta,r)}(\tilde{\tau})$.

By definitions 13.2 and 13.5 it is easily seen that we have already given the strategy to construct the moduli stack of objects of type (β, r) in $\mathcal{B}_p$ and $\mathcal{B}_p^{\mathbf{R}}$ in Chapter 3:

Theorem 13.7. *Let $\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})$ be the underlying scheme in Section 13.1 parametrizing $\tilde{\tau}$-semistable* *rigidified objects of type (β, r). Let $G := \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}(V)$ where V is as in Section 13.1. Let $\left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tau')}{G}\right]$ be the stack theoretic quotient of $\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})$ by G. There exists an isomorphism of groupoids*

$$\mathfrak{M}_{\mathcal{B}_p,ss}^{(\beta,r)}(\tilde{\tau}) \cong \left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{G}\right].$$

In particular $\mathfrak{M}_{\mathcal{B}_p,ss}^{(\beta,r)}(\tilde{\tau})$ is an Artin stack.

Proof. Use the fact that (for $F \in Coh(X)$) fixing β is equivalent to fixing the Hilbert polynomial P. Now replace τ'-stability and $\mathfrak{S}_{ss}^{P_2,r}(\tau')$ in Theorem 3.9 with $\tilde{\tau}$-stability and $\mathfrak{S}_{ss}^{\beta,r}(\tilde{\tau})$ respectively. The rest of the proof follows directly from proof of Theorem 3.9 and Corollary 3.11. □

Corollary 13.8. *Apply the proof of Theorem 13.7 to $\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})$ and $G = \mathrm{GL}(V)$ and obtain a natural isomorphism between $\mathfrak{M}_{\mathcal{B}_p^R,ss}^{(\beta,r)}(\tilde{\tau})$ and $\left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{\mathrm{GL}(V)}\right]$.*

One may use this natural isomorphism in order to obtain an alternative definition of the moduli stack of $\tilde{\tau}$-semistable rigidified objects of type (β, r) as the quotient stack $\left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{\mathrm{GL}(V)}\right]$

Corollary 13.9. *By Theorem 13.7 and Corollary 13.8 it is true that:*

$$\mathfrak{M}_{\mathcal{B}_p,ss}^{(\beta,r)}(\tilde{\tau}) = \left[\frac{\mathfrak{M}_{\mathcal{B}_{p,ss}^R}^{(\beta,r)}(\tilde{\tau})}{\mathrm{GL}_r(\mathbb{C})}\right] \tag{13.2}$$

Proposition 13.10. *The moduli stack, $\mathfrak{M}_{\mathcal{B}_p^R,ss}^{(\beta,r)}(\tilde{\tau})$, is a $\mathrm{GL}_r(\mathbb{C})$-torsor over $\mathfrak{M}_{\mathcal{B}_p,ss}^{(\beta,r)}(\tilde{\tau})$. It is true that locally in the flat topology, $\mathfrak{M}_{\mathcal{B}_p,ss}^{(\beta,r)}(\tilde{\tau}) \cong \mathfrak{M}_{\mathcal{B}_p^R,ss}^{(\beta,r)}(\tilde{\tau}) \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]$. This isomorphism does not hold true globally unless $r = 1$.*

Proof. Replace τ'-stability and $\mathfrak{S}_{ss}^{P_2,r}(\tau')$ in Proposition 3.6 with $\tilde{\tau}$-stability and $\mathfrak{S}_{ss}^{\beta,r}(\tilde{\tau})$ respectively. The rest of the proof follows directly from proof of Proposition 3.6. □

Via replacing $\tilde{\tau}$ with $\tau^\bullet$ stability one constructs $\mathfrak{M}_{ss,\mathcal{B}_p}^{(\beta,r)}(\tau^\bullet)$ similarly. We state a proposition which we need later for our computations.

Proposition 13.11. *(a). $\forall (\beta, d) \in \mathcal{C}(\mathcal{B}_p)$ we have natural stack isomorphisms $\mathfrak{M}_{ss,\mathcal{B}_p}^{(\beta,0)}(\tau^\bullet) \cong \mathfrak{M}_{ss}^{\beta}(\tau)$ (τ stands for Giseker semistability condition and $\mathfrak{M}_{ss}^{\beta}(\tau)$ stands for moduli stack of Giseker* *semistable coherent sheaves with $\mathcal{K}$-theory class β.) which is obtained by identifying $(F,0,0)$ with F, moreover $\mathfrak{M}^{(0,1)}_{ss,\mathcal{B}_p}(\tau^\bullet)\cong[Spec(\mathbb{C})/\mathbb{G}_m]$ with the unique point given by $(0,\mathbb{C},0)$. Furthermore, $\mathfrak{M}^{(\beta,2)}_{ss,\mathcal{B}_p}(\tau^\bullet)=\varnothing$ for $\beta\neq 0$.*

(b). $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet)\cong[\mathrm{Spec}(\mathbb{C})/\mathrm{GL}_2(\mathbb{C})]$ *with the unique point given by* $(0,\mathbb{C}^2,0)$.

Proof. The first two parts of part (a) of Proposition 13.11 are proved in [18] (Prop. 15.6). We start by proving the last part of (a). We know that every object $[(F,V,\phi)]=(\beta,2)$ fits in a short exact sequence

$$0\to(F,0,0)\to(F,V,\phi)\to(0,V,0)\to 0,$$

here $[(F,0,0)]=(\beta,0)$ and $[(0,V,0)]=(0,2)$. By Definition 12.13 $\tau^\bullet(F,0,0)=0>\tau^\bullet(0,V,0)=-1$ therefore $(F,0,0)$ $\tau^\bullet$-destabilizes (F,V,ϕ) $\forall[(F,V,\phi)]=(\beta,2)$ and this finishes the proof of last part of (a).

(b). Note that $(0,\mathbb{C}^2,0)$ is a unique point in $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet)$ which is made of two copies of $(0,\mathbb{C},0)$ which is the unique object in $\mathfrak{M}^{(0,1)}_{ss,\mathcal{B}_p}(\tau^\bullet)$. Moreover, the only nonzero sub-object that can destabilize $(0,\mathbb{C}^2,0)$ is $(0,\mathbb{C},0)$. There exists a short exact sequence:

$$0\to(0,\mathbb{C},0)\to(0,\mathbb{C}^2,0)\to(0,\mathbb{C},0)\to 0. \tag{13.3}$$

It is easily seen that $\tau^\bullet(0,\mathbb{C},0)=\tau^\bullet(0,\mathbb{C}^2,0)=-1$ and therefore the sub-object $(0,\mathbb{C},0)$ does not destabilize $(0,\mathbb{C}^2,0)$ and $(0,\mathbb{C}^2,0)$ is weak $\tau^\bullet$-semistable. Since the automorphisms of $(0,\mathbb{C}^2,0)$ are given by $\mathrm{GL}_2(\mathbb{C})$ then $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet)\cong[\mathrm{Spec}(\mathbb{C})/\mathrm{GL}_2(\mathbb{C})]$. □

Chapter 14

Stack function identities in Ringel Hall algebra

14.1 Stack functions for moduli stack of semistable sheaves

Definition 14.1. (Joyce and Song) [18] (Definition 2.5). Let $\mathfrak{S}$ be a $\mathbb{C}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R}, \rho)$ where $\mathfrak{R}$ is a finite type algebraic $\mathbb{C}$-stack with affine geometric stabilizers and $\rho : \mathfrak{R} \to \mathfrak{S}$ is a 1-morphism. Two pairs $(\mathfrak{R}, \rho)$ and $(\acute{\mathfrak{R}}, \acute{\rho})$ are called to be equivalent if there exists a 1-morphism $\iota : \mathfrak{R} \to \acute{\mathfrak{R}}$ such that in the diagram below:

$$\begin{array}{ccc} \mathfrak{R} & \xrightarrow{\iota} & \acute{\mathfrak{R}} \\ \rho \downarrow & & \downarrow \acute{\rho} \\ \mathfrak{S} & \xrightarrow{\cong} & \mathfrak{S} \end{array} \tag{14.1}$$

the two vertical maps are 2-isomorphic with each other. Write $[(\mathfrak{R}, \rho)]$ for the equivalence class of $(\mathfrak{R}, \rho)$. If $(\mathfrak{G}, \rho \mid_{\mathfrak{G}})$ represents a sub-pair of $(\mathfrak{R}, \rho)$ where $\mathfrak{G}$ is closed in $\mathfrak{R}$, then $(\mathfrak{G}, \rho \mid_{\mathfrak{G}})$ and $(\mathfrak{R}/\mathfrak{G}, \rho \mid_{\mathfrak{R}/\mathfrak{G}})$ are pairs of the same kind. Define $\underline{\mathrm{SF}}(\mathfrak{S})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes of $[\mathfrak{R}, \rho]$ subject to the relations

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{G}, \rho \mid_{\mathfrak{G}})] + [(\mathfrak{R}/\mathfrak{G}, \rho \mid_{\mathfrak{R}/\mathfrak{G}})] \tag{14.2}$$

for each subpair $(\mathfrak{G}, \rho \mid_{\mathfrak{G}})$ of $(\mathfrak{R}, \rho)$. Elements of $\underline{\mathrm{SF}}(\mathfrak{S})$ are called stack functions on $\mathfrak{S}$.

Definition 14.2. (Joyce and song) [18] (Definition 2.16.). Define the space of stack functions $\underline{\mathrm{S}\,\overline{\mathrm{F}}}(\mathfrak{S}, \chi, \mathbb{Q})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R}, \rho)]$ with the following relations imposed:

1. Given a sub-pair $(\mathfrak{G}, \rho\mid_{\mathfrak{G}}) \subset (\mathfrak{R}, \rho)$ we have $[(\mathfrak{R},\rho)] = [(\mathfrak{G}, \rho\mid_{\mathfrak{G}})] + [(\mathfrak{R}/\mathfrak{G}, \rho\mid_{\mathfrak{R}/\mathfrak{G}})]$ as above.

2. Let $\mathfrak{R}$ be a $\mathbb{C}$-stack of finite type with affine geometric stabilizers and let $\mathcal{U}$ denote a quasi-projective $\mathbb{C}$-variety and $\pi_{\mathfrak{R}} : \mathfrak{R} \times U \to \mathfrak{R}$ the natural projection and $\rho : \mathfrak{R} \to \mathfrak{S}$ a 1-morphism. Then $[(\mathfrak{R} \times \mathcal{U}, \rho \circ \pi_{\mathfrak{R}})] = \chi([\mathcal{U}])[(\mathfrak{R},\rho)]$.

3. Assume $\mathfrak{R} \cong [X/G]$ where X is a quasiprojective $\mathbb{C}$-variety and G a very special algebraic $\mathbb{C}$-group acting on X with maximal torus T^G, we have

$$[(\mathfrak{R},\rho)] = \sum_{Q \in \mathcal{Q}(G,T^G)} F(G, T^G, Q)[([X/Q], \rho \circ \iota^Q)], \tag{14.3}$$

where the rational coefficients $F(G, T^G, Q)$ have a complicated definition explained in [17] (Section 6.2). Here $\mathcal{Q}(G, T^G)$ is the set of closed $\mathbb{C}$-subgroups Q of T^G such that $Q = T^G \cap C(G)$ and $\iota^Q : [X/Q] \to \mathfrak{R} \cong [X/G]$ is the natural projection 1-morphism, where $C(G)$ denotes the center of the group G. Similarly we can define $\overline{\mathrm{SF}}(\mathfrak{S}, \chi, \mathbb{Q})$ by restricting the 1-morphisms ρ to be representable.

Remark 14.3. There exist notions of multiplication, pullback, pushforward of stack functions in $\underline{\mathrm{S}\,\overline{\mathrm{F}}}(\mathfrak{S}, \chi, \mathbb{Q})$ and $\overline{\mathrm{SF}}(\mathfrak{S}, \chi, \mathbb{Q})$ and to save space we do not construct them here. For further discussions look at (Joyce and Song) [18] (Definitions. 2.6, 2.7) and (Theorem. 2.9).

Now we restrict to the moduli stack of semistable sheaves and define Ringel-Hall identities for moduli stack of semistable sheaves as in [18].

Definition 14.4. (Joyce and Song) [18] (Definition 3.3). For $\alpha \in C(\mathcal{A})$ write $\mathfrak{M}^{\alpha}_{\mathcal{A}}$ for the substack of objects $F \in \mathcal{A}$ in class $\alpha \in K(\mathcal{A})$. Let $\mathfrak{Exact}_{\mathcal{A}}$ denote the moduli stack of short exact sequences $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$. For $i = 1, 2, 3$ let $\pi_i : \mathfrak{Exact} \to \mathfrak{M}_{\mathcal{A}}$ denote the 1-morphism of Artin stacks projecting the short exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ to E_i. Define bilinear operations $*$ on the stack function spaces $\underline{SF}$, $SF(\mathfrak{M}_{\mathcal{A}})$, $\underline{S\overline{F}}$ and $S\overline{F}(\mathfrak{M}_{\mathcal{A}}, \chi, \mathbb{Q})$ by

$$f * g = (\pi_2)_*((\pi_1 \times \pi_3)^*(f \otimes g)). \tag{14.4}$$

Definition 14.5. Let $(\tau, T, \leq)$ be a permissible stability condition on $\mathcal{A}$. Define the stack functions $\overline{\delta}^{\alpha}_{ss}(\tau) = \overline{\delta}^{\alpha}_{\mathfrak{M}^{\alpha}_{ss}}(\tau)$ in $\mathrm{SF}_{al}(\mathfrak{M}_{\mathcal{A}})$ (for definition of SF_{al} look at [18] (Definition 3.3) for $\alpha \in C(\mathcal{A})$. One thinks of $\overline{\delta}^{\alpha}_{\mathfrak{M}^{\alpha}_{ss}}(\tau)$ as the characteristic stack function of the component of the moduli stack of

τ-semistable objects in $\mathcal{A}$ whose class in $K(\mathcal{A})$ is α. Define elements $\bar{\epsilon}^{\alpha}(\tau)$ in $\mathrm{SF}_{al}(\mathfrak{M}_{\mathcal{A}})$

$$\bar{\epsilon}^{\alpha}(\tau) = \sum_{\substack{n\geq 1,\alpha_1,\cdots,\alpha_n\in C(\mathcal{A})\\ \alpha_1+\cdots+\alpha_n=\alpha\\ \tau(\alpha_i)=\tau(\alpha)\forall i}} \frac{(-1)^{n-1}}{n}\bar{\delta}_{ss}^{\alpha_1}(\tau)*\bar{\delta}_{ss}^{\alpha_2}(\tau)*\cdots*\bar{\delta}_{ss}^{\alpha_n}(\tau), \tag{14.5}$$

where $*$ is the Ringel-Hall multiplication defined in 14.4. Here each $\bar{\delta}_{ss}^{\alpha_i}(\tau)$ denotes the characteristic stack function associated to the moduli stack of objects $F \in \mathcal{A}$ in class $\alpha_i \subset C(\mathcal{A})$ as defined in [18] (Definition 3.3).

Next we briefly review the wall crossings in moduli stack of semistable sheaves under change of stability condition from (say) τ to $\tilde{\tau}$.

14.2 Wall crossings over moduli of semistable sheaves under change of stability condition

Definition 14.6. (Joyce and Song)[18] (Definition. 3.12.). Let $(\tau, T, \leq)$ and $(\tilde{\tau}, \tilde{T}, \leq)$ be two weak stability conditions on $\mathcal{A}$. Let $n \geq 1$ and $\alpha_1, \cdots, \alpha_n \in C(\mathcal{A})$. If for all $i = 1, \cdots, n-1$ we have either

(a). $\tau(\alpha_i) \leq \tau(\alpha_{i+1})$ and $\tilde{\tau}(\alpha_1 + \cdots + \alpha_i) > \tilde{\tau}(\alpha_{i+1} + \cdots + \alpha_n)$ or

(b). $\tau(\alpha_i) > \tau(\alpha_{i+1})$ and $\tilde{\tau}(\alpha_1 + \cdots + \alpha_i) \leq \tilde{\tau}(\alpha_{i+1} + \cdots + \alpha_n)$

then define $S(\alpha_1, \cdots, \alpha_n; \tau, \tilde{\tau}) = (-1)^r$ where r is the number of times that for some $i = 1, \cdots, n$ condition (a) is satisfied. Otherwise define $S(\alpha_1, \cdots, \alpha_n; \tau, \tilde{\tau}) = 0$.

The function $S(\alpha_1, \cdots, \alpha_n; \tau, \tilde{\tau})$ is a combinatorial ingredient that we need in order to compute the wall-crossing identities. We need another ingredient that we review here from [18] (Definition. 3.12.). Given $n \geq 1$ and $\alpha_1, \cdots, \alpha_n \in \mathcal{C}(\mathcal{A})$, choose two numbers l and m such that $1 \leq l \leq m \leq n$. Now for this choice choose numbers $0 = a_0 < a_1 < \cdots < a_m$ and $0 = b_0 < b_1 < \cdots < b_l = m$. Given m and $a_1, \cdots, a_m$, define elements $\beta_1, \cdots, \beta_m \in C(\mathcal{A})$ by $\beta_i = \alpha_{a_{i-1}+1} + \cdots \alpha_{a_i}$. Also given l and $b_1, \cdots, b_l$ define elements $\gamma_1, \cdots, \gamma_l \in C(\mathcal{A})$ by $\gamma_i = \beta_{b_{i-1}+1} + \cdots \beta_{b_i}$. Let Λ denote the set of choices $(l, m, a_1, \cdots, a_m, b_1, \cdots, b_l)$ for which the elements γ_i and β_i that we defined above satisfy the condition that $\tau(\beta_i) = \tau(\alpha_j)$ for $i = 1, \cdots, m$ and $a_{i-1} < j \leq a_i$ and $\tilde{\tau}(\gamma_i) = \tilde{\tau}(\alpha)$ for $i = 1, \cdots l$ (here $\alpha = \sum_i \alpha_i$). Given such $(l, m, a_1, \cdots, a_m, b_1, \cdots, b_l) \in \Lambda$ define:

$$U(\alpha_1, \cdots, \alpha_n; \tau, \tilde{\tau}) = \sum_{\Lambda} \frac{(-1)^{l-1}}{l} \prod_{i=1}^{l} S(\beta_{b_{i-1}+1}, \beta_{b_{i-1}+2}, \cdots, \beta_{b_i}; \tau, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}. \tag{14.6}$$

Joyce and Song give a formula to compute $\bar{\epsilon}^{\alpha}(\tilde{\tau})$ that uses the function U in (14.6). To state the theorem they define the notion of a stability condition dominating another stability condition as follows.

Definition 14.7. The triple $(\tilde{\tau}, \tilde{T}, \leq)$ is said to dominate $(\tau, T, \leq)$ if $\tau(\alpha) \leq \tau(\beta)$ implies $\tilde{\tau}(\alpha) \leq \tilde{\tau}(\beta)$, $\forall \alpha, \beta \in C(\mathcal{A})$.

With this notion we state the following theorem from [18].

Theorem 14.8. *(Joyce and Song) [18] (Theorem 3.13.). Let $(\tau, T, \leq)$, $(\acute{\tau}, \acute{T}, \leq)$, $(\tilde{\tau}, \tilde{T}, \leq)$ be permissible weak stability conditions on $\mathcal{A}$ such that $(\acute{\tau}, \acute{T}, \leq)$ dominates $(\tau, T, \leq)$ and $(\tilde{\tau}, \tilde{T}, \leq)$. Then $\forall \alpha \in C(\mathcal{A})$ we have*

$$\bar{\epsilon}^{\alpha}(\tilde{\tau}) = \sum_{n \geq 1} \sum_{\substack{(\alpha_1, \cdots, \alpha_n) \in C(\mathcal{A})^n: \\ \alpha_1 + \cdots + \alpha_n = \alpha}} U(\alpha_1, \cdots, \alpha_n; \tau, \tilde{\tau}) \bar{\epsilon}^{\alpha_1}(\tau) * \bar{\epsilon}^{\alpha_2}(\tau) * \cdots * \bar{\epsilon}^{\alpha_n}(\tau). \tag{14.7}$$

There are only finitely many nonzero terms in this equation.

Our introduction to background material on stack functions and Ringel Hall identities over the moduli stack of semistable sheaves ends here. We go back to the moduli stack of semistable objects in $\mathcal{B}_p$ and obtain similar identities.

14.2.1 Stack functions and similar identities over moduli stack of objects in $\mathcal{B}_p$

Consider the weak stability conditions on $\mathcal{B}_p$ given in 12.13. Note that $(\tau^{\bullet}, T^{\bullet} = \{-1, 0\}, \leq)$ and $(\tilde{\tau}, \tilde{T} = \{0, 1\}, \leq)$ are permissible stability conditions on $\mathcal{B}_p$, so similar to the above there exist elements $\bar{\delta}_{ss}^{(\beta,d)}(\tilde{\tau})$, $\bar{\delta}_{ss}^{(\beta,d)}(\tau^{\bullet})$ in the stack function space $\mathrm{SF}_{al}(\mathfrak{M}_{\mathcal{B}_p})$. Moreover, we can define

$\bar{\epsilon}^{(\beta,d)}(\tilde{\tau})$ and $\bar{\epsilon}^{(\beta,d)}(\tau^{\bullet})$. Also similar to (14.6) we can define the combinatorial ingredients S and U as follows. Consider the weak permissible stability conditions $(\tau^{\bullet}, T^{\bullet}, \leq)$ and $(\tilde{\tau}, \tilde{T}, \leq)$ and the dominating stability condition $(\tau^n, T^n = \{0\}, 0)$ in 12.13:

Definition 14.9. Let $n \geq 1$ and

$$(\beta_1, d_1), \cdots, (\beta_n, d_n) \in C(\mathcal{B}_p).$$

We define a number, $S((\beta_1, d_1), \cdots, (\beta_n, d_n); \tau^{\bullet}, \tilde{\tau})$, associated to the function U in (14.6) as follows. If for all $i = 1, \cdots, n$ we have either:

(a). $\tau^{\bullet}(\beta_i, d_i) \leq \tau^{\bullet}(\beta_{i+1}, d_{i+1})$ and

$$\tilde{\tau}((\beta_1, d_1) + \cdots + (\beta_i, d_i)) > \tilde{\tau}((\beta_{i+1}, d_{i+1}) + \cdots + (\beta_n, d_n)).$$

or

(b). $\tau^{\bullet}(\beta_i, d_i) > \tau^{\bullet}(\beta_{i+1}, d_{i+1})$ and

$$\tilde{\tau}((\beta_1, d_1) + \cdots + (\beta_i, d_i)) \leq \tilde{\tau}((\beta_{i+1}, d_{i+1}) + \cdots + (\beta_n, d_n)),$$

then define $S((\beta_1, d_1), \cdots, (\beta_n, d_n); \tau^{\bullet}, \tilde{\tau}) = (-1)^r$, where r is the number of times that for all $i = 1, \cdots, n-1$ condition (a) is satisfied and otherwise if for some $i = 1, \cdots, n-1$ neither (a) nor (b) is true, then set $S = 0$.

Given $n \geq 1$ and $(\beta_1, d_1), \cdots, (\beta_n, d_n)$ as above, choose two numbers l and m such that $1 \leq l \leq m \leq n$. Now for this choice choose numbers $0 = a_0 < a_1 < \cdots < a_m$ and $0 = b_0 < b_1 < \cdots < b_l = m$. Given such m and $a_1, \cdots, a_m$, define elements $\theta_1, \cdots, \theta_m \in C(\mathcal{B}_p)$ by $\theta_i = (\beta_{a_{i-1}+1}, d_{a_{i-1}+1}) + \cdots (\beta_{a_i}, d_{a_i})$ (To add two pairs just add them coordinate-wise in $C(\mathcal{B}_p)$). Also given such $l, b_1, \cdots, b_l$ define elements $\gamma_1, \cdots, \gamma_l \in C(\mathcal{B}_p)$ by $\gamma_i = \theta_{b_{i-1}+1} + \cdots \theta_{b_i}$. Let Λ denote the set of choices $(l, m, a_1, \cdots, a_m, b_1, \cdots, b_l)$ for which the two following conditions are satisfied:

(1). $\tau^{\bullet}(\theta_i) = \tau^{\bullet}(\beta_j, d_j)$ for $i = 1, \cdots, m$ and $a_{i-1} < j \leq a_i$.

(2). $\tilde{\tau}(\gamma_i) = \tilde{\tau}(\beta, d)$ for $i = 1, \cdots l$ (here $\beta = \sum_i \beta_i$ and $d = \sum_i d_i$). Now define:

$$U((\beta_1, d_1), \cdots, (\beta_n, d_n); \tau^\bullet, \tilde{\tau}) = \sum_{\Lambda} \frac{(-1)^{l-1}}{l} \prod_{i=1}^{l} S(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots, \theta_{b_i}; \tau^\bullet, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}. \tag{14.8}$$

By applying Theorem 3.13 in [18] we obtain a wall crossing identity over the moduli stack of objects in $\mathcal{B}_p$.

Proposition 14.10. *(Joyce and Song) [18] (Proposition 13.7). For all (β, d) in $\mathcal{C}(\mathcal{B}_p)$, the following identity holds in the Ringel Hall algebra of $\mathcal{B}_p$.*

1. There are only finitely many choices of $n \geq 1$ and $(\beta_i, d_i) \in \mathcal{C}(\mathcal{B}_p)$ for which the function U defined in (14.8) *is nonzero.*

2. For these nonzero terms the following identity holds in the Ringel Hall algebra of $\mathcal{B}_p$:

$$\bar{\epsilon}^{(\beta,d)}(\tilde{\tau}) = \sum_{n \geq 1} \sum_{\substack{((\beta_1,d_1),\cdots,(\beta_n,d_n)) \in \mathcal{C}(\mathcal{B}_p)^n: \\ (\beta_1,d_1)+\cdots+(\beta_n,d_n)=(\beta,d)}} U((\beta_1, d_1), \cdots (\beta_n, d_n); \tau^\bullet, \tilde{\tau}) \cdot \bar{\epsilon}^{(\beta_1,d_1)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\beta_n,d_n)}(\tau^\bullet). \tag{14.9}$$

Our introduction to stack functions and identities in Ringel Hall algebra of $\mathcal{B}_p$ ends here. For further discussions look at [18] (Propositions 15.6, 15.7). Now we apply this machinery to the case where the numerical class of objects is fixed to be $(\beta, 2)$.

Chapter 15

Lie algebra identities related to wallcrossing in rank 2

Our main goal is to compute the wall-crossing identity for the invariants of objects of type $(\beta, 2)$ in $\mathcal{B}_p$ by changing the stability condition from $\tau^{\bullet}$ to $\tilde{\tau}$. Based on above discussions one needs to take into account all possible decompositions of β and $d = 2$ into smaller pieces and compute their contributions. At first, keeping track of all possible decompositions may seem harder than it really is. However based on results obtained in Proposition 13.11 there exist restrictions that one exploits in favor of computational simplification. For example, $\mathfrak{M}^{(\beta,1)}_{ss,\mathcal{B}_p}(\tau^{\bullet})$ and $\mathfrak{M}^{(\beta,2)}_{ss,\mathcal{B}_p}(\tau^{\bullet})$ are both empty by [18] (Proposition 15.6) and the second part of (a) in 13.11 respectively. One needs to first break $d = 2$ into smaller dimensions and then decompose β. The only two possible ways to break $d = 2$ is to write $2 = 2 + 0$ and $2 = 1 + 1$. Now for each choice of decomposition of d one decomposes β into smaller classes β_i. For example for the case $2 = 2 + 0$, the decomposition of β into smaller classes produces elements in $C(\mathcal{B}_p)$ of type $(\beta_1, d_1), \cdots (\beta_n, d_n)$ where $\beta_1 + \cdots + \beta_n = \beta$ and $d_1 + \cdots + d_n = 2$, hence there exists a tuple in this sequence which is of type $(\beta_i, 2)$ and the remaining objects are of type $(\beta_j, 0)$. Now use Proposition 13.11 and note that $\mathfrak{M}^{(\beta_i,2)}_{ss}(\tau^{\bullet}) = \varnothing$ unless $\beta_i = 0$. Hence the set of numerical classes is given as $(\beta_1, 0), \cdots, (0, 2), \cdots, (\beta_n, 0)$. Similarly for the decomposition of type $2 = 1 + 1$ one obtains elements of type $(\beta_1, 0), \cdots, (\beta_{k-1}, 0), (0, 1), \cdots, (\beta_{m-1}, 0), (0, 1), \cdots, (\beta_n, 0)$ for $1 \leq k \neq m \leq n$. In order to ease the bookkeeping we use a re-parameterization of (β_i, d_i) which is consistent with work of Joyce and Song. For a decomposition $2 = 2 + 0$ define $(\psi_i, d_i) = (\beta_i, 0)$ for $i \leq k - 1$, and $(\psi_i, d_i) = (\beta_{i+1}, 0)$ for $i \geq k$. For decomposition of type $2 = 1 + 1$ define $(\psi_i, d_i) = (\beta_i, 0)$ for $i \leq k - 1$, $(\psi_i, d_i) = (\beta_{i+1}, 0)$ for $k \leq i \leq m - 1$ and $(\psi_i, d_i) = (\beta_{i+2}, 0)$ for

$i \geq m$. Equation 14.10 for the case of $(\beta, 2)$ is written as:

$$\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = \sum_{1\leq k\leq n} U((\psi_1,0),\cdots(\psi_{k-1},0),(0,2),(\psi_k,0),\cdots,(\psi_{n-1},0);\tau^\bullet,\tilde{\tau})$$
$$\cdot\,\bar{\epsilon}^{(\psi_1,0)}(\tau^\bullet)*\cdots*\bar{\epsilon}^{(\psi_{k-1},0)}(\tau^\bullet)*\bar{\epsilon}^{(0,2)}(\tau^\bullet)*\bar{\epsilon}^{(\psi_k,0)}(\tau^\bullet)*\cdots*\bar{\epsilon}^{(\psi_{n-1},0)}(\tau^\bullet)$$
$$+\sum_{\substack{k,m:\\ 1\leq k\neq m\leq n}} U((\psi_1,0),\cdots,(\psi_{k-1},0),(0,1),(\psi_k,0),\cdots,(\psi_{m-1},0),(0,1),(\psi_m,0),\cdots,(\psi_{n-2},0);\tau^\bullet,\tilde{\tau})$$
$$\cdot\,\bar{\epsilon}^{(\psi_1,0)}(\tau^\bullet)*\cdots*\bar{\epsilon}^{(\psi_{k-1},0)}(\tau^\bullet)*\bar{\epsilon}^{(0,1)}(\tau^\bullet)*\bar{\epsilon}^{(\psi_k,0)}(\tau^\bullet)$$
$$*\cdots*\bar{\epsilon}^{(\psi_{m-1},0)}(\tau^\bullet)*\bar{\epsilon}^{(0,1)}(\tau^\bullet)*\bar{\epsilon}^{(\psi_m,0)}(\tau^\bullet)*\cdots*\bar{\epsilon}^{(\psi_{n-2},0)}(\tau^\bullet). \tag{15.1}$$

Let **A** and **B** denote the first and second sums respectively on the right hand side of (15.1). Next we simplify **A**.

Remark 15.1. Here we calculate the function U's for our case. In doing so and to avoid notational confusion, we denote the function U appearing in **A** by $U_{\mathbf{A}}$ and the one in **B** by $U_{\mathbf{B}}$, moreover for the function S defined in 14.9 we use $S_{\mathbf{A}}$ and $S_{\mathbf{B}}$ if it appears in **A** or **B** respectively.

Simplification of A

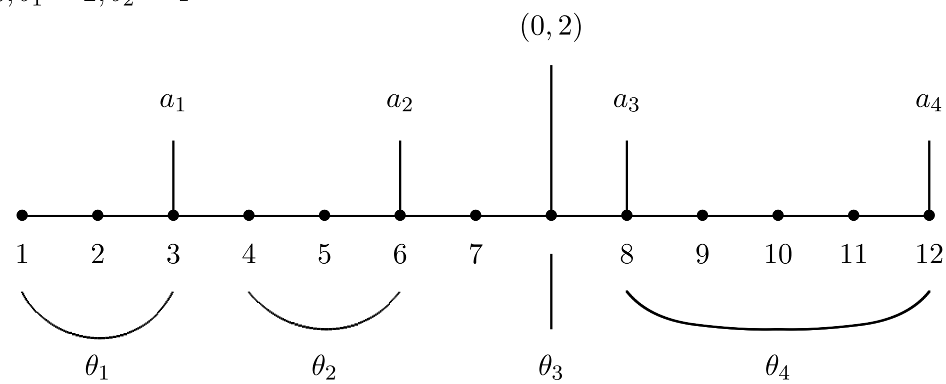
We recall the definition of $U_{\mathbf{A}}$ appearing in **A**,

$$U_{\mathbf{A}} = U((\psi_1,0),\cdots(\psi_{k-1},0),(0,2),(\psi_k,0),\cdots,(\psi_{n-1},0);\tau^\bullet,\tilde{\tau}) =$$
$$\sum_{\Lambda}\frac{(-1)^{l-1}}{l}\cdot\prod_{i=1}^{l} S_{\mathbf{A}}(\theta_{b_{i-1}+1},\theta_{b_{i-1}+2},\cdots\theta_{b_i};\tau^\bullet,\tilde{\tau})\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}. \tag{15.2}$$

Before we simplify this identity (to have a pictorial perspective of this calculation) we work out some examples. First we work out an example showing a configuration which is not allowed in our analysis, i.e a set of choices $(l, m, a_1, \cdots, a_m, b_1, \cdots, b_l) \notin \Lambda$ (the values for a_i and b_i in the next example do ***not*** contribute to the wall crossing formula).

Example 15.2. Let n in (15.2) be equal to 13. Choose $l = 2$ and $m = 4$. The sequence for case **A** is shown pictorially in the picture below in which each $\bullet_i$ represents the element $(\psi_i, d_i) \in C(\mathcal{B}_p)$. Assume that the term $(0, 2)$ appears in between positions 7 and 8 . Now we compute $U_{\mathbf{A}}$ for this choice of l and m . Choose the configuration $0 = a_0 < \cdots < a_4 = 13$ and $0 = b_0 < b_1 < b_2 = 4$ to be given respectively as: $a_0 = 0$, $a_1 = 3$, $a_2 = 6$, $a_3 = 9$ $a_4 = 13$

and $b_0 = 0, b_1 = 2, b_2 = 4$



Using the formula $\theta_i = (\psi_{a_{i-1}+1}, d_{a_{i-1}+1}) + \cdots + (\psi_{a_i}, d_{a_i})$ for $i = 1, \cdots, 4$, one computes the values of θ_i for $i = 1, \cdots, 4$ as follows:

$\theta_1 = (\psi_{a_0+1}, d_{a_0+1}) + \cdots + (\psi_{a_1}, d_{a_1}) = (\psi_1, d_1) + (\psi_2, d_2) + (\psi_3, d_3) = (\beta_1 + \beta_2 + \beta_3, 0)$.

$\theta_2 = (\psi_{a_1+1}, d_{a_1+1}) + \cdots + (\psi_{a_2}, d_{a_2}) = (\psi_4, d_4) + (\psi_5, d_5) + (\psi_6, d_6) = (\beta_4 + \beta_5 + \beta_6, 0)$.

$\theta_3 = (\beta_7 + \beta_8, 2)$.

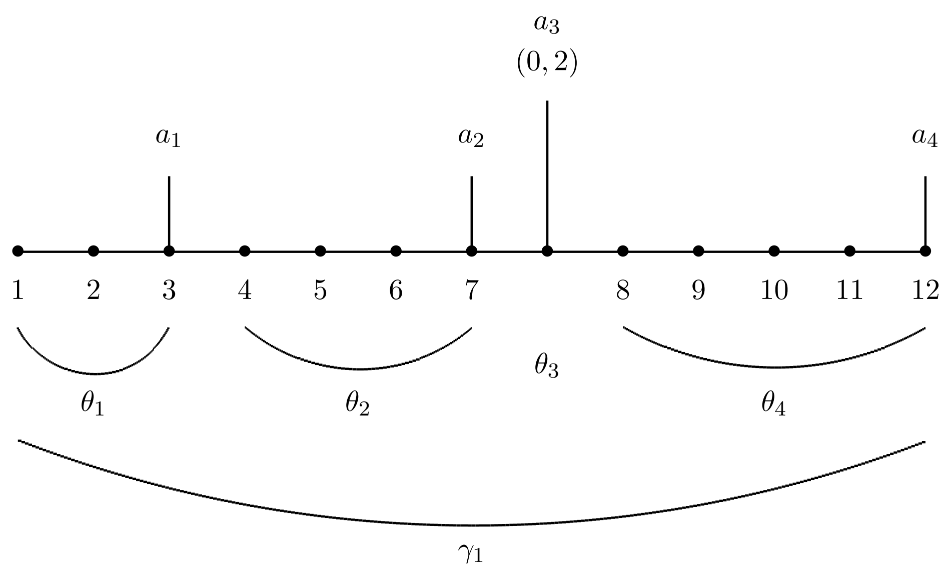
$\theta_4 = (\psi_{a_3}, d_{a_3}) + \cdots (\psi_{a_4-1}, d_{a_4-1}) = (\psi_8, d_8) + (\psi_9, d_9) + (\psi_{10}, d_{10}) + (\psi_{11}, d_{11}) + (\psi_{12}, d_{12}) = (\beta_9 + \beta_{10} + \beta_{11} + \beta_{12} + \beta_{13}, 0)$

Next step is to use the results obtained above and to compute γ_i's using the formula $\gamma_i = \theta_{b_{i-1}+1} + \cdots + \theta_{b_i}$ for $i = 1, 2$. We obtain the following: $\gamma_1 = \theta_{b_0+1} + \cdots + \theta_{b_1} = \theta_1 + \theta_2 = (\beta_1 + \cdots + \beta_6, 0)$. $\gamma_2 = \theta_{b_1+1} + \cdots + \theta_{b_2} = \theta_3 + \theta_4 = (\beta_7 + \cdots + \beta_{13}, 2)$.

From this computation, it is obvious that the condition $\tilde{\tau}(\gamma_i) = \tilde{\tau}(\beta, 2)$ for $i = 1, 2$ is not satisfied since in this case, $\tilde{\tau}(\gamma_1) = 0 \neq \tilde{\tau}(\gamma_2) = \tilde{\tau}(\beta, 2) = 1$. Therefore we need to have $l = 1$ in order to satisfy this condition. Moreover the second condition $\tau^\bullet(\theta_i) = \tau^\bullet(\beta_j, d_j)$ $\forall i = 1, \cdots 4$ and $\forall j, a_{i-1} < j \leq a_i$ is not satisfied since for $i = 3$ there exists $a_2 = 6 < j = 7 \leq 9 = a_3$ (a_3 corresponds to (β_9, d_9) or $\bullet_8 = (\psi_8, d_8)$ which corresponds to (β_9, d_9) because of re-parameterization) for which

$\tau^\bullet(\theta_3) = \tau^\bullet(\beta_7 + \beta_8, 2) = -1 \neq \tau^\bullet(\beta_7, 0) = 0$. So we need to choose a configuration in which $(0, 2)$ is the only term appearing in between a_{i-1} and a_i. (i.e the should exist some $p = 1, \cdots, 4$ so that $a_{p-1} = 7$ and $a_p = 8$).

Example 15.3. Fix $m = 4$ and $n = 13$. Let us compute the value of $S_{\mathbf{A}}$ for the configuration shown in the figure below:



Similar to Example 15.2 first we compute the values of θ_i's for $i = 1, \cdots, 4$.

$\theta_1 = (\psi_{a_0+1} d_{a_0+1}) + \cdots + (\psi_{a_1}, d_{a_1}) = (\psi_1 + \psi_2 + \psi_3, d_1 + d_2 + d_3) = (\beta_1 + \beta_2 + \beta_3, 0)$

$\theta_2 = (\psi_{a_1+1} d_{a_1+1}) + \cdots + (\psi_{a_2}, d_{a_2}) = (\psi_4 + \cdots + \psi_7, d_4 + \cdots d_7) = (\beta_4 + \cdots + \beta_7, 0)$

$\theta_3 = (0, 2)$

$\theta_4 = (\psi_{a_3} d_{a_3}) + \cdots + (\psi_{a_4-1}, d_{a_4-1}) = (\psi_8 + \cdots + \psi_{12}, d_8 + \cdots + d_{12}) = (\beta_9 + \cdots + \beta_{13}, 0)$

Next step is to compute γ_1 using the formula $\gamma_1 = \theta_1 + \cdots + \theta_4$.

$\gamma_1 = (\beta_1 + \cdots + \beta_{13}, 2)$. Consider Definition 14.9. If neither condition (a) nor (b) are satisfied, for any i, we set $S_{\mathbf{A}} = 0$, otherwise, $S_{\mathbf{A}}(\theta_1, \cdots, \theta_4; \tau^\bullet, \tilde{\tau}) = (-1)^r$ where r is equal to number of times that for $i = 1, \cdots, 4 - 1 = 3$, condition (a) is satisfied, i.e:

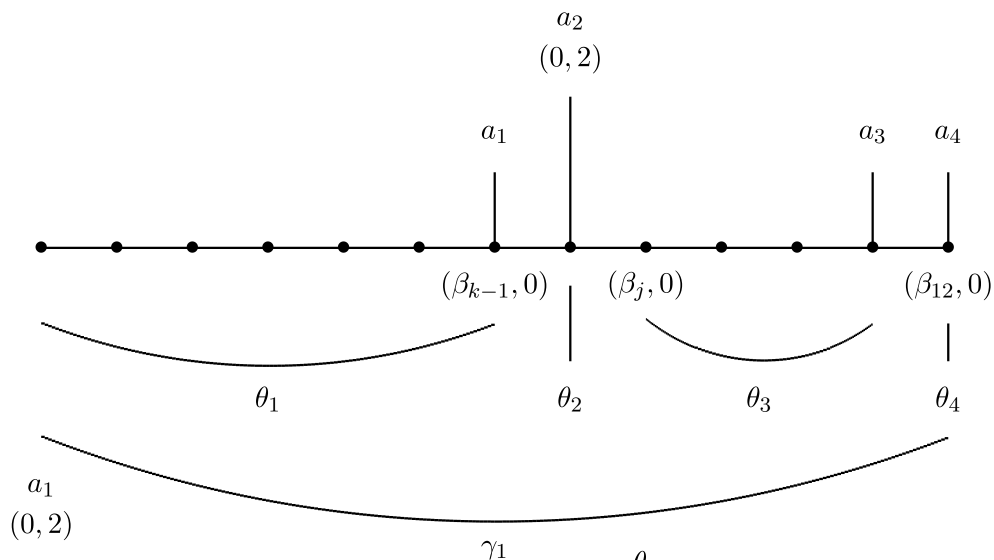
(a). $\tau^\bullet(\theta_i) \leq \tau^\bullet(\theta_{i+1})$ and $\tilde{\tau}(\theta_1 + \cdots + \theta_i) > \tilde{\tau}(\theta_{i+1} + \cdots + \theta_4)$.

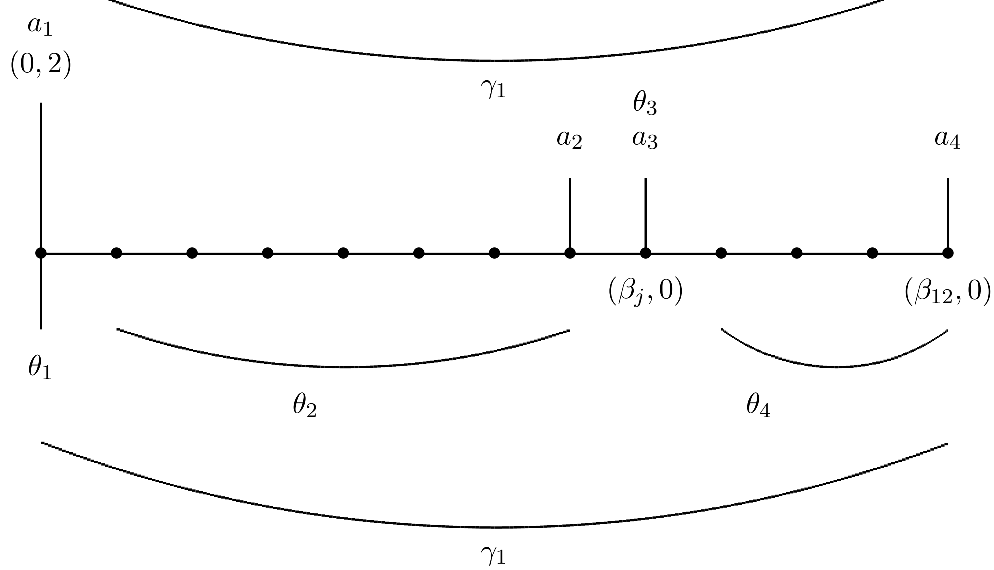
For $i = 1$, $0 = \tau^\bullet(\theta_1) \leq \tau^\bullet(\theta_2) = 0$ and $0 = \tilde{\tau}(\theta_1) \not> \tilde{\tau}(\theta_2 + \cdots + \theta_{11}) = 1$

For $i = 2$, $0 = \tau^\bullet(\theta_2) > \tau^\bullet(\theta_3) = -1$ and $0 = \tilde{\tau}(\theta_1 + \theta_2) \leq \tilde{\tau}(\theta_3 + \theta_4) = 1$

For $i = 3$, $-1 = \tau^{\bullet}(\theta_3) \leq \tau^{\bullet}(\theta_4) = 0$ and $1 = \tilde{\tau}(\theta_1 + \theta_2 + \theta_3) > \tilde{\tau}(\theta_4) = 0$

As it is seen, for $i = 1$ neither condition (a) nor (b) are satisfied therefore $S_{\mathbf{A}} = 0$. Considering the particular choice of $m = 4$ and $n = 13$ as in Example 15.3 the pictures below show the only allowable configurations for which one obtains nonzero values for $S_{\mathbf{A}}$:





Here we summarize the above observations and apply them to the general case:

1. In order to have $\tilde{\tau}(\gamma_i) = \tilde{\tau}(\beta, 2)$ for all $i = 1, \cdots, l$ one should set $l = 1$, [18] (Proposition 15.8). Therefore the set Λ reduces to the set of choices of m where $1 \leq m \leq n$.

2. It is clear that the only way that $\tau^{\bullet}(\theta_i) = \tau^{\bullet}(\beta_j, d_j)$ for $i = 1, \cdots, m$ and $a_{i-1} < j \leq a_i$ is that there exists some $p = 1, \cdots, m$ where $a_{p-1} = k - 1$ and $a_p = k$ (k =location of $(0, 2)$).

In (15.2) $\tau^\bullet(\theta_i) = 0$ for $i < p$ and $\tau^\bullet(\theta_p) = -1$ and $\tau^\bullet(\theta_i) = 0$ for $i > p$, therefore the following hold true:

1. $\tau^\bullet(\theta_i) = \tau^\bullet(\theta_{i+1}) = 0$ and $\tilde{\tau}(\theta_1 + \cdots + \theta_i) \not> \tilde{\tau}(\theta_{i+1} + \cdots + \theta_n)$ for $i < p-1$

2. $0 = \tau^\bullet(\theta_i) > \tau^\bullet(\theta_{i+1}) = -1$ and $0 = \tilde{\tau}(\theta_1 + \cdots + \theta_i) \leq \tilde{\tau}(\theta_{i+1} + \cdots + \theta_n) = 1$ for $i = p-1$

3. $\tau^\bullet(\theta_i) \leq \tau^\bullet(\theta_{i+1})$ and $\tilde{\tau}(\theta_1 + \cdots + \theta_i) > \tilde{\tau}(\theta_{i+1} + \cdots + \theta_n)$ for $i \geq p$

From this analysis one conclude that in (15.2) for $i < p-1$ neither condition (a) nor (b) are satisfied for $i = p-1$ condition (b) is satisfied and for $i \geq p$ condition (a) is satisfied (this implies $p = 1$ or $p = 2$). Finally similar to the above example, $p = 1$ when $k = 1$ and $p > 1$ when $k > 1$ and $S_{\mathbf{A}} = 0$ for $p > 2$. Now we can simplify $U_{\mathbf{A}}$ as follows:

$$
\begin{aligned}
\mathbf{A} = & \left(\sum_{\substack{p=1, \\ 0=a_0<a_1=1<a_2 \\ <\cdots<a_m}} \sum_m S_{\mathbf{A}}(\theta_1, \theta_2, \cdots, \theta_m; \tau^\bullet, \tilde{\tau}). \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!} \right) \cdot \bar{\epsilon}^{(0,2)} * \bar{\epsilon}^{(\psi_2,0)} * \cdots * \bar{\epsilon}^{(\psi_{n-1},0)} \\
& + \sum_{1<k\leq n} \left(\sum_{\substack{p=2, \\ 0=a_0<a_1=k-1<a_2=k< \\ \cdots<a_m}} \sum_m S_{\mathbf{A}}(\theta_1, \theta_2, \cdots, \theta_m; \tau^\bullet, \tilde{\tau}). \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!} \right) \\
& \cdot \bar{\epsilon}^{(\psi_1,0)} * \cdots * \bar{\epsilon}^{(\psi_{k-1},0)} * \bar{\epsilon}^{(0,2)} * \bar{\epsilon}^{(\psi_k,0)} * \cdots * \bar{\epsilon}^{(\psi_{n-1},0)} \\
& = \sum_{\substack{1\leq m\leq n, \\ 1=a_1<a_2<\cdots<a_m}} (-1)^{m-1} \cdot \prod_{i=2}^m \frac{1}{(a_i - a_{i-1})!} \cdot \bar{\epsilon}^{(0,2)} * \bar{\epsilon}^{(\psi_2,0)} * \cdots * \bar{\epsilon}^{(\psi_{n-1},0)} \\
& + \sum_{1<k\leq n} \frac{1}{(k-1)!} \cdot \sum_{1\leq m\leq n, k=a_2<a_3<\cdots<a_m} (-1)^{m-2} \cdot \prod_{i=3}^m \frac{1}{(a_i - a_{i-1})!} \\
& \cdot \bar{\epsilon}^{(\psi_1,0)} * \cdots * \bar{\epsilon}^{(\psi_{k-1},0)} * \bar{\epsilon}^{(0,2)} * \bar{\epsilon}^{(\psi_k,0)} * \cdots * \bar{\epsilon}^{(\psi_{n-1},0)}
\end{aligned}
\tag{15.3}
$$

Simplification of B

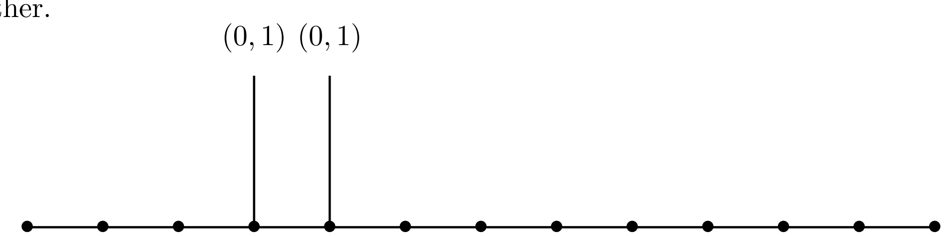
Recall the definition of $U_{\mathbf{B}}$ which appears in $\mathbf{B}$,

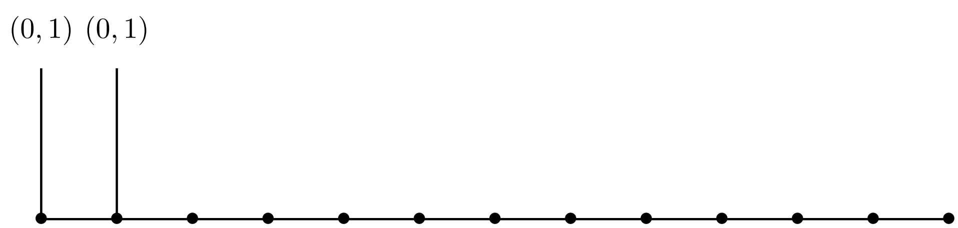
$$U_{\mathbf{B}} = U((\psi_1, 0), \cdots, (\psi_{k-1}, 0), (0,1), (\psi_k, 0), \cdots, (\psi_{k'-1}, 0), (0,1), (\psi_{k'}, 0), \cdots, (\psi_{n-2}, 0); \tau^{\bullet}, \tilde{\tau})$$
$$= \sum_{1 \le l \le m \le n} \frac{(-1)^{l-1}}{l} \cdot \prod_{i=1}^{l} S_{\mathbf{B}}(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots \theta_{b_i}; \tau^{\bullet}, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!} \tag{15.4}$$

To compute $S_{\mathbf{B}}$ we divide our analysis into three combinatorial cases based on how the $(0,1)$ elements are located in the diagrams:

15.1 Case 1

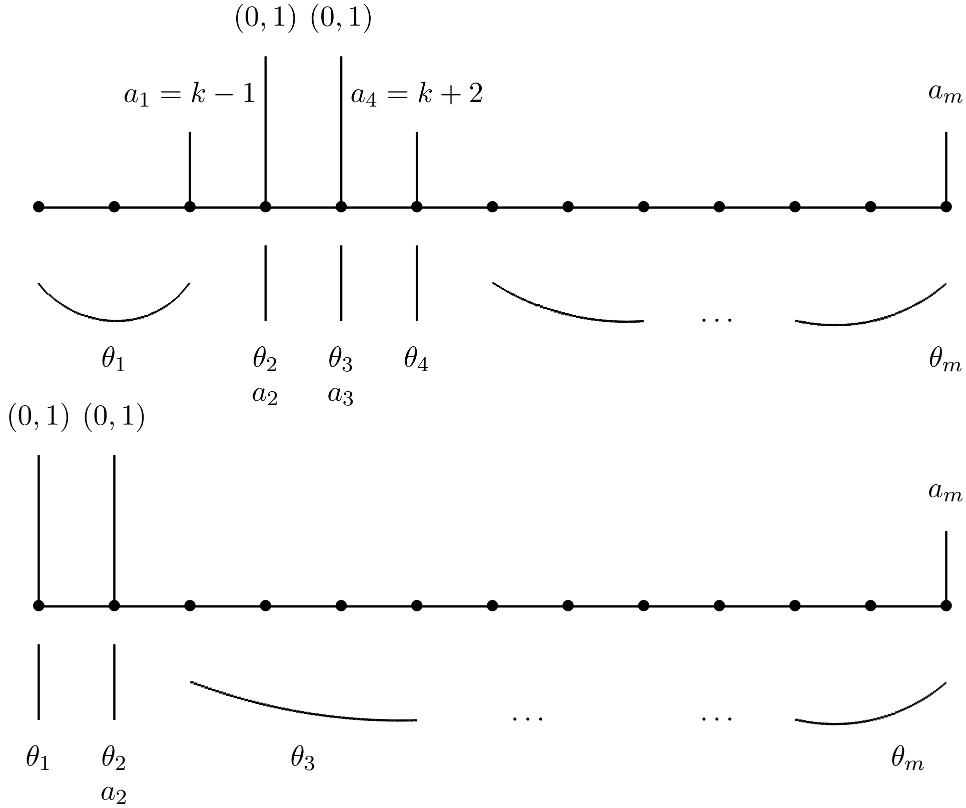
Definition 15.4. Case 1 represents the configurations where, the two $(0,1)$ elements occur adjacent to each other.





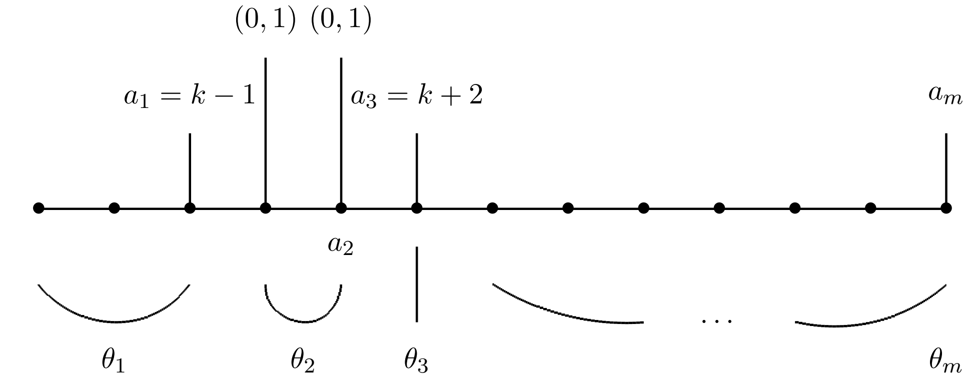
Now choose and distribute a_i in order to obtain equation (14.8). The following diagrams describe the two possible distribution types for a_i, we call them by Type 1 and Type 2. As shown in both Type 1 and Type 2 we assume that the first occurrence of a $(0,1)$ element is at k'th location. First assume $k > 1$. In that case $a_1 = k - 1$. Now for Type 1 we set $a_2 = k$ and $a_3 = k + 1$. One may freely choose any value for a_4 as far as $a_4 \geq k+2$. The diagram below depicts what the configuration would look like if one chooses $a_4 = k+2$. The remaining possible configurations (for $a_4 > k+2$) can be drawn similarly. The second diagram in Type 1 explains a situation where $k = 1$.

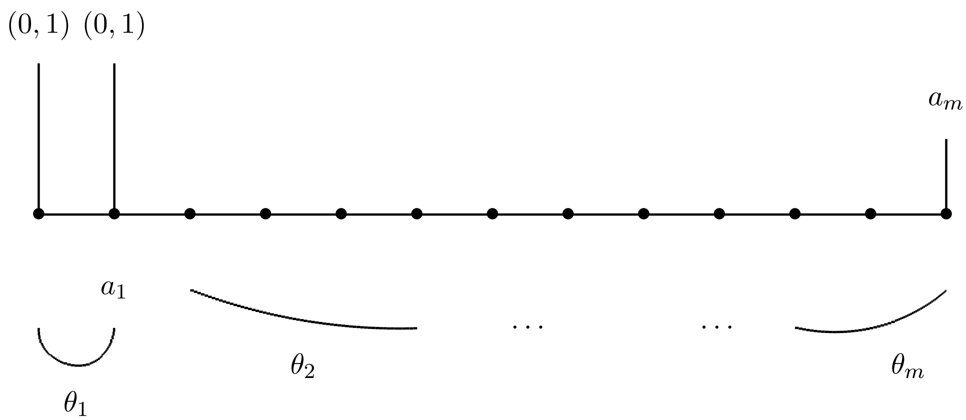
Type 1



After setting $a_1 = k-1$ what distinguishes Type 2 from Type 1 is the choice of a_2. In Type 2 (diagram below) we set $a_2 = k+1$ and a_3 can be chosen freely (similar to a_4 in Type 1) to have any value as long as $a_3 \geq k+2$. The first diagram in Type 2 depicts what the configuration would look like if $k > 1$ and one chooses $a_3 = k+2$. The remaining possible configurations (for $a_3 > k+2$) can be drawn similarly. The second diagram in Type 2 explains a situation where $k = 1$

Type 2





In this section, to avoid confusion we denote by $S_{\mathbf{B}1}$ the contribution to $S_{\mathbf{B}}$ of Case 1. First, we clarify the calculations for Case 1, via two simplified examples:

Example 15.5. Consider the case where $(\beta, 2) = (0, 2)$. The stack function identity obtained from the change of weak stability condition in category $\mathcal{B}_p$ is given by:

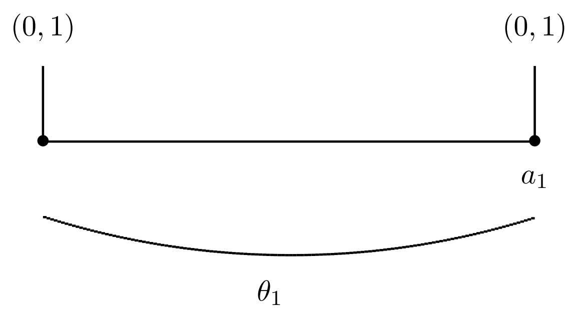
$$\bar{\epsilon}^{(0,2)}(\tilde{\tau}) = a \cdot \bar{\epsilon}^{(0,2)}(\tau^{\bullet}) + b \cdot \bar{\epsilon}^{(0,1)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,1)}(\tau^{\bullet}). \tag{15.5}$$

Here, a and b are combinatorial coefficients. Note that the second term on the right hand side of Equation (15.5) is the simplest possible example of the Case 1 which is described pictorially as:

We will show that $b = 0$. Similar to the general case discussed above, there exists two distrubution types for a_i's.

Type 1

Type 2



The function U for configuration in Type 1 is given as:

$$U = (\frac{-1}{2})S(\theta_1)S(\theta_2)\prod_{i=1}^{2}\frac{1}{(a_i - a_{i-1})!} = (\frac{-1}{2})(-1)^0(-1)^0\frac{1}{(a_1 - a_0)!}\frac{1}{(a_2 - a_1)!} = (\frac{-1}{2}). \quad (15.6)$$

Similarly the function U for configuration in Type 2 is given as:

$$U = S(\theta_1)\frac{1}{(a_1 - a_0)!} = (-1)^0\frac{1}{(a_1 - a_0)!} = \frac{1}{2}. \quad (15.7)$$

The second term on the right hand side of Equation (15.5) is obtained by adding the values of the function U obtained from configurations in Type 1 and Type 2:

$$(\frac{-1}{2})\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,1)}(\tau^{\bullet}) + \frac{1}{2}\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,1)}(\tau^{\bullet}) = 0. \tag{15.8}$$

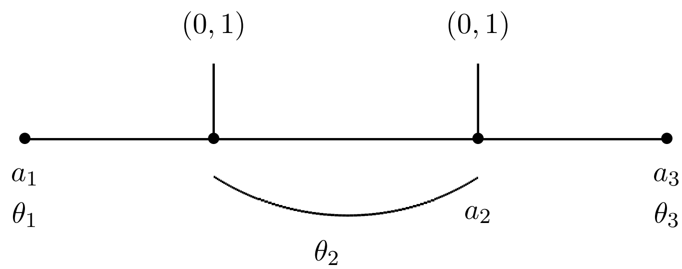
Hence in Equation (15.5) $b = 0$.

To clarify how this calculation can be generalized to the computation in Equation (15.20), we consider a more complicated example where our diagrams have 4 vertices.

Example 15.6.

Type 1

Type 2



Let us compute the function U for Type 1. First we set $l = 1$. In this case one needs to compute $S(\theta_1, \theta_2, \theta_3, \theta_4)$. However note that $\tau^{\bullet}(\theta_2) = -1 = \tau^{\bullet}(\theta_3)$ and $1 = \tilde{\tau}(\theta_1 + \theta_2) \not> \tilde{\tau}(\theta_3 + \theta_4) = 1$ hence neither condition (a) nor condition (b) in Definition 14.9 are satisfied and hence $S(\theta_1, \theta_2, \theta_3, \theta_4) = 0$. Now set $l = 2$ since for $l > 2$ one may always obtain a configuration in which $\tilde{\tau}(\gamma_i) = 0 \neq \tilde{\tau}(\beta, 2) = 1$ for some i and so $S_B = 0$ in that case. Therefore set $l = 2$ and obtain :

$$U_1 = \frac{(-1)^1}{2} S(\theta_1, \theta_2) S(\theta_3, \theta_4) \prod_{i=1}^{4} \frac{1}{(a_i - a_{i-1})!} = (\frac{1}{1})(-1)^0 \cdot (-1)^1 \frac{1}{(a_1 - a_0)!} = (\frac{-1}{2})(-1)^0(-1)^1 = \frac{1}{2}. \tag{15.9}$$

For Type 2, l needs to be set equal to 1 since similarly for $l > 1$ one may always obtain a configuration in which $\tilde{\tau}(\gamma_i) = 0 \neq \tilde{\tau}(\beta, 2) = 1$ for some i and so $S_B = 0$ in that case. Therefore, one obtains:

$$U_2 = \frac{(-1)^0}{1} S(\theta_1, \theta_2, \theta_3) \prod_{i=1}^{3} \frac{1}{(a_i - a_{i-1})!} = (\frac{1}{1})(-1)^1 \frac{1}{(a_2 - a_1)!} = (-1)^0(-1)^1 \frac{1}{2} = -\frac{1}{2}. \quad (15.10)$$

We emphasize that the presence of the term $\frac{1}{2}$ in Equation (15.10) is because here, $a_2 - a_1 = 2$. Therefore we obtain:

$$U((0,1),(0,1)) = U_1((0,1),(0,1)) + U_2((0,1),(0,1)) = -\frac{1}{2} + \frac{1}{2} = 0 \quad (15.11)$$

i.e, the contributions associated to Case 1 are obtained as:

$$\begin{aligned} &(\frac{-1}{2})\bar{\epsilon}^{(\beta_1,0)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(\beta_1,0)}(\tau^\bullet) \\ &+ \frac{1}{2}\bar{\epsilon}^{(\beta_1,0)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(\beta_2,0)}(\tau^\bullet) = 0 \end{aligned} \quad (15.12)$$

Here we compute $S_{\mathbf{B1}}$ coming from the fixed distributions of a_i's as shown in Type 1. Consider the first diagram in Type 1. We set for the variable l in (14.8), $l = 1$ or $l = 2$, (for $l > 2$, $S_{\mathbf{B1}} = 0$). If $l = 1$ then according to formula (14.8) we need to compute $S_{\mathbf{B1}}(\theta_1, \cdots, \theta_m)$. Note that $\tau^\bullet(\theta_2) = -1$ and $\tau^\bullet(\theta_3) = -1$ then $\tau^\bullet(\theta_2) \leq \tau^\bullet(\theta_3)$ however $\tilde{\tau}(\theta_1+\theta_2) \not> \tilde{\tau}(\theta_3+\cdots+\theta_m)$, hence neither condition (a) nor (b) in Definition 14.9 are satisfied and $S_{\mathbf{B1}}(\theta_1, \cdots, \theta_m) = 0$.

Now set $l = 2$. Setting $l = 2$ means that we need to choose $0 = b_0 < b_1 < b_2 = m$ so that b_i, $i = 0, 1, 2$, satisfy the conditions in Definition (14.8). Note that one can choose $b_1 = 1, \cdots, m$. However the only allowed choice for b_1 is to set $b_1 = 2$. We explain this fact further.

Set $b_1 = 1$, in that case $\gamma_1 = \theta_1$ and $\gamma_2 = \theta_2 + \cdots + \theta_m$. This configuration is not allowed, since for γ_1, $\tilde{\tau}(\gamma_1) = 0 \neq \tilde{\tau}(\beta, 2) = 1$. One easily observes that using similar arguments, the only allowable choice is to set $b_1 = 2$. Define:

$$U_1((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) :=$$
$$\sum_\Lambda \frac{-1}{2} S_{\mathbf{B1}}(\theta_1, \theta_2) \cdot S_{\mathbf{B1}}(\theta_3, \cdots, \theta_m) \cdot \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!}, \tag{15.13}$$

where by similar arguments $S_{\mathbf{B1}}(\theta_1, \theta_2) = (-1)^0 = 1$ and $S_{\mathbf{B1}}(\theta_3, \cdots, \theta_m) = (-1)^{(m-3)}$. Hence

$$U_1((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) = (-1) \cdot \sum_\Lambda \frac{1}{2}(-1)^{(m-3)} \cdot \prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!}$$
$$= (-1) \cdot \sum_\Lambda \frac{1}{2}(-1)^{(m-3)} \cdot \frac{1}{(a_3 - a_2)!} \cdot \frac{1}{(a_2 - a_1)!} \cdot \frac{1}{(a_1 - a_0)!} \cdot \prod_{i=4}^m \frac{1}{(a_i - a_{i-1})!}. \tag{15.14}$$

By looking at first diagram in Type 1, it is easy to see that $a_0 = 0, a_1 = k-1$, $a_2 = k$ and $a_3 = k+1$. Hence $(a_2 - a_1) = 1$ and $a_1 - a_0 = k - 1$. Now we use the result of Lemma 13.9 of [18] and rewrite this equation as follows:

$$U_1((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) =$$
$$(-\frac{1}{2}) \cdot \frac{1}{(a_3 - a_2)!} \cdot \frac{1}{(a_2 - a_1)!} \cdot \frac{1}{(a_1 - a_0)!} \sum_{1 \leq m \leq l} (-1)^{(m-3)} \cdot \prod_{i=4}^m \frac{1}{(a_i - a_{i-1})!} =$$
$$(-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(1+k))}}{(n-(1+k))!}. \tag{15.15}$$

A similar analysis is carried out for the second diagram in Type 1. The result would be equal to the one obtained in equation (15.15) for $k = 1$.

Now consider the first diagram in Type 2. Note that in this case $\theta_2 = (0,1) + (0,1) = (0,2)$. We can set $l = 1$ or $l = 2$. Setting $l = 2$ would result in obtaining a disallowed configuration, since there would always exist at least one γ_i for $i = 1, 2$ so that $\tilde{\tau}(\gamma_i) = 0 \neq \tilde{\tau}(\beta, 2) = 1$. Hence we set

$l = 1$.

Define:

$$U_2((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) := \sum_{\Lambda} S_{\mathbf{B}1}(\theta_1, \theta_2, \theta_3, \cdots, \theta_m) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}, \tag{15.16}$$

where by similar arguments, $S_{\mathbf{B}1}(\theta_1, \cdots, \theta_m) = (-1)^{(m-2)}$. Hence

$$\begin{aligned} U_2((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) &= \sum_{\Lambda} (-1)^{(m-2)} \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!} \\ = \sum_{\Lambda} (-1)^{(m-2)} \cdot \frac{1}{(a_2 - a_1)!} \cdot \frac{1}{(a_1 - a_0)!} \prod_{i=3}^{m} \frac{1}{(a_i - a_{i-1})!}. \end{aligned} \tag{15.17}$$

By looking at first diagram in Type 2, it is easy to see that $a_0 = 0, a_1 = k - 1$ and $a_2 = k + 1$, hence $(a_2 - a_1) = 2$ and $a_1 - a_0 = k - 1$. Now we use the result of Lemma 13.9 of [18] and rewrite this equation as follows:

$$\begin{aligned} &U_2((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) = \\ &\frac{1}{(a_1 - a_0)!} \cdot \frac{1}{(a_2 - a_1)!} \cdot \sum_{1 \le m \le l} (-1)^{(m-2)} \cdot \prod_{i=3}^{m} \frac{1}{(a_i - a_{i-1})!} = \frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(1+k))}}{(n-(1+k))!}. \end{aligned} \tag{15.18}$$

By Equation (14.8):

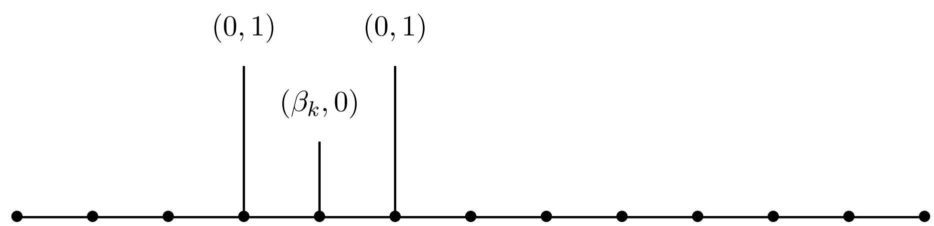
$$\begin{aligned} &U((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) = \\ &U_1((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) \\ &+ U_2((\beta_1, 0), \cdots, (0,1), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) \end{aligned} \tag{15.19}$$

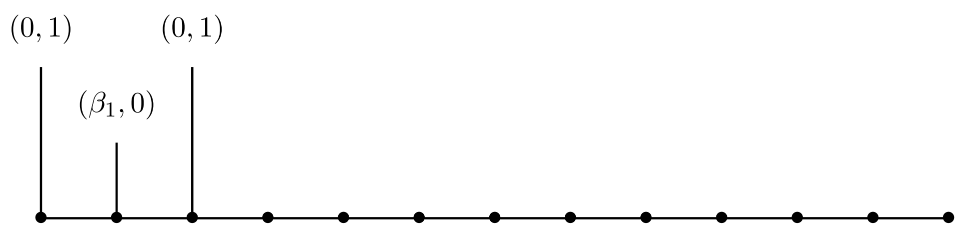
adding the values of the function $U_i,\ i = 1, 2$ obtained from the two distributions in Type 1 and Type 2, we obtain:

$$U((\beta_1, 0), \cdots, (0, 1), (0, 1), \cdots, (\beta_{n-2}, 0); \tau^{\bullet}, \tilde{\tau}) =$$
$$\frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-1-k)}}{(n-1-k)!} + (-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-1-k)}}{(n-1-k)!} = 0 \tag{15.20}$$

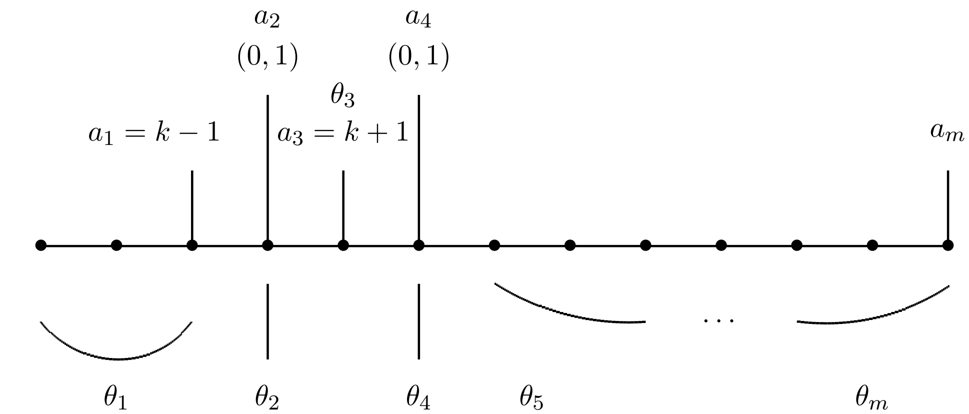
15.2 Case 2

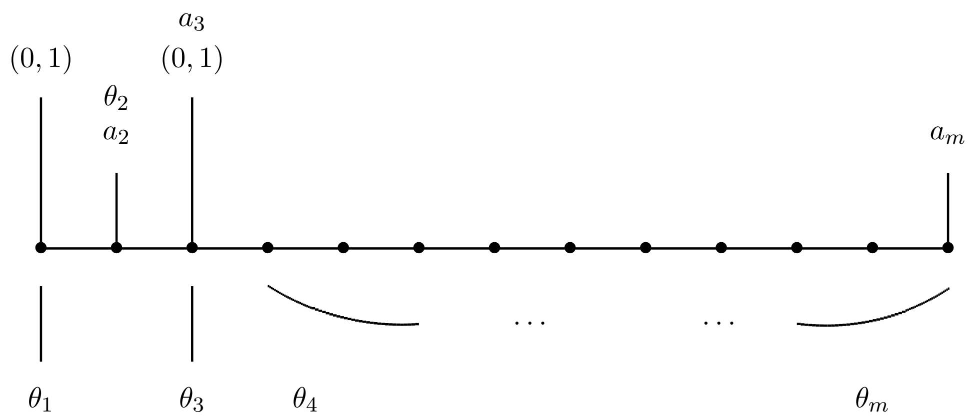
Definition 15.7. Case 2 represents the configurations where there exists some $1 \leq k \leq n$ for which there exists only one element of type $(\beta_k, 0)$ between the two elements of type $(0, 1)$ such that $\beta_k \neq 0$.





The set of allowable distributions for a_i's is given as:





Consider the first diagram in Case 2. Similar to Type 1 in Case 1, we can argue that the only possible value for l in both diagrams is $l = 2$. For $l = 1$, for example, consider θ_2 and θ_3 in the first diagram. Note that $\tau^\bullet(\theta_2) \leq \tau^\bullet(\theta_3)$ but $\tilde{\tau}(\theta_1 + \theta_2) \not> \tilde{\tau}(\theta_3 + \cdots + \theta_m)$ hence $S_{\mathbf{B}2}(\theta_1, \cdots, \theta_m) = 0$. Setting $l = 2$ means that we need to choose $0 = b_0 < b_1 < b_2 = m$ so that b_i, $i = 0, 1, 2$, satisfy the conditions in Definition (14.8). Note that one can choose $b_1 = 2$ or $b_1 = 3$. Set $b_1 = 2$. Define:

$$U_1((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) :=$$
$$\sum_{\Lambda} \frac{-1}{2} S_{\mathbf{B}1}(\theta_1, \theta_2) \cdot S_{\mathbf{B}1}(\theta_3, \cdots, \theta_m) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}, \tag{15.21}$$

where $S_{\mathbf{B2}}(\theta_1,\theta_2) = (-1)^0 = 1$ and $S_{\mathbf{B2}}(\theta_3,\cdots,\theta_m) = (-1)^{(m-4)}$. Hence

$$U_1((\beta_1,0),\cdots,(0,1),(\beta_k,0),(0,1),\cdots,(\beta_{n-2},0);\tau^{\bullet},\tilde{\tau}) = (-1)\cdot\sum_{\Lambda}\frac{1}{2}(-1)^{(m-4)}\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}. \tag{15.22}$$

Similar to before, we use the result of Lemma 13.9 of [18] and rewrite this equation as follows:

$$\begin{aligned}
&U_1((\beta_1,0),\cdots,(0,1),(\beta_k,0),(0,1),\cdots,(\beta_{n-2},0);\tau^{\bullet},\tilde{\tau}) = \\
&(-\frac{1}{2})\cdot\frac{1}{(a_4-a_3)!}\cdot\frac{1}{(a_3-a_2)!}\cdot\frac{1}{(a_2-a_1)!}\cdot\frac{1}{(a_1-a_0)!}\sum_{1\le m\le l}(-1)^{(m-4)}\cdot\prod_{i=5}^{m}\frac{1}{(a_i-a_{i-1})!} = \\
&(-\frac{1}{2})\cdot\frac{1}{(k+2-(k+1))!}\cdot\frac{1}{(k+1-(k))!}\cdot\frac{1}{(k-(k-1))!}\cdot\frac{1}{(k-1)!}\cdot\frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} = \\
&(-\frac{1}{2})\cdot\frac{1}{(k-1)!}\cdot\frac{(-1)^{(n-(k+2))}}{(n-(k+2))!}
\end{aligned} \tag{15.23}$$

Now Set $b_1 = 3$. Define:

$$\begin{aligned}
&U_2((\beta_1,0),\cdots,(0,1),(\beta_k,0),(0,1),\cdots,(\beta_{n-2},0);\tau^{\bullet},\tilde{\tau}) = \\
&\sum_{\Lambda}\frac{-1}{2}S_{\mathbf{B1}}(\theta_1,\theta_2,\theta_3)\cdot S_{\mathbf{B1}}(\theta_4,\cdots,\theta_m)\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!},
\end{aligned} \tag{15.24}$$

where $S_{\mathbf{B2}}(\theta_1,\theta_2,\theta_3) = (-1)^1 = -1$ and $S_{\mathbf{B2}}(\theta_4,\cdots,\theta_m) = (-1)^{(m-4)}$. Hence

$$U_2((\beta_1,0),\cdots,(0,1),(\beta_k,0),(0,1),\cdots,(\beta_{n-2},0);\tau^{\bullet},\tilde{\tau}) = \sum_{\Lambda}\frac{1}{2}(-1)^{(m-4)}\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}. \tag{15.25}$$

similar to before, we use the result of Lemma 13.9 of [18] and rewrite this equation as follows:

$$
\begin{aligned}
&U_2((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) = \\
&\frac{1}{2} \cdot \frac{1}{(a_3 - a_2)!} \frac{1}{(a_2 - a_1)!} \cdot \frac{1}{(a_1 - a_0)!} \sum_{1 \le m \le l} (-1)^{(m-4)} \cdot \prod_{i=5}^{m} \frac{1}{(a_i - a_{i-1})!} = \\
&\frac{1}{2} \cdot \frac{1}{(k+2-(k+1))!} \cdot \frac{1}{(k+1-(k))!} \frac{1}{(k-(k-1))!} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} = \\
&\frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!}.
\end{aligned}
\tag{15.26}
$$

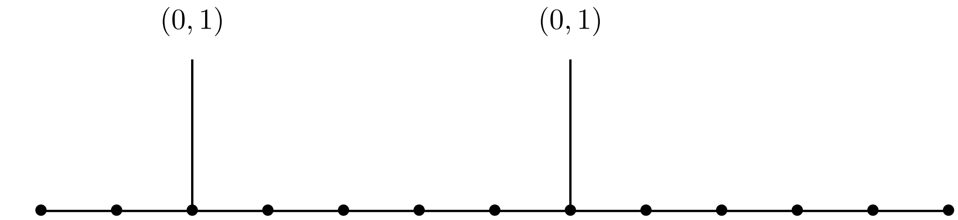
By adding the contributions due to the two choices of $b_1 = 2$ and $b_1 = 3$, we obtain

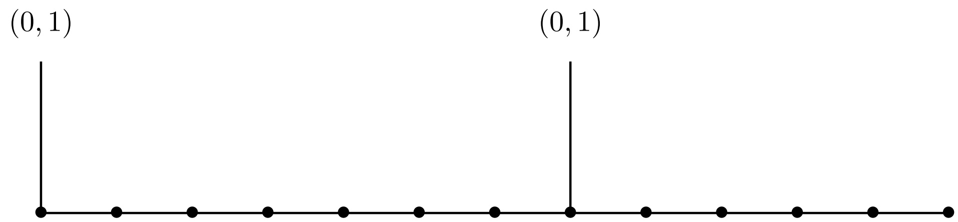
$$
\begin{aligned}
&U((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) = \\
&U_1((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) + \\
&U_2((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) = \\
&(-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} + \frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} = 0
\end{aligned}
\tag{15.27}
$$

A similar analysis is carried out for the second diagram in Case 2. The result would be equal to the one obtained in equation (15.26) for $k = 1$. However, setting $k = 1$ in Equation (15.26), will still make Equation (15.27) equal to zero.

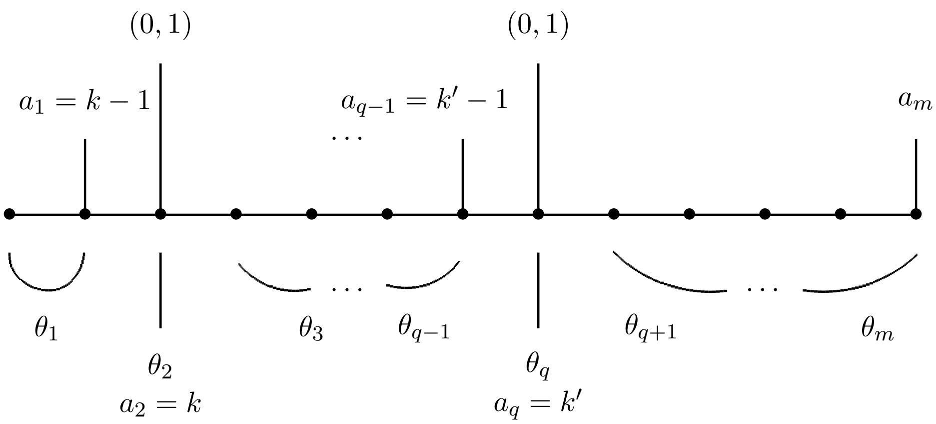
15.3 Case 3

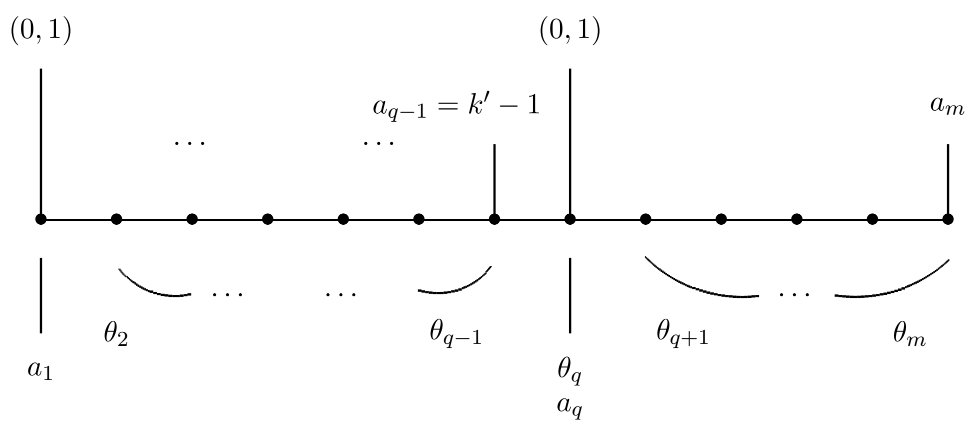
Definition 15.8. Case 3 represents the configurations where for some $1 \le k < k' \le n$, there exists at least 2 elements $(\beta_k, 0)$ and $(\beta_{k'}, 0)$ between the two elements of type $(0,1)$ such that $\beta_k \neq 0$ and $\beta_{k'} \neq 0$.





The possible set of distributions for a_i's is given as:





We compute the function U for the general case shown in the first diagram in Case 3. Using an argument similar to before, to get a nonzero value for $S_{\mathbf{B3}}$ we set $l = 2$.

$$U((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), \cdots, (\beta_{k'-2}, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) =$$
$$\sum_{\Lambda} \frac{-1}{2} \cdot \prod_{i=1}^{l} S_{\mathbf{B3}}(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots \theta_{b_i}; \tau^\bullet, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}$$
$$= \sum_{0=a_0<\cdots<a_m} \sum_{0=b_0<b_1<b_2=m} \frac{-1}{2} \cdot \prod_{i=1}^{l} S_{\mathbf{B3}}(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots \theta_{b_i}; \tau^\bullet, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}, \tag{15.28}$$

where a_i and b_i satisfy the conditions in Definition (14.8). Note that one can choose $b_1 = 2, \cdots, q-1$, hence the function U can be computed as follows:

$$U((\beta_1, 0), \cdots, (0,1), (\beta_k, 0), \cdots, (\beta_{k'-2}, 0), (0,1), \cdots, (\beta_{n-2}, 0); \tau^\bullet, \tilde{\tau}) =$$
$$\sum_{0=a_0<\cdots<a_m} \frac{1}{2} \Bigg[S_{\mathbf{B3}}(\theta_1, \theta_2) \cdot S_{\mathbf{B3}}(\theta_3, \cdots, \theta_m)$$
$$+ S_{\mathbf{B3}}(\theta_1, \theta_2, \theta_3) \cdot S_{\mathbf{B3}}(\theta_4, \cdots, \theta_m) + \cdots + S_{\mathbf{B3}}(\theta_1, \cdots, \theta_{q-1}) \cdot S_{\mathbf{B3}}(\theta_q, \cdots, \theta_m) \Bigg] \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}. \tag{15.29}$$

One sees that $S_{\mathbf{B3}}(\theta_1, \theta_2) = (-1)^0$ and $S_{\mathbf{B3}}(\theta_3, \cdots, \theta_m) = 0$ ($0 = \tau^\bullet(\theta_3) \leq \tau^\bullet(\theta_4)$ and $0 = \tilde{\tau}(\theta_3) \not> \tilde{\tau}(\theta_4 + \cdots + \theta_m) = 1$ hence nor (a) neither (b) in Definition 14.9 are satisfied). Following the same argument the terms $S_{\mathbf{B3}}(\theta_4, \cdots, \theta_m), \cdots S_{\mathbf{B3}}(\theta_{q-2}, \cdots, \theta_m)$ vanish. The non-vanishing of the term $S_{\mathbf{B3}}(\theta_{q-1}, \cdots, \theta_m)$ can be proved by observing that for θ_{q-1} and θ_q, $0 = \tau^\bullet(\theta_{q-1}) > \tau^\bullet(\theta_q) = -1$ and $0 = \tilde{(}\theta_{q-1}) \leq \tilde{\tau}(\theta_q + \cdots + \theta_m) = 1$ hence condition (b) in Definition 14.9 is satisfied and $S_{\mathbf{B3}}(\theta_{q-1}, \cdots, \theta_m) = (-1)^{(m-(q-1)-1)} = (-1)^{(m-q)}$. Finally $S_{\mathbf{B3}}(\theta_q, \cdots, \theta_m) = (-1)^{(m-q)}$. Now we use the fact that $S_{\mathbf{B3}}(\theta_1, \cdots, \theta_{q-2}) = (-1)^{(q-3)}$ and $S_{\mathbf{B3}}(\theta_1, \cdots, \theta_{q-1}) = (-1)^{(q-2)}$ hence the

Equation (15.29) can be rewritten as follows:

$$U_{\mathbf{B}} = U((\beta_1,0),\cdots,(0,1),(\beta_k,0),\cdots,(\beta_{k'-2},0),(0,1),\cdots,(\beta_{n-2},0);\tau^{\bullet},\tilde{\tau}) =$$
$$\sum_{0<a_0<\cdots<a_m} \frac{(-1)}{2}\cdot\left[(-1)^{(q-3)}\cdot(-1)^{(m-q)} + (-1)^{(q-2)}\cdot(-1)^{(m-q)}\right]\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!} = 0. \tag{15.30}$$

We conclude that the contributions in Cases 1, 2 and 3 are all equal to zero i.e in Equation (15.1):

$$\mathbf{B} = 0.$$

Recall that for $\mathbf{A}$ in Equation (15.3), the $(n-1)$'th $\mathcal{K}$-theory class, β_{n-1} was placed in the n'th spot, hence by change of variable $l-1$, the equation for $\mathbf{A}$ is given as:

$$\begin{aligned}\mathbf{A} &= \frac{(-1)^l}{(l)!}\cdot\bar{\epsilon}^{(0,2)}(\tau^{\bullet}) * \cdots * \bar{\epsilon}^{(\beta_l,0)}(\tau^{\bullet})\\ &+ \sum_{1\leq k\leq l}\frac{(-1)^{l-k}}{(k-1)!(l-k)!}\cdot\bar{\epsilon}^{(\beta_1,0)}(\tau^{\bullet}) * \cdots * \bar{\epsilon}^{(\beta_k,0)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,2)}(\tau^{\bullet}) * \bar{\epsilon}^{(\beta_{k+1},0)}(\tau^{\bullet}) * \cdots * \bar{\epsilon}^{(\beta_l,0)}(\tau^{\bullet})\\ &= \sum_{0\leq k\leq l}\frac{(-1)^{l-k}}{(k-1)!(l-k)!}\cdot\bar{\epsilon}^{(\beta_1,0)}(\tau^{\bullet}) * \cdots * \bar{\epsilon}^{(\beta_k,0)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,2)}(\tau^{\bullet}) * \bar{\epsilon}^{(\beta_{k+1},0)}(\tau^{\bullet}) * \cdots * \bar{\epsilon}^{(\beta_l,0)}(\tau^{\bullet})\end{aligned} \tag{15.31}$$

As we showed $\mathbf{B} = 0$. On the other hand the coefficients in (15.31) are precisely equal to those appearing on the right hand side of Equation (292) in [18]. By rewriting the product of stack functions in terms of a nested bracket we obtain an equation analogous to the computation of Joyce and Song in [18] (Proposition 15.10). Simply replace $\bar{\epsilon}^{(0,1)}(\tau^{\bullet})$ in Equation (292) in [18] with $\bar{\epsilon}^{(0,2)}(\tau^{\bullet})$ and obtain:

$$\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = \sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta}\frac{(-1)^l}{l!}[[\cdots[[\bar{\epsilon}^{(0,2)}(\tau^{\bullet}),\bar{\epsilon}^{(\beta_1,0)}(\tau^{\bullet})],\bar{\epsilon}^{(\beta_2,0)}(\tau^{\bullet})],\cdots],\bar{\epsilon}^{(\beta_l,0)}(\tau^{\bullet})] \tag{15.32}$$

Chapter 16

Calculation via Wallcrossing

Proposition 16.1. *(a). Let* $\nu^{(\beta,0)}_{\mathfrak{M}_{\mathcal{B}_p}}$ *and* $\nu^{\beta}_{\mathfrak{M}}$ *denote Behrend's constructible functions on the moduli stack of objects in* $\mathcal{B}_p$ *with fixed class* $(\beta, 0)$ *and the moduli stack of sheaves with second Chern character* β *respectively. The following identity holds true:*

$$\nu^{(\beta,0)}_{\mathfrak{M}_{\mathcal{B}_p}} \equiv \pi_0^*(\nu^{\beta}_{\mathfrak{M}}) \tag{16.1}$$

where π_0 *is the map* $\pi_0 : \mathfrak{M}^{(\beta,0)}_{\mathcal{B}_p} \to \mathfrak{M}^{\beta}$ *which sends* $(F, 0, 0)$ *with* $[(F, 0, 0)] = (\beta, 0)$ *to* F *with second Chern character* β.

Proof. This is proven in [18] (Proposition 13.12). □

Now we are ready to apply the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}$ defined in [18] (Section 13.4) to (15.32) and obtain the wall-crossing equation. First we study the image of $\bar{\epsilon}^{(\beta,2)}(\tilde{\tau})$, $\bar{\epsilon}^{(0,2)}(\tau^\bullet)$, $\bar{\epsilon}^{(\beta_i,0)}(\tau^\bullet)$ and $\bar{\epsilon}^{(0,1)}(\tau^\bullet)$ under the morphism $\tilde{\Psi}^{\mathcal{B}_p}$:

Definition 16.2. Define the invariant $\mathbf{B}^{ss}_p(X, \beta, 2, \tilde{\tau})$ associated to $\tilde{\tau}$-semistable objects of type $(\beta, 2)$ in $\mathcal{B}_p$ by

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\beta,2)}(\tilde{\tau})) = \mathbf{B}^{ss}_p(X, \beta, 2, \tilde{\tau}) \cdot \tilde{\lambda}^{(\beta,2)},$$

where $\tilde{\Psi}^{\mathcal{B}_p}$ is given by the Lie algebra morphism defined in [18] (Section 13.4).

Moreover, according to result of part (b) of Proposition 13.11 and the fact that $[\mathrm{Spec}(\mathbb{C})/\,\mathrm{GL}_2(\mathbb{C})]$

has dimension -4 we obtain the following:

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\delta}^{(0,2)}(\tau^{\bullet})) = \tilde{\lambda}^{(0,2)} \tag{16.2}$$

The next two identities that we list here are proved by Joyce and Song in [18] (13.5):

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,1)}(\tau^{\bullet})) = -\tilde{\lambda}^{(0,1)}. \tag{16.3}$$

Now suppose that $\beta = \sum_i \beta_i$ and β_i is indecomposable or (equivalently) there exist no strictly semistable sheaves with class β_i, then by [18] (13.5):

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\beta_i,0)}(\tau^{\bullet})) = -\overline{DT}^{\beta_i}(\tau)\tilde{\lambda}^{(\beta_i,0)} \tag{16.4}$$

where $\overline{DT}^{\beta_i}(\tau)$ is the generalized Donaldson-Thomas invariant defined by Joyce and Song in Definition (5.15) in [18]. To derive the wall crossing equation one writes the identity in (15.32) as an identity involving nested brackets of $\bar{\epsilon}^{(\beta_i,0)}(\tau^{\bullet})$. Now apply the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}$ to both sides of this equation and use the results obtained in (16.2), (16.3) and (16.4). One obtains the following equation:

$$\begin{aligned}&\mathbf{B}_p^{ss}(X,\beta,2,\tilde{\tau})\cdot\tilde{\lambda}^{(\beta,2)} = \\ &\sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta}\frac{(-1)^l}{l!}\cdot[[\cdots[[\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^{\bullet})),-\overline{DT}^{\beta_1}(\tau)\tilde{\lambda}^{(\beta_1,0)}],-\overline{DT}^{\beta_2}(\tau)\tilde{\lambda}^{(\beta_2,0)}],\cdots],-\overline{DT}^{\beta_l}(\tau)\tilde{\lambda}^{(\beta_l,0)}]\end{aligned} \tag{16.5}$$

16.1 Computation of $\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^{\bullet}))$

By part (b) of Proposition 13.11 the characteristic stack function of moduli stack of strictly $\tau^{\bullet}$-semistable objects in class $(0,2)$ is given by:

$$\bar{\delta}^{(0,2)}(\tau^{\bullet}) = \bar{\delta}(\mathfrak{M}^{(0,2),s}_{\mathcal{B}_p}(\tau^{\bullet})) = \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_2(\mathbb{C})}\right].$$

Joyce in [17] (6.2) has shown that given a stack function $\left[\left(\left[\frac{\mathcal{U}}{\mathrm{GL}_2(\mathbb{C})}\right],\nu\right)\right]$, where $\mathcal{U}$ is a quasi-projective variety, one has the following identity of stack functions:

$$\left[\left(\left[\frac{\mathcal{U}}{\mathrm{GL}_2(\mathbb{C})}\right],a\right)\right] = \\ F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m^2)\left[\left(\left[\frac{\mathcal{U}}{\mathbb{G}_m^2}\right],\mu\circ i_1\right)\right] + F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m)\left[\left(\left[\frac{\mathcal{U}}{\mathbb{G}_m}\right],\mu\circ i_2\right)\right], \tag{16.6}$$

where

$$\begin{aligned} F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m^2) &= \frac{1}{2} \\ F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m) &= -\frac{3}{4}, \end{aligned} \tag{16.7}$$

and $\mu\circ i_1$ and $\mu\circ i_2$ are the obvious embeddings. Substitute the values in (16.7) and obtain:

$$\bar{\delta}^{(0,2)}(\tau^\bullet) = \frac{1}{2}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}\right],\mu\circ i_1\right)\right] - \frac{3}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right],\mu\circ i_2\right)\right]. \tag{16.8}$$

In order to compute $\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^\bullet))$ one uses the definition of $\bar{\epsilon}^{(0,2)}(\tau^\bullet)$:

$$\bar{\epsilon}^{(0,2)}(\tau^\bullet) = \bar{\delta}^{(0,2)}(\tau^\bullet) - \frac{1}{2}\cdot\bar{\delta}^{(0,1)}(\tau^\bullet) * \bar{\delta}^{(0,1)}(\tau^\bullet). \tag{16.9}$$

Substitute the right hand side of (16.8) in (16.9) and obtain:

$$\bar{\epsilon}^{(0,2)}(\tau^\bullet) = \frac{1}{2}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}\right],\mu\circ i_1\right)\right] - \frac{3}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right],\mu\circ i_2\right)\right] - \frac{1}{2}\cdot\bar{\delta}^{(0,1)}(\tau^\bullet) * \bar{\delta}^{(0,1)}(\tau^\bullet). \tag{16.10}$$

Next we compute $\bar{\delta}^{(0,1)}(\tau^\bullet) * \bar{\delta}^{(0,1)}(\tau^\bullet)$ (which is equal to $\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet)$ since there exist no strictly $\tau^\bullet$-semistable objects in $\mathcal{B}_p$ with class $(0,1)$).

16.2 Computation of $\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,1)}(\tau^{\bullet})$

We know that $\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) \in \mathrm{SF}_{\mathcal{B}_p}$ is the stack function corresponding to the object $(0, \mathbb{C})$, in $\mathcal{C}(\mathcal{B}_p)$. Consider objects (F_i, V_i, ϕ_i) in $\mathcal{B}_p$ of type (β_i, d_i) for $i = 1, \cdots, 3$. In order to compute $\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,1)}(\tau^{\bullet})$ as in Definition 14.4 consider the moduli stack of exact sequences of objects in $\mathcal{B}_p$ of the form:

$$0 \to (F_1, V_1, \phi_1) \to (F_2, V_2, \phi_2) \to (F_3, V_3, \phi_3) \to 0 \tag{16.11}$$

and call it $\mathfrak{Exact}_{\mathcal{B}_p}$. Let $\pi_i : \mathfrak{Exact}_{\mathcal{B}_p} \to \mathfrak{M}_{B_p}(\tau^{\bullet})$ for $i = 1, 2, 3$ be the projection map that sends the exact sequence to the first, second and third objects respectively over the moduli stack of objects in $\mathcal{B}_p$. We also have the map $\pi_1 \times \pi_3 : \mathfrak{Exact}_{\mathcal{B}_p} \to \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})$. By Joyce's definition in [18] (Definition 3.3):

$$\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) * \bar{\epsilon}^{(0,1)}(\tau^{\bullet}) = \pi_2 * ((\pi_1 \times \pi_3)^*(\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) \otimes \bar{\epsilon}^{(0,1)}(\tau^{\bullet}))) \tag{16.12}$$

We also know that $\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) = [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m, \rho_1]$ where $\rho_1 : [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] \to \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})$ where $\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})$ denotes the moduli stack of $\tau^{\bullet}$-semistable objects in $\mathcal{B}_p$ with any given numerical class in $\mathcal{C}(\mathcal{B}_p)$. To make things more clear note that $\pi_2 * : SF(\mathfrak{Exact}_{\mathcal{B}_p}) \to SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}))$. Moreover

$$(\pi_1 \times \pi_3)^* : SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})) \to SF(\mathfrak{Exact}_{\mathcal{B}_p}). \tag{16.13}$$

and

$$\pi_2 * ((\pi_1 \times \pi_3)^*) : SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})) \to SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})) \tag{16.14}$$

Moreover by [18] (Definition 2.7):

$$\otimes : SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})) \times SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})) \to SF(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet})). \tag{16.15}$$

The first step is to compute $\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) \otimes \bar{\epsilon}^{(0,1)}(\tau^{\bullet})$. By definition:

$$\bar{\epsilon}^{(0,1)}(\tau^{\bullet}) \otimes \bar{\epsilon}^{(0,1)}(\tau^{\bullet}) = [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m, \rho_1] \otimes [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m, \rho_2] = [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m \times \mathrm{Spec}(\mathbb{C})/\mathbb{G}_m, \rho_1 \times \rho_2]. \tag{16.16}$$

Therefore

$$\begin{aligned} &(\pi_1 \times \pi_3)^*([\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m \times \mathrm{Spec}(\mathbb{C})/\mathbb{G}_m, \rho_1 \times \rho_2]) \\ &= [(\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m \times \mathrm{Spec}(\mathbb{C})/\mathbb{G}_m) \times_{\rho_1 \times \rho_2, \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}), \pi_1 \times \pi_3} \mathfrak{Exact}_{\mathcal{B}_p}, \Phi] \end{aligned} \tag{16.17}$$

where Φ is the map that embeds the resulting stack in $\mathfrak{Exact}_{\mathcal{B}_p}$. By pushing forward along π_2 we obtain:

$$\begin{aligned} &\pi_2 * ((\pi_1 \times \pi_3)^*)([\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m \times \mathrm{Spec}(\mathbb{C})/\mathbb{G}_m, \rho_1 \times \rho_2]) \\ &= [(\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m \times \mathrm{Spec}(\mathbb{C})/\mathbb{G}_m) \times_{\rho_1 \times \rho_2, \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}), \pi_1 \times \pi_3} \mathfrak{Exact}_{\mathcal{B}_p}, \pi_2 \circ \Phi] \end{aligned} \tag{16.18}$$

Let us denote by $\mathcal{Z}$ the fibered product

$$([\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] \times [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m]) \times_{\rho_1 \times \rho_2, \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}), \pi_1 \times \pi_3} \mathfrak{Exact}_{\mathcal{B}_p}$$

simply put, the above formulation is described by the following diagram:

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{\quad \Phi \quad} & \mathfrak{Exact}_{\mathcal{B}_p} & \xrightarrow{\pi_2} & \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \\ \downarrow & & \downarrow{\scriptstyle \pi_1 \times \pi_3} & & \\ [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] \times [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] & \xrightarrow{\rho_1 \times \rho_2} & \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}) & & \end{array} \tag{16.19}$$

Let $\mathfrak{M}^{(0,1)}_{\mathcal{B}_p}(\tau^\bullet)$ denote the moduli stack of objects in $\mathcal{B}_p$ with class $(0,1)$. Note that the embedding of $[\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m]$ in $\mathfrak{M}_{\mathcal{B}_p}$ is obtained by the following composite morphism.

$$[\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] \to \mathfrak{M}^{(0,1)}_{\mathcal{B}_p}(\tau^\bullet) \to \mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet) \tag{16.20}$$

Lemma 16.3. *The product* $\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet)$ *is given as*

$$\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{A}^1 \rtimes \mathbb{G}_m^2}, \iota\right)\right] \tag{16.21}$$

where ι *is defined to be the corresponding embedding.*

Proof. Let $E_i \in \mathfrak{M}^{(0,1)}_{\mathcal{B}_p}(\tau^\bullet)$ for $i = 1, 3$ and $E_2 \in \mathfrak{M}^{(0,2)}_{\mathcal{B}_p}(\tau^\bullet)$. Consider the exact sequence in $\mathfrak{Exact}_{\mathcal{B}_p}$:

$$0 \to E_1 \to E_2 \to E_3 \to 0 \tag{16.22}$$

By definition this exact sequence can be written as a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} \otimes \mathcal{O}_X(-n) & \longrightarrow & \mathbb{C}^{\oplus 2} \otimes \mathcal{O}_X(-n) & \longrightarrow & \mathbb{C} \otimes \mathcal{O}_X(-n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array} \tag{16.23}$$

Hence the set of extensions and the automorphism of diagram (16.23) can be equivalently studied by considering the simplified diagram:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^{\oplus 2} \longrightarrow \mathbb{C} \longrightarrow 0. \tag{16.24}$$

As pointed out in [18] (page 155), given an exact sequence of vector spaces

$$0 \to U \to U \oplus V \to V \to 0 \tag{16.25}$$

the automorphism group of this exact sequence is given by the algebraic group $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \rtimes$

$\mathrm{Hom}(V, U)$ with multiplication defined by:

$$(\gamma, \delta, \epsilon).(\gamma', \delta', \epsilon') = (\gamma \circ \gamma', \delta \circ \delta', \gamma \circ \epsilon' + \epsilon' \circ \delta).$$

for elements $\gamma, \gamma' \in \mathrm{GL}(U)$, $\delta, \delta' \in \mathrm{GL}(V)$ and $\epsilon, \epsilon' \in \mathrm{Hom}(V, U)$. Alternatively $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \rtimes \mathrm{Hom}(V, U)$ is isomorphic to the subgroup of $\mathrm{GL}(U \oplus V)$ given by elements of the form $\begin{pmatrix} \gamma & \epsilon \\ 0 & \delta \end{pmatrix}$. Now replace U and V in (16.25) by $\mathbb{C}$ and find that the automorphism group of the exact sequence in (16.24) is given by $(\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ which is canonically isomorphic to $\mathbb{G}_m^2 \rtimes \mathbb{A}^1$. The set of exact sequences in (16.22) up to isomorphism is given by $\mathrm{Ext}^1(E_3, E_1)$. Consider the action of $\mathbb{G}_m^2 \rtimes \mathbb{A}^1$ on $\mathrm{Ext}^1(E_3, E_1)$ induced by the identification of $\mathrm{Ext}^1(E_3, E_1)$ with isomorphism classes of exact sequences of the form in (16.24). This action is given by $(\gamma, \delta, \epsilon) : e \mapsto \gamma\mu^{-1}e$ where $\gamma, \delta \in \mathbb{G}_m$ and $e \in \mathrm{Ext}^1(E_3, E_1)$. On the other hand, since $(0, \mathbb{C}) \cong E_1 \cong E_3 \cong (0, \mathbb{C})$ then $\mathrm{Ext}^1(E_3, E_1) = 0$ and the quasi projective variety parametrizing the split extensions in the exact sequence (16.22) is given by $\mathrm{Spec}(\mathbb{C})$. Hence one obtains:

$$\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{Hom}(E_3, E_1) \rtimes \mathbb{G}_m^2}, \iota\right)\right] = \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{A}^1 \rtimes \mathbb{G}_m^2}, \iota\right)\right] \tag{16.26}$$

This finishes the proof of 16.3. □

Consider the notation in Definition 14.2. Let $G = \mathrm{GL}_2(\mathbb{C})$. The maximal torus of G is given by $\mathbb{G}_m^2 \cong T^G \subset \mathrm{GL}_2(\mathbb{C})$. The set $\mathcal{Q}(G, T^G)$ in Definition 14.2 consists of T^G and $\mathbb{G}_m$ given by elements of the form

$$\mathbb{G}_m = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \tag{16.27}$$

where $g \in \mathbb{C}^*$. Given $G = \mathbb{A}^1 \rtimes \mathbb{G}_m^2$ for $T^G = \{0\} \times \mathbb{G}_m^2$ and $\mathbb{G}_m \subset T^G$ given as (16.27) by a computation of Joyce and Song in [18] (page 158):

$$F(G, T^G, T^G) = 1 \tag{16.28}$$

and

$$F(G, T^G, \mathbb{G}_m) = -1. \tag{16.29}$$

Therefore by Definition 14.4 and Lemma 16.3:

$$\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{A}^1 \rtimes \mathbb{G}_m^2}, \iota\right)\right] = -\left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}, e_2\right)\right] + \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}, e_1\right)\right], \tag{16.30}$$

where $e_1 = \mu \circ i_1$ and $e_2 = \mu \circ i_2$ denote the corresponding embedding maps. Since $\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = \bar{\delta}^{(0,1)}(\tau^\bullet) * \bar{\delta}^{(0,1)}(\tau^\bullet)$, by substituting the right hand side of (16.30) in (16.10) one obtains:

$$\begin{aligned}\bar{\epsilon}^{(0,2)}(\tau^\bullet) &= \frac{1}{2}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}\right], \mu \circ i_1\right)\right] - \frac{3}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \mu \circ i_2\right)\right] \\ &- \frac{1}{2}\left(-\left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}, \mu \circ i_2\right)\right] + \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}, \mu \circ i_1\right)\right]\right) = -\frac{1}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \mu \circ i_2\right)\right].\end{aligned} \tag{16.31}$$

Now apply the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}$ to $\bar{\epsilon}^{(0,2)}(\tau^\bullet)$. By definition and Equation (16.31):

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^\bullet)) = \chi^{na}\left(\frac{-1}{4}\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], (\mu \circ i_2)^* \nu_{\mathfrak{M}^{(0,2)}_{\mathcal{B}_p}}\right)\tilde{\lambda}^{(0,2)}. \tag{16.32}$$

Note that by Proposition 13.11 $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet) \cong [\mathrm{Spec}(\mathbb{C})/\,\mathrm{GL}_2(\mathbb{C})]$ and hence $\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]$ has relative dimension 3 over $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet)$. Moreover, $\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]$ is given by a single point with Behrend's multiplicity -1 and

$$(\mu \circ i_2)^* \nu_{\mathfrak{M}^{(0,2)}_{\mathcal{B}_p}})\tilde{\lambda}^{(0,2)} = (-1)^3 \cdot \nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]},$$

therefore:

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^{\bullet})) = \chi^{na}\left(\frac{-1}{4}\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], (-1)^3 \cdot \nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]}\right)\tilde{\lambda}^{(0,2)} = (-1)^1 \cdot (-1)^3 \cdot \frac{-1}{4}\tilde{\lambda}^{(0,2)} = \frac{-1}{4}\tilde{\lambda}^{(0,2)}. \tag{16.33}$$

The wall-crossing identity for $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ is simplified as follows:

$$\mathbf{B}_p^{ss}(X,\beta,2,\tilde{\tau}) \cdot \tilde{\lambda}^{(\beta,2)} = \sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta} \frac{-1}{4} \cdot \frac{(1)}{l!}\prod_{i=1}^{l}\overline{DT}^{\beta_i}(\tau) \cdot [[\cdots[[\tilde{\lambda}^{(0,2)}, \tilde{\lambda}^{(\beta_1,0)}], \tilde{\lambda}^{(\beta_2,0)}], \cdots], \tilde{\lambda}^{(\beta_l,0)}]. \tag{16.34}$$

Now we use the fact that by definition the generators $\tilde{\lambda}^{(\beta,d)}$ satisfy the following property:

$$[\tilde{\lambda}^{(\beta,d)}, \tilde{\lambda}^{(\gamma,e)}] = (-1)^{\bar{\chi}_{\mathcal{B}_p}((\beta,d),(\gamma,e))}\bar{\chi}_{\mathcal{B}_p}((\beta,d),(\gamma,e))\tilde{\lambda}^{(\beta+\gamma,d+e)} \tag{16.35}$$

this enables us to simplify (16.34) as follows:

$$\begin{aligned}&\mathbf{B}_p^{ss}(X,\beta,2,\tilde{\tau}) \cdot \tilde{\lambda}^{(\beta,2)} = \\ &\sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta} \frac{-1}{4} \cdot \frac{(1)}{l!} \cdot \prod_{i=1}^{l}\left(\overline{DT}^{\beta_i}(\tau) \cdot \bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots+\beta_{i-1},2),(\beta_i,0))\right) \\ &\cdot (-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0))+\sum_{i=1}^{l}\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots\beta_{i-1},2),(\beta_i,0))} \cdot \tilde{\lambda}^{(\beta,2)}\end{aligned} \tag{16.36}$$

by canceling $\tilde{\lambda}^{(\beta,2)}$ from both sides we obtain the wallcrossing equation and this finishes our computation:

$$\mathbf{B}_p^{ss}(X,\beta,2,\tilde{\tau}) =$$
$$\sum_{1\le l,\beta_1+\cdots+\beta_l=\beta} \frac{-1}{4}\cdot\left[\frac{(1)}{l!}\cdot\prod_{i=1}^{l}\left(\overline{DT}^{\beta_i}(\tau)\cdot\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots+\beta_{i-1},2),(\beta_i,0))\right.\right.$$
$$\left.\left.\cdot(-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0))+\sum_{i=1}^{l}\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots\beta_{i-1},2),(\beta_i,0))}\right)\right]. \tag{16.37}$$

Chapter 17

Partial progress on direct calculations

In this chapter we introduce a direct approach to calculation of invariants of objects in $\mathcal{B}_p$ in some specific examples ***without using*** any wallcrossing computation. As a result it will be seen that our results in this chapter verify the results obtained in identity (16.37) through an example.

Example 17.1. Computation of $\mathbf{B}_p^{ss}(X,[\mathbb{P}^1],2,\tilde{\tau})$ where X is given by total space of $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-1)\to\mathbb{P}^1$.

We compute the invariant of $\tilde{\tau}$-semistable objects $(F,\mathbb{C}^2,\phi_{\mathbb{C}^2})$ of type $([\mathbb{P}^1],2)$ in $\mathcal{B}_p$. Note that In this case F has rank 1 and $p(n)=n+\chi(F)$. Assume $\chi(F)=r$. In this case by computations in [19] and [11] the only semistable sheaf, F, with $\mathrm{ch}_2(F)=[\mathbb{P}^1]$ is given by $\mathcal{O}_{\mathbb{P}^1}(r-1)$ which is a stable sheaf. First we give description of $\mathfrak{M}^{(2,[\mathbb{P}^1])}_{ss,\mathcal{B}_p}(\tilde{\tau})$. By definition an object of type $([\mathbb{P}^1,2])$ in $\mathcal{B}_p$ is identified by a complex $\mathcal{O}_X(-n)^{\oplus 2}\to\iota_*\mathcal{O}_{\mathbb{P}^1}(r-1)$ where $\iota:\mathbb{P}^1\hookrightarrow X$ (from now on we suppress ι_* in our notation). By the constructions in Section 13.1 the parameter scheme of $\tilde{\tau}$-semistable objects is obtained by choosing two sections (s_1,s_2) such that $s_i\in\mathrm{H}^0((\mathcal{O}_{\mathbb{P}^1}(n+r-1))$ for $i=1,2$. More over since $\mathcal{O}_{\mathbb{P}^1}(r-1)$ is a stable sheaf, its stabilizer is given by $\mathbb{G}_m$.

An important point to note is that given a $\tilde{\tau}$-semistable object $(F,\mathbb{C}^2,\phi_{\mathbb{C}^2})$ one is always able to obtain a an exact sequence of the form

$$0\to(F,\mathbb{C},\phi_{\mathbb{C}})\to(F,\mathbb{C}^2,\phi_{\mathbb{C}^2})\to(0,\mathbb{C},0)\to 0$$

for every object in the moduli stack and since $\tilde{\tau}(F,\mathbb{C}^2,\phi_{\mathbb{C}})=1\leq\tilde{\tau}(0,\mathbb{C},0)=1$ then one concludes that all the objects parametrized by the moduli stack are given by extensions of rank 1 $\tilde{\tau}$-stable objects and hence all objects are $\tilde{\tau}$-strictly-semistable. Moreover, note that giving a $\tilde{\tau}$-semistable object of the form $\mathcal{O}_X(-n)^{\oplus 2} \xrightarrow{(s_1,s_2)} F$ is equivalent to requiring the condition that $(s_1, s_2) \neq (0,0)$, since other wise one may be able to obtain a an exact sequence:

$$0 \to (\mathbb{C}^2, 0, 0) \to (\mathbb{C}^2, F, 0) \to (0, F, 0) \to 0$$

such that $\tilde{\tau}(\mathbb{C}^2, 0, 0) = 1 > \tilde{\tau}(0, F, 0) = 0$, hence $(\mathbb{C}^2, 0, 0)$ weakly destabilizes $(\mathbb{C}^2, F, 0)$ and one obtains a contradiction. Now use Theorem 13.7 and obtain

$$\mathfrak{M}^{(2,[\mathbb{P}^1])}_{ss,\mathcal{B}_p}(\tilde{\tau}) = \left[\frac{(\mathrm{H}^0((\mathcal{O}_{\mathbb{P}^1}(n+r-1)))^{\oplus 2}\backslash\{0\}/\mathbb{G}_m}{\mathrm{GL}_2(\mathbb{C})}\right] \cong \left[\frac{\mathbb{P}(\mathrm{H}^0((\mathcal{O}_{\mathbb{P}^1}(n+r-1))^{\oplus 2})}{\mathrm{GL}_2(\mathbb{C})}\right]. \tag{17.1}$$

Now we need to compute the element of the Hall algebra $\bar{\epsilon}^{(\beta,2)}(\tilde{\tau})$. By applying Definition 14.5 to $\mathfrak{M}^{([\mathbb{P}^1],2)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ we obtain:

$$\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = \bar{\delta}^{([\mathcal{P}^1],2)}_{ss}(\tilde{\tau}) - \frac{1}{2}\sum_{\beta_k+\beta_l=[\mathbb{P}^1]} \bar{\delta}^{(\beta_k,1)}_{s}(\tilde{\tau}) * \bar{\delta}^{(\beta_l,1)}_{s}(\tilde{\tau}). \tag{17.2}$$

Now we use a stratification strategy in order to decompose $\mathfrak{M}^{([\mathbb{P}^1],2)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ into a disjoint union of strata as follows: Given the fact that the objects in the moduli stack under study are of type $([\mathbb{P}^1], 2)$ one would immediately see that the only possible decomposition for a $\tilde{\tau}$-semistable object of type $([\mathbb{P}^1], 2)$ is given by decomposition of its class as $([\mathbb{P}^1], 2) = ([\mathbb{P}^1], 1) + (0, 1)$. This means that a strictly $\tilde{\tau}$-semistable object of type $([\mathbb{P}^1], 2)$ is either given by (split or non-split) extensions of object of type $([\mathbb{P}^1], 1)$ by objects of type $(0, 1)$ or it is given by the extensions with reversed order, i.e the extensions of objects of type $(0, 1)$ by objects of type $([\mathbb{P}^1], 1)$. Our stratification technique involves a study of the parametrizing moduli stacks for these objects depending on what extensions are used to produce the objects. We decompose $\mathfrak{M}^{([\mathbb{P}^1],2)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ into a disjoint union of split and non-split strata.

Definition 17.2. Define $\mathfrak{M}^{([\mathbb{P}^1],2)}_{sp,\mathcal{B}_p}(\tilde{\tau}) \subset \mathfrak{M}^{([\mathbb{P}^1],2)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ to be the locally closed stratum over which an object of type $([\mathbb{P}^1], 2)$ is given by split extensions involving objects of type $([\mathbb{P}^1], 1)$ and $(0, 1)$. Define $\mathfrak{M}^{([\mathbb{P}^1],2)}_{nsp,\mathcal{B}_p}(\tilde{\tau}) \subset \mathfrak{M}^{([\mathbb{P}^1],2)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ to be a locally closed stratum over which an object of type $([\mathbb{P}^1], 2)$ is given by non-split extensions involving objects of type $([\mathbb{P}^1], 1)$ and $(0, 1)$.

Now we study the structure of each stratum separately.

17.1 Stacky structure of $\mathfrak{M}^{([\mathbb{P}^1],2)}_{sp,\mathcal{B}_p}(\tilde{\tau})$

It is easy to see that any $\tilde{\tau}$-semistable objects of type $([\mathbb{P}^1],2)$ given as

$$\mathcal{O}_X(-n)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(r-1) \cong \big(\mathcal{O}_X(-n)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(r-1)\big) \oplus (\mathcal{O}_X(-n) \to 0)$$

has the property that the sections s_1, s_2 for this object are linearly dependent on one another. Hence by discussion in Section 13.1 the underlying parameter scheme of $\tilde{\tau}$-semistable objects of this given form is given by chooing a nonzero section of $\mathcal{O}_{\mathbb{P}^1}(n+r-1)$ in other words we obtain $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1))\backslash\{0\}$. Now we need to take the quotient of this space by the stabilizer group of points. We know that the condition required for a $\tilde{\tau}$-semistable object $\mathcal{O}_X(-n)^{\oplus 2} \xrightarrow{(s_1,s_2)} F$ to be given by split extensions of rank 1 objects is that s_1 and s_2 are linearly dependent on one another. Now pick such an object given by $\mathcal{O}_X(-n)^{\oplus 2} \xrightarrow{(s_1,0)} F$. The automorphisms of this object are given by the group which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{O}_X(-n)^{\oplus 2} & \xrightarrow{(s_1,0)} & \mathcal{O}_{\mathbb{P}^1}(r-1) \\ \cong \big\downarrow & & \big\downarrow \cong \\ \mathcal{O}_X(-n)^{\oplus 2} & \xrightarrow{(s_1,0)} & \mathcal{O}_{\mathbb{P}^1}(r-1) \end{array}$$

Hence it is seen that the left vertical map needs to be given by a subgroup of $\mathrm{GL}_2(\mathbb{C})$ which preserves s_1, i.e the Borel subgroup of $\mathrm{GL}_2(\mathbb{C})$ whose elements are given by 2×2 upper triangular matrices $\begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix}$ where $k_1, k_3 \in \mathbb{G}_m$ and $k_2 \in \mathbb{A}^1$. Having fixed one of the automorphisms via fixing k_1, k_2, k_3, it is seen that by the commutativity of the square diagram the right vertical map needs to be given by multiplication by k_1 which is an element of $\mathbb{G}_m$. Note that one needs to take the quotient of the parameter scheme by all isomorphisms between any two objects in the split stratum, not just the automorphisms of one fixed representative. In general for an object to live in the split stratum one requires the sections (s_1, s_2) to be given by $(s_1, a\cdot s_1)$. We observed that fixing a representative for a split object of rank 2 (such as fixing $(s_1, s_2) = (s_1, 0)$ as above) would tell us that its automorphisms are given by $\mathbb{G}_m^2 \rtimes \mathbb{A}^1$. Hence taking into account all possible representatives implies that the stabilizer group of objects in spit stratum is given by $\mathbb{G}_m^2 \rtimes \mathbb{A}^1 \times \mathbb{G}_m$. Hence we obtain

$$\mathfrak{M}_{sp,\mathcal{B}_p}^{([\mathbb{P}^1],2)}(\tilde{\tau}) = \left[\frac{\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1))\backslash\{0\}}{\mathbb{G}_m^2 \rtimes \mathbb{A}^1 \times \mathbb{G}_m}\right] = \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2 \rtimes \mathbb{A}^1}\right]. \tag{17.3}$$

17.2 Stacky structure of $\mathfrak{M}_{nsp,\mathcal{B}_p}^{([\mathbb{P}^1],2)}(\tilde{\tau})$

In this case all the objects in $\mathfrak{M}_{nsp,\mathcal{B}_p}^{([\mathbb{P}^1],2)}(\tilde{\tau})$ are given by non-split extensions of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-n) & \longrightarrow & \mathcal{O}_X^{\oplus 2}(-n) & \longrightarrow & \mathcal{O}_X(-n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle (s_1,s_2)} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(r-1) & \xrightarrow{\cong} & F & \longrightarrow & 0 & \longrightarrow & 0, \end{array} \tag{17.4}$$

Note that switching the place of $\mathcal{O}_{\mathbb{P}^1}(r-1)$ and 0 in the bottom row of diagram (17.4) would produce a split extension. Now in order to obtain non-split extensions one needs to choose two sections s_1, s_2 such that s_1 and s_2 are linearly independent. The set of all linearly independent choices of s_1 and s_2 spans a two dimensional subspace of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1))$ which is given by the Grassmanian:

$$\mathrm{G}(2, n+r).$$

Now we need to take the quotient of this scheme by the stabilizer group of points in the stratum. We know that the condition required for a $\tilde{\tau}$-semistable object $\mathcal{O}_X(-n)^{\oplus 2} \xrightarrow{(s_1,s_2)} F$ to be given by nonsplit extensions of rank 1 objects is that s_1 and s_2 are linearly independent. Now pick such an object given by $\mathcal{O}_X(-n)^{\oplus 2} \xrightarrow{(s_1,s_2)} F$. The automorphisms of this object are given by the group which makes the following diagram commutative:

$$
\begin{array}{ccc}
\mathcal{O}_X(-n)^{\oplus 2} & \xrightarrow{(s_1,s_2)} & \mathcal{O}_{\mathbb{P}^1}(r-1) \\
\cong \big\downarrow & & \big\downarrow \cong \\
\mathcal{O}_X(-n)^{\oplus 2} & \xrightarrow{(s_1,s_2)} & \mathcal{O}_{\mathbb{P}^1}(r-1)
\end{array}
$$

Hence it is seen that the left vertical map needs to be given by a subgroup of $\mathrm{GL}_2(\mathbb{C})$ whose elements are given by 2×2 diagonal matrices of the form $\begin{pmatrix} k_1 & 0 \\ 0 & k_1 \end{pmatrix}$ where $k_1 \in \mathbb{G}_m$. Having fixed one of the automorphisms via fixing k_1, it is seen that by the commutativity of the square diagram the right vertical map needs to be given by multiplication by k_1 which is an element of $\mathbb{G}_m$.

Hence we obtain

$$
\mathfrak{M}^{([\mathbb{P}^1],2)}_{nsp,\mathcal{B}_p}(\tilde{\tau}) = \left[\frac{\mathrm{G}(2,n+r)}{\mathbb{G}_m}\right]. \tag{17.5}
$$

Now we compute $\sum_{\beta_k+\beta_l=[\mathbb{P}^1]} \bar{\delta}^{(\beta_k,1)}_s(\tilde{\tau}) * \bar{\delta}^{(\beta_l,1)}_s(\tilde{\tau})$ appearing on the right hand side of (17.2). We use the fact that

$$
\sum_{\beta_k+\beta_l=[\mathbb{P}^1]} \bar{\delta}^{(\beta_k,1)}_s(\tilde{\tau}) * \bar{\delta}^{(\beta_l,1)}_s(\tilde{\tau}) = \bar{\delta}^{([\mathbb{P}^1],1)}_s(\tilde{\tau}) * \bar{\delta}^{(0,1)}_s(\tilde{\tau}) + \bar{\delta}^{(0,1)}_s(\tilde{\tau}) * \bar{\delta}^{([\mathbb{P}^1],1)}_s(\tilde{\tau}) \tag{17.6}
$$

and compute each term on the right hand side of (17.6) separately.

Remark 17.3. As we described above there exists an action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbf{S} := \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1))^{\oplus 2}))$. This action induces an action of the corresponding Lie algebra on the tangent space given by the map:

$$
\mathcal{O}_{\mathbf{S}} \otimes \mathfrak{K} \to T_{\mathbf{S}}, \tag{17.7}
$$

where $\mathfrak{K}$ denotes the Lie algebra associated to the group $\mathrm{GL}_2(\mathbb{C})$. The dimension of the automorphism group of objects representing the elements of $\mathbf{S}$ is given by the dimension of the stabilizer (in $\mathrm{GL}_2(\mathbb{C})$) group of these elements, which itself is given by the dimension of the kernel of the map in (17.7). On the other hand, the dimension of the kernel of the map in (17.7) is an upper-semicontinious function. Therefore by the usual arguments, we obtain a stratification of $\mathbf{S}$ which induces a stratification of $\left[\frac{\mathbf{S}}{\mathrm{GL}_2(\mathbb{C})}\right]$ into locally closed strata such that over each stratum the dimension of the stabilizer group is constant as we vary over points inside that stratum. Hence in Definition 17.2 we stated without proof that the defined strata are locally closed in $\mathfrak{M}^{([\mathbb{P}^1],2)}_{ss,\mathcal{B}_p}(\tilde{\tau})$.

17.3 Computation of $\bar{\delta}_s^{(\beta_k,1)}(\tilde{\tau}) * \bar{\delta}_s^{(\beta_l,1)}(\tilde{\tau})$ in general cases

Background:

In this section we describe the computation of the Ringel hall product of the stack functions $\bar{\delta}_s^{(\beta_k,1)}(\tilde{\tau}) * \bar{\delta}_s^{(\beta_l,1)}(\tilde{\tau})$ for β_k and β_l in general cases and later we specialize to our specific example. Similar to discussions in section 16.2, let $\pi_i : \mathfrak{Exact}_{\mathcal{B}_p} \to \mathfrak{M}_{B_p}(\tilde{\tau})$ for $i = 1, 2, 3$ be the projection map that sends the exact sequence to the first,second and third objects respectively over moduli stack of objects in $\mathcal{B}_p$. We also have the map $\pi_1 \times \pi_3 : \mathfrak{Exact}_{\mathcal{B}_p} \to \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) \times \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau})$. By Joyce's definition in [18]:

$$\delta_{ss}^{(\beta_k,1)}(\tilde{\tau}) * \delta_{ss}^{(\beta_l,1)}(\tilde{\tau}) = \pi_2 * ((\pi_1 \times \pi_3)^*(\delta^{(\beta_k,1)}(\tilde{\tau}) \otimes \delta^{(\beta_l,1)}(\tilde{\tau}))) \tag{17.8}$$

Suppose that $\delta^{(\beta_k,1)} = [\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})/\mathbb{G}_m, \rho_1]$ and $\delta^{(\beta_l,1)} = [\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})/\mathbb{G}_m, \rho_3]$ where $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})$ and $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})$ denote some underlying parameter schemes and

$$\rho_1 : \left[\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})/\mathbb{G}_m\right] \to \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}),$$

and

$$\rho_3 : \left[\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})/\mathbb{G}_m\right] \to \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}).$$

Remark 17.4. To have a clear picture of our computation one may choose $\beta_k = [\mathbb{P}^1]$ and $\beta_l = 0$ and see that $\mathcal{M}^{([\mathbb{P}^1],1)}(\tilde{\tau}) := \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n + r - 1)))$ and $\mathcal{M}^{(0,1)}(\tilde{\tau}) := \mathrm{Spec}(\mathbb{C})$. However as we explained above, in this section we choose to carry out the computation in more generality and later substitute for $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})$ and $\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})$.

Let us denote by $\mathcal{Z}'$ the fibered product

$$\left(\left[\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})/\mathbb{G}_m\right] \times \left[\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})/\mathbb{G}_m\right]\right) \times_{\rho_1\times\rho_3, \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau})\times\mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}), \pi_1\times\pi_3} \mathfrak{Exact}_{\mathcal{B}_p}$$

the identity in (17.8) is described by the following diagram:

$$\begin{array}{ccccc} \mathcal{Z}' & \xrightarrow{\quad\Phi\quad} & \mathfrak{Exact}_{\mathcal{B}_p} & \xrightarrow{\pi_2} & \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) \\ \downarrow & & \downarrow{\scriptstyle \pi_1\times\pi_3} & & \\ \left[\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})/\mathbb{G}_m\right] \times \left[\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})/\mathbb{G}_m\right] & \xrightarrow{\rho_1\times\rho_1} & \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) \times \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) & & \end{array} \tag{17.9}$$

We compute the product of stack functions in (17.8) by computing it over the $\mathbb{C}$-points of $\bar{\delta}^{(\beta_k,1)}(\tilde{\tau})$ and $\bar{\delta}^{(\beta_l,1)}(\tilde{\tau})$ (these are induced from $\mathbb{C}$-points of $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})$ and $\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})$) and then integrating over all points in $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau}) \times \mathcal{M}^{(\beta_l,1)}(\tilde{\tau})$.

Consider the stack function

$$\delta_1 = \left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \rho_1 \circ \iota_1\right),$$

with

$$\iota_1 : \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \to \left[\frac{\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})}{\mathbb{G}_m}\right].$$

Moreover let

$$\delta_3 = \left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \rho_3 \circ \iota_3\right),$$

with

$$\iota_3 : \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \to \left[\frac{\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})}{\mathbb{G}_m}\right].$$

Note that δ_1 and δ_3 are the stack functions associated to $\mathbb{C}$-points of $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})$ and $\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})$ respectively. Let $E_1 \in \mathfrak{M}_{\mathcal{B}_p}^{(\beta_k,1)}(\tilde{\tau})$ and $E_3 \in \mathfrak{M}_{\mathcal{B}_p}^{(\beta_l,1)}(\tilde{\tau})$ and $E_2 \in \mathfrak{M}_{\mathcal{B}_p}^{(\beta,2)}(\tilde{\tau})$. Consider the exact sequence in $\mathfrak{Exact}_{\mathcal{B}_p}$:

$$0 \to E_1 \to E_2 \to E_3 \to 0 \tag{17.10}$$

Similar to computations in Section 16.2, the automorphism group of the extension 17.10 is given by $\mathrm{Hom}(E_3,E_1)\rtimes\mathbb{G}_m^2$. The element $(g_1,g_2)\in\mathbb{G}_m^2$ acts on $\mathrm{Ext}^1(E_3,E_1)$ by multiplication by $g_2^{-1}g_1$ and the action of $\mathrm{Hom}(E_3,E_1)$ on $\mathrm{Ext}^1(E_3,E_1)$ is trivial. If the extensions in (17.10) are non-split, then the parametrizing scheme of such extensions is obtained by $\mathbb{P}(\mathrm{Ext}^1(E_3,E_1))$ and for split extensions, it is obtained by $\mathrm{Spec}(\mathbb{C})$. In case of nonsplit extensions, the stabilizer group of the action of $\mathbb{G}_m^2$ is given by $\mathbb{G}_m$ and for split extensions, the stabilizer group of the action of $\mathbb{G}_m^2$ is $\mathbb{G}_m^2$ itself, hence:

$$\delta_1 * \delta_3 = \left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{Hom}(E_3,E_1)\rtimes\mathbb{G}_m^2}\right],\mu_1\right)+\left(\left[\frac{\mathbb{P}(\mathrm{Ext}^1(E_3,E_1))}{\mathrm{Hom}(E_3,E_1)\rtimes\mathbb{G}_m}\right],\mu_3\right). \tag{17.11}$$

Definition 17.5. Let $\mathfrak{R}$ be a $\mathbb{C}$-scheme. Consider $\mathcal{B}G$ given as a quotient stack $\left[\frac{\mathrm{Spec}(\mathbb{C})}{G}\right]$. Define motivic integration over $\mathfrak{R}$ as an identity in the motivic ring of stack functions:

$$\int_{\mathfrak{R}}\left[\frac{\mathrm{Spec}(\mathbb{C})}{G}\right]d\mu_m := \left[\frac{\mathfrak{R}}{G}\right]. \tag{17.12}$$

Moreover assume that $\mathbf{P}\to\mathfrak{R}$ is a vector bundle over $\mathfrak{R}$. Then define:

$$\int_{\mathfrak{R}}\left[\frac{\mathbf{P}}{G}\right]d\mu_m := \int_{\mathfrak{R}}\chi(\mathbf{P})\cdot\left[\frac{\mathrm{Spec}(\mathbb{C})}{G}\right]d\mu_m := \chi(\mathbf{P})\cdot\left[\frac{\mathfrak{R}}{G}\right], \tag{17.13}$$

where $\chi(\mathbf{P})$ denotes the topological Euler characteristic of $\mathbf{P}$. Here The measure μ_m is the map sending constructible sets on $\mathfrak{R}$ to the their corresponding elements in the Grothendieck group of schemes.

Now integrate Equation (17.11) over $\mathcal{M}^{(\beta_k,1)}(\tilde{\tau}) \times \mathcal{M}^{(\beta_l,1)}(\tilde{\tau})$:

$$\begin{aligned}
&\bar{\delta}_s^{(\beta_k,1)}(\tilde{\tau}) * \bar{\delta}_s^{(\beta_l,1)}(\tilde{\tau}) = \int_{(E_1,E_3)\in\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})\times\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})} \delta_1 * \delta_3 = \\
&\int_{(E_1,E_3)\in\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})\times\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})} \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{Hom}(E_3,E_1)\rtimes\mathbb{G}_m^2}\right] d\mu_m \\
&+ \int_{(E_1,E_3)\in\mathcal{M}^{(\beta_k,1)}(\tilde{\tau})\times\mathcal{M}^{(\beta_l,1)}(\tilde{\tau})} \left[\frac{\mathbb{P}(\mathrm{Ext}^1(E_3,E_1))}{\mathrm{Hom}(E_3,E_1)\rtimes\mathbb{G}_m}\right] d\mu_m.
\end{aligned} \tag{17.14}$$

Remark 17.6. At this stage It is important to point out that in what follows we intend to compute the product of characteristic stack functions of $\mathcal{M}^{([\mathbb{P}^1],1)}(\tilde{\tau}) := \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))$ and $\mathcal{M}^{(0,1)}(\tilde{\tau}) := \mathrm{Spec}(\mathbb{C})$ however we will not use the motivic integration in Definition 17.5. We rather do the computations directly by computing the corresponding base parameter schemes and taking their quotients by the stabilizer group of their points.

17.4 Computation of $\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})$

Given $\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) = (\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right], \rho_1)$ and $\bar{\delta}_s^{(0,1)}(\tilde{\tau}) = (\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \rho_2)$ consider the diagram:

$$\begin{array}{ccccc}
\mathcal{Z}'_{12} & \xrightarrow{\Phi} & \mathfrak{Exact}_{\mathcal{B}_p} & \xrightarrow{\pi_2} & \left[\frac{\mathbb{P}(\mathrm{H}^0((\mathcal{O}_{\mathbb{P}^1}(n+r-1))^{\oplus 2})}{\mathrm{GL}_2(\mathbb{C})}\right] \\
\downarrow & & \downarrow{\scriptstyle \pi_1\times\pi_3} & & \\
\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right] \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] & \xrightarrow{\rho_1\times\rho_2} & \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) \times \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) & &
\end{array} \tag{17.15}$$

Here $\mathcal{Z}'_{12}$ is given by the scheme parametrizing the set of commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-n) & \longrightarrow & \mathcal{O}_X^{\oplus 2}(-n) & \longrightarrow & \mathcal{O}_X(-n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle (s_1,s_2)} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(r-1) & \xrightarrow{\cong} & F & \longrightarrow & 0 & \longrightarrow & 0,
\end{array} \tag{17.16}$$

Note that the extensions in (17.16) have the possibility of being split or non-split. Hence we consider each case separately and introduce a new notation:

Definition 17.7. Let $[\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})]_{sp}$ denote the stratum of $(\pi_2 \circ \Phi)_* \mathcal{Z}'_{12}$ over which the points are represented by split extensions given by coomutative diagram in (17.16).

Definition 17.8. Let $[\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})]_{nsp}$ denote the stratum of $(\pi_2 \circ \Phi)_* \mathcal{Z}'_{12}$ over which the points are represented by non-split extensions given by coomutative diagram in (17.16).

Now we compute $[\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})]_{sp}$. This amounts to choosing sections s_1, s_2 so that s_1 and s_2 are linearly depending on one another. The scheme parametrizing the nonzero sections s_1 is given by $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))$. Now we take the quotient of this scheme by the stabilizer group of points. Similar to arguments in Lemma 16.3 given any point in $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))$ represented by the extension in diagram (17.16) its stabilizer group is given by the semi-direct product $\mathbb{G}_m^2 \rtimes \mathrm{Hom}(E_3, E_1)$ where each factor of $\mathbb{G}_m$ amounts to the stabilizer group of objects given as $E_3 := \mathcal{O}_X(-n) \to 0$ and $E_1 := \mathcal{O}_X(-n) \to \mathcal{O}_{\mathbb{P}^1}(r-1)$ respectively. Note that the extra factor of $\mathbb{A}^1$ will not appear as a part of the stabilizer group since by the given description of E_1 and E_3 we know that $\mathrm{Hom}(E_3, E_1) = 0$ for every such E_1 and E_3. We obtain the following conclusion:

$$[\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})]_{sp} = \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right]. \tag{17.17}$$

Now we compute $[\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})]_{nsp}$. This amounts to choosing s_1, s_2 so that s_1 and s_2 are linearly independent and the extension in diagram (17.16) becomes non-split . Note that for any fixed value of s_1 one has $\mathbb{P}^1$ worth of choices for s_2. Now we need to consider all possible choices of s_1 and in doing so we require the pair s_1, s_2 to remain linearly independent. This gives the flag variety $\mathrm{F}(1,2,n+r)$. Hence we obtain:

$$[\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau})]_{nsp} = \left[\frac{\mathrm{F}(1,2,n+r)}{\mathbb{G}_m}\right]. \tag{17.18}$$

Note that the factor of $\mathbb{G}_m$ in the denominator of (17.18) is due to the fact that we have used one of the $\mathbb{G}_m$ factors in projectivising the bundle of s_2-choices over the Grassmanian. We finish this section by summarizing our computation. By (17.17) and (17.18) one obtains:

$$\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau}) = \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right] + \left[\frac{\mathrm{F}(1,2,n+r)}{\mathbb{G}_m}\right] \tag{17.19}$$

17.5 Computation of $\bar{\delta}_s^{(0,1)}(\tilde{\tau}) * \bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau})$

Now change the order of $\bar{\delta}_s^{(0,1)}(\tilde{\tau})$ and $\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau})$ and obtain a diagram

$$\begin{array}{ccccc} & \mathcal{Z}'_{21} & \xrightarrow{\Phi} & \mathfrak{Exact}_{\mathcal{B}_p} & \xrightarrow{\pi_2} \left[\frac{\mathbb{P}(\mathrm{H}^0((\mathcal{O}_{\mathbb{P}^1}(n+r-1))^{\oplus 2})}{\mathrm{GL}_2(\mathbb{C})}\right] \\ & \downarrow & & \downarrow{\scriptstyle \pi_1\times\pi_3} & \\ \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \times & \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right] & \xrightarrow{\rho_2\times\rho_1} & \mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau})\times\mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}) & \end{array} \tag{17.20}$$

Here $\mathcal{Z}'_{21}$ is given by the scheme parametrizing the set of commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-n) & \longrightarrow & \mathcal{O}_X^{\oplus 2}(-n) & \longrightarrow & \mathcal{O}_X(-n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (0,s_2)} & & \downarrow & & \\ 0 & \longrightarrow & 0 & \xrightarrow{\cong} & F & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(r-1) & \longrightarrow & 0, \end{array} \tag{17.21}$$

Note that the computation in this case is easier since the only possible extensions of the form given in (17.21) are the split extensions. The computation in this case is similar to computation in (17.17) except that one needs to take into account that over any point represented by an extension (as in diagram (17.21)) of $E_1 := \mathcal{O}_X(-n) \to 0$ and $E_3 := \mathcal{O}_X(-n) \to \mathcal{O}_{\mathbb{P}^1}(r-1)$ we have $\mathrm{Hom}(E_3, E_1) \cong \mathbb{A}^1$. Hence by similar discussions we obtain :

$$\bar{\delta}_s^{([\mathbb{P}^1],1)}(\tilde{\tau}) * \bar{\delta}_s^{(0,1)}(\tilde{\tau}) = \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2 \rtimes \mathbb{A}^1}\right]. \tag{17.22}$$

17.6 Computation of $\bar{\epsilon}^{(\beta,2)}(\tilde{\tau})$

By (17.2), (17.3), (17.5), (17.6), (17.19) and (17.22) we obtain:

$$\begin{aligned}\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = & \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2 \rtimes \mathbb{A}^1}\right] + \left[\frac{\mathrm{G}(2,n+r)}{\mathbb{G}_m}\right] \\ & - \frac{1}{2}\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right] - \frac{1}{2}\cdot\left[\frac{\mathrm{F}(1,2,n+r)}{\mathbb{G}_m}\right] \\ & - \frac{1}{2}\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2 \rtimes \mathbb{A}^1}\right]. \end{aligned} \tag{17.23}$$

Now use the decomposition used by Joyce and Song in [18] (page 158) and write

$$\begin{aligned}&\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2 \rtimes \mathbb{A}^1}\right] = \\ &F(G,\mathbb{G}_m^2,\mathbb{G}_m^2)\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right] + F(G,\mathbb{G}_m^2,\mathbb{G}_m)\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right], \end{aligned} \tag{17.24}$$

where $F(G,\mathbb{G}_m^2,\mathbb{G}_m^2) = 1$ and $F(G,\mathbb{G}_m^2,\mathbb{G}_m) = -1$. Equation (17.23) simplifies as follows:

$$\begin{aligned}\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = & \cancel{\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right]} - \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right] \\ & + \left[\frac{\mathrm{G}(2,n+r)}{\mathbb{G}_m}\right] - \cancel{\frac{1}{2}\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right]} \\ & \frac{1}{2}\cdot\left[\frac{\mathrm{F}(1,2,n+r)}{\mathbb{G}_m}\right] - \cancel{\frac{1}{2}\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m^2}\right]} \\ & + \frac{1}{2}\cdot\left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right] \end{aligned} \tag{17.25}$$

Now use Definition 17.5 and write:

$$\left[\frac{\mathrm{G}(2,n+r)}{\mathbb{G}_m}\right] = \chi(\mathrm{G}(2,n+r))\cdot\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \tag{17.26}$$

and

$$\begin{aligned}
&\left[\frac{\mathrm{F}(1,2,n+r)}{\mathbb{G}_m}\right] = \\
&= \chi(\mathrm{F}(1,2,n+r)) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \\
&= \chi(\mathbb{P}^1) \cdot \chi(\mathrm{G}(2,n+r)) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \\
&= 2 \cdot \chi(\mathrm{G}(2,n+r)) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]
\end{aligned} \tag{17.27}$$

where the equality in the third line is due the fact that the topological Euler characteristic of a vector bundle over a base variety is equal to the Euler characteristic of its fibers times the Euler characteristic of the base. By (17.23) and (17.27) we obtain:

$$\begin{aligned}
&-\frac{1}{2} \cdot \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right] + \chi(\mathrm{G}(2,n+r)) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \\
&- 2 \cdot \frac{1}{2} \cdot \chi(\mathrm{G}(2,n+r)) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \\
&= -\frac{1}{2} \cdot \left[\frac{\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))}{\mathbb{G}_m}\right] = -\frac{1}{2}\chi(\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n+r-1)))) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right] \\
&= -\frac{1}{2}(n+r) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right].
\end{aligned} \tag{17.28}$$

17.7 Computation of the invariant

Now apply the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}$ to $\bar{\epsilon}^{([\mathbb{P}^1],2)}(\tilde{\tau})$. By definition:

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{([\mathbb{P}^1],2)}(\tilde{\tau})) = \chi^{na}(-\frac{1}{2}(n+r) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], (\mu \circ i_2)^* \nu_{\mathfrak{M}^{(0,2)}_{\mathcal{B}_p}})\tilde{\lambda}^{([\mathbb{P}^1],2)}. \tag{17.29}$$

Note that by Equation (17.1) $\mathfrak{M}^{(2,[\mathbb{P}^1])}_{ss,\mathcal{B}_p}(\tilde{\tau}) = \left[\frac{\mathbb{P}(\mathrm{H}^0((\mathcal{O}_{\mathbb{P}^1}(n+r-1))^{\oplus 2})}{\mathrm{GL}_2(\mathbb{C})}\right]$ and hence $\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]$ has relative dimension $-1-(2n+2r-5) = 4-2n-2r$ over $\mathfrak{M}^{(2,[\mathbb{P}^1])}_{ss,\mathcal{B}_p}(\tilde{\tau})$. Moreover, $\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]$ is given by a single point with Behrend's multiplicity -1 and

$$(\mu \circ i_2)^* \nu_{\mathfrak{M}^{(0,2)}_{\mathcal{B}_p}})\tilde{\lambda}^{([\mathbb{P}^1],2)} = (-1)^{4-2n-2r} \cdot \nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]} = \nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]},$$

therefore:

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{([\mathbb{P}^1],2)}(\tau^\bullet)) = \chi^{na}\left(-\frac{1}{2}(n+r) \cdot \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]}\right)\tilde{\lambda}^{([\mathbb{P}^1],2)} = (-1)^1 \cdot \frac{-1}{2}(n+r) \cdot \tilde{\lambda}^{([\mathbb{P}^1],2)}. \tag{17.30}$$

Finally by Definition 16.2 we obtain:

$$\mathbf{B}_p^{ss}(X, \beta, 2, \tilde{\tau}) = \frac{1}{2}(n+r). \tag{17.31}$$

Note that it is easily seen that substituting $([\mathbb{P}^1], 2)$ for $(\beta, 2)$ in identity (16.37) would give the same answer as in (17.31).

Chapter 18

Objects in $\mathcal{B}_p$ versus highly frozen triples

As we showed in chapters 16 and 17, exploiting the nice properties of the auxiliary category $\mathcal{B}_p$ and the wall-crossing machinery of Kontsevich-Soibelman [22] and Joyce-Song [18] enables one to obtain a relationship between invariants of $\tilde{\tau}$-semistable objects of type $(\beta, 2)$ in $\mathcal{B}_p$ and the generalized Donaldson-Thomas invariants. In this chapter we investigate the relationship between $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ and highly frozen triples with a given stability condition. The outcome is to claim that the invariants of the highly frozen triples with this given stability condition are related to generalized Donaldson-Thomas invariants. For the choice of stability condition we use the stability condition associated to Joyce-Song pairs [18] (Definition 5.18):

Definition 18.1. Consider a highly frozen triple (E, F, ϕ, ψ) of rank r as in Definition 2.9. Let p_F denote the reduced Hilbert polynomial of F with respect to the ample line bundle $\mathcal{O}_X(1)$. This triple is said to be $\hat{\tau}$-limit stable if:

1. $p_{\acute{F}} \leq p_F$ for all proper subsheaves $F' \subset F$ such that $\acute{F} \neq 0$.
2. If ϕ factors through $\acute{F}$ ($\acute{F}$ a proper subsheaf of F), then $p_{\acute{F}} < p_F$.

Remark 18.2. It is obviously seen that the $\hat{\tau}$-limit-stable highly frozen triples behave differently than $q(m) \to \infty$ τ'-limit-stable highly frozen triples in Section 2.1. The τ'-limit-stable highly frozen triples are higher rank analog of PT stable pairs in [28] and $\hat{\tau}$-limit-stable highly frozen triples are higher rank analog of Joyce-Song stable pairs [18] (Chapter 5).

Remark 18.3. The construction of the moduli stack of $\hat{\tau}$-limit-stable highly frozen triples is followed by replacing τ'-stability with $\hat{\tau}$-limit-stability in Chapter 3.

Definition 18.4. Given a highly frozen triple (E, F, ϕ, ψ) as in Definition 2.9 fix the Chern character of F to be equal to β. Let $\beta_2 = \mathrm{Ch}_2(F)$ denote the second Chern character of F (fixing β

results in fixing β_2). Let $P_\beta = \chi(F(m)) = m\int_{\beta_2} c_1(L) + \chi(F)$ denote the Hilber polynomial of F (L is a fixed polarization over X). Define $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ to be the moduli stack of $\hat{\tau}$-limit-stable highly frozen triples of type (P,r).

Theorem 18.5. *The moduli stack* $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ *is a principal* $\mathrm{GL}_r(\mathbb{C})$ *bundle over* $\bigcup_{\beta|_{P_\beta=P}}\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau})$.

Proof. First prove that there exists a map $\pi^{\hat{\tau}}_{\tilde{\tau}} : \mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau}) \to \bigcup_{\beta|_{P_\beta=P}}\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau})$:

Let $p \in \mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})(\mathrm{Spec}(\mathbb{C}))$ be represented by (E,F,ϕ,ψ) as in Definition 2.9. Now forget the choice of isomorphism $\psi : E \xrightarrow{\cong} \mathcal{O}_X^{\oplus r}(-n)$ and obtain (E,F,ϕ) which itself is represented by a complex $I^\bullet := [V\otimes\mathcal{O}_X(-n)\to F]$ such that $E\cong V\otimes\mathcal{O}_X(-n)$ for V a $\mathbb{C}^r$-vector space. Now use [18] (Page 185) and identify the complex $I^\bullet$ with an object (F,V,ϕ_V) of type (β,r) in $\mathcal{B}_p$. Now one needs the following lemma:

Lemma 18.6. *The highly frozen triple* (E,F,ϕ,ψ) *is* $\hat{\tau}$*-limit-stable if and only if the associated* (F,V,ϕ_V) *of type* (β,r) *is* $\tilde{\tau}$*-semistable.*

Proof. 1. $\hat{\tau}$-limit-stability$\Rightarrow$ $\tilde{\tau}$-semistability:

One proves the claim by contradiction. Suppose (F,V,ϕ_V) is not $\tilde{\tau}$-semistable. Then there exists a subobject (F',V,ϕ'_V), a quotient object $(Q,0,0)$ and an exact sequence

$$0\to(F',V,\phi'_V)\to(F,V,\phi_V)\to(Q,0,0)\to 0,$$

such that $\tilde{\tau}(F',V,\phi'_V) = 1$ and $\tilde{\tau}(Q,0,0) = 0$. Now use the identification of (F,V,ϕ_V) and (F',V,ϕ'_V) with the complexes $I^\bullet := V\otimes\mathcal{O}_X(-n)\to F$ and $I'^\bullet := V\otimes\mathcal{O}_X(-n)\to F'$ respectively [18] (Page 185) and consider the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mathcal{O}_X^{\oplus r}(-n) & \longrightarrow & F'\\
\cong\downarrow & & \hookdownarrow\\
\mathcal{O}_X(-n) & \longrightarrow & F\\
\downarrow & & \downarrow\\
0 & \longrightarrow & Q\\
\downarrow & & \downarrow\\
0 & & 0
\end{array} . \tag{18.1}$$

From the right vertical short exact sequence in diagram (18.1) it is seen that $P(F') = P(F) - P(Q)$ and $rk(F') = rk(F) - rk(Q)$. Note that F and Q are both objects in $\mathcal{A}_p$ and since $\mathcal{A}_p$ is an abelian category it contains kernels and hence $p(F') = p$. Hence we obtain a contradiction with $\hat{\tau}$-limit-stability of (E, F, ϕ, ψ).

2. $\tilde{\tau}$-semistability$\Rightarrow$ $\hat{\tau}$-limit-stability:

Similarly suppose (E, F, ϕ, ψ) is not $\hat{\tau}$-limit-stable. Then there exists a proper nonzero subsheaf $F' \subset F$ such that ϕ factors through F' and $p(F') = p(F) = p$. Now obtain the diagram in (18.1) and consider the right vertical short exact sequence. By the same reasoning as above $p(Q) = p$. Hence $Q \in \mathcal{A}_p$ and the complex $0 \to Q$ represents an object in $\mathcal{B}_p$ given by $(Q, 0, 0)$ with $\tilde{\tau}(Q, 0, 0) = 0$. Hence (F, V, ϕ_V) is not $\tilde{\tau}$-semistable which contradicts the assumption.

Now in order to show that $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ is a principal $\mathrm{GL}_r(\mathbb{C})$ bundle over $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau})$ replace $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ in proof of Proposition 3.6 with $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ and $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau})$ respectively. This finishes the proof of Lemma 18.6 as well as Theorem 18.5. □

Remark 18.7. Let $\nu^{(P_\beta,r)}_{\mathfrak{M}_{\mathrm{HFT}}}$ denote Behrend's constructible function on $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$. Let $\nu^{(\beta,r)}_{\mathfrak{M}_{\mathcal{B}_p,ss}}$ denote Behrend's constructible function on $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau})$. By Theorem 18.5 the following identity holds true:

$$\nu^{(P_\beta,r)}_{\mathfrak{M}_{\mathrm{HFT}}} = (-1)^{r^2} \cdot (\pi^{\hat{\tau}}_{\tilde{\tau}})^* \nu^{(\beta,r)}_{\mathfrak{M}_{\mathcal{B}_{p,ss}}}, \tag{18.2}$$

where the map $\pi^{\hat{\tau}}_{\tilde{\tau}}$ is defined as in Theorem 18.5.

Assuming that there exists a well-defined deformation-obstruction theory over $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ then the identity (18.2) provides one with a way to relate the invariants of $\hat{\tau}$-limit-stable frozen triples to invariants of $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$.

As we showed in Section 2.1, a τ'-limit-stable highly frozen triple is thought of as higher rank analog of a PT stable pair [28] while a $\hat{\tau}$-limit-stable highly frozen triple is thought of as a higher rank analog of Joyce-Song stable pair [18] (Chapter 5). We constructed a well-behaved deformation obstruction theory for higher rank PT pairs in Chapter 6 and our constructions depend heavily on the properties of τ'-limit stability. Our methods in Chapter 6 only hold true for PT-stability, hence we do not know if there exists any well-behaved deformation obstruction theory for higher rank JS pairs (i.e $\hat{\tau}$-limit-stable highly frozen triples). Hence the remaining statements in this chapter are all conjectural:

Definition 18.8. Define the invariant of $\hat{\tau}$-limit-stable highly frozen triples of rank r as follows:

$$\mathrm{HFT}(X, P_\beta, r, \hat{\tau}) = \chi(\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau}), \nu^{(P_\beta,r)}_{\mathfrak{M}_{\mathrm{HFT}}}). \tag{18.3}$$

Corollary 18.9. *By Definition 18.3 and* (18.2)*:*

$$\mathrm{HFT}(X, P_\beta, r, \hat{\tau}) = \chi(\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau}), \nu^{(P_\beta,r)}_{\mathfrak{M}_{\mathrm{HFT}}}) = (-1)^{r^2} \cdot \chi(\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p,ss}(\tilde{\tau}), \nu^{(\beta,r)}_{\mathfrak{M}_{\mathcal{B}_p,ss}}) = (-1)^{r^2} \cdot \boldsymbol{B}^{ss}_p(X, \beta, r, \tilde{\tau}), \tag{18.4}$$

where $\boldsymbol{B}^{ss}_p(X, \beta, r, \tilde{\tau})$ denotes the invariant of $\tilde{\tau}$-semistable objects of type (β, r) in $\mathcal{B}_p$.

Remark 18.10. Assuming that there exists a well-behaved deformation obstruction theory for $\mathfrak{M}^{(P_\beta,r)}_{s,\mathrm{HFT}}(\hat{\tau})$, then it follows that the invariants defined in 18.8 are deformation invariants.

Corollary 18.11. *Let $r = 2$. By Corollary 18.9 the invariants of rank 2 $\hat{\tau}$-limit-stable highly frozen triples can be expressed in terms of generalized Donaldson-Thomas invariants.*

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