INTERFERENCE CHANNELS WITH HALF-DUPLEX SOURCE COOPERATION

BY

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THESIS

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ABSTRACT

The performance gain of allowing half-duplex source cooperation is studied for Gaussian interference channels. The source cooperation is in-band, meaning that each source can listen to the other source's transmission, but there is no independent channel between the sources; half-duplex assumes that, at each time instant, the sources can either transmit or listen but cannot do both. This assumption differs from some previous works on source cooperation. When the cooperation is bidirectional and the channel gains are symmetric, the sum capacity is characterized within a constant. When the cooperation is unidirectional, from the primary to the secondary, it is essentially a cognitive channel. By requiring the primary to achieve a rate at most a constant from its link capacity, the best possible rate for the secondary is characterized within a constant. A general coding scheme is proposed for this type of channel. In the first step, only one source transmits and the other source listens. The active source can send data to its destination, share information with the other nodes, or relay data from the other source to the other destination. In the second step, both sources transmit. The shared information from the previous step and the interference channel together can be viewed as a virtual channel. On this virtual channel, the sources can do beamforming for the shared messages, and the destinations can partially cancel the interference, achieving better rates compared with the original interference channel.

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CHAPTER 1

INTRODUCTION

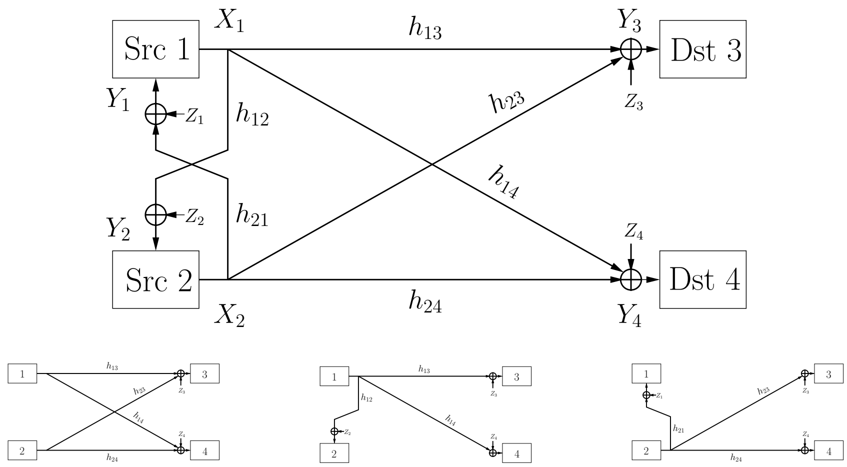
A basic characteristic of the wireless medium is its *broadcast* nature. This manifests itself as interference when multiple users try to share the medium. An active area of research which investigates efficient schemes for managing interference has focused on interference channels [3, 8, 9, 10]. However, the broadcast feature is also a blessing in disguise in that the same transmission could be heard by multiple receivers, opening up the possibility of cooperation. Traditionally, the cooperation aspect has been investigated separately using relay channels in which only one source-destination pair is present [7]. Recently, the role of cooperation in managing interference has come under scrutiny. [4, 5, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] is an incomplete list of references.

In this thesis we investigate the two-user interference channel, where the two sources may not only transmit but also receive (Figure 1.1). This ability to receive will allow the sources to cooperate. However, to be realistic about the gains that can be derived from this cooperation, we impose two key restrictions:

- *In-band cooperation.* No extra orthogonal band is available for the source nodes to transmit to each other over; all transmission and reception must happen over the same band. Thus, the sources cooperate by transmitting and receiving over the same band that is originally available for the interference channel.

- *Half-duplex operation.* Each source node may either transmit or receive at a time but cannot do both. This respects the limitations of current hardware technology.

The in-band cooperation assumption here is a key difference from [13], which considered cooperation over conferencing links orthogonal to the orig-



(a) mode A (b) mode B (c) mode C

Figure 1.1: Interference channel with half-duplex source cooperation. The sources can work in three modes: (A) both sources transmit, (B) source 1 transmits while source 2 receives, and (C) source 2 transmits and source 1 receives.

inal channel. Our model is identical to the one in [16] except that the full-duplex mode of operation was studied there.

We focus on two scenarios. In the first case, the cooperation is bidirectional and the channel gains are symmetric. The sum capacity of this channel is characterized within a constant. In the second case, the cooperation is unidirectional; i.e., source 2 can listen to source 1's transmission but not the other way around, which is essentially a cognitive channel. We call source 1 and destination 3 the primary user and source 2 and destination 4 the secondary user. One interesting question we want to ask about this cognitive channel is, what rate can the secondary achieve without affecting the primary user's performance much? We call the best such rate the cognitive capacity of the channel and characterize it up to a constant. This definition will be made precise in later sections.

The coding scheme is quite general and can be applied to all interference channels with half-duplex source cooperation. The key idea is to turn the half-duplex cooperation problem to a virtual channel problem. A virtual channel is an interference channel with rate-limited bit-pipes between the two sources and from each source to the destination where it causes interference. This virtual channel is similar to the channel considered in [22] except that there they do not have bit-pipes from sources to destinations.

The coding scheme for the virtual channel is an extension of the superposition coding scheme for the interference channels [10]. In addition to public and private messages, we further introduce cooperative private and pre-shared public messages. Cooperative private messages are shared over the bit-pipes between the two sources so they can be sent using source beamforming. Pre-shared public messages are shared over the bit-pipes from the sources to the destinations so the signals corresponding to such messages can be canceled at the other destination and do not cause interference.

To reduce the original channel to a virtual channel, we schedule the transmission in two steps. In the first step, only one source transmits and the other source listens. The active source can send data to its destination, share information with the other nodes, or relay data from the other source to the other destination. In the second step, both sources transmit. The shared information from the previous step and the interference channel together is indeed a virtual channel, and the scheme mentioned above is applied to this channel. In the end, we optimize over the scheduling parameters to get the best achievable rate.

The rest of the paper is organized as follows. In chapter 2, we formally state the two problems, and in chapter 3 the main results about the sum capacity and the cognitive capacity are given. The general coding scheme is described in chapter 4. Chapter 5 deals with the symmetric case, and chapter 6 is for the cognitive case. In both cases, we start by examining the corresponding linear deterministic model and use the intuition there to solve the Gaussian model.

CHAPTER 2

PROBLEM STATEMENT

2.1 The Symmetric Case

The Gaussian interference channel with bidirectional source cooperation is depicted in Figure 1.1.

The source nodes 1 and 2 want to communicate with destination nodes 3 and 4, respectively. Without loss of generality, we assume the channel is normalized; i.e., the additive noise processes $(Z_{it}), i = 1, 2, 3, 4$ are independent $\mathcal{CN}(0, 1)$, i.i.d. over time, and the codeword (X_{it}) at source i satisfies the power constraint

$$\frac{1}{N} \sum_{t=1}^{N} E\left[|X_{it}|^2\right] \leq 1, i = 1, 2.$$

Further, we assume the channel is symmetric, i.e., $|h_{13}|^2 = |h_{24}|^2 = \texttt{SNR}, |h_{14}|^2 = |h_{23}|^2 = \texttt{INR}, |h_{12}|^2 = |h_{21}|^2 = \texttt{CNR}$.

As the cooperation is half-duplex, the sources can work in one of the following three modes. In mode A, both sources transmit. The nodes receive

$$\begin{aligned}
Y_{1t} &= 0, \\
Y_{2t} &= 0, \\
Y_{3t} &= h_{13}X_{1t} + h_{23}X_{2t} + Z_{3t}, \\
Y_{4t} &= h_{14}X_{1t} + h_{24}X_{2t} + Z_{4t}.
\end{aligned}$$

In mode B, source 1 transmits and source 2 listens. Then

$$\begin{aligned}
Y_{1t} &= 0, \\
Y_{2t} &= h_{12}X_{1t} + Z_{2t}, \\
Y_{3t} &= h_{13}X_{1t} + Z_{3t},
\end{aligned}$$

$$Y_{4t} = h_{14}X_{1t} + Z_{4t}.$$

In mode C, source 2 transmits, source 1 listens, and

$$\begin{aligned} Y_{1t} &= h_{21}X_{2t} + Z_{1t}, \\ Y_{2t} &= 0, \\ Y_{3t} &= h_{23}X_{2t} + Z_{3t}, \\ Y_{4t} &= h_{24}X_{2t} + Z_{4t}. \end{aligned}$$

A block length-L codebook of rate (R_1, R_2) for the channel consists of a schedule function $\varphi(t) \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and a sequence of encoding functions f_{it} and decoding functions g_{i+2}, $i = 1, 2$, $t = 1, 2, \ldots, L$. The source messages $W_i \in \{1, 2, \ldots, 2^{LR_i}\}$, $i = 1, 2$ are independent and uniformly distributed. The sources transmit $X_{it} = f_{it}(W_i, Y_i^{t-1})$, where $Y_i^{t-1} = (Y_{i1}, \ldots, Y_{i(t-1)})$. Note that the encoding functions are causal. Further, the encoding functions also satisfy the schedule set out by $\varphi(t)$; i.e., we have $X_{2t} = f_{2t}(W_2, Y_2^{t-1}) = 0$ when $\varphi(t) = \mathrm{B}$ and $X_{1t} = f_{1t}(W_1, Y_1^{t-1}) = 0$ when $\varphi(t) = \mathrm{C}$. Denote the proportion of time spent on mode A as $\varphi(A)$ and define the scheduling parameter $\delta = \frac{2\varphi(A)}{1-\varphi(A)}$ or $\varphi(A) = \frac{\delta}{2+\delta}$, which is proportional to $\varphi(A)$ and specifies roughly how the resource is allocated among the three modes. Destination-$(i+2)$ estimates the message intended for it as $\hat{W}_i = g_{i+2}(Y_{i+2}^L)$, $i = 1, 2$. We say that a rate pair (R_1, R_2) is *achievable* if there is sequence of rate (R_1, R_2) codebooks such that as $L \to \infty$,

$$P(\hat{W}_i \neq W_i) \to 0, i = 1, 2.$$

The capacity region $\mathscr{C}$ is the collection of all achievable (R_1, R_2). The *sum-capacity* C_{sum} of the channel is defined as the largest $R_1 + R_2$ such that $(R_1, R_2) \in \mathscr{C}$. In Section 3 we will provide a characterization of the sum-capacity within a constant.

2.2 The Cognitive Case

The Gaussian interference channel with unidirectional source cooperation is depicted in Figure 2.1. This channel has no cooperation link from source 2

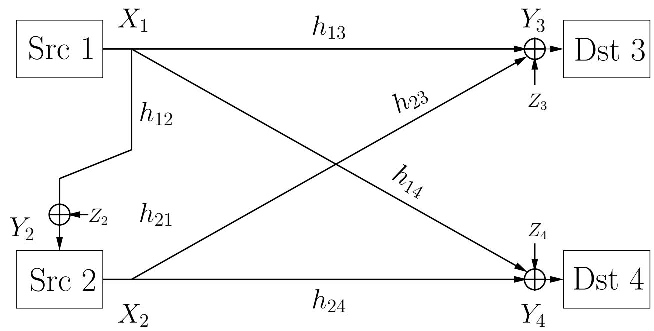


Figure 2.1: Interference channel with unidirectional half-duplex source cooperation.

to source 1.

The source nodes 1 and 2 want to communicate with destination nodes 3 and 4, respectively. Without loss of generality, we assume the channel is normalized; i.e., the additive noise processes $(Z_{it}), i = 2, 3, 4$ are independent $\mathcal{CN}(0, 1)$, i.i.d. over time, and the codeword (X_{it}) at source i satisfies the power constraint

$$\frac{1}{N}\sum_{t=1}^{N} E\left[|X_{it}|^2\right] \leq 1, i = 1, 2.$$

Here, we assume that the channel gains are asymmetric in general. We can view source 1 as the primary user and source 2 as the secondary user, and the secondary can listen to the primary's transmission and adapt its behavior accordingly. Hence, this case corresponds to the cognitive scenario.

As there is only one-side half-duplex cooperation, the sources can work in one of the following two modes. In mode A, both sources transmit. The nodes receive

$$\begin{aligned}
Y_{1t} &= 0, \\
Y_{2t} &= 0, \\
Y_{3t} &= h_{13}X_{1t} + h_{23}X_{2t} + Z_{3t}, \\
Y_{4t} &= h_{14}X_{1t} + h_{24}X_{2t} + Z_{4t}.
\end{aligned}$$

In mode B, source 1 transmits and source 2 listens. Then

$$Y_{1t} = 0,$$

$$Y_{2t} = h_{12}X_{1t} + Z_{2t},$$
$$Y_{3t} = h_{13}X_{1t} + Z_{3t},$$
$$Y_{4t} = h_{14}X_{1t} + Z_{4t}.$$

Let $\mathtt{SNR}_1 = |h_{13}|^2, \mathtt{SNR}_2 = |h_{24}|^2, \mathtt{INR}_1 = |h_{23}|^2, \mathtt{INR}_2 = |h_{14}|^2, \mathtt{CNR} = |h_{12}|^2$.

The codebook definition is similar to that in the symmetric case except that now $\varphi(t) \in A, B$ and the encoding function f_{1t} is only a function of W_1, as Y_{1t} is always 0. Define the scheduling parameter $\delta = \frac{\varphi(B)}{1-\varphi(B)}$ or $\varphi(B) = \frac{\delta}{1+\delta}$. This definition of δ is a little bit different from the one for the symmetric case, as it is now proportional to $\varphi(B)$, which is more convenient for presenting the result. In this case, rather than the sum capacity, we are more interested in another question from the cognitive perspective: what can the secondary achieve if we do not sacrifice the primary's performance? This motivates us to give the following definition.

Definition 2.2.1 *Let $C_0 = \log(1 + \mathtt{SNR}_1)$ be the capacity achieved by source 1 when $X_{2t} = 0, \forall t$. Then R_0-capacity for the secondary user is defined as*

$$C_{R_0} = \max_{\substack{(R_1,R_2)\in\mathscr{C} \\ R_1 \geq C_0 - R_0}} R_2.$$

This definition specifies the best secondary performance, given the primary backs off less than R_0 from its link capacity. In Section 3, the R_0-capacity is characterized when R_0 is larger than some constant.

To see why we introduce a back-off in the primary rate, consider the Z-channel where $\mathtt{CNR} = \mathtt{INR}_2 = 0, \mathtt{SNR}_1 = \mathtt{SNR}_2 \gg \mathtt{INR}_1 \approx 0$. If no back-off is allowed, i.e., if we insist that $R_1 = C_0$, then destination 3 must first decode the interference, or the secondary's message, and then its own message. So we achieve only $R_2 = \log(1+\frac{\mathtt{INR}_1}{1+\mathtt{SNR}_1}) \approx 0$ bits. But if the primary can back off its rate by some constant, the secondary can send its message as long as it does not cause significant interference at destination 3. As $\mathtt{INR}_1 \approx 0$ and $\mathtt{INR}_2 = 0$, the secondary can essentially achieve nontrivial rate $R_2 \approx \log(1 + \mathtt{SNR}_2)$. Notice that the gap between the two is unbounded when $\mathtt{SNR}_2$ scales to ∞. Since we are more interested in the high-SNR region and would want to characterize capacity only up to a constant, the definition above with back-off better serves our purpose.

We further remark that this definition is not a constant gap characterization of the upper-right corner point of the capacity region $\mathscr{C}$. In some channel parameter settings, with the help of the secondary, the primary can do strictly better than C_0.

CHAPTER 3

RESULTS

The main result of this thesis is the approximate characterization of the sum capacity of the symmetric case and the R_0-capacity of the cognitive case for R_0 larger than some constant. We will state them in the following two theorems in this section and also show the gains we can get from half-duplex cooperation. In Section 5.1 and 6.1, we will study the corresponding linear deterministic model to motivate the schemes we use, and in Section 5.2 and 6.2 we sketch proofs of the two theorems with details taken up in the appendices.

3.1 The Symmetric Case

Theorem 3.1 *Define* $\overline{C_{\mathsf{sum}}} = \max_\delta \overline{C_{\mathsf{sum}}}(\delta) = \max_\delta \min(u_1, u_2, u_3, u_4)$, *where*

$$\begin{aligned}
u_1 =& \frac{2}{2+\delta}\Big[\delta \log(1+x) + \log(1+x+z)\Big] \\
u_2 =& \frac{1}{2+\delta}\Big[\delta \log(1+2x+2y) + \log(1+x) + \log(1+x+y+z) \\
& + \delta \log(1+\frac{x}{1+y})\Big] \\
u_3 =& \frac{2}{2+\delta}\Big[\delta \max\{\log(1+y+\frac{2x+y}{1+y}), \log(1+2y)\} + \log(1+x+y+z)\Big] \\
u_4 =& \frac{1}{2+\delta}\Big[\delta \log(1+4x+4y+x^2+y^2-2xy\cos\theta) + 2\log(1+x+y)\Big].
\end{aligned}$$

Then the sum capacity C_{sum} *of the symmetric channel defined in section 2.1 satisfies* $\overline{C_{\mathsf{sum}}} - 17 \le C_{\mathsf{sum}} \le \overline{C_{\mathsf{sum}}} + 3$.

To demonstrate the gains from cooperation, we now plot the generalized degree of freedom [9] of the sum capacity. Here, we use the natural

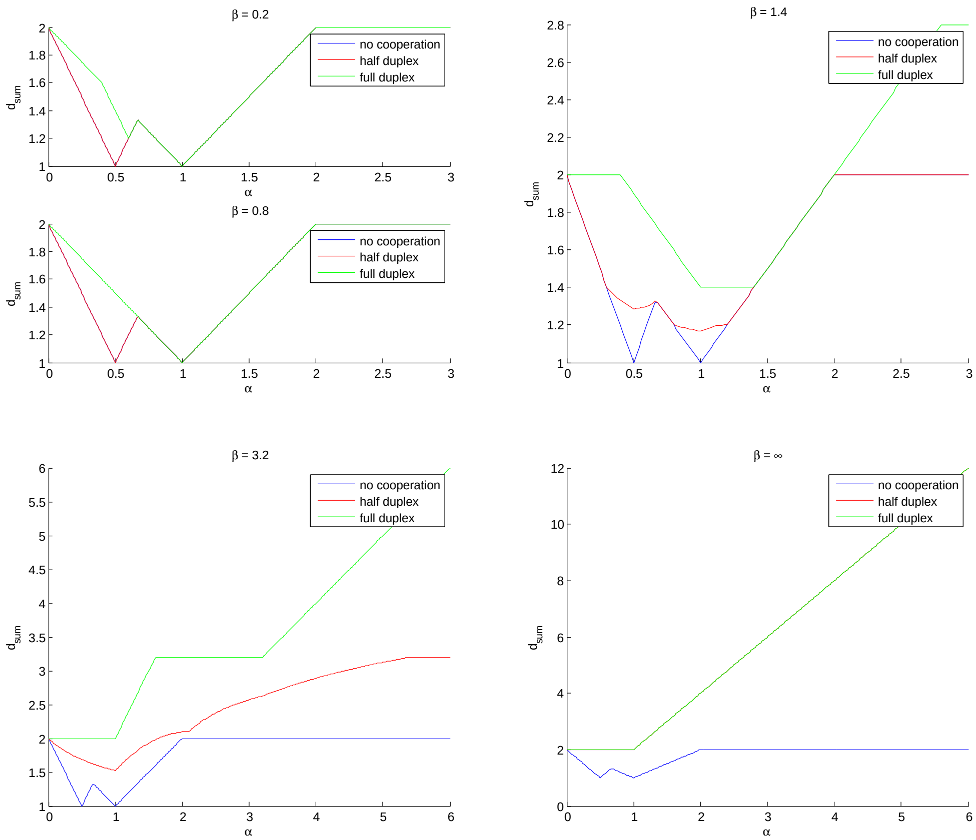


Figure 3.1: Sum capacity of the interference channel with half-duplex source cooperation.

generalization of the original definition given by [22]. Let

$$\lim_{\texttt{SNR}\to\infty}\frac{\log\texttt{INR}}{\log\texttt{SNR}}=\alpha,\ \lim_{\texttt{SNR}\to\infty}\frac{\log\texttt{CNR}}{\log\texttt{SNR}}=\beta.$$

Then the generalized degree of freedom for fixed α, β is

$$d_{sum}(\alpha,\beta)=\lim_{\substack{\text{fix}(\alpha,\beta)\\ \texttt{SNR}\to\infty}}\frac{C_{\mathsf{sum}}}{\log\texttt{SNR}}.$$

d_{sum} is well-defined for $\alpha \neq 1$. When $\alpha = 1$, d_{sum} can take two different values, and we need to treat them separately.

1. $h_{13}h_{24} = h_{14}h_{23}$. Consider the cut-set bound with sources on one side and destinations on the other. The upper bound on the sum capacity of the interference channel reduces to the capacity of a degenerated

multiple input multiple output (MIMO) point-to-point channel. As the latter channel is of degree of freedom 1, Hence, we can get only $d_{sum} = 1$.

2. $h_{13}h_{24} \neq h_{14}h_{23}$. For this setting, the channel is well-conditioned and d_{sum} is a continuous function with respect to α at the $\alpha = 1$.

In Figure 3.1, we show some plots of d_{sum} against α for different $\beta' s$ under the assumption $h_{13}h_{24} \neq h_{14}h_{23}$, which is usually the case, and we also compare it with the result for full-duplex source cooperation [16]. In [16], the sources are allowed both to listen and transmit at the same time instant. For full-duplex cooperation, only one mode is used: both sources transmit and listen. So the resulted d_{sum} is a piecewise linear function of α. For half-duplex cooperation, however, we need to schedule over the three modes properly; and the optimization involved makes each piece a smoothed curve rather than a linear segment. From the plots, we can see that half-duplex cooperation is helpful only when $\beta > 1$, while full-duplex cooperation is helpful for all $\beta > 0$. Moreover, when β is large enough (as in $\beta = 3.2$), by having half-duplex cooperation the sum capacity can be strictly better than that of the usual interference channel. When $\beta = \infty$, the sources can get to know both messages in a negligible amount of time with either half-duplex or full-duplex cooperation. So the channel essentially become a two-source-antenna broadcast channel, and both would have the same sum capacity.

3.2 The Cognitive Case

Theorem 3.2 *Define* $\overline{C_{R_0}} = \max_\delta \overline{C_{R_0}}(\delta) = \max_\delta \min(u_1, u_2, u_3, u_4)$, *where*

$$
\begin{aligned}
u_1 =& \frac{1}{1+\delta}\log(1+x_2)+1 \\
u_2 =& \frac{1}{1+\delta}\Big[\log(1+2x_2+2y_2) - \log(1+x_1) + \delta\log(1+\frac{y_2+z}{1+x_1}) \\
&+ \log(1+\frac{x_1}{1+y_2})\Big] + 2 + R_0 \\
u_3 =& \frac{1}{1+\delta}\left[\log(1+2x_1+2y_1) - \log(1+x_1) + \log(1+\frac{x_2}{1+y_1})\right] + 2 + R_0 \\
u_4 =& \frac{1}{1+\delta}\Big[\log(1+2x_1+2y_1) - \log(1+x_1) + \log(1+\frac{x_1}{1+y_2}) - \log(1+x_1)
\end{aligned}
$$

$$+ \max(\log(1 + y_2 + \frac{2x_2 + y_2}{1 + y_1}), \log(1 + 2y_2)) + \delta \log(1 + \frac{y_2 + z}{1 + x_1})\Big]$$
$$+ 3 + 2R_0.$$

Then when $R_0 \geq 7$, the R_0-capacity C_{R_0} of the cognitive channel defined in section 2.2 satisfies $\overline{C_{R_0}} - 23 - 2R_0 \leq C_{R_0} \leq \overline{C_{R_0}}$.

To demonstrate the gains from cooperation, we now plot the generalized degree of freedom [9] of the R_0-capacity. Similar to above, let

$$\lim_{\mathtt{SNR}_1 \to \infty} \frac{\log \mathtt{SNR}_2}{\log \mathtt{SNR}_1} = n_2', \lim_{\mathtt{SNR}_1 \to \infty} \frac{\log \mathtt{INR}_1}{\log \mathtt{SNR}_1} = \alpha_1',$$
$$\lim_{\mathtt{SNR}_1 \to \infty} \frac{\log \mathtt{INR}_2}{\log \mathtt{SNR}_1} = \alpha_2', \lim_{\mathtt{SNR}_1 \to \infty} \frac{\log \mathtt{CNR}}{\log \mathtt{SNR}_1} = \beta'.$$

Then the generalized degree of freedom for fixed $n_1', \alpha_1', \alpha_2', \beta'$ is

$$d_{cog}(n_2', \alpha_1', \alpha_2', \beta') = \lim_{\mathtt{SNR}_1 \to \infty} \frac{C_{R_0}}{\log \mathtt{SNR}_1}.$$

Unlike the symmetric case, this limit always exists; hence, d_{cog} is well-defined. In fact, when $|h_{13}||h_{24}| = |h_{23}||h_{14}|$, we have d_{cog} the same as that of an interference channel without cooperation. This is essentially saying that cooperation is not quite helpful when the absolute value of the channel gains are aligned. Phases do not matter here. Figure 3.2 shows two typical plots of d_{cog} against α_1' for various β' while n_2', α_2' are held fixed.

In our model, when $\beta' = 0$, it corresponds to an interference channel without cooperation. The above plot shows that when $\beta' \leq \alpha_2' \vee 1$, the generalized degree of freedom is the same as that of $\beta' = 0$. Hence, cooperation is not very helpful unless it is above the threshold. This behavior is the same as what happens for the symmetric channel case. On the other hand, when $\beta' = \infty$, the cooperation link is so strong that the secondary can decode the primary's message in a negligible amount of time. This case is equivalent to the cognitive radio channel model in [12], where the secondary is assumed to know both messages. One interesting thing to notice is that when $n_2 \leq \alpha_1 \leq n_1$, the d_{cog} is always 0, even with infinite cooperation. In fact, in this region, what destination 4 gets from source 2 is only a noisy version of what destination 3 gets from source 2. So destination 3 can also decode W_2. On the other hand, as we require the primary to achieve a

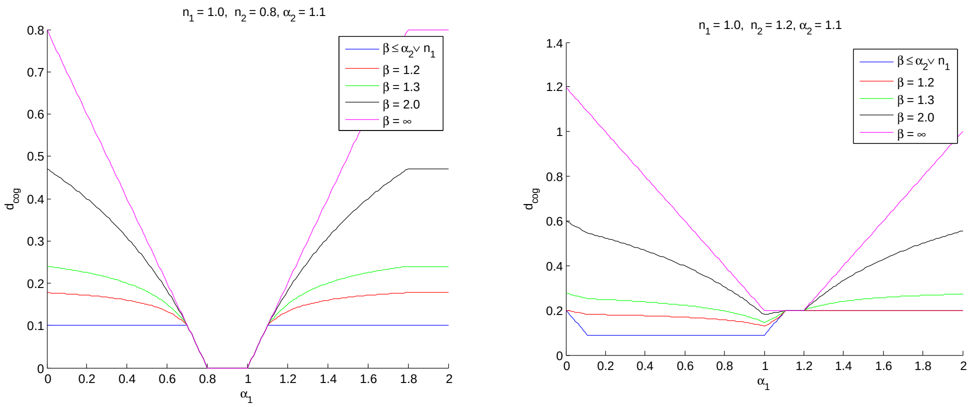


Figure 3.2: Cognitive capacity of the interference channel with half-duplex source cooperation.

rate near its link capacity and the interference at destination 3 is weak, the allowable rate for W_2 is negligible in the high-SNR region.

CHAPTER 4

ACHIEVABILITY

Our coding scheme turns the original channel into a virtual two-user interference channel (Figure 4.1) with rate-limited (noiseless) bit-pipes between the two sources and from each source to the destination node where it causes interference. The bit-pipes are realized by operating in modes B and C (Figure1.1b and 1.1c) where only one of the source nodes transmits while the other receives. In these modes, the transmitting source sends data to its own destination and, in addition, sends messages to the other nodes in order to realize the noiseless links, as will become clear in the sequel. We will first describe a coding scheme and characterize an achievable rate region for the virtual channel. Then we will use this characterization to obtain an achievable rate region for the two-user interference channel with half-duplex source cooperation.

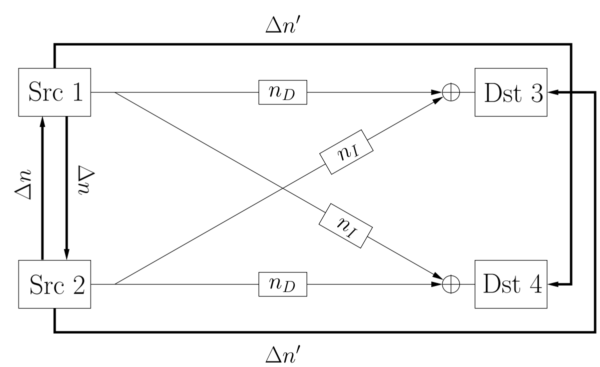


Figure 4.1: Interference channel with bit-pipes. The rate-limited bit pipes (shown in bold) run between the two sources and from each source to the destination node where it causes interference.

4.1 Interference Channel with Bit-Pipes

We denote the *virtual channel* in Figure 4.1 by

$$\mathrm{IF}^{\mathrm{coop}}(p_{Y_3,Y_4|X_1,X_2}, \mathsf{C}_{12}, \mathsf{C}_{21}, \mathsf{C}_{14}, \mathsf{C}_{23}),$$

where C_{ij} are the rates of the bit-pipes between node $i \in \{1,2\}$ and node $j \in \{1,2,3,4\}$. For this new channel, we limit ourselves to block-coding schemes of the following type:

1. First, the sources send at most $L\mathsf{C}_{ij}$ bits over the bit-pipes, where L is the block length. These bits are functions of only the message of the source sending the bits.

2. Then, the sources transmit over the interference channel with each of their channel inputs (of block length L) being functions of their message and the bits exchanged in the first step. For the Gaussian channel, these transmissions are required to satisfy average power constraints of unity.

In the rest of the section, we first discuss an achievable region for the virtual channel[1] $\mathcal{R}_{\mathsf{virtual}}(\mathsf{C}_{12}, \mathsf{C}_{21}, \mathsf{C}_{14}, \mathsf{C}_{23})$. Then using this result, an achievable region for the half-duplex channel will be presented.

Our coding scheme for this virtual channel is a generalization of the superposition coding scheme of Han-Kobayashi for interference channels. The scheme of Han and Kobayashi in this context may be interpreted as follows. Each source node transmits its information in two parts:

- *public message* is decoded by both destinations (even though it is meant for only one of the destinations),

- *private message* is decoded by only one of the destinations, the one to which it is intended.

Our scheme also uses superposition coding and involves two additional parts, each of which takes advantage of one of the two types of bit-pipes available.

1. *cooperative private message.* These messages are shared in advance between the sources over the bit-pipes between them. The messages are

[1] We drop the channel $p_{Y_3,Y_4|X_1,X_2}$ from the notation since the channel will be clear from the context.

then sent out cooperatively by the two sources. But they are decoded by only the intended destination. Below, we will use superposition coding and beamforming for transmitting these messages.

2. *pre-shared public message.* Each source shares this type of message with the unintended destination in advance over the bit-pipes to that destination. This ensures that, when it appears as interference in the transmission over the interference channel, the destination can treat it as known interference while decoding.

In slightly greater detail, our coding scheme is as follows:
Encoding: Source $i \in \{1, 2\}$ divides its message into four parts

$$m_i = (m_{W_i}, m_{U_i}, m_{V_i}, m_{V_i'}),$$

where W stands for (noncooperative) *public*, U for (noncooperative) *private*, V for *cooperative private*, and V' for *pre-shared public*. First, m_{V_i} is shared with the other source and $m_{V_i'}$ is shared with the other destination over the bit-pipes. Superposition codewords are then transmitted over the interference channel. A random codebook construction for these codewords is as follows:

1. At source $i \in \{1, 2\}$, generate the pre-shared public codeword $X^L_{V_i'}(m_{V_i'})$ independently according to distribution $p(x^L_{V_i'}) = \prod_{t=1}^{L} p(x_{V_i',t})$, where $m_{V_i'} \in \{1, 2, \ldots, 2^{L(R_{V_i'}-\epsilon)}\}$.

2. At source i, for each $m_{V_i'}$, generate the public codeword $X^L_{W_i}(m_{V_i}, m_{W_i})$ independently according to distribution

$$p(x^L_{W_i} | x^L_{V_i'}(m_{V_i'})) = \prod_{t=1}^{L} p(x_{W_i,t} | x_{V_i,t}(m_{V_i'})),$$

where $m_{W_i} \in \{1, 2, \ldots, 2^{L(R_{W_i}-\epsilon)}\}$.

3. At source i, for each pair of $(m_{W_i}, m_{V_i'})$, generate the private codeword $X^L_{U_i}(m_{U_i}, m_{W_i}, m_{V_i'})$ according to distribution

$$p(x^L_{U_i} | x^L_{W_i}(m_{W_i}, m_{V_i'}), x^L_{V_i'}(m_{V_i'})) = \prod_{t=1}^{L} p(x_{U_i,t} | x_{W_i,t}(m_{W_i}, m_{V_i'}), x_{V_i',t}(m_{V_i'})),$$

where $m_{U_i} \in \{1, 2, \ldots, 2^{L(R_{U_i}-\epsilon)}\}$.

4. Generate, for $i \in \{1, 2\}$, the *auxiliary* cooperative private codewords $V_i^L(m_{V_i})$, according to distribution $p_{v_i^L} = \prod_{t=1}^{L} p(v_{i,t})$, where $m_{V_i} \in \{1, 2, \ldots, 2^{L(R_{V_i}-\epsilon)}\}$. For every pair (m_{V_1}, m_{V_2}), define the cooperative private codewords $(X_{V_1}^L, X_{V_2}^L)(m_{V_1}, m_{V_2})$ according to distribution $p(x_{V_1}^L, x_{V_2}^L | v_1^L(m_{V_1}), v_2^L(m_{V_2})) = \prod_{t=1}^{L} p(x_{V_1,t}, x_{V_2,t} | v_{1,t}(m_{V_1}), v_{2,t}(m_{V_2}))$.

5. Superimpose the codewords to form the transmit codewords

$$X_1^L(m_{W_1}, m_{U_1}, m_{V_1'}, m_{V_1}, m_{V_2}) = X_{U_1}^L(m_{U_1}, m_{W_1}, m_{V_1'}) + X_{V_1}^L(m_{V_1}, m_{V_2}),$$
$$X_2^L(m_{W_2}, m_{U_2}, m_{V_2'}, m_{V_2}, m_{V_1}) = X_{U_2}^L(m_{U_2}, m_{W_2}, m_{V_2'}) + X_{V_2}^L(m_{V_1}, m_{V_2}).$$

Decoding: Destination 3 looks for a unique $(m_{W_1}, m_{U_1}, m_{V_1}, m_{V_1'})$ such that $(Y_3^L, X_{V_1'}^L(m_{V_1'}), X_{W_1}^L(m_{W_1}, m_{V_1'}), X_{U_1}^L(m_{U_1}, m_{W_1}, m_{V_1'}), V_1^L(m_{V_1}), X_{W_2}^L(\hat{m}_{W_2}), X_{V_2'}^L(m_{V_2'}))$ is jointly typical, for some $\hat{m}_{W_2}$. Note that $m_{V_2'}$ is available to destination 3 via the bit-pipe from source 2. Destination 4 uses the same decoding rule with index 1 and 2 exchanged.

Theorem 4.1 *The rate pair $(R_{W_1}+R_{U_1}+R_{V_1}+R_{V_1'}, R_{W_2}+R_{U_2}+R_{V_2}+R_{V_2'})$ is achievable if $R_{W_1}, R_{W_2}, R_{U_1}, R_{U_2}, R_{V_1}, R_{V_2}, R_{V_1'}, R_{V_2'}$ are nonnegative reals which satisfy the following constraints.*

Constraints at destination 3:

$$R_{U_1} \leq I(X_{U_1}; Y_3 | X_{W_1}, V_1, X_{V_1'}, X_{W_2}, X_{V_2'})$$
$$R_{W_1} + R_{U_1} \leq I(X_{W_1}, X_{U_1}; Y_3 | V_1, X_{V_1'}, X_{W_2}, X_{V_2'})$$
$$R_{V_1'} + R_{W_1} + R_{U_1} \leq I(X_{W_1}, X_{U_1}, X_{V_1'}; Y_3 | V_1, X_{W_2}, X_{V_2'})$$
$$R_{V_1} \leq I(V_1; Y_3 | X_{W_1}, X_{U_1}, X_{V_1'}, X_{W_2}, X_{V_2'})$$
$$R_{V_1} + R_{U_1} \leq I(X_{U_1}, V_1; Y_3 | X_{W_1}, X_{V_1'}, X_{W_2}, X_{V_2'})$$
$$R_{V_1} + R_{W_1} + R_{U_1} \leq I(X_{W_1}, X_{U_1}, V_1; Y_3 | X_{V_1'}, X_{W_2}, X_{V_2'})$$
$$R_{V_1} + R_{V_1'} + R_{W_1} + R_{U_1} \leq I(X_{W_1}, X_{U_1}, V_1, X_{V_1'}; Y_3 | X_{W_2}, X_{V_2'})$$
$$R_{W_2} + R_{U_1} \leq I(X_{W_2}, X_{U_1}; Y_3 | X_{W_1}, V_1, X_{V_1'}, X_{V_2'})$$
$$R_{W_2} + R_{W_1} + R_{U_1} \leq I(X_{W_2}, X_{W_1}, X_{U_1}; Y_3 | V_1, X_{V_1'}, X_{V_2'})$$
$$R_{W_2} + R_{V_1'} + R_{W_1} + R_{U_1} \leq I(X_{W_2}, X_{W_1}, X_{U_1}, X_{V_1'}; Y_3 | V_1, X_{V_2'})$$
$$R_{W_2} + R_{V_1} \leq I(X_{W_2}, V_1; Y_3 | X_{W_1}, X_{U_1}, X_{V_1'}, X_{V_2'})$$

$$R_{W_2} + R_{V_1} + R_{U_1} \le I(X_{W_2}, X_{U_1}, V_1; Y_3 | X_{W_1}, X_{V_1'}, X_{V_2'})$$
$$R_{W_2} + R_{V_1} + R_{W_1} + R_{U_1} \le I(X_{W_2}, X_{W_1}, X_{U_1}, V_1; Y_3 | X_{V_1'}, X_{V_2'})$$
$$R_{W_2} + R_{V_1} + R_{V_1'} + R_{W_1} + R_{U_1} \le I(X_{W_1}, X_{U_1}, X_{V_1'}, V_1, X_{W_2}; Y_3 | X_{V_2'})$$

Constraints at destination 4: Above, with index 1 and 2 exchanged.

Constraints at sources:

$$R_{V_1'} \le \mathsf{C}_{14}, \quad R_{V_2'} \le \mathsf{C}_{23}, \quad R_{V_1} \le \mathsf{C}_{12}, \quad R_{V_2} \le \mathsf{C}_{21}$$

for some

$$\begin{aligned} &p(x_{W_1}, x_{U_1}, x_{V_1}, x_{V_1'}, x_{W_2}, x_{U_2}, x_{V_2}, x_{V_2'}, v_1, v_2) \\ =&p(x_{V_1'}, x_{W_1}, x_{U_1}) p(x_{V_2'}, x_{W_2}, x_{U_2}) p(v_1) p(v_2) p(x_{V_1}, x_{V_2} | v_1, v_2). \end{aligned}$$

For the Guassian channel, the joint distribution must satisfy

$$\mathrm{Var}\,(X_{U_i}) + \mathrm{Var}\,(X_{V_i}) \le 1, \qquad i \in \{1, 2\}.$$

We denote this rate region by $\mathcal{R}_{\mathsf{virtual}}(\mathsf{C}_{12}, \mathsf{C}_{21}, \mathsf{C}_{14}, \mathsf{C}_{23})$.

Proof. The proof is omitted since it follows from standard arguments for superposition coding. □

4.2 Achievability for Half-Duplex Channel

Now we give a scheme for the original channel. The rate region will be given in terms of $\mathcal{R}_{\mathsf{virtual}}$ in Theorem 4.1. Our coding scheme consists of a sequence of blocks. Each block is $\lceil \delta_{\mathrm{A}} L \rceil + \lceil \delta_{\mathrm{B}} L \rceil + \lceil \delta_{\mathrm{C}} L \rceil$ long ($\delta_{\mathrm{A}}, \delta_{\mathrm{B}}, \delta_{\mathrm{C}} \ge 0$). Let us denote, $L_{\mathrm{A}} = \lceil \delta_{\mathrm{A}} L \rceil$, $L_{\mathrm{B}} = \lceil \delta_{\mathrm{B}} L \rceil$, and $L_{\mathrm{C}} = \lceil \delta_{\mathrm{C}} L \rceil$. In each block, the first $1, 2, \ldots, L_{\mathrm{B}}$ and $L_{\mathrm{B}} + 1, L_{\mathrm{B}} + 2, \ldots, L_{\mathrm{B}} + L_{\mathrm{C}}$, respectively, are operated in modes B and C, respectively. The rest L_{A} long duration is in mode A. During the mode B and C phases of each block, we will realize the bit-pipes of the virtual channel, which allow us to implement our coding scheme for the virtual channel during the mode A phase. In addition to realizing the bit-pipes of the virtual channel, during the mode B amd C phases, the sources

(i) send data to their own destinations, (ii) send data to the other source to be *relayed* to the intended destination in a future phase/block, and (iii) relay data received from the other source during a previous phase/block to its intended destination.

In the mode B phase, source node 1 uses superposition coding to send messages to each of the other nodes. In particular, it sends at a rate of $R_{1\mathrm{B}}$ to destination 3, at a rate $\frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{B}}}\mathsf{C}_{12} + \Delta\mathsf{R}_{123}$ to the other source (node 2), and at a rate of $\frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{B}}}\mathsf{C}_{14} + \frac{\delta_{\mathrm{C}}}{\delta_{\mathrm{B}}}\Delta\mathsf{R}_{214}$ to destination node 4. The transmissions at rates $\frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{B}}}\mathsf{C}_{12}$ and $\frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{B}}}\mathsf{C}_{14}$ are used to realize the bit-pipes originating from source node 1 to nodes 2 and 4, respectively, in the virtual channel. Similarly, the following mode C phase realizes the bit-pipes originating from source node 2. With these bit-pipes in place, the following mode A phase is effectively transformed into a virtual channel. The transmission at rate $\Delta\mathsf{R}_{123}$ is meant to be relayed by source node 2 to destination node 3 in the following mode C phase, and the transmission at rate $\frac{\delta_{\mathrm{C}}}{\delta_{\mathrm{B}}}\Delta\mathsf{R}_{214}$ is of the data that node 1 received from source node 2 in the mode C phase of the previous block intended to be relayed to destination node 4. Similarly, in the mode C phase, source node 2 sends using superposition coding at rates $R_{2\mathrm{C}}$, $\frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{C}}}\mathsf{C}_{21} + \Delta\mathsf{R}_{214}$, and $\frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{C}}}\mathsf{C}_{23} + \frac{\delta_{\mathrm{B}}}{\delta_{\mathrm{C}}}\Delta\mathsf{R}_{123}$ to nodes 4, 1, and 3, respectively. Note that for the first block, in the mode B phase, there is no relay data available for node 1 to relay to node 4. But, by increasing the number of blocks, the resulting deficit in rate can be made as small as desired.

For the Gaussian channel, we will use the natural ordering of users for superposition coding. To denote all possiblities together, we adopt the following notation. Let

$$\begin{aligned}
\tilde{\mathsf{R}}_3^{\mathrm{B}} &= R_{1\mathrm{B}}, & \tilde{\mathsf{R}}_4^{\mathrm{C}} &= R_{2\mathrm{C}}, \\
\tilde{\mathsf{R}}_2^{\mathrm{B}} &= \frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{B}}}\mathsf{C}_{12} + \Delta\mathsf{R}_{123}, & \tilde{\mathsf{R}}_1^{\mathrm{C}} &= \frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{C}}}\mathsf{C}_{21} + \Delta\mathsf{R}_{214}, \\
\tilde{\mathsf{R}}_4^{\mathrm{B}} &= \frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{B}}}\mathsf{C}_{14} + \frac{\delta_{\mathrm{C}}}{\delta_{\mathrm{B}}}\Delta\mathsf{R}_{214}, \text{ and} & \tilde{\mathsf{R}}_3^{\mathrm{C}} &= \frac{\delta_{\mathrm{A}}}{\delta_{\mathrm{C}}}\mathsf{C}_{23} + \frac{\delta_{\mathrm{B}}}{\delta_{\mathrm{C}}}\Delta\mathsf{R}_{123}.
\end{aligned}$$

Then, by superposition coding, the above rates are achievable if there are permutations ϕ^{B} of $\{2,3,4\}$ and ϕ^{C} of $\{1,3,4\}$, and a joint distribution $p(\tilde{u}_1^{\mathrm{B}})p(\tilde{u}_2^{\mathrm{B}})p(\tilde{u}_3^{\mathrm{B}})p(x_1|\tilde{u}_1^{\mathrm{B}},\tilde{u}_2^{\mathrm{B}},\tilde{u}_3^{\mathrm{B}})p(\tilde{u}_1^{\mathrm{C}})p(\tilde{u}_2^{\mathrm{C}})p(\tilde{u}_3^{\mathrm{C}})p(x_2|\tilde{u}_1^{\mathrm{C}},\tilde{u}_2^{\mathrm{C}},\tilde{u}_3^{\mathrm{C}})$, (which satisfies the condition $\mathrm{Var}\,(X_1)$, $\mathrm{Var}\,(X_2) \leq 1$ for the Gaussian case) such that

$$\sum_{j=1}^{i} \tilde{\mathsf{R}}^{\mathrm{B}}_{\phi^{\mathrm{B}}(j)} \leq I(\tilde{U}^{\mathrm{B}}_1, \ldots, \tilde{U}^{\mathrm{B}}_i; Y_{\phi^{\mathrm{B}}(i)}), \qquad i \in \{1, 2, 3\}, \tag{4.1}$$

$$\sum_{j=1}^{i} \tilde{\mathsf{R}}^{\mathrm{C}}_{\phi^{\mathrm{C}}(j)} \leq I(\tilde{U}^{\mathrm{C}}_1, \ldots, \tilde{U}^{\mathrm{C}}_i; Y_{\phi^{\mathrm{C}}(i)}), \qquad i \in \{1, 2, 3\}. \tag{4.2}$$

Thus, we have proved the following theorem:

Theorem 4.2 *The rate pair* (R_1, R_2) *is achievable for the half-duplex channel, where*

$$R_1 = \frac{\delta_{\mathrm{A}} R_{1\mathrm{A}} + \delta_{\mathrm{B}} R_{1\mathrm{B}} + \delta_{\mathrm{B}} \Delta\mathsf{R}_{123}}{\delta_{\mathrm{A}} + \delta_{\mathrm{B}} + \delta_{\mathrm{C}}},$$
$$R_2 = \frac{\delta_{\mathrm{A}} R_{2\mathrm{A}} + \delta_{\mathrm{C}} R_{2\mathrm{C}} + \delta_{\mathrm{C}} \Delta\mathsf{R}_{214}}{\delta_{\mathrm{A}} + \delta_{\mathrm{B}} + \delta_{\mathrm{C}}},$$

for parameters as defined in the above discussion such that (4.1) *and* (4.2) *hold and*

$$(R_{1\mathrm{A}}, R_{2\mathrm{A}}) \in \mathcal{R}_{\mathsf{virtual}}(\mathsf{C}_{12}, \mathsf{C}_{21}, \mathsf{C}_{14}, \mathsf{C}_{23}).$$

CHAPTER 5

SYMMETRIC CASE

5.1 Symmetric Case: LDM

In this section, we will study the linear deterministic model (LDM) of the symmetric half duplex source cooperation problem.

5.1.1 Channel Model and Sum Capacity

The corresponding linear deterministic channel [1] is parameterized by non-negative integers

$$n_D = \lfloor \log \mathtt{SNR} \rfloor^+, n_I = \lfloor \log \mathtt{INR} \rfloor^+, n_C = \lfloor \log \mathtt{CNR} \rfloor^+.$$

The channel is depicted in Figure 5.1. Let $n = \max\{n_D, n_I, n_C\}$ and $S \in \mathbb{F}_2^{n \times n}$ be the shift matrix

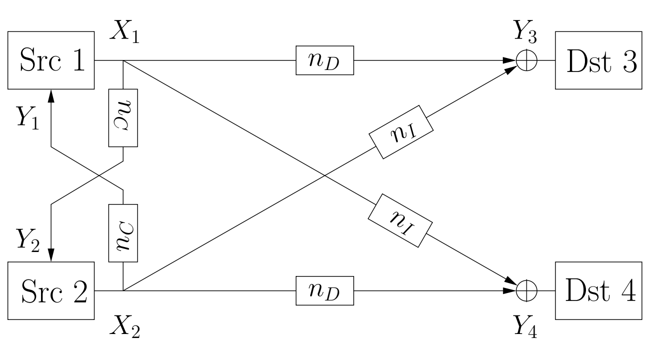


Figure 5.1: Linear deterministic interference channel with half-duplex source cooperation.

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n \times n}$$

where $\mathbb{F}_2$ is the finite field with two elements. The channel inputs X_{1t}, X_{2t} are n-length vectors over $\mathbb{F}_2$. As above, the sources can work in one of the three modes. In mode A, both sources transmit. The nodes receive:

$$\begin{aligned} Y_{1t} &= 0, \\ Y_{2t} &= 0, \\ Y_{3t} &= S^{n-n_D} X_{1t} \oplus S^{n-n_I} X_{2t}, \\ Y_{4t} &= S^{n-n_D} X_{2t} \oplus S^{n-n_I} X_{1t}. \end{aligned}$$

In mode B, source 2 listens. Then,

$$\begin{aligned} Y_{1t} &= 0, \\ Y_{2t} &= S^{n-n_C} X_{1t}, \\ Y_{3t} &= S^{n-n_D} X_{1t}, \\ Y_{4t} &= S^{n-n_I} X_{1t}. \end{aligned}$$

In mode C, source 1 listens and

$$\begin{aligned} Y_{1t} &= S^{n-n_C} X_{2t}, \\ Y_{2t} &= 0, \\ Y_{3t} &= S^{n-n_I} X_{2t}, \\ Y_{4t} &= S^{n-n_D} X_{2t}. \end{aligned}$$

Theorem 5.1 *The sum capacity of the interference channel in Figure 5.1 is*

$$C_{sum} = \max_{\delta \geq 0} \min\{l_1(\delta), l_2(\delta), l_3(\delta), l_4(\delta)\},$$

where

$$
\begin{aligned}
l_1(\delta) &= \begin{cases} \frac{2(1+\delta)}{2+\delta}\max\{n_D,n_I\}, & n_D \neq n_I \\ n_D, & n_D = n_I \end{cases} \\
l_2(\delta) &= \frac{2}{2+\delta}\left(\delta n_D + \max\{n_D,n_C\}\right), \\
l_3(\delta) &= \frac{1}{2+\delta}(\delta\max\{2n_D - n_I, n_I\} + n_D \\
&\quad + \max\{n_D,n_I,n_C\}), \\
l_4(\delta) &= \frac{2}{2+\delta}\left(\delta\max\{n_I, n_D - n_I\} + \max\{n_D,n_C\}\right).
\end{aligned}
$$

The parameter δ is just a scheduling parameter. Its meaning will become clearer when we describe the coding scheme and the the converse. Note that when $n_I = n_D$ or $n_C \leq n_D$, the sum capacity reduces to that of the interference channel without cooperation. Hence, it can be achieved with the optimal interference channel scheme. In the following discussions, we assume $n_I \neq n_D$ and $n_C > n_D$.

5.1.2 Coding Scheme

To characterize the sum capacity, we will need to consider only symmetric schemes. The induced virtual channel is also symmetric. The symmetric virtual channel has an interference channel determined by (n_D, n_I), and its bit-pipes have rates $\mathsf{C}_{12} = \mathsf{C}_{21} = \mathsf{C}_{ss,}$ say, and $\mathsf{C}_{14} = \mathsf{C}_{23} = \mathsf{C}_{sd,}$ say. We denote this type of virtual channel by $\text{IF}^{\text{coop}}((n_D, n_I), \mathsf{C}_{ss}, \mathsf{C}_{sd})$.

To choose the auxiliary random variables in Theorem 4.1 for this symmetric virtual channel, let $n = n_D \vee n_I$ (by which we mean $\max(n_D, n_I)$). For source $i \in \{1, 2\}$, we define the public, pre-shared, and private auxiliary random variables W_i, V_i', U_i, respectively, to be independent and with identical alphabets $\mathbb{F}_2^n$. While the public and pre-shared auxiliary random variables are uniformly distributed over their alphabets, the upper $n - (n_D - n_I)^+$ elements of the private auxiliary random variables are fixed to be 0 and the lower $(n_D - n_I)^+$ elements uniformly distributed over $\mathbb{F}_2^{(n_D - n_I)^+}$. In Theorem 4.1, we set

$$X_{V_i'} = V_i',$$

$$X_{W_i} = V_i' + W_i,$$
$$X_{U_i} = V_i' + W_i + U_i.$$

Note that the private auxiliary random variable U_i occupies the lower $(n_D - n_I)^+$ levels, so that it does not appear at the other destination. This choice is similar to the choice made in [9] for the (noncooperative) intereference channel.

For the cooperative private codebook, we choose the auxiliary random variables $V_i, i = 1, 2$ independently (of each other and all the other auxiliary random variables) distributed uniformly over $\mathbb{F}_2^n$. We choose (X_{V_1}, X_{V_2}) as deterministic functions of (V_1, V_2) such that the beamforming is a *zero-forcing* beamforming. In other words, our choice will be such that the interfering signals (e.g., V_1 at destination 2) are canceled out by beamforming. Specifically, we require

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} S^{n-n_D} & S^{n-n_I} \\ S^{n-n_I} & S^{n-n_D} \end{bmatrix} \begin{bmatrix} X_{V_1} \\ X_{V_2} \end{bmatrix}.$$

As the channel matrix is invertible, we may always find such X_{V_i} for arbitrary V_i. Source i sends $X_{U_i}^L + X_{V_i}^L, i = 1, 2$. The induced channel (after removing the unintended pre-shared public signals which the receivers know in advance) $p_{Y_3,Y_4|V_1',V_2',W_1,W_2,U_1,U_2,V_1,V_2}$ is

$$Y_3 = S^{n-n_D}(W_1 + U_1 + V_1') + S^{n-n_I} W_2 + V_1$$
$$Y_4 = S^{n-n_D}(W_2 + U_2 + V_2') + S^{n-n_I} W_1 + V_2.$$

We will choose symmetric rates for the four types of messages: i.e., $R_{V_1'} = R_{V_2'} = R_{V'}$, and so on. When $n_I < n_D$, we will naturally set $\mathsf{C}_{sd} = 0$ in the superposition coding in modes B and C. Hence, we have $R_{V'} = 0$. By Theorem 4.1, the rate pair $(R_W + R_U + R_V, R_W + R_U + R_V)$ is achievable if

$$2R_W + R_V + R_U \leq n_D$$
$$R_U + R_W \leq \max\{n_I, n_D - n_I\}$$
$$R_U \leq n_D - n_I$$

with $R_W \geq 0, R_U \geq 0, 0 \leq R_V \leq \mathsf{C}_{ss}$. When $n_I > n_D$, we set $R_U = 0$, and by

Theorem 4.1, the rate pair $(R_W + R_V + R'_V, R_W + R_V + R'_V)$ is achievable if

$$\begin{aligned} 2R_W + R_V + R_{V'} &\le n_I \\ R_W + R_{V'} &\le n_D \end{aligned}$$

with $R_W \ge 0, 0 \le R_V \le \mathsf{C}_{ss}, 0 \le R'_V \le \mathsf{C}_{sd}$. By the Fourier-Motzkin elimination, we arrive at

Theorem 5.2 *The following is an achievable sum rate $R_{sum}^{virtual}$ for the channel $IF^{coop}((n_D, n_I), \mathsf{C}_{ss}, \mathsf{C}_{sd})$.*

1. *When $n_I < n_D, \mathsf{C}_{sd} = 0$,*

$$R_{sum}^{virtual} = 2\min\left\{ \begin{array}{c} n_D, \\ n_D - \frac{1}{2}n_I + \frac{1}{2}\mathsf{C}_{ss}, \\ \max\{n_I, n_D - n_I\} + \mathsf{C}_{ss} \end{array} \right\}.$$

2. *When $n_I > n_D$,*

$$R_{sum}^{virtual} = 2\min\left\{ \begin{array}{c} n_D + \mathsf{C}_{ss}, \\ \frac{n_I + \mathsf{C}_{ss} + \mathsf{C}_{sd}}{2}, \\ n_I \end{array} \right\}.$$

Now we can show the achievability of the sum capacity C_{sum} using a symmetric version of the scheme in Section 4.2. We set $\delta_B = \delta_C = 1, \delta_A = \delta$. For superposition coding in modes B and C, the sources set the data rates $R_{1B} = R_{1C} = n_D$ and choose the shared rates $\mathsf{C}_{12} = \mathsf{C}_{21} = \mathsf{C}_{ss}$, $\mathsf{C}_{14} = \mathsf{C}_{23} = \mathsf{C}_{sd}$ and relay rates $\Delta\mathsf{R}_{123} = \Delta\mathsf{R}_{214} = \Delta R$. The constraints (4.1) and (4.2) translate to

$$\begin{aligned} \delta\mathsf{C}_{ss} + \Delta R &\le (n_C - n_D)^+, \\ \delta\mathsf{C}_{sd} + \Delta R &\le (n_I - n_D)^+, \\ \delta\mathsf{C}_{ss} + \delta\mathsf{C}_{sd} + 2\Delta R &\le (\max\{n_I, n_C\} - n_D)^+. \end{aligned}$$

By Theorem 4.2, the sum rate achieved by this scheme is

$$R_{\text{sum}} = \max_{\delta \geq 0} \frac{1}{2+\delta}(2n_D + 2\Delta R + \delta R_{\text{sum}}^{\text{virtual}}(n_D, n_I, \mathsf{C}_{ss}, \mathsf{C}_{sd})).$$

It is not hard to verify that with the following choice of $\mathsf{C}_{ss}, \mathsf{C}_{sd}, \Delta R$, the above constraints are satisfied and $R_{\text{sum}} = C_{\text{sum}}$.

1. $n_I < n_D < n_C$. $\mathsf{C}_{ss} = (n_C - n_D)/\delta$, $\mathsf{C}_{sd} = 0$, and $\Delta R = 0$.

2. $n_D < n_I \leq n_C$. $\mathsf{C}_{sd} = 0$.

   (a) $n_C - n_D \leq \delta n_I$. $\mathsf{C}_{ss} = (n_C - n_D)/\delta$ and $\Delta R = 0$.

   (b) $n_C - n_D > \delta n_I$. $\mathsf{C}_{ss} = n_I$ and

   $$\Delta R = \min\left(\frac{n_C - n_D - \delta n_I}{2}, n_I - n_D\right).$$

3. $n_D < n_C < n_I$.

   (a) $n_I - n_D \leq \delta n_I$ or $n_C - n_D \leq \delta(n_I - n_D)$. $\mathsf{C}_{ss} = (n_C - n_D)/\delta, \mathsf{C}_{sd} = (n_I - n_C)/\delta$ and $\Delta R = 0$.

   (b) $n_I - n_D > \delta n_I$ and $n_C - n_D > \delta(n_I - n_D)$. $\mathsf{C}_{ss} = n_I - n_D, \mathsf{C}_{ss} + \mathsf{C}_{sd} = n_I$ and

   $$\Delta R = \min\left(n_C - n_D - \delta(n_I - n_D), \frac{n_I - n_D - \delta n_I}{2}\right).$$

Remark: Primarily, cooperation enables better rates of transmission over the interference channel. When both n_C and n_I are large relative to n_D, relaying also comes into play. In the Gaussian model, we will also consider the regions separately as above and the power for each signal will be set according to the intuition provided by the linear deterministic channel.

5.2 The Symmetric Case: Gaussian Model

We follow the intuition from the linear deterministic channel and use a symmetric version of the coding scheme in section 4 here as well. The auxiliary random variables in Theorem 4.1 for the induced symmetric virtual channel are chosen as follows: for source $i = 1, 2$, we define the auxiliary random variables W_i, U_i, V_i' to be independent, zero-mean Gaussian random variables with variances $\sigma_W^2, \sigma_U^2, \sigma_{V'}^2$, respectively. In Theorem 4.1, we set

$$\begin{aligned} X_{V_i'} &= V_i', \\ X_{W_i} &= V_i' + W_i, \\ X_{U_i} &= V_i' + W_i + U_i. \end{aligned}$$

The choice of σ_U^2 will be such that it appears at lower than the noise power at the destination where it causes interference. Following the intuition from the linear deterministic case, we will employ *zero-forcing beamforming* for the cooperative private messages. We choose V_1, V_2 to be independent (of each other and all previously defined auxiliary random variables), identically distributed, zero-mean Gaussian random variables with variance σ_V^2. When the channel matrix is invertible, $X_{V_i}, i = 1, 2$ are chosen such that

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} \begin{bmatrix} X_{V_1} \\ X_{V_2} \end{bmatrix}.$$

where $X_{V_i}, i = 1, 2$ are correlated Gaussians with variance

$$\mathrm{Var}\left(X_{V_i}\right) = \frac{\mathtt{SNR} + \mathtt{INR}}{\mathtt{SNR}^2 + \mathtt{INR}^2 - 2\mathtt{SNR}\,\mathtt{INR}\cos\theta}\sigma_V^2.$$

When the channel matrix is not invertible, we will set $\sigma_V^2 = 0$, and $X_{V_1} = X_{V_2} = 0$ (i.e., there will be no cooperative private message). The variance parameters must satisfy the power constraint

$$\sigma_W^2 + \sigma_U^2 + \sigma_{V'}^2 + \mathrm{Var}\left(X_{V_i}\right) \leq 1, \quad i = 1, 2.$$

The destinations receive (with the unintended, pre-shared public signals canceled)

$$\begin{aligned} Y_3 &= h_{13}(W_1 + U_1 + V_1') + h_{23}W_2 + V_1 + h_{23}U_2 + Z_3, \\ Y_4 &= h_{24}(W_2 + U_2 + V_2') + h_{24}W_1 + V_2 + h_{14}U_1 + Z_4. \end{aligned}$$

We set the rates for the four types of messages to be symmetric, i.e., $R_{W_1} = R_{W_2} = R_W$, say, and so on. Also, in Theorem 4.2, we set $\mathsf{C}_{12} = \mathsf{C}_{21} = \mathsf{C}_{ss}$, say, $\mathsf{C}_{14} = \mathsf{C}_{23} = \mathsf{C}_{sd}$, say, and $\Delta\mathsf{R}_{123} = \Delta\mathsf{R}_{214} = \Delta R$, say.

In appendix A, we show that with the choice of auxiliary random variables above, there are power and rate allocations such that a sum-rate of $\overline{C_{\mathsf{sum}}}$ (defined in Theorem 3.1) is achievable within a constant. Specifically,

Theorem 5.3 $C_{\mathsf{sum}} \geq \overline{C_{\mathsf{sum}}} - 17.$

We are also able to show an upperbound to the sum-rate. In appendix B, we prove the following theorem.

Theorem 5.4 *Let*

$$\begin{aligned}
Cut(\delta) =& \frac{1}{2+\delta}\Big[\delta\log(1+xP_{1A}) + \delta\log(1+xP_{2A}) \\
& \log(1+(x+z)P_{1B}) + \log(1+(x+z)P_{2C})\Big] \\
Z(\delta) =& \frac{1}{2+\delta}\Big[\delta\log(1+2xP_{1A}+2yP_{2A}) + \log(1+xP_{1B}) \\
& + \log(1+(x+y+z)P_{2C}) + \delta\log(1+\frac{xP_{2A}}{1+yP_{2A}})\Big] \\
V(\delta) =& \frac{1}{2+\delta}\Big[\delta\log\left(1+yP_{2A}+\frac{2xP_{1A}+yP_{2A}}{1+yP_{1A}}\right) + \log(1+(x+y+z)P_{1B}) \\
& + \delta\log\left(1+yP_{1A}+\frac{2xP_{2A}+yP_{1A}}{1+yP_{2A}}\right) + \log(1+(x+y+z)P_{2C})\Big] \\
Cut'(\delta) =& \frac{1}{2+\delta}\Big[\delta\log(1+2(x+y)(P_{1A}+P_{2A}) + P_{1A}P_{2A}(x^2+y^2-2xy\cos\theta)) \\
& + \log(1+(x+y)P_{\mathrm{1B}}) + \log(1+(x+y)P_{\mathrm{2C}})\Big].
\end{aligned}$$

Define $\overline{C_{sum}^{HD}} = \max_{\delta,P_{1A},P_{1B}} \min(Cut(\delta), Z(\delta), V(\delta), Cut'(\delta))$*, where the maximization is over all nonnegative* $\delta, P_{1A}, P_{1B}, P_{2A}, P_{2C}$ *which satisfy the power constraints*

$$\frac{\delta P_{1A}+P_{1B}}{2+\delta} \leq 1 \ \textit{and}\ \frac{\delta P_{2A}+P_{2C}}{2+\delta} \leq 1.$$

Then

$$C_{\mathsf{sum}} \leq \overline{C_{\mathsf{sum}}^{\mathsf{HD}}} \leq \overline{C_{\mathsf{sum}}} + 3.$$

The above two theorems together imply Theorem 3.1.

CHAPTER 6

THE COGNITIVE CASE

6.1 The Cognitive Case: LDM

6.1.1 Channel Model and Cognitive Capacity

The corresponding linear deterministic model is parameterized by the non-negative integers

$$n_1 = \lfloor \log \mathtt{SNR}_1 \rfloor^+, n_2 = \lfloor \log \mathtt{SNR}_2 \rfloor^+, \alpha_1 = \lfloor \log \mathtt{INR}_1 \rfloor^+,$$
$$\alpha_1 = \lfloor \log \mathtt{INR}_2 \rfloor^+, \beta = \lfloor \log \mathtt{CNR} \rfloor^+$$

The channel is depicted in Figure 6.1. Let $n = \max\{n_1, n_2, \alpha_1, \alpha_2, \beta\}$ and S be the shift matrix defined as in Section 5.1. The channel inputs X_{1t}, X_{2t} are n-length vectors over $\mathbb{F}_2$. As the cooperation is only unidirectional, the sources can work in mode A and B. In mode A, both sources transmit. The nodes receive:

$$\begin{aligned}
Y_{1t} &= 0,\\
Y_{2t} &= 0,\\
Y_{3t} &= S^{n-n_1} X_{1t} \oplus S^{n-\alpha_1} X_{2t},\\
Y_{4t} &= S^{n-n_2} X_{2t} \oplus S^{n-\alpha_2} X_{1t}.
\end{aligned}$$

In mode B, source 2 listens. Then,

$$\begin{aligned}
Y_{1t} &= 0,\\
Y_{2t} &= S^{n-\beta} X_{1t},\\
Y_{3t} &= S^{n-n_1} X_{1t},\\
Y_{4t} &= S^{n-\alpha_2} X_{1t}.
\end{aligned}$$

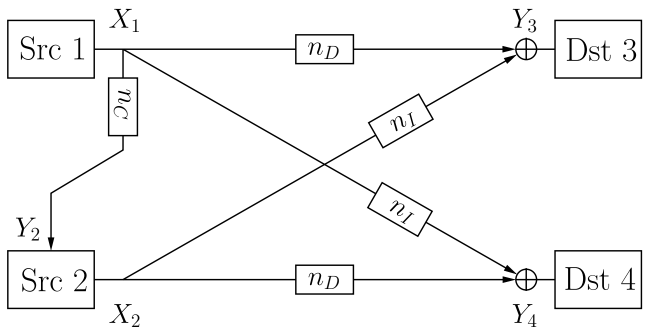


Figure 6.1: Linear deterministic interference channel with unidirectional half-duplex source cooperation.

For this channel, source 1 is the primary and source 2 is the secondary, and as mentioned in Section 2, we would like to know the best rate the secondary can get when the primary is communicating at its link capacity, which is $R_1 = n_1$. So here, we will define the cognitive capacity for this linear deterministic model, which is similar to the R_0-capacity.

Definition 6.1.1 *Assume the capacity region of the channel in Figure 6.1 is $\mathscr{C}$. The cognitive capacity is defined as*

$$C_{cog} = \max_{\substack{(R_1,R_2)\in\mathscr{C} \\ R_1=n_1}} R_2.$$

Note that in this definition, the primary does not need to back off as in the R_0-capacity. This difference is because the linear deterministic model is a coarser description of the true channel. In fact, it characterizes the channel capacity only up to degree of freedom; hence, a constant difference is negligible in this model. But the idea is essentially the same.

Theorem 6.1 *The cognitive capacity C_{cog} of channel in Figure 6.1 is given by*

$$C_{\mathsf{cog}} = \max_{\delta\geq 0} \min(u_1, u_2, u_3, u_4),$$

where

$$u_1 = \frac{1}{1+\delta} n_2$$
$$u_2 = \frac{1}{1+\delta}[n_2 \vee \alpha_2 - \alpha_2 \wedge n_1 + \delta(\beta \vee \alpha_2 \vee n_1 - n_1)]$$

$$u_3 = \frac{1}{1+\delta}[(\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+]$$
$$u_4 = \frac{1}{1+\delta}[(\alpha_1 - n_1)^+ - \alpha_2 \wedge n_1 + (n_2 - \alpha_1) \vee \alpha_2 + \delta(\beta \vee \alpha_2 \vee n_1 - n_1)].$$

The parameter δ is again the scheduling parameter as defined for the cognitive case in Section 2.

For comparison, we also summarize here the result for cognitive capacity of the interference channel without cooperation.

Proposition 6.1.1 *The cognitive capacity of linear deterministic interference channel parameterized by* $n_1, n_2, \alpha_1, \alpha_2$ *is*

$$C_{\mathsf{cog}}^{\mathsf{IFC}} = \min(v_1, v_2, v_3, v_4),$$

where

$$\begin{aligned} v_1 &= n_2 \\ v_2 &= n_2 \vee \alpha_2 - \alpha_2 \wedge n_1 \\ v_3 &= (\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ \\ v_4 &= (\alpha_1 - n_1)^+ - \alpha_2 \wedge n_1 + (n_2 - \alpha_1) \vee \alpha_2. \end{aligned}$$

Proof. The capacity region of the linear deterministic interference channel [2] is given by the set of (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\le n_1 \\ R_2 &\le n_2 \\ R_1 + R_2 &\le (n_1 - \alpha_2)^+ + n_2 \vee \alpha_2 \\ R_1 + R_2 &\le (n_2 - \alpha_1)^+ + n_1 \vee \alpha_1 \\ R_1 + R_2 &\le \alpha_1 \vee (n_1 - \alpha_2) + \alpha_2 \vee (n_2 - \alpha_1) \\ 2R_1 + R_2 &\le n_1 \vee \alpha_1 + (n_1 - \alpha_2)^+ + \alpha_2 \vee (n_2 - \alpha_1) \\ R_1 + 2R_2 &\le n_2 \vee \alpha_2 + (n_2 - \alpha_1)^+ + \alpha_1 \vee (n_1 - \alpha_2). \end{aligned}$$

Evaluating the inequalities at $R_1 = n_1$, the maximum R_2 gives the cognitive capacity above. □

Using the notation in the proposition, we can rewrite the cognitive capacity of the cognitive channel as

$$C_{cog} = \max_{\delta} \frac{1}{1+\delta} \min(v_1, v_2 + \delta(\beta \vee \alpha_2 \vee n_1 - n_1), v_3, v_4 + \delta(\beta \vee \alpha_2 \vee n_1 - n_1)).$$

When $\beta = 0$, clearly the cognitive channel reduces to the original interference channel and $C_{cog}(\beta = 0) = C_{cog}^{IFC}$. When $\beta \leq \alpha_2 \vee n_1$, we can see that $C_{cog}(\beta) = C_{cog}(\beta = 0) = C_{cog}^{IFC}$. Moreover, when the channel is aligned, i.e., $n_1 + n_2 = \alpha_1 + \alpha_2$, we have

$$C_{cog} \leq \max_{\delta} u_3 = v_3 = \max(n_1, n_2, \alpha_1, \alpha_2) - n_1 = C_{cog}^{IFC}.$$

In both cases, the cooperation link is useless in this region and the optimal interference channel scheme is enough. In the following discussions, we assume $\beta > \alpha_2 \vee n_1$, $n_1 + n_2 \neq \alpha_1 + \alpha_2$, and

$$C_{cog} = \max_{\delta} \frac{1}{1+\delta} \min(v_1, v_2 + \delta(\beta - n_1), v_3, v_4 + \delta(\beta - n_1)).$$

6.1.2 Coding Scheme

We consider general asymmetric schemes for this channel. Compared with the symmetric case, we have several differences: (a) the interference channel is asymmetric and is determined by $(n_1, \alpha_1, n_2, \alpha_2)$; (b) for the virtual channel, as $n_{21} = 0$, we have $\mathsf{C}_{21} = 0$.

In our coding scheme, we will not use the pre-shared message. So we set $\mathsf{C}_{14} = \mathsf{C}_{23} = 0$. Hence, we can denote the channel as $\text{IF}^{\text{coop}}(n_1, \alpha_1, n_2, \alpha_2, \mathsf{C}_{12})$. Moreover, relay is also not used in this case, and we set the relay rates $\Delta\mathsf{R}_{123} = \Delta\mathsf{R}_{214} = 0$. To meet $R_1 = n_1$ in the cognitive capacity definition, our scheme chooses $R_{1B} = R_{1A} = n_1$.

To choose the auxiliary random variables in Theorem 4.1 for this symmetric virtual channel, let $n = n_1 \vee \alpha_1 \vee n_2 \vee \alpha_2$. For source $i \in \{1, 2\}$, we define the public and private auxiliary random variables W_i, U_i, respectively, to be independent and with identical alphabets $\mathbb{F}_2^n$. While the public auxiliary random variables are uniformly distributed over their alphabets, the upper $n - (n_i - \alpha_i)^+$ elements of the private auxiliary random variables are fixed to be 0 and the lower $(n_i - \alpha_i)^+$ elements are uniformly distributed over

$\mathbb{F}_2^{(n_i-\alpha_i)^+}$. In Theorem 4.1, we set $V_i' = 0$ and

$$X_{W_i} = W_i,$$
$$X_{U_i} = W_i + U_i.$$

Note that U_i occupies the lower $(n_i - n_i)^+$ levels, so that it does not appear at the other destination.

For the cooperative private codebook, we set the auxiliary random variable $V_2 = 0$ and choose V_1 independent of the auxiliary random variables and distributed over $\mathbb{F}_2^n$ with the upper $n-k$ elements being 0 and the lower k elements uniformly distributed over $\mathbb{F}_2^k$. The choice of k will be specified later. We choose (X_{V_1}, X_{V_2}) as deterministic functions of V_1 such that the beamforming is a *zero-forcing* beamforming. In other words, our choice will be such that the signals cancel each other at destination 2. Specifically, we require

$$\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{bmatrix} S^{n-n_1} & S^{n-\alpha_1} \\ S^{n-\alpha_2} & S^{n-n_2} \end{bmatrix} \begin{bmatrix} X_{V_1} \\ X_{V_2} \end{bmatrix}. \tag{6.1}$$

For this scheme to be feasible, k is chosen such that for arbitrary V_1 in $\mathbb{F}_2^n$ with the upper $n-k$ elements being 0, there exists (X_{V_1}, X_{V_2}) satisfying the above equation. We call such k *realizable* and we have the following lemma.

Lemma 6.1 *For a channel with parameters* $(n_1, n_2, \alpha_1, \alpha_2)$*, the biggest realizable* k *is* $[n_1 - (\alpha_2 - n_2)^+] \vee [\alpha_1 - (n_2 - \alpha_2)^+]$.

Proof. Clearly we have $k \leq n_1 \vee \alpha_1$. Assume $\alpha_2 \geq n_2$. As $V_2 = 0$ and the upper $\alpha_2 - n_2$ bits of V_2 and X_{V_1} are the same, those bits of X_{V_1} must be zero. After removing the corresponding first $\alpha_2 - n_2$ columns, the channel matrix is equivalent to a channel with parameters $(n_1 - (\alpha_2 - n_2), n_2, \alpha_1, n_2)$. Hence, we have $k \leq (n_1 - (\alpha_2 - n_2)) \vee \alpha_1$. Ignoring the all-zero rows of this new channel matrix, it is not hard to see that it is of full row rank and for any $V_1 \in \mathbb{F}_2^k$ with its upper $n-k$ elements being 0, where $k = (n_1 - (\alpha_2 - n_2)) \vee \alpha_1$, there exists X_{V_1}, X_{V_2} satisfying (6.1). Hence, the maximum realizable k is $(n_1 - (\alpha_2 - n_2)) \vee \alpha_1$. A similar argument can be made for $\alpha_2 < n_2$; and combining the two, we have the lemma. □

So in our scheme, we set $k = [n_1 - (\alpha_2 - n_2)^+] \vee [\alpha_1 - (n_2 - \alpha_2)^+]$.

Source 1 sends $X_{U_1}^L + X_{V_1}^L$ and source 2 sends $X_{U_2}^L$. The induced channel $p_{Y_3,Y_4|W_1,W_2,U_1,U_2,V_1}$ is

$$\begin{aligned}
Y_3 &= S^{n-n_1}(W_1 + U_1) + S^{n-\alpha_1}W_2 + V_1 \\
Y_4 &= S^{n-n_2}(W_2 + U_2) + S^{n-\alpha_2}W_1.
\end{aligned}$$

By Theorem 4.1, the rate pair $(R_{W_1}+R_{U_1}+R_{V_1}, R_{W_2}+R_{U_2})$ is achievable if the rates $R_{W_1}, R_{U_1}, R_{V_1}, R_{W_2}, R_{U_2}$ are nonnegative and they satisfy the following conditions:

$$\begin{aligned}
R_{W_1} + R_{U_1} + R_{W_2} + R_{V_1} &\leq \max(\alpha_1, n_1) \\
R_{U_1} + R_{W_2} + R_{V_1} &\leq \max(\alpha_1, k) \\
R_{W_1} + R_{U_1} + R_{V_1} &\leq \max(n_1, k) \\
R_{W_1} + R_{U_1} &\leq n_1 \\
R_{U_1} + R_{W_2} &\leq \max(n_1 - \alpha_2, \alpha_1) \\
R_{U_1} + R_{V_1} &\leq k \\
R_{U_1} &\leq (n_1 - \alpha_2)^+ \\
R_{V_1} &\leq \mathsf{C}_{12} \\
R_{W_1} + R_{W_2} + R_{U_2} &\leq \max(\alpha_2, n_2) \\
R_{W_1} + R_{U_2} &\leq \max(n_2 - \alpha_1, \alpha_2) \\
R_{W_2} + R_{U_2} &\leq n_2 \\
R_{U_2} &\leq (n_2 - \alpha_1)^+.
\end{aligned}$$

Set $R_1 = R_{W_1} + R_{U_1} + R_{V_1} = n_1$ and $R_2 = R_{W_2} + R_{U_2}$. Applying Fourier-Motzkin elimination to the above inequalities, we get

Theorem 6.2 *The following is an achievable cognitive rate for the channel $IF^{coop}(n_1, \alpha_1, n_2, \alpha_2, \mathsf{C}_{12})$,*

$$R_{\mathsf{cog}}^{\mathsf{virtual}} = \min(v_1, v_2 + \mathsf{C}_{12}, v_3, v_4 + \mathsf{C}_{12})$$

in which $v_i, i = 1, 2, 3, 4$ are defined in Proposition 6.1.1.

With this theorem in hand, showing the achievability of the cognitive capacity for the original half-duplex channel C_{cog} is quite straightforward. We set $\delta_B = \delta, \delta_C = 0, \delta_A = 1$. For the superposition coding in mode B, source

1 sets rate $R_{1\text{B}} = n_1$ and the shared rate $\frac{\mathsf{C}_{12}}{\delta} = \beta - n_1$ or $\mathsf{C}_{12} = \delta(\beta - n_1)$. As $R_{1\text{B}} = R_{1A} = n_1$, the total rate for the primary is $R_1 = n_1$. So by Theorem 4.2, the cognitive rate achieved by the secondary is

$$R_{\mathsf{cog}} = \max_{\delta \geq 0} \frac{1}{1+\delta} R_{\mathsf{cog}}^{\mathsf{virtual}} = \max_{\delta \geq 0} \min(u_1, u_2, u_3, u_4),$$

where u_1, u_2, u_3, u_4 were defined in Theorem 6.1.

6.1.3 An Interpretation of the Scheme

For the interesting region $\beta > \alpha_2 \vee n_1$ and $n_1 + n_2 \neq \alpha_1 + \alpha_2$, we can obtain a simple interpretation of the scheme by optimizing over δ. Let

$$\begin{aligned} C_{cog}(\delta) =& \frac{1}{1+\delta} \min(v_1, v_2 + \delta(\beta - n_1), v_3, v_4 + \delta(\beta - n_1)) \\ =& \frac{1}{1+\delta} \min(v_1 \wedge v_3, v_2 \wedge v_4 + \delta(\beta - n_1)). \end{aligned}$$

Define $\delta_0 = \frac{v_1 \wedge v_3 - v_1 \wedge v_2 \wedge v_3 \wedge v_4}{\beta - n_1} \geq 0$. When $\delta \geq \delta_0$,

$$C_{cog}(\delta) = \frac{1}{1+\delta}[v_1 \wedge v_3] \leq \frac{1}{1+\delta_0}[v_1 \wedge v_3].$$

When $0 \leq \delta < \delta_0$, we must have $\delta_0 > 0$, which means $v_1 \wedge v_3 > v_1 \wedge v_2 \wedge v_3 \wedge v_4$; hence, $v_2 \wedge v_4 = v_1 \wedge v_2 \wedge v_3 \wedge v_4$.

$$\begin{aligned} C_{cog}(\delta) &= \frac{1}{1+\delta}[v_2 \wedge v_4 + \delta(\beta - 1)] \\ &\leq \max\left(v_2 \wedge v_4, \frac{1}{1+\delta_0}[v_2 \wedge v_4 + \delta_0(\beta - 1)]\right) \\ &= \max\left(v_1 \wedge v_2 \wedge v_3 \wedge v_4, \frac{1}{1+\delta_0}[v_1 \wedge v_3]\right). \end{aligned}$$

The second inequality is due to the fact that $C_{cog}(\delta)$ is a monotone function in this region and its maximum is achieved at the end points. Noting $v_2 \wedge v_4 = v_1 \wedge v_2 \wedge v_3 \wedge v_4$, we get the last equality.

In summary,

$$C_{cog}(\delta) \leq \max\left(v_1 \wedge v_2 \wedge v_3 \wedge v_4, \frac{1}{1+\delta_0}[v_1 \wedge v_3]\right)$$

$$C_{cog} = \max_{\delta} C_{cog}(\delta) = \max\left(v_1 \wedge v_2 \wedge v_3 \wedge v_4, \frac{1}{1+\delta_0}[v_1 \wedge v_3]\right).$$

The equality is achieved by taking either $\delta = 0$ or $\delta = \delta_0$. As defined in Section 6.1.1, $C_{cog}^{IFC} = v_1 \wedge v_2 \wedge v_3 \wedge v_4$. Now if we let $\alpha_2 = 0$, the interference channel reduces to the corresponding Z-channel and we can define its cognitive capacity as

$$C_{cog}^{Z} = C_{cog}^{IFC}(\alpha_2 = 0) = v_1 \wedge v_2 \wedge v_3 \wedge v_4|_{\alpha_2=0} = v_1 \wedge v_3,$$

then the equation above can be rewritten as

$$C_{cog} = \max\left(C_{cog}^{IFC}, \frac{1}{1+\delta_0} C_{cog}^{Z}\right).$$

Using this expression, we can now get a new interpretation of our scheme . It contains two optional schemes. One is the optimal scheme for the interference channel that achieves its cognitive capacity. In the second scheme, the secondary first listens in mode B long enough to collect information of the interference from source 1 during mode A. In each time instant, it gets $\beta - n_1$ bits. Then in mode A, it uses this information to perform dirty paper coding to fully "cancel" the interference. So the original channel is now equivalent to a Z-channel and C_{cog}^{Z} is achieved for the secondary. The amount of information needed to cancel interference is $C_{cog}^{Z} - C_{cog}^{IFC}$; hence, the time to listen is $\delta_0 = \frac{C_{cog}^{Z} - C_{cog}^{IFC}}{\beta - n_1}$, as defined above. It is easy to see that this scheme achieves rate $\frac{1}{1+\delta_0} C_{cog}^{Z}$, and our scheme picks the better of the two and achieves C_{cog}.

6.1.4 Converse

To prove the converse, we need the following theorem.

Theorem 6.3 *The capacity region $\mathscr{C}$ is contained within $\bigcup_{\delta} \mathscr{C}(\delta)$, where $\mathscr{C}(\delta)$ is the set of rate pairs (R_1, R_2) satisfying*

$$\begin{aligned} R_2 &\leq \frac{1}{1+\delta} n_2 \\ R_1 + R_2 &\leq \frac{1}{1+\delta}[\max(n_2, \alpha_2) + \delta \max(\beta, \alpha_2, n_1) + (n_1 - \alpha_2)^+] \end{aligned}$$

$$R_1 + R_2 \leq \frac{1}{1+\delta}[\max(\alpha_1, n_1) + \delta n_1 + (n_2 - \alpha_1)^+]$$

$$2R_1 + R_2 \leq \frac{1}{1+\delta}[\max(\alpha_1, n_1) + \delta + (n_1 - \alpha_2)^+ + \max(n_2 - \alpha_1, \alpha_2)$$
$$+ \delta \max(\beta, \alpha_2, n_1)].$$

For schemes with scheduling parameter δ, $\mathscr{C}(\delta)$ can be shown as an outer bound on the achievable rate region. The first upper bound is proved by assuming no interference. The second and third upper bounds are proved along the lines of the Z-channel bound in [9], and the last bound has similarities to the $2R_1 + R_2$ upper bound in the same reference. The full details are provided for the Gaussian model.

By evaluating the upper bounds with $R_1 = n_1$ and optimizing over δ, we get an upper bound on R_2 that matches the cognitive capacity given in Theorem 6.1.

6.2 The Cognitive Case: Gaussian Model

We follow the intuition in the previous section to approximately characterize the R_0-capacity of the Gaussian cognitive channel. The auxiliary random variables for the virtual channel in Theorem 4.1 are chosen as follows: For source $i = 1, 2$, we define respectively the public and the private auxiliary random variables W_i and U_i to be independent, zero-mean Gaussian random variables with variances $\sigma^2_{W_i}, \sigma^2_{U_i}$, respectively. In Theorem 4.1, we define

$$X_{W_i} = W_i,$$
$$X_{U_i} = W_i + U_i.$$

The choice of $\sigma^2_{U_i}$ will be such that it appears at lower than the noise power at the destination where it may cause interference. Following the intuition from the linear deterministic case, we will employ *zero-forcing beamforming* for the cooperative private messages. We choose $V_2 = 0$ and V_1 to be independent (of each other and all previously defined auxiliary random variables), identically distributed, zero-mean Gaussian random variables with variance $\sigma^2_{V_1}$. When the channel matrix is invertible, we will impose the following zero-forcing condition, which will ensure that the cooperative signal cancels out at destination 4.

$$\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} \begin{bmatrix} X_{V_1} \\ X_{V_2} \end{bmatrix}.$$

In this case, $X_{V_i}, i = 1, 2$ are correlated Gaussian random variables with variances

$$\begin{aligned} \text{Var}\left(X_{V_1}\right) &= \frac{|h_{24}|^2}{|h_{13}h_{24} - h_{14}h_{23}|^2}\sigma_{V_1}^2 \\ &= \frac{\mathtt{SNR}_2}{\mathtt{SNR}_1\mathtt{SNR}_2 + \mathtt{INR}_1\mathtt{INR}_2 - 2\sqrt{\mathtt{SNR}_1\mathtt{SNR}_2\mathtt{INR}_1\mathtt{INR}_2}\cos\theta}\sigma_{V_1}^2, \quad (6.2) \\ \text{Var}\left(X_{V_2}\right) &= \frac{|h_{14}|^2}{|h_{13}h_{24} - h_{14}h_{23}|^2}\sigma_{V_1}^2 \\ &= \frac{\mathtt{INR}_2}{\mathtt{SNR}_1\mathtt{SNR}_2 + \mathtt{INR}_1\mathtt{INR}_2 - 2\sqrt{\mathtt{SNR}_1\mathtt{SNR}_2\mathtt{INR}_1\mathtt{INR}_2}\cos\theta}\sigma_{V_1}^2. \quad (6.3) \end{aligned}$$

When the channel matrix is singular, we set[1] $\sigma_{V_1}^2 = 0$; i.e., there is no cooperative provate message. The variance parameters must satisfy the power constraint

$$\text{Var}\left(X_{U_i}\right) + \text{Var}\left(X_{V_i}\right) \leq 1, \quad i = 1, 2.$$

The destinations receive

$$\begin{aligned} Y_3 =& h_{13}(W_1 + U_1) + h_{23}W_2 + V_1 + h_{23}U_2 + Z_3 \\ Y_4 =& h_{24}(W_2 + U_2) + h_{24}W_1 + h_{14}U_1 + Z_4. \end{aligned}$$

In Theorem 4.2, as mentioned earlier, we set $\mathsf{C}_{21} = \mathsf{C}_{14} = \mathsf{C}_{23} = \Delta\mathsf{R}_{123} = \Delta\mathsf{R}_{214} = 0$; i.e., only C_{12} is nonzero, in general.

In appendix D, we show that with the above choice of auxiliary random variables, there are power and rate allocations under which we achieve an R_1 which is within R_0 of the point-to-point capacity $C_0 = \log(1 + \mathtt{SNR}_1)$ of the primary link and an R_2 which is within a constant of $\overline{C_{R_0}}$ as defined in Theorem 3.2. Specifically, we prove that

[1] In fact, in a region where the channel matrix is ill-conditioned, we do not employ cooperative private messages.

Theorem 6.4 *If $R_0 > 7$,*

$$C_{R_0} \geq \overline{C_{R_0}} - 23 - 2R_0.$$

To prove the converse part of Theorem 3.2, we need a theorem that is similar to Theorem 6.3. We prove the following in appendix E.

Theorem 6.5 *The capacity region $\mathscr{C}$ is contained within $\bigcup_\delta \mathscr{C}(\delta)$, where $\mathscr{C}(\delta)$ is the set of rate pairs (R_1, R_2) satisfying*

$$\begin{aligned}
R_2 \leq & \frac{1}{1+\delta}\log(1+x_2P_{2A}) \\
R_1 + R_2 \leq & \frac{1}{1+\delta}\Big[\log(1+2x_2P_{2A}+2y_2P_{1A}) + \delta\log(1+(x_1+y_2+z)P_{1B}) \\
& + \log(1+\frac{x_1P_{1A}}{1+y_2P_{1A}})\Big] \\
R_1 + R_2 \leq & \frac{1}{1+\delta}\Big[\log(1+2x_1P_{1A}+2y_1P_{2A}) + \delta\log(1+x_1P_{1B}) \\
& + \log(1+\frac{x_2P_{2A}}{1+y_1P_{2A}})\Big] \\
2R_1 + R_2 \leq & \frac{1}{1+\delta}\Big[\log(1+2x_1P_{1A}+2y_1P_{2A}) + \delta\log(1+x_1P_{1B}) \\
& + \log(1+\frac{x_1P_{1A}}{1+y_2P_{1A}}) + \log(1+y_2P_{1A}+\frac{2x_2P_{2A}+y_2P_{1A}}{1+y_1P_{2A}}) \\
& + \delta\log(1+(x_1+y_2+z)P_{1B})\Big]
\end{aligned}$$

with power constraint

$$\frac{P_{1A}+\delta P_{1B}}{1+\delta} \leq 1, \quad \frac{P_{2A}}{1+\delta} \leq 1, \quad P_{2B} = 0.$$

By setting the power terms to be their maximum possible value, i.e., $P_{iA} = 1+\delta, P_{1B} = \frac{1+\delta}{\delta}, i = 1, 2$, we get a new outer bound on the capacity region. The following lemma is shown in appendix C.

Lemma 6.2 *The capacity region $\mathscr{C}$ is contained within $\bigcup_\delta \mathscr{C}(\delta)$, where $\mathscr{C}(\delta)$ is the set of rate pairs (R_1, R_2) satisfying*

$$\begin{aligned}
R_2 \leq & \frac{1}{1+\delta}\log(1+x_2) + 1 \\
R_1 + R_2 \leq & \frac{1}{1+\delta}\Big[\log(1+2x_2+2y_2) + \delta\log(1+(x_1+y_2+z))
\end{aligned}$$

$$
\begin{aligned}
&+\log(1+\frac{x_1}{1+y_2})\Big] + 2 \\
R_1 + R_2 \leq &\frac{1}{1+\delta}\left[\log(1+2x_1+2y_1) + \delta\log(1+x_1) + \log(1+\frac{x_2}{1+y_1})\right] + 2 \\
2R_1 + R_2 \leq &\frac{1}{1+\delta}\Big[\log(1+2x_1+2y_1) + \delta\log(1+x_1) + \log(1+\frac{x_1}{1+y_2}) \\
&+\max(\log(1+y_2+\frac{2x_2+y_2}{1+y_1}), \log(1+2y_2)) \\
&+\delta\log(1+(x_1+y_2+z))\Big] + 3
\end{aligned}
$$

Setting $R_1 = \log(1+x_1) - R_0$ in this lemma, we get $C_{R_0} \leq \overline{C_{R_0}}$.

APPENDIX A

PROOF OF THEOREM 5.3

We prove this sum-rate achievability result in two steps. Instead of directly comparing $\overline{C_{\mathsf{sum}}}$ with the rate achievable by the coding scheme in section 4, we will first show that the $\overline{C_{\mathsf{sum}}}$ is within a constant of $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$, a quantity we define below, inspired by the result for the linear deterministic model. We will then prove that the coding scheme in section 4 can be used to achieve a sum-rate which is within a constant of $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$. Specifically, we prove the following two lemmas, which together imply Theorem 5.3.

Lemma A.1 *Let* $n_D = \lfloor \log \mathtt{SNR} \rfloor^+, n_I = \lfloor \log \mathtt{INR} \rfloor^+, n_C = \lfloor \log \mathtt{CNR} \rfloor^+$. *Define*

$$\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}} = \max_{\delta} \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta) = \max_{\delta} \min(u_1' - 6, u_2' - 4, u_3', u_4' - 4, u_4 - 10)$$

where

$$u_1' = \frac{2}{2+\delta}\left(\delta n_D + \max\{n_D, n_C\}\right)$$
$$u_2' = \frac{1}{2+\delta}\left(\delta \max\{2n_D - n_I, n_I\} + n_D + \max\{n_D, n_I, n_C\}\right)$$
$$u_3' = \frac{2}{2+\delta}\left(\delta \max\{n_I, n_D - n_I\} + \max\{n_D, n_C\}\right)$$
$$u_4' = \frac{2(1+\delta)}{2+\delta}\max\{n_D, n_I\}$$

and u_4 *is as defined in Theorem 3.1. Then* $\overline{C_{\mathsf{sum}}} \leq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}} + 10$.

Lemma A.2 $C_{\mathsf{sum}} \geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}} - 7$.

Note that in the definition of $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ we have preserved the term u_4 rather than have all the terms as functions of n_D, n_I, and n_C. The reason for this is that the linear deterministic model is too coarse to model the channel phase information. When the channel matrix becomes ill-conditioned, the term u_4 may dominate $\overline{C_{\mathsf{sum}}}$ and also have a large gap with respect to u_4'.

A.1 Proof of Lemma A.1

We let $\lfloor \log x \rfloor^+ = n_D, \lfloor \log y \rfloor^+ = n_I, \lfloor \log z \rfloor^+ = n_C$. Observe that

1. $\lfloor \log x \rfloor^+ \leq (\log x)^+ \leq \lfloor \log x \rfloor^+ + 1$.
2. $\lfloor \log x \rfloor^+ \leq \log(1+x) \leq 1 + (\log x)^+ \leq 2 + \lfloor \log x \rfloor^+$.

We can show that

$$\begin{aligned}
\log(1+2x+2y) &\leq \log 5 + (\log(x \vee y))^+ \leq \log 5 + (\log x)^+ \vee (\log z)^+ \\
&\leq \log 5 + 1 + n_D \vee n_I \\
\log(1+y+\frac{2x+y}{1+y}) &\leq \log(1+3y+y^2+2x) - \log(1+y) \\
&\leq \log 7 + (\log y)^+ \vee (\log y^2)^+ \vee (\log x)^+ - n_I \\
&\leq \log 7 + (2n_I+2) \vee (n_D+1) - n_I \\
&\leq \log 7 + n_I \vee (n_D - n_I) + 2.
\end{aligned}$$

Using these, it is easy to verify that

$$\begin{aligned}
u_1 &\leq u_1' + \frac{2}{2+\delta}(2\delta + \log 3 + 1) \leq u_1' + 4 \leq u_1' - 6 + 10 \\
u_2 &\leq u_2' + \frac{1}{2+\delta}((\log 5 + 1)\delta + 2 + 3 + (\log 3 + 1)\delta) \leq u_2' + \log 15 + 2 \\
&\leq u_2' - 4 + 10 \\
u_3 &\leq u_3' + \frac{2}{2+\delta}((\log 7 + 2)\delta + 3) \leq u_3' + 2(\log 7 + 2) \leq u_3' + 10 \\
u_4 &\leq u_4' + \frac{1}{2+\delta}((\log 13 + 2)\delta + 2(\log 3 + 1)) \leq u_4' + \log 13 + 2 \\
&\leq u_4' - 4 + 10.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\overline{C_{\mathsf{sum}}} &= \max_{\delta} \min(u_1, u_2, u_3, u_4) \\
&\leq \max_{\delta} \min(u_1' - 6, u_2' - 4, u_3', u_4' - 4, u_4 - 10) + 10 = \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}} + 10.
\end{aligned}$$

A.2 Proof of Lemma A.2

To simplify notation, let

$$\beta_1 = \frac{x^2 + y^2 - 2xy\cos\theta}{x(x+y)}$$
$$\beta_2 = \frac{x^2 + y^2 - 2xy\cos\theta}{y(x+y)}$$

Then, for the auxiliary random variables in section 5.2, we have $\sigma_V^2 = \beta_1 x \mathrm{Var}\left(X_V\right)$. Let us also note some useful facts about β_1 and β_2.

1. $\beta_1 x = \beta_2 y$.
2. $\frac{1}{2} \leq \frac{x}{y} \leq 2 \rightarrow \beta_i \leq 3, i = 1, 2$
3. $\frac{x}{y} \geq 2 \rightarrow \beta_1 \geq \frac{1}{6}, \frac{y}{x} \geq 2 \rightarrow \beta_2 \geq \frac{1}{6}$.

To prove the lemma, we will consider five different regions which together cover all possibilities.

Region 1: $z \leq x$ or $z \leq 1$ or $y \leq 1$.

In this region, we do not use any cooperation ($\delta_B = \delta_C = 0$ in Theorem 4.2). The scheme reduces to Han and Kobayashi's scheme for the interference channel. Using this scheme, we have the following achievable sum-rate for the interference channel.

Lemma A.3 $\overline{R^{\mathsf{IFC}}_{\mathsf{sum}}} - 6$ *is achievable for the interference channel, where*

$$\overline{R^{\mathsf{IFC}}_{\mathsf{sum}}} = \min \begin{pmatrix} (2n_D - n_I) \vee n_I \\ 2(n_I \vee (n_D - n_I)) \\ 2n_D \end{pmatrix}.$$

Proof. We know from [9] that the sum capacity of the interference channel is upper bounded by $\overline{C^{\mathsf{IFC}}_{\mathsf{sum}}}$ defined below.
When $\mathtt{SNR} \geq \mathtt{INR}$,

$$\overline{C^{\mathsf{IFC}}_{\mathsf{sum}}} = \min \begin{pmatrix} \log(1 + \mathtt{SNR}) + \log(1 + \frac{\mathtt{SNR}}{1+\mathtt{INR}}) \\ 2\log(1 + \mathtt{INR} + \frac{\mathtt{SNR}}{1+\mathtt{INR}} \end{pmatrix}.$$

When $\mathtt{SNR} < \mathtt{INR}$,

$$\overline{C^{\mathsf{IFC}}_{\mathsf{sum}}} = \min \begin{pmatrix} \log(1 + \mathtt{SNR} + \mathtt{INR}) \\ 2\log(1 + \mathtt{SNR}) \end{pmatrix}.$$

In either case, it is not hard to verify that $\overline{C^{\mathsf{IFC}}_{\mathsf{sum}}} > \overline{R^{\mathsf{IFC}}_{\mathsf{sum}}} - 4$. As $\overline{C^{\mathsf{IFC}}_{\mathsf{sum}}}$ can be achieved within 2 bits when $\mathtt{SNR} \geq \mathtt{INR}$ [9] and achieved exactly when $\mathtt{SNR} < \mathtt{INR}$ [6], we can conclude that $\overline{R^{\mathsf{IFC}}_{\mathsf{sum}}} - 6$ is achievable for the interference channel. □

When $z \leq x$ or $z \leq 1$ or $y \leq 1$, we can verify that $\overline{C^{\mathsf{LDM}}_{\mathsf{sum}}} = \max_\delta \min(u'_1 - 6, u'_2 - 4, u'_3, u'_4 - 4) \leq \max_\delta \min(u'_1, u'_2, u'_3, u'_4) \leq \overline{R^{\mathsf{IFC}}_{\mathsf{sum}}}$. By the above lemma, $\overline{C^{\mathsf{LDM}}_{\mathsf{sum}}}$ can be achieved within 6 bits in this region.

Region 2: $\frac{1}{2} \leq \frac{x}{y} \leq 2, z > x, z > 1$, and $y > 1$.

If $x < 1$, we have $y \leq 2x < 2$, and we get $n_D = n_I = 0$. Hence $\overline{C^{\mathsf{LDM}}_{\mathsf{sum}}} = 0$, which can be achieved trivially. Hence, let us take $x \geq 1$.

In this region, we have $n_I - 2 \leq n_D \leq n_I + 2$ and $\beta_1 \leq 3$. We set $\mathsf{C}_{sd} = 0, \Delta R = 0$. In modes B and C, each source uses power $1 - \frac{1}{x}$ to send data to its own destination and uses power $\frac{1}{x}$ to share bits with the other source. Under the natural order for superposition coding, the following rates are achievable.

$$R_{\mathrm{B}} = R_{\mathrm{C}} = \log\left(1 + \frac{(1-\frac{1}{x})x}{2}\right) = \log(1+x) - 1 \geq (n_D - 1)^+$$
$$\delta \mathsf{C}_{ss} = \log\left(1 + \frac{z}{x}\right) \geq \log\left(\frac{z}{x}\right) \geq (n_C - n_D - 1)^+.$$

We set $R_{\mathrm{B}} = R_{\mathrm{C}} = (n_D - 1)^+$ and $\delta \mathsf{C}_{ss} = (n_C - n_D - 1)^+$.

For the virtual channel, we set rates $R_U = R_{V'} = 0$ and powers $\sigma^2_W = \frac{1}{2}, \mathrm{Var}\,(X_V) = \frac{1}{2}$. Then, destination 3 gets W_1, W_2, V_1 with powers $\frac{x}{2}, \frac{y}{2}, \frac{\beta_1 x}{2}$, respectively, and destination 4 gets W_2, W_1, V_2 with powers $\frac{x}{2}, \frac{y}{2}, \frac{\beta_1 x}{2}$, respectively. From Theorem 4.1, nonnegative rates which satisfy the following conditions are achievable[1]

$$
\begin{aligned}
R_W &\leq \log\left(1 + \frac{x}{2}\right) \\
R_V &\leq \log\left(1 + \frac{\beta_1 x}{2}\right) \\
R_V + R_W &\leq \log\left(1 + \frac{x + \beta_1 x}{2}\right) \\
2R_W &\leq \log\left(1 + \frac{x + y}{2}\right) \\
R_W + R_V &\leq \left(1 + \frac{y + \beta_1 x}{2}\right) \\
2R_W + R_V &\leq \log\left(1 + \frac{x + y + \beta_1 x}{2}\right) \\
R_V &\leq \mathsf{C}_{ss}.
\end{aligned}
$$

The above conditions are met if the following reduced set of constraints are satisfied.

$$
\begin{aligned}
2R_W + R_V &\leq \log\left(1 + \frac{y}{2}\right) \\
R_W + R_V &\leq \log\left(1 + \frac{x}{2}\right) \\
R_V &\leq \log\left(1 + \frac{\beta_1 x}{2}\right) \wedge \mathsf{C}_{ss}.
\end{aligned}
$$

and $R_W, R_V \geq 0$. This implies that nonnegative rates satisfying the conditions below are also achievable.

$$
\begin{aligned}
2R_W + R_V &\leq (n_I - 1)^+ \\
R_W + R_V &\leq (n_D - 1)^+ \\
R_V &\leq \log\left(1 + \frac{\beta_1 x}{2}\right) \wedge \mathsf{C}_{ss}
\end{aligned}
$$

[1]Redundant conditions are not listed here. Also, conditions corresponding to error events which involve an unwanted message along with zero-rate messages are also not listed. For example, the rate constraint on $R_{W_2} + R_{U_1}$ is avoided since it corresponds to the error event of destination 3 making an error on the unwanted message m_{W_2} and the message m_{U_1}, which is absent in this case.

Let $R_{\mathrm{A}} = 2(R_W + R_V)$, the sum-rate achieved by the virtual channel. We have

$$R_{\mathrm{A}} = \min\left\{\begin{array}{c} (2n_D - 2)^+ \\ (n_I + \mathsf{C}_{ss} - 1)^+ \\ (n_I + \log(1 + \frac{\beta_1 x}{2}) - 1)^+ \end{array}\right\}.$$

We achieve a sum-rate of $R_{\mathsf{sum}} = \frac{1}{2+\delta}(\delta R_{\mathrm{A}} + R_{\mathrm{B}} + R_{\mathrm{C}})$. Observing that

1. $n_D \wedge n_I \geq \max(n_D, n_I) - 2$,
2. $\max(2n_D - n_I, n_I) \leq \max(n_D, n_I) + 2$
   $n_C = \max(n_D, n_C) \geq \max(n_D, n_I, n_C) - 2$,
3. $\log(1 + \frac{\beta_1 x}{2}) = \log(1 + \frac{x^2+y^2-2xy\cos\theta}{2(x+y)})$
   $\geq \log(1 + x + y + x^2 + y^2 - 2xy\cos\theta) - \log(2(x+y))$
   $\geq \log(1 + 4x + 4y + x^2 + y^2 - 2xy\cos\theta) - 2 - (\max(n_D, n_I) + 3)$, and
4. $2\log(1 + x + y) \leq 2(\log 3 + 1 + \max(n_D, n_I))$,

we get

$$R_{\mathsf{sum}} \geq \min\left\{\begin{array}{c} u_4' - \frac{6\delta+6}{2+\delta} \\ u_2' - \frac{4\delta+5}{2+\delta} \\ u_4 - \frac{6\delta+8+2\log 3}{2+\delta} \end{array}\right\} \geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta) - 2.$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved within 2 bits in this region.

Region 3: $2y < x < z$ and $y > 1$.

In this region, $\beta_1 \geq \frac{1}{6}$ is a finite constant bounded away from 0. We set $\mathsf{C}_{sd} = 0, \Delta R = 0$. As in the previous region, in modes B and C, each source uses power $1 - \frac{1}{x}$ to send data to its own destination and uses power $\frac{1}{x}$ to share bits with the other source. Under the natural order for superposition coding, the following rates are achievable.

$$R_{\mathrm{B}} = R_{\mathrm{C}} = \log\left(1 + \frac{(1-\frac{1}{x})x}{2}\right) = \log(1+x) - 1 \geq (n_D - 1)^+$$
$$\delta\mathsf{C}_{ss} = \log\left(1 + \frac{z}{x}\right) \geq \log\left(\frac{z}{x}\right) \geq (n_C - n_D - 1)^+.$$

We set $R_{\mathrm{B}} = R_{\mathrm{C}} = (n_D - 1)^+$ and $\delta\mathsf{C}_{ss} = (n_C - n_D - 1)^+$.

For the virtual channel, we take $R_{V'} = 0$ and set powers $\sigma_W^2 = \frac{1}{3}, \sigma_U^2 = \frac{1}{3y}, \mathrm{Var}(X_V) = \frac{1}{3}$. So destination 3 receives W_1, U_1, W_2, V_1, U_2 with powers $\frac{x}{3}, \frac{x}{3y}, \frac{y}{3}, \frac{\beta_1 x}{3}, \frac{1}{3}$, respectively, and destination 4 gets W_2, U_2, W_1, V_2, U_1 with powers $\frac{x}{3}, \frac{x}{3y}, \frac{y}{3}, \frac{\beta_1 x}{3}, \frac{1}{3}$, respectively. It is easy to verify that the following constraints on nonnegative rates imply all the relevant rate constraints in Theorem 4.1.

$$\begin{aligned}
2R_W + R_U + R_V &\le \log\left(1 + \frac{\beta_1 x}{4}\right) \\
R_U + R_W &\le \log\left(1 + \frac{\frac{x}{y} + y}{4}\right) \\
R_U &\le \log\left(1 + \frac{x}{4y}\right) \\
R_V &\le \mathsf{C}_{ss}.
\end{aligned}$$

Hence, the following nonnegative rates are achievable.

$$\begin{aligned}
2R_W + R_U + R_V &\le (n_D - 2 - \log 6)^+ \\
R_U + R_W &\le (\max(n_D - n_I, n_I) - 3)^+ \\
R_U &\le (n_D - n_I - 3)^+ \\
R_V &\le \mathsf{C}_{ss}.
\end{aligned}$$

Setting $R_{\mathrm{A}} = 2(R_W + R_U + R_V)$, we can achieve

$$R_{\mathrm{A}} = \min \left\{ \begin{array}{c} (2n_D - 4 - 2\log 6)^+ \\ (2\max(n_D - n_I, n_I) + 2\mathsf{C}_{ss} - 6)^+ \\ (2n_D - n_I + \mathsf{C}_{ss} - 5 - \log 6)^+ \end{array} \right\}.$$

We get

$$\begin{aligned}
R_{\mathsf{sum}} &= \frac{1}{2+\delta}(\delta R_{\mathrm{A}} + R_{\mathrm{B}} + R_{\mathrm{C}}) \\
&\ge \min \left\{ \begin{array}{c} u_4' - \frac{(4+2\log 6)\delta + 2}{2+\delta} \\ u_3' - \frac{6\delta+4}{2+\delta} \\ u_2' - \frac{(5+\log 6)\delta+3}{2+\delta} \end{array} \right\} \ge \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta) - 6.
\end{aligned}$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved within 6 bits in this region.

Region 4: $2x < y \le z$ and $y > 1$.

In this region, $\beta_2 \ge \frac{1}{6}$ and we set $\mathsf{C}_{sd} = 0$. We further divide this region into two subregions depending on whether $x \ge 1$ or $x < 1$. Moreover, when n_C is small, we will use only the cooperative private signal to improve the virtual channel sum-rate. But when n_C is big enough to achieve the cut-set bound of the virtual channel, we need to use relaying in modes B and C ($\Delta R > 0$) to further increase the achievable rate. In each subregion, we consider two cases – one involving relaying in modes B and C ($\Delta R > 0$) and the other not ($\Delta R = 0$).

Subregion 1: $x < 1 < y \le z$. As $n_D = 0$, no (significant) direct transmission of data from source to destination is possible; all data must pass through the other source. It can happen in one of two ways: relaying in modes B and C, and cooperative private message for the virtual channel. We consider these two cases.

Case 1: $y^\delta \ge z$. In this case, we do not relay in modes B and C (i.e., $\Delta R = 0$). In modes B and C, the sources use all their power to send data to the other source. Then

$$\delta \mathsf{C}_{ss} = \log(1 + z) \ge n_C.$$

For the virtual channel, each source relays the shared data to the other destination and the direct link signals are treated as interference. It is easy to see that we can achieve

$$\begin{aligned} R_{\mathrm{A}} &= 2\min(\log(1 + \frac{y}{1+x}), \mathsf{C}_{ss}) \\ &\ge 2\min((n_I - 1), \mathsf{C}_{ss}) \ge 2(\frac{n_C}{\delta} - 2), \end{aligned}$$

where the last inequalty follows from the fact that the condition $y^\delta \ge z$ implies that $\delta n_I + \delta \ge n_C$. The sum-rate achieved is

$$R_{\mathsf{sum}} = \frac{1}{2+\delta}\delta R_{\mathrm{A}} \ge u_1' - \frac{4\delta}{2+\delta} \ge \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta).$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved in this case.

Case 2: $y^\delta \le z$. In mode B and C, each source uses powers $\frac{1}{2}\sqrt{\frac{y^\delta}{z}}$ and $\frac{1}{2}$, respectively, to share bits with the other source and the other destination, respectively. With the natural order for superposition coding, the following rates are achievable.

$$\begin{aligned}
\Delta R &= \log\left(1 + \frac{\frac{y}{2}}{1+\frac{1}{2}\sqrt{\frac{y^{\delta+2}}{z}}}\right) \\
&\ge \log\left(1+\frac{y}{2}\right) - \log\left(1+\frac{1}{2}\sqrt{\frac{y^{\delta+2}}{z}}\right) \\
&\ge n_I - 1 - \left(\log\sqrt{\frac{y^{\delta+2}}{z}} - 1\right)^+ - 1 \\
&\ge n_I - 2 - \left(\frac{\delta+2}{2}n_I - \frac{1}{2}n_C + \frac{\delta}{2}\right)^+ \\
&\ge n_I - 2 - \left(\frac{\delta+2}{2}n_I - \frac{1}{2}n_C\right)^+ - \frac{\delta}{2} \\
&\ge \min(n_I, \frac{1}{2}(n_C - \delta n_I)) - \frac{\delta}{2} - 2 \\
\Delta R + \delta \mathsf{C}_{ss} &= \log(1 + \frac{1}{2}\sqrt{y^\delta z}) \\
&\ge (\log\sqrt{y^\delta z} - 1)^+ \\
&\ge \log\sqrt{y^\delta z} - 1 \\
&\ge \frac{1}{2}(n_C + \delta n_I) - 1.
\end{aligned}$$

We set

$$\begin{aligned}
\delta \mathsf{C}_{ss} &= \left(\delta n_I - \frac{3}{2}\right)^+ \\
\Delta R &= \left(\min(n_I, \frac{1}{2}(n_C - \delta n_I)) - \frac{\delta}{2} - 2\right)^+,
\end{aligned}$$

which is easily seen to be a valid choice. For the virtual channel, we use the same scheme as in case 1 and we achieve

$$R_{\mathrm{A}} = 2\min((n_I - 1), \mathsf{C}_{ss}) = 2\left(n_I - 1 - \frac{3}{2\delta}\right).$$

Then, the achieved sum-rate is

$$R_{\mathsf{sum}} = \frac{1}{2+\delta}(\delta R_{\mathrm{A}} + 2\Delta R)$$

$$\geq \frac{1}{2+\delta}(2\delta n_I - 2\delta - 3 + 2\Delta R)$$
$$\geq \min\left\{\begin{array}{c} u_4' \\ u_2' \end{array}\right\} - \frac{3\delta+7}{2+\delta} \geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta).$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved in this case.

Subregion 2: $1 \leq x < y \leq z$. We consider two cases.

Case 1: $y^\delta x \geq z$. This is the case when n_C is small. We set $\Delta R = 0$. As in regions 2 and 3, in modes B and C, each source uses power $1 - \frac{1}{x}$ to send data to its own destination and uses power $\frac{1}{x}$ to share bits with the other source. Under the natural order for superposition coding, the following rates are achievable.

$$R_{\mathrm{B}} = R_{\mathrm{C}} = \log\left(1 + \frac{(1-\frac{1}{x})x}{2}\right) = \log(1+x) - 1 \geq (n_D - 1)^+$$
$$\delta \mathsf{C}_{ss} = \log\left(1 + \frac{z}{x}\right) \geq \log\left(\frac{z}{x}\right) \geq (n_C - n_D - 1)^+.$$

We set $R_{\mathrm{B}} = R_{\mathrm{C}} = (n_D - 1)^+$ and $\delta \mathsf{C}_{ss} = (n_C - n_D - 1)^+$.

For the virtual channel, we choose $R_U = R_{V'} = 0$ and set powers $\sigma_W^2 = \frac{1}{2}, \mathrm{Var}(X_V) = \frac{1}{2}$. Note that $\beta_1 x = \beta_2 y$. So destination 3 receives W_1, W_2, V_1 with powers $\frac{x}{2}, \frac{y}{2}, \frac{\beta_2 y}{2}$, respectively, and destination 4 gets W_2, W_1, V_2 with powers $\frac{x}{2}, \frac{y}{2}, \frac{\beta_2 y}{2}$, respectively. It is easy to verify that the following constraints on nonnegative rates imply all the rate constraints in Theorem 4.1.

$$2R_W + R_V \leq \log\left(1 + \frac{\beta_2 y}{2}\right)$$
$$R_W \leq \log\left(1 + \frac{x}{2}\right)$$
$$R_V \leq \mathsf{C}_{ss}.$$

Hence, the following nonnegative rates are achievable.

$$2R_W + R_V \leq (n_I - \log 6 - 2)^+$$
$$R_W \leq (n_D - 1)^+$$
$$R_V \leq \mathsf{C}_{ss}.$$

Setting $R_{\mathrm{A}} = 2(R_W + R_V)$, we can achieve

$$R_{\mathrm{A}} = \min \left\{ \begin{array}{c} (2n_I - 2\log 6 - 4)^+ \\ (n_I + \mathsf{C}_{ss} - \log 6 - 2)^+ \\ (2n_D + 2\mathsf{C}_{ss} - 2)^+ \end{array} \right\}$$
$$\geq \min \left\{ \begin{array}{c} (n_I + \frac{n_C - n_D - 1}{\delta} - 2\log 6 - 5)^+ \\ (2n_D + 2\frac{n_C - n_D - 1}{\delta} - 2)^+ \end{array} \right\}.$$

The second inequality is due to the fact the condition $y^\delta x \geq z$ implies that $n_C - n_D - 1 \leq \delta(n_I + 1)$. Thus, the following is an achievable sum-rate

$$R_{\mathsf{sum}} = \frac{1}{2+\delta}(\delta R_{\mathrm{A}} + R_{\mathrm{B}} + R_{\mathrm{C}})$$
$$\geq \min \left\{ \begin{array}{c} u_2' - \frac{(5+2\log 6)\delta + 3}{2+\delta} \\ u_1' - \frac{2\delta+4}{2+\delta} \end{array} \right\}$$
$$\geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}} - 7.$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved within 7 bits in this case.

Case 2: $y^\delta x \leq z$. In this case, n_C is large enough to require relaying in modes B and C. Unlike case 1, for modes B and C, sources use power of $\frac{1}{3}$ to send data to its own destination, and $\frac{1}{3}\sqrt{\frac{y^\delta}{xz}}$ and $\frac{1}{3x}$, respectively, to send to the other source and the other destination, respectively. With the natural order for superposition coding, the following are achievable.

$$R_{\mathrm{B}} = R_{\mathrm{C}} = \log\left(1 + \frac{\frac{x}{3}}{1 + \frac{1}{3} + \frac{1}{3}\sqrt{\frac{xy^\delta}{z}}}\right) \geq \log(1 + \frac{x}{5}) \geq (n_D - \log 5)^+$$
$$\Delta R = \log\left(1 + \frac{\frac{y}{3x}}{1 + \frac{1}{3}\sqrt{\frac{y^{\delta+2}}{xz}}}\right)$$
$$\geq \left(\log\frac{y}{x} - \log 3\right)^+ - \left(\log\sqrt{\frac{y^{\delta+2}}{xz}} - \log 3\right)^+ - 1$$
$$\geq n_I - n_D - 1 - \log 3$$
$$- (\frac{1}{2}((\delta+2)n_I - n_D - n_C + \delta) + 1 - \log 3)^+ - 1$$
$$\geq n_I - n_D - 2 - \log 3 - \frac{1}{2}((\delta+2)n_I - n_D - n_C)^+ - \frac{1}{2}\delta$$
$$\geq \min(n_I - n_D, \frac{1}{2}(n_C - n_D - \delta n_I)) - 2 - \log 3 - \frac{1}{2}\delta$$

$$\begin{aligned}
\Delta R + \delta \mathsf{C}_{ss} &= \log\left(1 + \frac{1}{3}\sqrt{\frac{zy^\delta}{x}}\right) \\
&\geq (\frac{1}{2}\log(\frac{zy^\delta}{x}) - \log 3)^+ \\
&\geq \frac{1}{2}\log(\frac{zy^\delta}{x}) - \log 3 \\
&\geq \frac{1}{2}(n_C + \delta n_I - n_D - 1) - \log 3.
\end{aligned}$$

We set

$$\begin{aligned}
R_{\text{B}} = R_{\text{C}} &= (n_D - \log 5)^+ \\
\delta \mathsf{C}_{ss} &= (\delta n_I - 1 - \log 3)^+ \\
\Delta R &= (\min(n_I - n_D, \frac{1}{2}(n_C - n_D - \delta n_I)) - 2 - \log 3 - \frac{1}{2}\delta)^+.
\end{aligned}$$

Since the condition $y^\delta x \leq z$ implies that $\delta n_I - 1 \leq n_C - n_D$, it is easy to see that the above choice is valid.

For the virtual channel, the scheme is the same as in case 1, which gives

$$\begin{aligned}
R_{\text{A}} &= \min\left\{\begin{array}{c} (2n_I - 2\log 6 - 4)^+ \\ (n_I + \mathsf{C}_{ss} - \log 6 - 2)^+ \\ (2n_D + 2\mathsf{C}_{ss} - 2)^+ \end{array}\right\} \\
&\geq 2n_I - 2\log 6 - 4 - \frac{2 + 2\log 3}{\delta}.
\end{aligned}$$

Thus, the following is an achievable sum-rate.

$$\begin{aligned}
R_{\mathsf{sum}} &= \frac{1}{2+\delta}(\delta R_{\text{A}} + R_{\text{B}} + R_{\text{C}} + 2\Delta R) \\
&= \frac{2}{2+\delta}(\delta n_I - (2 + \log 6)\delta + n_D - 1 - \log 3 - \log 5 \Delta R) \\
&= \frac{2}{2+\delta}(\delta n_I + n_D + \min(n_I - n_D, \frac{1}{2}(n_C - n_D - \delta n_I)) \\
&\quad - (\frac{5}{2} + \log 6)\delta - 3 - 2\log 3 - \log 5) \\
&\geq \min\left\{\begin{array}{c} u_4' \\ u_2' \end{array}\right\} - \frac{(5 + 2\log 6)\delta + 6 + 4\log 3 + 2\log 5}{2+\delta} \\
&\geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta) - (1 + 2\log 6) \\
&\geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta) - 7.
\end{aligned}$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved within 7 bits in this case.

Region 5: $x < z < y, 2x < y$, and $z > 1$.

In this region, $\beta_2 \geq \frac{1}{6}$. We again subdivide this region into two subregions depending on whether or not $x \geq 1$. Moreover, when n_C and n_I are small, we will use only the cooperative private signal and the pre-shared public signal to improve the virtual channel sum-rate. But when n_C, n_I are big enough to achieve the cut-set bound of the virtual channel, we need to use relaying in modes B and C (i.e., $\Delta R > 0$) to further improve the achievable rate. In each subregion, we will consider both cases.

Subregion 1: $x < 1 < z < y$. As in subregion 1 of region 4, since $n_D = 0$, no (significant) direct transmission of data from source to destination is possible; and all data must pass through the other source. It can happen in one of two ways: relaying in modes B and C, and cooperative private message for the virtual channel. Again, we consider two cases.

Case 1: $y^\delta \geq z$. The same scheme that we used in region 4, subregion 1, case 1 (i.e., $2x < y$, $x < 1 < y \leq z$ and $y^\delta \geq z$), applies here.

Case 2: $y^\delta < z$. In modes B and C, each source uses powers $\frac{1}{2}$ and $\frac{1}{2\sqrt{y^{1+\delta}}}$ to share bits with the other source and the other destination, respectively. Under the natural order of superposition coding, the following rates are achievable.

$$\begin{aligned}
\delta \mathsf{C}_{ss} + \Delta R &= \log\left(1 + \frac{\frac{z}{2}}{1 + \frac{z}{2\sqrt{y^{1+\delta}}}}\right) \\
&\geq \log\left(1 + \frac{z}{2}\right) - \log\left(1 + \frac{z}{2\sqrt{y^{1+\delta}}}\right) \\
&\geq n_C - 1 - \left(\log \frac{z}{\sqrt{y^{1+\delta}}} - 1\right)^+ - 1 \\
&\geq n_C - 2 - \left(n_C - \frac{1+\delta}{2} n_I\right)^+ \\
&= \min\left(n_C, \frac{1+\delta}{2} n_I\right) - 2 \\
\Delta R &= \log\left(1 + \frac{1}{2}\sqrt{y^{1-\delta}}\right) \\
&\geq \frac{1-\delta}{2} n_I - 1.
\end{aligned}$$

We set

$$\delta \mathsf{C}_{ss} = \delta n_I - 3$$
$$\Delta R = \min\Big(n_C - \delta n_I, \frac{1-\delta}{2} n_I\Big) - 1.$$

It is easy to see that the above choice is valid. For the virtual channel, let us use the same scheme as in case 1, and we have

$$R_{\mathrm{A}} = 2\min((n_I - 1), \mathsf{C}_{ss}) \geq 2\Big(n_I - 1 - \frac{3}{\delta}\Big).$$

The sum-rate achieved is

$$\begin{aligned} R_{\mathsf{sum}} &= \frac{1}{2+\delta}\Big(\delta R_{\mathrm{A}} + 2\Delta R\Big) \\ &\geq \frac{2}{2+\delta}\Big(\delta n_I - \delta - 3 + \Delta R\Big) \\ &\geq \min\left\{ \begin{array}{c} u_1' \\ u_2' \end{array} \right\} - \frac{2(\delta+4)}{2+\delta} \geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta). \end{aligned}$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved in this case.

Subregion 2: $1 \leq x < z < y$.

Case 1: $y \leq xy^{\delta}$ or $n_C - n_D + 1 \leq \delta(n_I - n_D)$. The condition $y \leq xy^{\delta}$ leads to $n_I \leq n_D + \delta n_I + \delta + 1$. In mode B, C, each source uses power $1 - 1/x$ to send data to its own destination and $1/x - 1/z$ and $1/z$ to share bits with the other source and the other destination, respectively. Under the natural order of superposition coding, the following rates are achievable.

$$R_{\mathrm{B}} = R_{\mathrm{C}} = \log(1+x) - 1 \geq (n_D - 1)^+$$
$$\delta \mathsf{C}_{ss} = \log(1 + \frac{z}{x}) - 1 \geq (n_C - n_D - 2)^+$$
$$\delta \mathsf{C}_{sd} = \log(1 + \frac{y}{z}) \geq (n_I - n_C - 1)^+.$$

We set the rates to the right-hand sides above.

For the virtual channel, we take $R_U = 0$ and set powers $\sigma_W^2 = \frac{1}{3}, \sigma_{V'}^2 = \frac{1}{3}, \mathrm{Var}\,(X_V) = \frac{1}{3}$. Note that $\beta_1 x = \beta_2 y$. Hence, destination 3 gets W_1, W_2, V_1, V_1' with powers $\frac{x}{3}, \frac{y}{3}, \frac{\beta_2 y}{3}, \frac{x}{3}$, respectively, and destination 4 gets W_2, W_1, V_2, V_2' with power $\frac{x}{3}, \frac{y}{3}, \frac{\beta_2 y}{3}, \frac{x}{3}$, respectively. It is easy to verify (using the fact that

$\beta_2 \leq (x+y)/y$) that the following constraints on nonnegative rates imply all the rate constraints in Theorem 4.1.

$$\begin{aligned} 2R_W + R_V + R_{V'} &\leq \log\left(1 + \frac{\beta_2 y}{3}\right) \\ R_W + R_{V'} &\leq \log\left(1 + \frac{x}{3}\right) \\ R_{V'} &\leq \mathsf{C}_{sd} \\ R_V &\leq \mathsf{C}_{ss}. \end{aligned}$$

Hence, the following nonnegative rates are achievable.

$$\begin{aligned} 2R_W + R_V + R_{V'} &\leq (n_I - \log 3 - \log 6)^+ \\ R_W + R_{V'} &\leq (n_D - \log 3)^+ \\ R_{V'} &\leq \mathsf{C}_{sd} \\ R_V &\leq \mathsf{C}_{ss}. \end{aligned}$$

Setting $R_{\mathrm{A}} = 2(R_W + R_V + R_{V'})$, we can achieve

$$R_{\mathrm{A}} = \min \left\{ \begin{array}{c} (2n_D + 2\mathsf{C}_{ss} - 2\log 3)^+ \\ (n_I + \mathsf{C}_{ss} + \mathsf{C}_{sd} - \log 3 - \log 6)^+ \\ (2n_I - 2\log 3 - 2\log 6)^+ \end{array} \right\}.$$

Since $y \leq xy^\delta$ or $n_C - n_D + 1 \leq \delta(n_I - n_D)$ holds, we have either

$$\begin{aligned} \mathsf{C}_{ss} &= \frac{(n_C - n_D - 2)^+}{\delta} \leq \frac{(\delta(n_I - n_D) - 3)^+}{\delta} \leq n_I - n_D, \quad \text{or} \\ \mathsf{C}_{ss} + \mathsf{C}_{sd} &\leq \frac{(n_I - n_D - 3)^+}{\delta} \leq \frac{(\delta n_I + \delta - 2)^+}{\delta} \leq n_I + 1. \end{aligned}$$

Therefore,

$$R_{\mathrm{A}} = \min \left\{ \begin{array}{c} 2n_D + 2\mathsf{C}_{ss} - 2\log 3 - 2\log 6 \\ n_I + \mathsf{C}_{ss} + \mathsf{C}_{sd} - 2\log 3 - 2\log 6 - 1 \end{array} \right\}.$$

Thus, we have an achievable sum-rate of

$$R_{\mathsf{sum}} = \frac{1}{2+\delta}(\delta R_{\mathrm{A}} + R_{\mathrm{B}} + R_{\mathrm{C}})$$

$$\geq \min \left\{ \begin{array}{c} u_1' - \frac{(2\log 3 + 2\log 6)\delta + 6}{2+\delta} \\ u_2' - \frac{(2\log 3 + 2\log 6 + 1)\delta + 5}{2+\delta} \end{array} \right\} \geq \overline{C^{\mathsf{LDM}}_{\mathsf{sum}}}(\delta) - 7.$$

Hence, $\overline{C^{\mathsf{LDM}}_{\mathsf{sum}}}$ can be achieved within 7 bits in this case.

Case 2: $y \geq xy^\delta$ and $n_C - n_D + 1 \geq \delta(n_I - n_D)$. In modes B and C, each source uses a power of $\frac{1}{3}$ to send data to its own destination and $\frac{1}{3x}$ and $\frac{1}{3\sqrt{y^{1+\delta}x^{1-2\delta}}}$ to share bits with the other source and the other destination, respectively. Note that this is a valid power allocation since we may prove that for this case, $y^{1+\delta}x^{1-2\delta} \geq 1$. Under the natural order for superposition coding, the following rates are achievable

$$\begin{aligned}
R_{\mathrm{B}} = R_{\mathrm{C}} &= \log\left(1 + \frac{\frac{2}{3}}{1 + \frac{1}{3} + \frac{x}{3\sqrt{y^{1+\delta}x^{1-2\delta}}}}\right) \\
&\geq \log\left(1 + \frac{x}{5}\right) \geq (n_D - \log 5)^+ \\
\delta \mathsf{C}_{ss} + \Delta R &= \log\left(1 + \frac{\frac{z}{3x}}{1 + \frac{z}{3\sqrt{y^{1+\delta}x^{1-2\delta}}}}\right) \\
&\geq \log\left(1 + \frac{z}{3x}\right) - \log\left(1 + \frac{z}{3\sqrt{y^{1+\delta}x^{1-2\delta}}}\right) \\
&\geq n_C - n_D - 1 - \log 3 - \left(\log \frac{z}{\sqrt{y^{1+\delta}x^{1-2\delta}}} - \log 3\right)^+ - 1 \\
&\geq n_C - n_D - 2 - \log 3 \\
&\quad - \left(n_C + 1 - \frac{1+\delta}{2} n_I - \frac{1-2\delta}{2} n_D + \frac{\delta}{2} - \log 3\right)^+ \\
&\geq n_C - n_D - 2 - \log 3 - \left(n_C - \frac{1+\delta}{2} n_I - \frac{1-2\delta}{2} n_D\right)^+ - \frac{\delta}{2} \\
&= \min\left(n_C - n_D, \frac{1+\delta}{2} n_I - \frac{1+2\delta}{2} n_D\right) - 2 - \log 3 - \frac{\delta}{2} \\
\delta \mathsf{C}_{sd} + \Delta R &= \log\left(1 + \frac{1}{3}\sqrt{\frac{y^{1-\delta}}{x^{1-2\delta}}}\right) \\
&\geq \left(\frac{1}{2}\log \frac{y^{1-\delta}x^\delta}{x^{1-\delta}} - \log 3\right)^+ \\
&\geq \frac{1}{2}\log \frac{y^{1-\delta}x^\delta}{x^{1-\delta}} - \log 3 \\
&\geq \frac{1-\delta}{2} n_I - \frac{1-2\delta}{2} n_D - \frac{1-\delta}{2} - \log 3.
\end{aligned}$$

We set

$$\begin{aligned}
\delta\mathsf{C}_{ss} &= \delta(n_I - n_D) - 3 - \log 3 - \frac{\delta}{2} \\
\delta\mathsf{C}_{sd} &= \delta n_D - \frac{3}{2} + \frac{\delta}{2} - \log 3 \\
\Delta R &= \min\left(n_C - n_D - \delta(n_I - n_D), \frac{1-\delta}{2} n_I - \frac{1}{2} n_D\right) + 1.
\end{aligned}$$

It is easy to verify that the above choice is valid using the fact that the condition $y \geq xy^\delta$ implies that $n_I + 1 \geq n_D + \delta n_I$ and $1 - \delta \geq 0$.

For the virtual channel, we use the same scheme as in case 1 and achieve

$$\begin{aligned}
R_{\mathrm{A}} = \min &\left\{ \begin{array}{c} 2n_D + 2\mathsf{C}_{ss} - 2\log 3 \\ n_I + \mathsf{C}_{ss} + \mathsf{C}_{sd} - \log 3 - \log 6 \\ 2n_I - 2\log 3 - 2\log 6 \end{array} \right\} \\
&\geq 2n_I - 2\log 3 - 2\log 6 - \frac{\frac{9}{2} + 2\log 3}{\delta} \\
&\geq 2n_I - 2\log 3 - 2\log 6 - \frac{6 + 2\log 3}{\delta}.
\end{aligned}$$

Hence, the total achievable sum rate is

$$\begin{aligned}
R_{sum} &= \frac{1}{2+\delta}(\delta R_A + R_B + R_C + 2\Delta R) \\
&\geq \frac{2}{2+\delta}(\delta n_I + n_D - (\log 3 + \log 6)\delta - 3 - \log 3 - \log 5 + \Delta R) \\
&\geq \min\left\{ \begin{array}{c} u_1' \\ u_2' \end{array} \right\} - \frac{2}{2+\delta}((\log 3 + \log 6)\delta + 2 + \log 3 + \log 5) \\
&\geq \overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}(\delta) - 6.
\end{aligned}$$

Hence, $\overline{C_{\mathsf{sum}}^{\mathsf{LDM}}}$ can be achieved within 6 bits in this case.

APPENDIX B

PROOF OF THEOREM 5.4

We will show that for fixed $\delta \geq 0$, $Cut(\delta), Z(\delta), V(\delta), Cut'(\delta)$ are upper-bounds to the sum rate. Let $P_{i,t} = |X_{i,t}|^2$, $i = 1, 2$, and $t = 1, 2, \ldots, N$. We define the average power in the different modes as follows:

$$P_{i\mathrm{A}} = \frac{2+\delta}{\delta N} \sum_{t \in \mathrm{A}} P_{1,t}, \quad P_{i\mathrm{B}} = \frac{2+\delta}{N} \sum_{t \in \mathrm{B}} P_{1,t}, \quad P_{i\mathrm{C}} = \frac{2+\delta}{N} \sum_{t \in \mathrm{B}} P_{1,t}, \quad i = 1, 2.$$

By power constraint, we have $\frac{\delta P_{i\mathrm{A}} + P_{i\mathrm{B}} + P_{i\mathrm{C}}}{2+\delta} \leq 1, i = 1, 2.$ We further define

$$V_{1\mathrm{A}}^L = h_{13} X_{1\mathrm{A}}^L + Z_{1\mathrm{A}}^L, \qquad U_{1\mathrm{A}}^L = h_{14} X_{1\mathrm{A}}^L + Z_{2\mathrm{A}}^L,$$
$$V_{2\mathrm{A}}^L = h_{24} X_{2\mathrm{A}}^L + Z_{2\mathrm{A}}^L, \qquad U_{2\mathrm{A}}^L = h_{23} X_{2\mathrm{A}}^L + Z_{1\mathrm{A}}^L.$$

1. $Cut(\delta)$

$$\begin{aligned}
&L(R_1 - \frac{\epsilon}{2}) \\
\leq& I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{3\mathrm{C}}^L, Y_{2\mathrm{B}}^L) \\
\leq& I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{3\mathrm{C}}^L, Y_{2\mathrm{B}}^L, Y_{1\mathrm{C}}^L, W_2) \\
\overset{1}{=}& I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{3\mathrm{C}}^L, X_2^L, Y_{2\mathrm{B}}^L, Y_{1\mathrm{C}}^L, W_2) \\
=& I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Z_{3\mathrm{C}}^L, X_2^L, Y_{2\mathrm{B}}^L, Y_{1\mathrm{C}}^L, W_2) \\
=& I(W_1; V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, X_2^L, Y_{2\mathrm{B}}^L, Y_{1\mathrm{C}}^L | W_2) \\
=& H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, X_2^L, Y_{2\mathrm{B}}^L, Y_{1\mathrm{C}}^L | W_2) - H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, X_1^L, X_2^L, Y_{2\mathrm{B}}^L, Y_{1\mathrm{C}}^L | W_1, W_2) \\
=& H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, X_2^L, Y_{2\mathrm{B}}^L, Z_{1\mathrm{C}}^L | W_2) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, X_1^L, X_2^L, Z_{2\mathrm{B}}^L, Z_{1\mathrm{C}}^L | W_1, W_2) \\
\leq& H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, X_2^L, Y_{2\mathrm{B}}^L | W_2) + H(Z_{1\mathrm{C}}^L) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L) \\
&- H(Z_{2\mathrm{B}}^L, Z_{1\mathrm{C}}^L | W_1, W_2) - H(X_1^L, X_2^L | Z_{2\mathrm{B}}^L, Z_{1\mathrm{C}}^L, W_1, W_2) \\
=& H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{2\mathrm{B}}^L | W_2) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{2\mathrm{B}}^L) \\
\leq& H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{2\mathrm{B}}^L) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{2\mathrm{B}}^L) \\
\leq& H(V_{1A}^L) + H(Y_{3\mathrm{B}}^L, Y_{2\mathrm{B}}^L) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{2\mathrm{B}}^L).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&R_1 + R_2 - \epsilon \\
\leq &\frac{1}{2+\delta}\Big[\delta \log(1 + xP_{1A}) + \delta \log(1 + xP_{2A}) \\
&+ \log(1 + (x+z)P_{1B}) + \log(1 + (x+z)P_{2C})\Big].
\end{aligned}
$$

2. $Z(\delta)$

$$
\begin{aligned}
&I(W_1; Y_{3\text{A}}^L, Y_{3\text{B}}^L, Y_{3\text{C}}^L) \\
\leq &I(W_1; , Y_{3\text{A}}^L, Y_{3\text{B}}^L, Y_{3\text{C}}^L, Y_{4\text{C}}^L, Y_{1\text{C}}^L) \\
= &H(Y_{3\text{A}}^L, Y_{3\text{B}}^L, Y_{3\text{C}}^L, Y_{4\text{C}}^L, Y_{1\text{C}}^L) - H(Y_{3\text{A}}^L, Y_{3\text{B}}^L, Y_{3\text{C}}^L, Y_{4\text{C}}^L, Y_{1\text{C}}^L|W_1) \\
= &H(Y_{3\text{A}}^L, Y_{3\text{B}}^L, Y_{3\text{C}}^L, Y_{4\text{C}}^L, Y_{1\text{C}}^L) - H(U_{2\text{A}}^L, Y_{3\text{C}}^L, Y_{4\text{C}}^L, Y_{1\text{C}}^L|W_1) - H(Z_{3\text{B}}^L)
\end{aligned}
$$

$$
\begin{aligned}
&I(W_2; , Y_{4\text{A}}^L, Y_{4\text{B}}^L, Y_{4\text{C}}^L) \\
\leq &I(W_2; , Y_{4\text{A}}^L, Y_{4\text{B}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{2\text{B}}^L, Y_{1\text{C}}^L|W_1) \\
= &I(W_2; , Y_{4\text{A}}^L, Y_{4\text{B}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, X_1^L, Y_{2\text{B}}^L, Y_{1\text{C}}^L|W_1) \\
= &I(W_2; , V_{2\text{A}}^L, Z_{4\text{B}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, X_1^L, Y_{2\text{B}}^L, Y_{1\text{C}}^L|W_1) \\
= &I(W_2; , V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, X_1^L, Y_{2\text{B}}^L, Y_{1\text{C}}^L|W_1) \\
\leq &H(V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, X_1^L, Y_{2\text{B}}^L, Y_{1\text{C}}^L|W_1) \\
&- H(V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, X_1^L, X_2^L, Y_{2\text{B}}^L, Y_{1\text{C}}^L|W_1, W_2) \\
\leq &H(V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Z_{2\text{B}}^L, Y_{1\text{C}}^L|W_1) \\
&- H(Z_{4\text{A}}^L, Z_{4\text{C}}^L, Z_{3\text{C}}^L, X_1^L, X_2^L, Z_{2\text{B}}^L, Z_{1\text{C}}^L|W_1, W_2) \\
\leq &H(V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{1\text{C}}^L|W_1) + H(Z_{2\text{B}}^L) - H(Z_{4\text{A}}^L, Z_{4\text{C}}^L, Z_{3\text{C}}^L, Z_{2\text{B}}^L, Z_{1\text{C}}^L) \\
&- H(X_1^L, X_2^L|Z_{2\text{B}}^L, Z_{1\text{C}}^L, W_1, W_2) \\
= &H(V_{2\text{A}}^L, U_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{1\text{C}}^L|W_1) - H(U_{2\text{A}}^L|V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{1\text{C}}^L, W_1) \\
&- H(Z_{4\text{A}}^L, Z_{4\text{C}}^L, Z_{3\text{C}}^L, Z_{1\text{C}}^L) \\
\leq &H(V_{2\text{A}}^L, U_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{1\text{C}}^L|W_1) - H(U_{2\text{A}}^L|X_2^L, V_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{1\text{C}}^L, W_1) \\
&- H(Z_{4\text{A}}^L, Z_{4\text{C}}^L, Z_{3\text{C}}^L, Z_{1\text{C}}^L) \\
\leq &H(V_{2\text{A}}^L, U_{2\text{A}}^L, Y_{4\text{C}}^L, Y_{3\text{C}}^L, Y_{1\text{C}}^L|W_1) - H(Z_{3\text{A}}^L) - H(Z_{4\text{A}}^L, Z_{4\text{C}}^L, Z_{3\text{C}}^L, Z_{1\text{C}}^L) \\
\leq &H(V_{2\text{A}}^L|U_{2\text{A}}^L) + H(U_{2\text{A}}^L, Y_{3\text{C}}^L, Y_{4\text{C}}^L, Y_{1\text{C}}^L|W_1) - H(Z_{3\text{A}}^L, Z_{4\text{A}}^L, Z_{4\text{C}}^L, Z_{3\text{C}}^L, Z_{1\text{C}}^L)
\end{aligned}
$$

$$
\begin{aligned}
&L(R_1+R_2-\epsilon)\\
\leq &H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{3\mathrm{C}}^L, Y_{4\mathrm{C}}^L, Y_{1\mathrm{C}}^L) + H(V_{2\mathrm{A}}^L | U_{2\mathrm{A}}^L)\\
&- H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{A}}^L, Z_{4\mathrm{C}}^L, Z_{3\mathrm{C}}^L, Z_{1\mathrm{C}}^L)\\
\leq &H(Y_{3\mathrm{A}}^L) + H(Y_{3\mathrm{B}}^L) + H(Y_{3\mathrm{C}}^L, Y_{4\mathrm{C}}^L, Y_{1\mathrm{C}}^L) + H(V_{2\mathrm{A}}^L | U_{2\mathrm{A}}^L)\\
&- H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{A}}^L, Z_{4\mathrm{C}}^L, Z_{3\mathrm{C}}^L, Z_{1\mathrm{C}}^L).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&H(Y_{3\mathrm{A}}^L) - H(Z_{3\mathrm{A}}^L)\\
\leq &\sum_A H(Y_{3i}) - H(Z_{3i})\\
\leq &\sum_A \log(1 + (\sqrt{xP_{1,i}} + \sqrt{yP_{2,i}})^2)\\
\leq &\sum_A \log(1 + 2xP_{1i} + 2yP_{2i})\\
\leq &\frac{\delta L}{2+\delta} \log(1 + 2xP_{1\mathrm{A}} + 2yP_{2\mathrm{A}})
\end{aligned}
$$

$$
\begin{aligned}
&H(V_{2\mathrm{A}}^L | U_{2\mathrm{A}}^L) - H(Z_{4\mathrm{A}}^L)\\
\leq &H(V_{2\mathrm{A}}^L - cU_{2\mathrm{A}}^L) - H(Z_{4\mathrm{A}}^L) \qquad (c = \frac{h_{24}h_{23}^* P_{2\mathrm{A}}}{1+yP_{2\mathrm{A}}})\\
\leq &\sum_A H(V_{2i} - cU_{2i}) - H(Z_{4i})\\
\leq &\sum_A H(\frac{h_{24}}{1+yP_{2A}} X_{2i} + Z_{4i} - cZ_{3i}) - H(Z_{4i})\\
\leq &\sum_A \log(1 + c^2 + \frac{xP_{2i}}{1+yP_{2A}})\\
\leq &\frac{\delta L}{2+\delta} \log(1 + c^2 + \frac{xP_{2A}}{(1+yP_{2A})^2})\\
= &\frac{\delta L}{2+\delta} \log(1 + \frac{(xP_{2A})^2}{(1+yP_{2A})^2} + \frac{xP_{2A}}{(1+yP_{2A})^2})\\
= &\frac{\delta L}{2+\delta} \log(1 + \frac{xP_{2A}}{1+yP_{2A}}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&R_1 + R_2 - \epsilon\\
\leq &\frac{1}{2+\delta}\Big[\delta \log(1 + 2xP_{1A} + 2yP_{2A}) + \log(1 + xP_{1B})
\end{aligned}
$$

$$+\log(1+(x+y+z)P_{2C})+\delta\log(1+\frac{xP_{2A}}{1+yP_{2A}})\Big]$$

3. $V(\delta)$

$$\begin{aligned}
&I(W_1;Y_{3\mathrm{A}}^L,Y_{3\mathrm{B}}^L,Y_{3\mathrm{C}}^L)\\
\leq &I(W_1;Y_{3\mathrm{A}}^L,Y_{3\mathrm{B}}^L,Y_{3\mathrm{C}}^L,Y_{4\mathrm{B}}^L,U_{1\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)\\
= &H(Y_{3\mathrm{A}}^L,Y_{3\mathrm{B}}^L,Y_{3\mathrm{C}}^L|Y_{4\mathrm{B}}^L,U_{1\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)+H(U_{1\mathrm{A}}^L,Y_{4\mathrm{B}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)\\
&-H(Y_{3\mathrm{A}}^L,Y_{3\mathrm{B}}^L,Y_{3\mathrm{C}}^L,Y_{4\mathrm{B}}^L,U_{1\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L|W_1)
\end{aligned}$$

The last term can be bounded as follows

$$\begin{aligned}
&H(Y_{3\mathrm{A}}^L,Y_{3\mathrm{B}}^L,Y_{3\mathrm{C}}^L,Y_{4\mathrm{B}}^L,U_{1\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L|W_1)\\
= &H(U_{2\mathrm{A}}^L,Y_{3\mathrm{C}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L|W_1)+H(Z_{4\mathrm{A}}^L)+H(Z_{3\mathrm{B}}^L)+H(Z_{4\mathrm{B}}^L)\\
= &\sum_{t=1}^{L}H(U_{2\mathrm{A}t},Y_{3\mathrm{C}t},Y_{2\mathrm{B}t},Y_{1\mathrm{C}t}|U_{2\mathrm{A}}^{t-1},Y_{3\mathrm{C}}^{t-1},Y_{2\mathrm{B}}^{t-1},Y_{1\mathrm{C}}^{t-1},W_1)\\
&+H(Z_{4\mathrm{A}}^L,Z_{3\mathrm{B}}^L,Z_{4\mathrm{B}}^L)\\
= &\sum_{t=1}^{L}H(U_{2\mathrm{A}t},Y_{3\mathrm{C}t},Y_{1\mathrm{C}t}|U_{2\mathrm{A}}^{t-1},Y_{3\mathrm{C}}^{t-1},Y_{2\mathrm{B}}^{t-1},Y_{1\mathrm{C}}^{t-1})+H(Z_{2\mathrm{B}t})\\
&+H(Z_{4\mathrm{A}}^L,Z_{3\mathrm{B}}^L,Z_{4\mathrm{B}}^L)\\
\geq &\sum_{t=1}^{L}H(U_{2\mathrm{A}t},Y_{3\mathrm{C}t},Y_{2\mathrm{B}t},Y_{1\mathrm{C}t}|U_{2\mathrm{A}}^{t-1},Y_{3\mathrm{C}}^{t-1},Y_{2\mathrm{B}}^{t-1},Y_{1\mathrm{C}}^{t-1})-H(Y_{2\mathrm{B}t}|Y_{2\mathrm{B}}^{t-1})\\
&+H(Z_{2\mathrm{B}}^L,Z_{4\mathrm{A}}^L,Z_{3\mathrm{B}}^L,Z_{4\mathrm{B}}^L)\\
= &H(U_{2\mathrm{A}}^L,Y_{3\mathrm{C}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)-H(Y_{2\mathrm{B}}^L)+H(Z_{2\mathrm{B}}^L,Z_{4\mathrm{A}}^L,Z_{3\mathrm{B}}^L,Z_{4\mathrm{B}}^L).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&I(W_2;Y_{4\mathrm{A}}^L,Y_{4\mathrm{B}}^L,Y_{4\mathrm{C}}^L)\\
\leq &I(W_2;Y_{4\mathrm{A}}^L,Y_{4\mathrm{B}}^L,Y_{4\mathrm{C}}^L,Y_{3\mathrm{C}}^L,U_{2\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)\\
= &H(Y_{4\mathrm{A}}^L,Y_{4\mathrm{B}}^L,Y_{4\mathrm{C}}^L|Y_{3\mathrm{C}}^L,U_{2\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)+H(U_{2\mathrm{A}}^L,Y_{3\mathrm{C}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L)\\
&-H(Y_{4\mathrm{A}}^L,Y_{4\mathrm{B}}^L,Y_{4\mathrm{C}}^L,Y_{3\mathrm{C}}^L,U_{2\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L|W_2)
\end{aligned}$$

and

$$H(Y_{4\mathrm{A}}^L,Y_{4\mathrm{B}}^L,Y_{4\mathrm{C}}^L,Y_{3\mathrm{C}}^L,U_{2\mathrm{A}}^L,Y_{2\mathrm{B}}^L,Y_{1\mathrm{C}}^L|W_2)$$

$$\geq H(U^L_{1\mathrm{A}}, Y^L_{4\mathrm{B}}, Y^L_{2\mathrm{B}}, Y^L_{1\mathrm{C}}) - H(Y^L_{1\mathrm{C}}) + H(Z^L_{1\mathrm{C}}, Z^L_{3\mathrm{A}}, Z^L_{4\mathrm{C}}, Z^L_{3\mathrm{C}}).$$

Hence,

$$\begin{aligned}
&L(R_1 + R_2 - \epsilon) \\
\leq &H(Y^L_{3\mathrm{A}}|U^L_{1\mathrm{A}}) + H(Y^L_{3\mathrm{B}}|Y^L_{4\mathrm{B}}, Y^L_{2\mathrm{B}}) + H(Y^L_{3\mathrm{C}}|Y^L_{1\mathrm{C}}) + H(Y^L_{2\mathrm{B}}) \\
&+ H(Y^L_{4\mathrm{A}}|U^L_{2\mathrm{A}}) + H(Y^L_{4\mathrm{B}}|Y^L_{2\mathrm{B}}) + H(Y^L_{4\mathrm{C}}|Y^L_{3\mathrm{C}}, Y^L_{1\mathrm{C}}) + H(Y^L_{1\mathrm{C}}) \\
&- H(Z^L_{2\mathrm{B}}, Z^L_{3\mathrm{B}}, Z^L_{4\mathrm{B}}, Z^L_{4\mathrm{A}}, Z^L_{1\mathrm{C}}, Z^L_{3\mathrm{C}}, Z^L_{4\mathrm{C}}, Z^L_{3\mathrm{A}}) \\
= &H(Y^L_{3\mathrm{A}}|U^L_{1\mathrm{A}}) + H(Y^L_{3\mathrm{B}}, Y^L_{2\mathrm{B}}, Y^L_{4\mathrm{B}}) + H(Y^L_{4\mathrm{A}}|U^L_{2\mathrm{A}}) + H(Y^L_{4\mathrm{C}}, Y^L_{1\mathrm{C}}, Y^L_{3\mathrm{C}}) \\
&- H(Z^L_{2\mathrm{B}}, Z^L_{3\mathrm{B}}, Z^L_{4\mathrm{B}}, Z^L_{4\mathrm{A}}, Z^L_{1\mathrm{C}}, Z^L_{3\mathrm{C}}, Z^L_{4\mathrm{C}}, Z^L_{3\mathrm{A}}).
\end{aligned}$$

Notice that

$$\begin{aligned}
&H(Y^L_{3\mathrm{A}}|U^L_{1\mathrm{A}}) - H(Z^L_{3\mathrm{A}}) \\
\leq &H(Y^L_{3\mathrm{A}} - cU^L_{1\mathrm{A}}) - H(Z^L_{3\mathrm{A}}) \qquad\qquad (c = \frac{h_{13}h^*_{14}P_{1\mathrm{A}}}{1 + yP_{1\mathrm{A}}}) \\
\leq &\sum_A H(Y_{3i} - cU_{1i}) - H(Z_{3i}) \\
= &\sum_A H(\frac{h_{13}}{1 + yP_{1A}} X_{1i} + h_{23}X_{2i} + Z_{3i} - cZ_{4i}) \\
\leq &\sum_A \log\left(1 + c^2 + \frac{x}{(1 + yP_{1A})^2} P_{1i} + yP_{2i} + \frac{2\sqrt{xyP_{1i}P_{2i}}}{1 + yP_{1A}}\right) \\
\leq &\sum_A \log\left(1 + c^2 + \frac{x}{(1 + yP_{1A})^2} P_{1i} + yP_{2i} + \frac{xP_{1i} + yP_{2i}}{1 + yP_{1A}}\right) \\
\leq &\frac{\delta L}{2 + \delta} \log\left(1 + c^2 + \frac{x}{(1 + yP_{1A})^2} P_{1A} + yP_{2A} + \frac{xP_{1A} + yP_{2A}}{1 + yP_{1A}}\right) \\
= &\frac{\delta L}{2 + \delta} \log\left(1 + yP_{2A} + \frac{2xP_{1A} + yP_{2A}}{1 + yP_{1A}}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&R_1 + R_2 - \epsilon \\
\leq &\frac{1}{2 + \delta}\Big[\delta \log\left(1 + yP_{2A} + \frac{2xP_{1A} + yP_{2A}}{1 + yP_{1A}}\right) + \log(1 + (x + y + z)P_{1B}) \\
&+ \delta \log\left(1 + yP_{1A} + \frac{2xP_{2A} + yP_{1A}}{1 + yP_{2A}}\right) + \log(1 + (x + y + z)P_{2C})\Big].
\end{aligned}$$

4. $Cut'(\delta)$

$$\begin{aligned}
&L(R_1+R_2-\epsilon)\\
\le& I(W_1,W_2;Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C})\\
=& H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C})\\
&-H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C}|W_1,W_2)\\
\le& H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C})\\
&-H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C}|Y^L_{2B},Y^L_{1C},W_1,W_2)\\
=& H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C})\\
&-H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C},X^L_1,X^L_2|Y^L_{2B},Y^L_{1C},W_1,W_2)\\
=& H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C})-H(Z^L_{3A},Z^L_{3B},Z^L_{3C},Z^L_{4A},Z^L_{4B},Z^L_{4C})\\
&-H(X^L_1,X^L_2|Y^L_{2B},Y^L_{1C},W_1,W_2)\\
=& H(Y^L_{3A},Y^L_{3B},Y^L_{3C},Y^L_{4A},Y^L_{4B},Y^L_{4C})-H(Z^L_{3A},Z^L_{3B},Z^L_{3C},Z^L_{4A},Z^L_{4B},Z^L_{4C})\\
\le& H(Y^L_{3A},Y^L_{4A})+H(Y^L_{3B},Y^L_{4B})+H(Y^L_{3C},,Y^L_{4C})\\
&-H(Z^L_{3A},Z^L_{3B},Z^L_{3C},Z^L_{4A},Z^L_{4B},Z^L_{4C})
\end{aligned}$$

The covariance matrix of $[Y_{3i},Y_{4i}]^T$ is $K=\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}$, where

$$\begin{aligned}
K_{11}&=1+xP_{1i}+yP_{2i}+2Re(h_{13}h^*_{23}\rho)\sqrt{P_{1i}P_{2i}}\\
K_{12}&=h_{13}h^*_{14}P_{1i}+h_{23}h^*_{24}P_{2i}+h_{13}h^*_{24}\rho\sqrt{P_{1i}P_{2i}}+h_{23}h^*_{14}\rho^*\sqrt{P_{1i}P_{2i}}\\
K_{21}&=h^*_{13}h_{14}P_{1i}+h^*_{23}h_{24}P_{2i}+h^*_{13}h_{24}\rho^*\sqrt{P_{1i}P_{2i}}+h^*_{23}h_{14}\rho\sqrt{P_{1i}P_{2i}}\\
K_{22}&=1+xP_{2i}+yP_{1i}+2Re(h_{14}h^*_{24}\rho)\sqrt{P_{1i}P_{2i}}.
\end{aligned}$$

Then

$$\begin{aligned}
&H(Y_{3i},Y_{4i})-H(Z_{3i},Z_{4i})=\log(\det K)\\
=&1+(x^2+y^2)(1-|\rho|^2)P_{1i}P_{2i}+(x+y)(P_{1i}+P_{2i})\\
&+2Re(h_{13}h^*_{23}\rho)\sqrt{P_{1i}P_{2i}}+2Re(h_{14}h^*_{24}\rho)\sqrt{P_{1i}P_{2i}}\\
&-2Re(h_{13}h^*_{23}h^*_{14}h_{24})(1-|\rho|^2)P_{1i}P_{2i}\\
\le&1+(x+y)(P_{1i}+P_{1i})+4\sqrt{xy}|\rho|\sqrt{P_{1i}P_{1i}}\cos\frac{\theta}{2}\\
&+(x^2+y^2-2xy\cos\theta)(1-|\rho|^2)P_{1i}P_{2i}
\end{aligned}$$

$$\leq \log(1 + 2(x + y)(P_{1i} + P_{2i}) + P_{1i}P_{2i}(x^2 + y^2 - 2xy\cos\theta))$$

$$\begin{aligned}
&H(Y_{3\mathrm{A}}^L, Y_{4\mathrm{A}}^L) - H(Z_{3\mathrm{A}}^L, Z_{4\mathrm{A}}^L) \\
\leq &\sum_A H(Y_{3i}, Y_{4i}) - H(Z_{3i}, Z_{4i}) \\
\leq &\sum_A \log(1 + 2(x + y)(P_{1i} + P_{2i}) + P_{1i}P_{2i}(x^2 + y^2 - 2xy\cos\theta)) \\
\leq &\frac{\delta L}{2 + \delta} \log(1 + 2(x + y)(P_{1A} + P_{2A}) + P_{1A}P_{2A}(x^2 + y^2 - 2xy\cos\theta)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&R_1 + R_2 - \epsilon \\
\leq &\frac{1}{2 + \delta}\Big[\delta \log(1 + 2(x + y)(P_{1A} + P_{2A}) + P_{1A}P_{2A}(x^2 + y^2 - 2xy\cos\theta)) \\
&+ \log(1 + (x + y)P_{1\mathrm{B}}) + \log(1 + (x + y)P_{2\mathrm{C}})\Big].
\end{aligned}$$

It remains to show that $\overline{C_{\mathsf{sum}}^{\mathsf{HD}}} \leq \overline{C_{\mathsf{sum}}} + 3$. By power constraint, we have $P_{1A} \leq \frac{2+\delta}{\delta}, P_{2A} \leq \frac{2+\delta}{\delta}, P_{1B} \leq 2+\delta, P_{2C} \leq 2+\delta$. In $Cut(\delta), Z(\delta)$, and $Cut'(\delta)$, each term is a monotone increasing function of $P_{iA}, P_{iB}, P_{iC}, i = 1, 2$, so

$$\begin{aligned}
Cut(\delta) \leq &\frac{1}{2 + \delta}\Big[\delta \log(1 + x\frac{2 + \delta}{\delta}) + \delta \log(1 + x\frac{2 + \delta}{\delta}) \\
&\log(1 + (x + z)(2 + \delta)) + \log(1 + (x + z)(2 + \delta))\Big] \\
Z(\delta) \leq &\frac{1}{2 + \delta}\Big[\delta \log(1 + 2x\frac{2 + \delta}{\delta} + 2y\frac{2 + \delta}{\delta}) + \log(1 + x(2 + \delta)) \\
&+ \log(1 + (x + y + z)(2 + \delta)) + \delta \log(1 + \frac{x\frac{2+\delta}{\delta}}{1 + y\frac{2+\delta}{\delta}})\Big] \\
Cut'(\delta) \leq &\frac{1}{2 + \delta}\Big[\delta \log(1 + 2(x + y)(\frac{2 + \delta}{\delta} + \frac{2 + \delta}{\delta}) \\
&+ (\frac{2 + \delta}{\delta})^2(x^2 + y^2 - 2xy\cos\theta)) + \log(1 + (x + y)(2 + \delta)) \\
&+ \log(1 + (x + y)(2 + \delta))\Big].
\end{aligned}$$

In $V(\delta)$, observe that

$$\begin{aligned}
&1 + yP_{2A} + \frac{2xP_{1A} + yP_{2A}}{1 + yP_{1A}} \\
&\leq 1 + y\frac{2+\delta}{\delta} + \frac{2xP_{1A} + y\frac{2+\delta}{\delta}}{1 + yP_{1A}} \\
&\leq \max \left\{ \begin{array}{c} 1 + y\frac{2+\delta}{\delta} + \frac{(2x+y)\frac{2+\delta}{\delta}}{1+y\frac{2+\delta}{\delta}} \\ 1 + 2y\frac{2+\delta}{\delta} \end{array} \right\}.
\end{aligned}$$

So we have

$$\begin{aligned}
V(\delta) \leq & \frac{1}{2+\delta}\Bigg[\delta \log \left(\max \left\{ \begin{array}{c} 1 + y\frac{2+\delta}{\delta} + \frac{(2x+y)\frac{2+\delta}{\delta}}{1+y\frac{2+\delta}{\delta}} \\ 1 + 2y\frac{2+\delta}{\delta} \end{array} \right\} \right) \\
& + \log(1 + (x+y+z)(2+\delta)) \\
& + \delta \log \left(\max \left\{ \begin{array}{c} 1 + y\frac{2+\delta}{\delta} + \frac{(2x+y)\frac{2+\delta}{\delta}}{1+y\frac{2+\delta}{\delta}} \\ 1 + 2y\frac{2+\delta}{\delta} \end{array} \right\} \right) \\
& + \log(1 + (x+y+z)(2+\delta))\Bigg].
\end{aligned}$$

Comparing them term by term with $u_i, i = 1, 2, 3, 4$, we get

$$\begin{aligned}
Cut(\delta) - u_1 &\leq \frac{1}{2+\delta}\left[\delta \log \frac{2+\delta}{\delta} + \delta \log \frac{2+\delta}{\delta} + \log(2+\delta) + \log(2+\delta)\right] \\
Z(\delta) - u_2 &\leq \frac{1}{2+\delta}\left[\delta \log \frac{2+\delta}{\delta} + \log(2+\delta) + \log(2+\delta) + \delta \log \frac{2+\delta}{\delta}\right] \\
V(\delta) - u_3 &\leq \frac{1}{2+\delta}\left[\delta \log \frac{2+\delta}{\delta} + \log(2+\delta) + \delta \log \frac{2+\delta}{\delta} + \log(2+\delta)\right] \\
Cut'(\delta) - u_4 &\leq \frac{1}{2+\delta}\left[\delta \log \left(\frac{2+\delta}{\delta}\right)^2 + \log(2+\delta) + \log(2+\delta)\right].
\end{aligned}$$

For $\delta \geq 0$,

$$\frac{\delta}{2+\delta}\log(\frac{2+\delta}{\delta}) \leq \frac{1}{e \ln 2}, \quad \frac{1}{2+\delta}\log(2+\delta) \leq \frac{1}{e \ln 2}.$$

So we can conclude that

$$\begin{aligned}
\overline{C_{\mathsf{sum}}^{\mathsf{HD}}} &= \max_{\delta} \min(Cut(\delta), Z(\delta), V(\delta), Cut'(\delta)) \\
&\leq \max_{\delta} \min(u_1, u_2, u_3, u_4) + \frac{4}{e \ln 2} \leq \overline{C_{\mathsf{sum}}} + 3.
\end{aligned}$$

APPENDIX C

PROOF OF LEMMA 6.2

The power constraint implies that we have $P_{1A} \leq 1+\delta, P_{2A} \leq 1+\delta, P_{1B} \leq \frac{1+\delta}{\delta}$. In the upper bound of R_2 and $R_1 + R_2$, each term is a monotone increasing function of P_{1A}, P_{2A}, P_{1B}. So

$$
\begin{aligned}
R_2 \leq & \frac{1}{1+\delta} \log(1 + x_2(1+\delta)) \leq \frac{1}{1+\delta} \log(1+x_2) + \frac{1}{1+\delta} \log(1+\delta) \\
R_1 + R_2 \leq & \frac{1}{1+\delta} \Big[\log(1 + 2x_2(1+\delta) + 2y_2(1+\delta)) \\
& + \delta \log(1 + (x_1 + y_2 + z) \frac{1+\delta}{\delta}) + \log(1 + \frac{x_1(1+\delta)}{1 + y_2(1+\delta)}) \Big] \\
\leq & \frac{1}{1+\delta} \Big[\log(1 + 2x_2 + 2y_2) + \delta \log(1 + (x_1 + y_2 + z)) \\
& + \log(1 + \frac{x_1}{1+y_2}) \Big] + \frac{\delta}{1+\delta} \log(\frac{1+\delta}{\delta}) + \frac{2}{1+\delta} \log(1+\delta) \\
R_1 + R_2 \leq & \frac{1}{1+\delta} \Big[\log(1 + 2x_1(1+\delta) + 2y_1(1+\delta)) + \delta \log(1 + x_1 \frac{1+\delta}{\delta}) \\
& + \log(1 + \frac{x_2(1+\delta)}{1 + y_1(1+\delta)}) \Big] \\
\leq & \frac{1}{1+\delta} \left[\log(1 + 2x_1 + 2y_1) + \delta \log(1 + x_1) + \log(1 + \frac{x_2}{1+y_1}) \right] \\
& + \frac{\delta}{1+\delta} \log(\frac{1+\delta}{\delta}) + \frac{2}{1+\delta} \log(1+\delta).
\end{aligned}
$$

In the upper bound for $2R_1 + R_2$, observe that

$$
\begin{aligned}
& 1 + y_2 P_{1A} + \frac{2x_2 P_{2A} + y_2 P_{1A}}{1 + y_1 P_{2A}} \\
& \leq 1 + y_2(1+\delta) + \frac{2x_2 P_{2A} + y_2(1+\delta)}{1 + y_1 P_{2A}} \\
& \leq \max \left\{ \begin{array}{c} 1 + y_2(1+\delta) + \frac{(2x_2+y_2)(1+\delta)}{1+y_1(1+\delta)} \\ 1 + 2y_2(1+\delta) \end{array} \right\}
\end{aligned}
$$

$$\leq (1+\delta)\max\left\{\begin{array}{c} 1+y_2+\frac{2x_2+y_2}{1+y_1} \\ 1+2y_2 \end{array}\right\}.$$

So we have

$$\begin{aligned}
2R_1+R_2 \leq & \frac{1}{1+\delta}\Big[\log(1+2x_1(1+\delta)+2y_1(1+\delta))+\delta\log(1+x_1\frac{1+\delta}{\delta}) \\
& +\log(1+\frac{x_1(1+\delta)}{1+y_2(1+\delta)}) \\
& +\log\left(\max\left\{\begin{array}{c} 1+y_2(1+\delta)+\frac{(2x_2+y_2)(1+\delta)}{1+y_1(1+\delta)} \\ 1+2y_2(1+\delta) \end{array}\right\}\right) \\
& +\delta\log(1+(x_1+y_2+z)\frac{1+\delta}{\delta})\Big] \\
\leq & \frac{1}{1+\delta}\Big[\log(1+2x_1+2y_1)+\delta\log(1+x_1)+\log(1+\frac{x_1}{1+y_2}) \\
& +\max(\log(1+y_2+\frac{2x_2+y_2}{1+y_1}),\log(1+2y_2)) \\
& +\delta\log(1+(x_1+y_2+z))\Big]+\frac{2\delta}{1+\delta}\log(\frac{1+\delta}{\delta}) \\
& +\frac{3}{1+\delta}\log(1+\delta).
\end{aligned}$$

We finish the proof by noticing that for $\delta \geq 0$,

$$\frac{\delta}{1+\delta}\log(\frac{1+\delta}{\delta}) \leq \frac{1}{e\ln 2} \quad \text{and} \quad \frac{1}{1+\delta}\log(1+\delta) \leq \frac{1}{e\ln 2}.$$

APPENDIX D

PROOF OF THEOREM 6.4

As in the sum-rate case, we will prove this achievability result in two steps. Instead of directly comparing $\overline{C_{R_0}}$ with the rate achievable by the coding scheme in section 4, we will first show that the $\overline{C_{R_0}}$ is within a constant of $\overline{C_{R_0}^{\mathsf{LDM}}}$, a quantity we define below inspired by the result for the linear deterministic model. We will then prove that the coding scheme in section 4 can be used to achieve an R_1 which is within R_0 of the point-to-point capacity $C_0 = \log(1 + \mathtt{SNR}_1)$ of the primary link and an R_2 which is within a constant of $\overline{C_{R_0}^{\mathsf{LDM}}}$. Specifically, we prove the following two lemmas, which together imply Theorem 6.4

Lemma D.1 *Let* $n_1 = \lfloor \log \mathtt{SNR}_1 \rfloor^+, n_2 = \lfloor \log \mathtt{SNR}_2 \rfloor^+, \alpha_1 = \lfloor \log \mathtt{INR}_1 \rfloor^+, \alpha_2 = \lfloor \log \mathtt{INR}_2 \rfloor^+, \beta = \lfloor \log \mathtt{CNR} \rfloor^+$. *Define*

$$\overline{C_{R_0}^{\mathsf{LDM}}} = \max_{\delta} \overline{C_{R_0}^{\mathsf{LDM}}}(\delta) = \max_{\delta>0} \min(u'_1 - 10 - 2R_0, u'_2 - 5 - R_0, u'_3 - 5 - R_0, u'_4),$$

where

$$\begin{aligned}
u'_1 =& \frac{1}{1+\delta} n_2 \\
u'_2 =& \frac{1}{1+\delta}[n_2 \vee \alpha_2 - \alpha_2 \wedge n_1 + \delta(\beta \vee \alpha_2 \vee n_1 - n_1)] \\
u'_3 =& \frac{1}{1+\delta}[(\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+] \\
u'_4 =& \frac{1}{1+\delta}[(\alpha_1 - n_1)^+ - \alpha_2 \wedge n_1 + (n_2 - \alpha_1) \vee \alpha_2 + \delta(\beta \vee \alpha_2 \vee n_1 - n_1)].
\end{aligned}$$

Then $\overline{C_{R_0}} < \overline{C_{R_0}^{\mathsf{LDM}}} + 13 + 2R_0$.

Lemma D.2 *For* $R_0 > 7$, $(R_1, R_2) = (C_0 - R_0, \overline{C_{R_0}^{LDM}} - 10)$ *is achievable.*

D.1 Proof of Lemma D.1

As in the proof of Lemma A.1, we can show that

$$\begin{aligned}
u_1 \leq& u_1' + \frac{2}{1+\delta} + 1 \leq u_1' + 3 \\
u_2 \leq& u_2' + \frac{1}{1+\delta}[(\log 5 + 1) + \delta(\log 4 + 1) + (\log 3 + 1)] + 2 + R_0 \\
\leq& u_2' + 8 + R_0 \\
u_3 \leq& u_3' + \frac{1}{1+\delta}[(\log 5 + 1) + (\log 3 + 1)] + 2 + R_0 \leq u_3' + 8 + R_0 \\
u_4 \leq& u_4' + \frac{1}{1+\delta}[(\log 5 + 1) + (\log 3 + 1) + (\log 7 + 1) + \delta(\log 4 + 1)] + 3 + 2R_0 \\
\leq& u_4' + 13 + 2R_0.
\end{aligned}$$

So we get

$$\begin{aligned}
\overline{C_{R_0}} &= \max_{\delta} \min(u_1, u_2, u_3, u_4) \\
&\leq \max_{\delta} \min(u_1' - 10 - 2R_0, u_2' - 5 - R_0, u_3' - 5 - R_0, u_4') + 13 + 2R_0 \\
&\leq \overline{C_{R_0}^{LDM}} + 13 + 2R_0.
\end{aligned}$$

D.2 Proof of Lemma D.2

First we will first prove the following result for the interference channel.

Lemma D.3 *For $R_0 \geq 7$, $\overline{C_{cog}^{IFC-LDM}} \leq C_{R_0}^{IFC} + 1$, where*

$$\overline{C_{cog}^{IFC-LDM}} = \min \begin{pmatrix} n_2 \\ n_2 \vee \alpha_2 - \alpha_2 \wedge n_1 \\ (\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ \\ (\alpha_1 - n_1)^+ - \alpha_2 \wedge n_1 + (n_2 - \alpha_1) \vee \alpha_2 \end{pmatrix}$$

and $C_{R_0}^{IFC}$ is the R_0-capacity for the interference channel.

Proof. Let $\overline{C^{IFC}}$ be the outer bound to the interference channel capacity region derived in [9]. From the achievability result there, we know that given

$R_1 = \log(1+\texttt{SNR}_1) - R_0,\ R_2$ is achievable, if

$$(\log(1+\texttt{SNR}_1) - R_0 + 1, R_2 + 1) \in \overline{C^{IFC}}.$$

By taking the maximum of such R_2, we get $C^{IFC}_{R_0}$. Now we will show the result by considering the weak, mixed, and strong interference regions separately.

1. $\texttt{SNR}_1 \geq \texttt{INR}_2, \texttt{SNR}_2 \geq \texttt{INR}_1$. In this region,

$$\overline{C^{IFC-LDM}_{cog}} = \min \begin{pmatrix} (n_2-\alpha_2)^+ \\ (\alpha_1-n_1)^+ + (n_2-\alpha_1-\alpha_2)^+ \end{pmatrix}$$

and the outer bound $\overline{C^{IFC}}$ is the set of (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq \log(1+x_1) \\ R_2 &\leq \log(1+x_2) \\ R_1 + R_2 &\leq \log(1+x_1) + \log(1+\frac{x_2}{1+y_2}) \\ R_1 + R_2 &\leq \log(1+x_2) + \log(1+\frac{x_1}{1+y_1}) \\ R_1 + R_2 &\leq \log(1+y_1+\frac{x_1}{1+y_2}) + \log(1+y_2+\frac{x_2}{1+y_1}) \\ 2R_1 + R_2 &\leq \log(1+x_1+y_1) + \log(1+y_2+\frac{x_2}{1+y_1}) + \log(\frac{1+x_1}{1+y_2}) \\ R_1 + 2R_2 &\leq \log(1+x_2+y_2) + \log(1+y_1+\frac{x_1}{1+y_2}) + \log(\frac{1+x_2}{1+y_1}). \end{aligned}$$

So when $R_1 = \log(1+\texttt{SNR}_1) - R_0$, R_2 is achievable if

$$\begin{aligned} R_2 + 1 \leq& \log(1+x_2) \\ R_2 + 1 \leq& \log(1+\frac{x_2}{1+y_2}) + R_0 - 1 \\ R_2 + 1 \leq& \log(1+x_2) + \log(1+\frac{x_1}{1+y_1}) - \log(1+x_1) + R_0 - 1 \\ R_2 + 1 \leq& \log(1+y_1+\frac{x_1}{1+y_2}) + \log(1+y_2+\frac{x_2}{1+y_1}) \\ &- \log(1+x_1) + R_0 - 1 \\ R_2 + 1 \leq& \log(1+x_1+y_1) + \log(1+y_2+\frac{x_2}{1+y_1}) - \log(1+y_2) \\ &- \log(1+x_1) + 2R_0 - 2 \end{aligned}$$

$$2R_2 + 2 \leq \log(1 + x_2 + y_2) + \log(1 + y_1 + \frac{x_1}{1 + y_2}) + \log(\frac{1 + x_2}{1 + y_1})$$
$$- \log(1 + x_1) + R_0 - 1.$$

The following rate is achievable

$$\begin{aligned}
R_2 &\leq n_2 - 1 \\
R_2 &\leq n_2 \vee \alpha_2 - \alpha_2 - 2 + R_0 - 2 \\
R_2 &\leq n_2 + n_1 \vee \alpha_1 - \alpha_1 - 2 - n_1 - 2 + R_0 - 2 \\
R_2 &\leq n_1 \vee (\alpha_1 + \alpha_2) - \alpha_2 - 2 + n_2 \vee (\alpha_1 + \alpha_2) - \alpha_1 - 2 - n_1 - 2 + R_0 - 2 \\
R_2 &\leq n_1 \vee \alpha_1 + n_2 \vee (\alpha_1 + \alpha_2) - \alpha_1 - 2 - \alpha_2 - 2 - n_1 - 2 + 2R_0 - 3 \\
2R_2 &\leq n_2 \vee \alpha_2 + n_1 \vee (\alpha_1 + \alpha_2) - \alpha_2 - 2 + n_2 - \alpha_1 - 2 - n_1 - 2 + R_0 - 2.
\end{aligned}$$

The following rate is achievable

$$\begin{aligned}
R_2 &\leq (n_2 - \alpha_2)^+ - 1 \\
R_2 &\leq (\alpha_1 - n_1)^+ + (n_2 - \alpha_1 - \alpha_2)^+ - 8 + R_0.
\end{aligned}$$

Hence, $\overline{C_{cog}^{IFC-LDM}} \leq C_{R_0}^{IFC} + 1$.

2. $\mathtt{SNR}_1 \geq \mathtt{INR}_2, \mathtt{SNR}_2 \leq \mathtt{INR}_1$. In this region,

$$\overline{C_{cog}^{IFC-LDM}} = \min \begin{pmatrix} (n_2 - \alpha_2)^+ \\ (\alpha_1 - n_1)^+ \end{pmatrix}$$

and the outer bound $\overline{C^{IFC}}$ is the set of (R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\leq \log(1 + x_1) \\
R_2 &\leq \log(1 + x_2) \\
R_1 + R_2 &\leq \log(1 + x_1) + \log(1 + \frac{x_2}{1 + y_2}) \\
R_1 + R_2 &\leq \log(1 + x_1 + y_1) \\
R_1 + 2R_2 &\leq \log(1 + x_2 + y_2) + \log(1 + y_1 + \frac{x_1}{1 + y_2}) + \log(1 + \frac{x_2}{1 + y_1}).
\end{aligned}$$

So when $R_1 = \log(1 + \mathtt{SNR}_1) - R_0$, R_2 is achievable if

$$R_2 + 1 \leq \log(1 + x_2)$$

$$\begin{aligned}
R_2 + 1 &\leq \log(1 + \frac{x_2}{1+y_2}) + R_0 - 1 \\
R_2 + 1 &\leq \log(1 + x_1 + y_1) - \log(1 + x_1) + R_0 - 1 \\
2R_2 + 2 &\leq \log(1 + x_2 + y_2) + \log(1 + y_1 + \frac{x_1}{1+y_2}) + \log(1 + \frac{x_2}{1+y_1}) \\
&\quad - \log(1 + x_1) + R_0 - 1.
\end{aligned}$$

The following rate is achievable

$$\begin{aligned}
R_2 \leq& n_2 - 1 \\
R_2 \leq& n_2 \vee \alpha_2 - \alpha_2 - 2 + R_0 - 2 \\
R_2 \leq& n_1 \vee \alpha_1 - n_1 - 2 + R_0 - 2 \\
2R_2 \leq& n_2 \vee \alpha_2 + n_1 \vee (\alpha_1 + \alpha_2) - \alpha_2 - 2 + n_2 \vee \alpha_1 - \alpha_1 - 2 - n_1 \\
&- 2 + R_0 - 3.
\end{aligned}$$

The following rate is achievable

$$\begin{aligned}
R_2 &\leq (n_2 - \alpha_2)^+ - 1 \\
R_2 &\leq (\alpha_1 - n_1)^+ - 8 + R_0.
\end{aligned}$$

Hence, $\overline{C_{cog}^{IFC-LDM}} \leq C_{R_0}^{IFC} + 1$.

3. $\texttt{SNR}_1 \leq \texttt{INR}_2, \texttt{SNR}_2 \geq \texttt{INR}_1$. In this region,

$$\overline{C_{cog}^{IFC-LDM}} = \min \begin{pmatrix} n_2 \\ n_2 \vee \alpha_2 - n_1 \\ (\alpha_1 - n_1)^+ + n_2 - \alpha_1 \\ (\alpha_1 - n_1)^+ - n_1 + (n_2 - \alpha_1) \vee \alpha_2 \end{pmatrix}$$

and the outer bound $\overline{C^{IFC}}$ is the set of (R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\leq \log(1 + x_1) \\
R_2 &\leq \log(1 + x_2) \\
R_1 + R_2 &\leq \log(1 + x_2) + \log(1 + \frac{x_1}{1+y_1}) \\
R_1 + R_2 &\leq \log(1 + x_2 + y_2) \\
2R_1 + R_2 &\leq \log(1 + x_1 + y_1) + \log(1 + y_2 + \frac{x_2}{1+y_1}) + \log(1 + \frac{x_1}{1+y_2}).
\end{aligned}$$

So when $R_1 = \log(1+\texttt{SNR}_1) - R_0$, R_2 is achievable if

$$\begin{aligned}
R_2 + 1 &\le \log(1+x_2)\\
R_2 + 1 &\le \log(1+x_2) + \log(1+\frac{x_1}{1+y_1}) - \log(1+x_1) + R_0 - 1\\
R_2 + 1 &\le \log(1+x_2+y_2) - \log(1+x_1) + R_0 - 1\\
R_2 + 1 &\le \log(1+x_1+y_1) + \log(1+y_2+\frac{x_2}{1+y_1}) + \log(1+\frac{x_1}{1+y_2})\\
&\qquad - 2\log(1+x_1) + 2R_0 - 2.
\end{aligned}$$

The following rate is achievable

$$\begin{aligned}
R_2 \le& n_2 - 1\\
R_2 \le& n_2 + n_1 \vee \alpha_1 - \alpha_1 - 2 - n_1 - 2 + R_0 - 2\\
R_2 \le& n_2 \vee \alpha_2 - n_1 - 2 + R_0 - 2\\
R_2 \le& n_1 \vee \alpha_1 + n_2 \vee (\alpha_1+\alpha_2) - \alpha_1 - 2 + n_1 \vee \alpha_2 - \alpha_2 - 2 - 2n_1\\
&- 4 + 2R_0 - 3.
\end{aligned}$$

The following rate is achievable

$$\begin{aligned}
R_2 &\le n_2 - 1\\
R_2 &\le (\alpha_1 - n_1)^+ + n_2 - \alpha_1 - 6 + R_0\\
R_2 &\le n_2 \vee \alpha_2 - n_1 - 4 + R_0\\
R_2 &\le (\alpha_1 - n_1)^+ - n_1 + (n_2 - \alpha_1) \vee \alpha_2 - 11 + 2R_0.
\end{aligned}$$

Hence, $\overline{C_{cog}^{IFC-LDM}} \le C_{R_0}^{IFC} + 1$.

4. $\texttt{SNR}_1 \le \texttt{INR}_2, \texttt{SNR}_2 \le \texttt{INR}_1$. In this region,

$$\overline{C_{cog}^{IFC-LDM}} = \min \begin{pmatrix} n_2 \\ n_2 \vee \alpha_2 - n_1 \\ (\alpha_1 - n_1)^+ \end{pmatrix}$$

and the outer bound $\overline{C^{IFC}}$ (which in this case is achievable without any gap) is the set of (R_1, R_2) satisfying

$$R_1 \le \log(1+x_1)$$

$$R_2 \leq \log(1 + x_2)$$
$$R_1 + R_2 \leq \log(1 + x_1 + y_1)$$
$$R_1 + R_2 \leq \log(1 + x_2 + y_2).$$

So when $R_1 = \log(1 + \texttt{SNR}_1) - R_0$, R_2 is achievable if

$$R_2 + 1 \leq \log(1 + x_2)$$
$$R_2 + 1 \leq \log(1 + x_1 + y_1) - \log(1 + x_1) + R_0 - 1$$
$$R_2 + 1 \leq \log(1 + x_2 + y_2) - \log(1 + x_1) + R_0 - 1.$$

The following rate is achievable

$$R_2 \leq n_2 - 1$$
$$R_2 \leq n_1 \vee \alpha_1 - n_1 - 2 + R_0 - 2$$
$$R_2 \leq n_2 \vee \alpha_2 - n_1 - 2 + R_0 - 2.$$

The following rate is achievable

$$R_2 \leq n_2 - 1$$
$$R_2 \leq (\alpha_1 - n_1)^+ - 4 + R_0$$
$$R_2 \leq n_2 \vee \alpha_2 - n_1 - 4 + R_0.$$

Hence, $\overline{C_{cog}^{IFC-LDM}} \leq C_{R_0}^{IFC} + 1$.

□

Now we are ready to show that $\overline{C_{R_0}^{LDM}}$ can be achieved within a constant for $R_0 \geq 7$. We consider four separate regions.

Region 1: $z \leq x_1 \vee y_2$ or $y_2 \leq 1$.

In this region, $\beta \leq \alpha_2 \vee n_1$ or $\alpha_2 = 0$. In both cases, we have $u_2' \geq u_1', u_4' \geq u_3'$. Hence,

$$\overline{C_{R_0}^{LDM}}(\delta) \leq \min(u_1' - 101 - 2R_0, u_3' - 5 - R_0)$$

$$\leq \overline{C_{cog}^{IFC-LDM}} - 5 - R_0$$
$$\leq C_{R_0}^{IFC}.$$

Hence, $\overline{C_{R_0}}$ can be achieved in this region.
For the following regions, we will assume $z > x_1 \vee y_2$ and $y_2 > 1$.

Region 2: $x_1 \leq 1$ or $x_2 \leq 1$.

When $x_1 \leq 1$, the primary link capacity $\log(1+x_1)$ is of a constant smaller than 7 and will be 0 after backing off R_0. Hence, the secondary achieves $\log(1+x_2)$, which is the best possible; and R_0-capacity is achieved without gap. When $x_2 \leq 1$, the secondary can at most achieve $\log(1+x_2) \leq 2$, which is only a constant. Hence, by letting $R_2 = 0$, the R_0-capacity is achieved within 2 bits in this region. For the following regions, we also assume $x_1 > 1$ and $x_2 > 1$.

Region 3: $y_1 \leq 1$, which implies that $\alpha_1 = 0$. We consider two subregions.

(1) $\frac{1}{4} \leq \frac{x_1 x_2}{y_1 y_2} \leq 4$.

In this region, the channel gains are aligned, the cooperation is not very helpful, and the interference channel scheme suffices to achieve the upper bound within a constant.

By condition $\frac{x_1 x_2}{y_1 y_2} \leq 4$, we have $n_1 + n_2 \leq \alpha_2 + 4$. Then it can be shown that

1. $n_2 \vee \alpha_2 - \alpha_2 \wedge n_1 \geq \alpha_2 - n_1 \geq n_2 - 4$.
2. $(\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ = n_2$.
3. $(\alpha_1 - n_1)^+ - \alpha_2 \wedge n_1 + (n_2 - \alpha_1) \vee \alpha_2 \geq -n_1 + \alpha_2 \geq n_2 - 4$.

which gives $\overline{C_{R_0}^{\mathsf{LDM}}} = \max_\delta u_1' - 10 - 2R_0 = n_2 - 10 - 2R_0$ and $\overline{C_{cog}^{IFC-LDM}} \geq n_2 - 4$. So we get

$$\overline{C_{R_0}^{\mathsf{LDM}}} \leq \overline{C_{cog}^{IFC-LDM}} - 6 - 2R_0 \leq C_{R_0}^{IFC}.$$

Hence, $\overline{C_{R_0}^{\mathsf{LDM}}}$ can be achieved in this region.

(2) $\frac{x_1x_2}{y_1y_2} \geq 4$ or $\frac{x_1x_2}{y_1y_2} \leq \frac{1}{4}$.

For this region, in the scheme from section 4, we set $\delta_{\rm A} = 1, \delta_{\rm B} = \delta$, and $\delta_C = 0$; i.e., the secondary receiver will listen for part of the time and then transmit for the rest of time, when it cooperates with the primary by using some of the information it gathered during the time it listened. The mode of cooperation is through cooperative private messages. For simplicity, we will require that $R_{1\rm B}, R_{1\rm A} \geq \log(1+x_1) - R_0$.

In mode B, source 1 uses power $\frac{1}{x_1}$ to share bits with source 2 and power $1-\frac{1}{x_1}$ to send data to destination 3. So, under the natural order for superposition coding, the following rates are achievable

$$R_{1B} \geq \log(1+\frac{(1-\frac{1}{x_1})x_1}{2}) = \log(1+x_1) - 1$$
$$\frac{\mathsf{C}_{12}}{\delta} \geq \log(1+\frac{z}{x_1}) \geq (\log(\frac{z}{x_1}))^+ \geq (\beta - n_1 - 1)^+ \geq \beta - n_1 - 1$$

Hence, we need at least $R_0 \geq 1$.

For the virtual channel, source 1 uses three messages W_1, U_1, V_1 and source 2 uses only message U_2. Let

$$\beta_1 = \frac{x_1x_2 + y_1y_2 - 2\sqrt{x_1x_2y_1y_2}\cos\theta}{x_1x_2}$$
$$\beta_2 = \frac{x_1x_2 + y_1y_2 - 2\sqrt{x_1x_2y_1y_2}\cos\theta}{y_1y_2}$$

By (6.2) and (6.3), we have $\sigma^2_{V_1} = \mathrm{Var}\,(X_{V_1})\,\beta_1 x_1 = \mathrm{Var}\,(X_{V_2})\,\beta_2 y_1$ and $\mathrm{Var}\,(X_{V_2})\,x_2 = \mathrm{Var}\,(X_{V_1})\,y_2$. It is not hard to see that

$$\frac{x_1x_2}{y_1y_2} \geq 4 \text{ implies that } \beta_1 \geq \frac{1}{4}, \quad \text{and} \quad \frac{x_1x_2}{y_1y_2} \leq \frac{1}{4} \text{ implies that } \beta_2 \geq \frac{1}{4}.$$

At source 1 we allocate powers $\sigma^2_{W_1} = \frac{1}{3}, \sigma^2_{U_1} = \frac{1}{3y_2}, \mathrm{Var}\,(X_{V_1}) = \frac{1}{3}(1 \wedge \frac{x_2}{y_2})$ and at source 2 $\sigma^2_{U_2} = \frac{1}{3}, \mathrm{Var}\,(X_{V_2}) = \frac{y_2}{x_2}\mathrm{Var}\,(X_{V_1}) = \frac{1}{3}(1 \wedge \frac{y_2}{x_2})$. Destination 1 gets W_1, U_1, V_1 with power $\frac{x_1}{3}, \frac{x_1}{3y_2}, \frac{\beta_1 x_1}{3}(1 \wedge \frac{x_2}{y_2})$, respectively, and U_2 with power $\frac{y_1}{3} \leq \frac{1}{3}$, which is treated as noise. Destination 2 gets U_2, W_1, U_1 with powers $\frac{x_2}{3}, \frac{y_2}{3}, \frac{1}{3}$, respectively, and U_1 is treated as noise.

To simplify the constraints at the destinations, we first prove the following lemma.

Lemma D.4 *When* $\frac{x_1x_2}{y_1y_2} \geq 4$ *or* $\frac{x_1x_2}{y_1y_2} \leq \frac{1}{4}$*, we have* $\beta_1 x_1(1 \wedge \frac{x_2}{y_2}) \geq \frac{1}{4}[x_1(1 \wedge \frac{x_2}{y_2})] \vee [y_1(1 \wedge \frac{y_2}{x_2})] \stackrel{def}{=} \frac{\tilde{k}}{4}$.

Proof. If $\frac{x_1x_2}{y_1y_2} \geq 4$, we have $\beta_1 \geq \frac{1}{4}$ and $x_1 \geq 4\frac{y_1y_2}{x_2}$. Hence

$$\beta_1 x_1(1 \wedge \frac{x_2}{y_2}) \geq \frac{1}{4}x_1(1 \wedge \frac{x_2}{y_2})$$
$$\beta_1 x_1(1 \wedge \frac{x_2}{y_2}) \geq \beta_1 \frac{4y_1y_2}{x_2}(1 \wedge \frac{x_2}{y_2}) \geq y_1(1 \wedge \frac{y_2}{x_2}) \geq \frac{1}{4}y_1(1 \wedge \frac{y_2}{x_2})$$

If $\frac{x_1x_2}{y_1y_2} \leq \frac{1}{4}$, we can rewrite the LHS as

$$\beta_1 x_1(1 \wedge \frac{x_2}{y_2}) = \beta_2 \frac{y_1y_2}{x_2}(1 \wedge \frac{x_2}{y_2}) = \beta_2 y_1(1 \wedge \frac{y_2}{x_2}).$$

Now, using the fact that $\beta_2 \geq \frac{1}{4}$ and $y_1 \geq 4\frac{x_1x_2}{y_2}$ when $\frac{x_1x_2}{y_1y_2} \leq \frac{1}{4}$, we can show similarly that

$$\beta_2 y_1(1 \wedge \frac{y_2}{x_2}) \geq \frac{1}{4}[x_1(1 \wedge \frac{x_2}{y_2})] \vee [y_1(1 \wedge \frac{y_2}{x_2})].$$

□

Using this lemma it is easy to verify that the following constraints on nonnegative rates imply all the relevant constraints in Theorem 4.1.

$$\begin{aligned}
R_{W_1} + R_{U_1} + R_{V_1} &\leq \log(1 + \frac{x_1}{4}) \\
R_{U_1} + R_{V_1} &\leq \log(1 + \frac{x_1(1 \wedge \frac{x_2}{y_2})}{16}) \\
R_{U_1} &\leq \log(1 + \frac{x_1}{4y_2}) \\
R_{V_1} &\leq \mathsf{C}_{12} \\
R_{W_1} + R_{U_2} &\leq \log(1 + \frac{x_2 + y_2}{4}) \\
R_{U_2} &\leq \log(1 + \frac{x_2}{4}).
\end{aligned}$$

First, we will get the condition on R_0 such that $R_{1A} = \log(1 + x_1) - R_0$ is supported by the above constraints. Set $R_2 = 0$. In the worst case, we have

$\mathsf{C}_{12} = 0$ when $R_{V_1} = 0$. So at least we can achieve $R_{1A} = R_{W_1} + R_{U_1}$, where nonnegative R_{W_1} and R_{U_1} satisfy the constraints

$$\begin{aligned}
R_{W_1} + R_{U_1} &\leq \log(1 + \frac{x_1}{4}) \\
R_{U_1} &\leq \log(1 + \frac{x_1}{16y_2}) \\
R_{W_1} &\leq \log(1 + \frac{x_2 + y_2}{4}).
\end{aligned}$$

Hence, a rate R_{1A} which is the minimum of $\log(1 + \frac{x_1}{4})$ and $\log(1 + \frac{x_1}{16y_2}) + \log(1 + \frac{x_2+y_2}{4})$ is acheivable. Thus, we may conclude that $R_{1A} = (\log(1 + x_1) - R_0)^+$ is achievable when $R_0 \geq 7$.

Now, in the original constraints, eliminating V_1, U_2 with $R_{1\mathrm{A}} = R_{W_1} + R_{U_1} + R_{V_1}$ and $R_{2\mathrm{A}} = R_{U_2}$ and setting $R_{1A} = (\log(1 + x_1) - R_0)^+$, we get

$$\begin{aligned}
(\log(1 + x_1) - R_0)^+ &\leq \log(1 + \frac{x_1}{4}) \\
-R_{W_1} &\leq \log(1 + \frac{x_1(1 \wedge \frac{x_2}{y_2})}{16}) - (\log(1 + x_1) - R_0)^+ \\
0 \leq R_{U_1} &\leq \log(1 + \frac{x_1}{4y_2}) \\
R_{W_1} + R_{U_1} &\leq (\log(1 + x_1) - R_0)^+ \\
R_{W_1} + R_{U_1} &\geq (\log(1 + x_1) - R_0)^+ - \mathsf{C}_{12} \\
R_{W_1} + R_{2A} &\leq \log(1 + \frac{x_2 + y_2}{4}) \\
0 &\leq R_{W_1} \\
0 \leq R_{2A} &\leq \log(1 + \frac{x_2}{4}).
\end{aligned}$$

By the choice of R_0, we already have

$$(\log(1 + x_1) - R_0)^+ \leq \log(1 + \frac{x_1}{4}).$$

Using the inequalities

$$\begin{aligned}
n_i \leq \log x_i &< n_i + 1 \\
\log(1 + x_i) &< n_i + 2 \\
\log(1 + \frac{x_i}{a}) &\geq n_i - \log a,
\end{aligned}$$

we can rewrite the constraints as follows:

$$\begin{aligned}
-R_{W_1} &\leq ([n_1 - (\alpha_2 - n_2)^+]^+ - 5)^+ - (n_1 + 2 - R_0)^+ \\
0 \leq R_{U_1} &\leq (n_1 - \alpha_2)^+ - 3)^+ \\
R_{W_1} + R_{U_1} &\leq (n_1 - R_0)^+ \\
R_{W_1} + R_{U_1} &\geq \min((n_1 + 2 - R_0)^+ - \mathsf{C}_{12}, (n_1 - R_0)^+) \\
R_{W_1} + R_{2A} &\leq (n_2 \vee \alpha_2 - 2)^+. \\
0 &\leq R_{W_1} \\
0 \leq R_{2A} &\leq (n_2 - 2)^+.
\end{aligned}$$

By Fourier-Motzkin, the following rate is achievable

$$\begin{aligned}
0 \leq R_{2A} &\leq n_2 - 2 \\
R_{2A} &\leq n_2 \vee \alpha_2 - n_1 \wedge \alpha_2 + \mathsf{C}_{12} - 7 + R_0 \\
R_{2A} &\leq n_2 \vee \alpha_2 - (\alpha_2 - n_2)^+ - 9 + R_0.
\end{aligned}$$

With $R_0 \geq 7$, the above conditions can be simplified as

$$\begin{aligned}
0 \leq R_{2A} &\leq n_2 - 2 \\
R_{2A} &\leq n_2 \vee \alpha_2 - n_1 \wedge \alpha_2 + \mathsf{C}_{12} - 7 + R_0.
\end{aligned}$$

Hence, we can achieve $R_2 = \max_{\delta} R_2(\delta)$, where

$$R_2(\delta) = \frac{1}{1+\delta} R_{\mathrm{2A}} \geq \min(u_1' - 2, u_2' - 7 + R_0 - 1) \geq \overline{C_{R_0}^{\mathsf{LDM}}}(\delta).$$

Hence, $\overline{C_{R_0}^{\mathsf{LDM}}}$ can be achieved in this region.

Region 4: $x_i > 1, y_i > 1, z > 1, i = 1, 2.$ We again consider two subregions.

(1) $\frac{1}{4} \leq \frac{x_1 x_2}{y_1 y_2} \leq 4.$

In this region, the channel gains are aligned, the cooperation is not very helpful, and the interference channel scheme suffices to achieve the upper bound within a constant. The condition $\frac{1}{4} \leq \frac{x_1 x_2}{y_1 y_2} \leq 4$ implies that $\alpha_1 + \alpha_2 - 4 \leq$

$n_1 + n_2 \leq \alpha_1 + \alpha_2 + 4$. Observing

$$(\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ = \max(\alpha_1 - n_1, n_2 - \alpha_1, n_2 - n_1, 0),$$

it can be shown that

1. $n_2 \vee \alpha_2 - \alpha_2 \wedge n_1 \geq (\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ - 4$.
2. $(\alpha_1 - n_1)^+ - \alpha_2 \wedge n_1 + (n_2 - \alpha_1) \vee \alpha_2 \geq (\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ - 4$,

which gives

$$\begin{aligned}\overline{C_{R_0}^{\mathsf{LDM}}}(\delta) &\leq \min(u_1' - 10 - 2R_0, u_3' - 5 - R_0)\\ &\leq \min(n_2 - 10 - 2R_0, (\alpha_1 - n_1)^+ + (n_2 - \alpha_2)^+),\\ \overline{C_{cog}^{IFC-LDM}} &\geq \min(n_2, (\alpha_1 - n_1)^+ + (n_2 - \alpha_1)^+ - 4).\end{aligned}$$

So we get

$$\begin{aligned}\overline{C_{R_0}^{\mathsf{LDM}}}(\delta) &\leq \overline{C_{cog}^{IFC-LDM}} - 1 - R_0\\ &\leq C_{R_0}^{IFC}.\end{aligned}$$

Hence, $\overline{C_{R_0}^{\mathsf{LDM}}}$ can be achieved in this region.

(2) $\frac{x_1 x_2}{y_1 y_2} \geq 4$ or $\frac{x_1 x_2}{y_1 y_2} \leq \frac{1}{4}$.

As in region 3, in the scheme from section 4, we set $\delta_{\mathrm{A}} = 1, \delta_{\mathrm{B}} = \delta$, and $\delta_C = 0$. Here also, cooperation is achieved through cooperative private messages. For simplicity, we will require that $R_{1\mathrm{B}}, R_{1\mathrm{A}} \geq \log(1 + x_1) - R_0$.

In mode B, source 1 uses power $\frac{1}{x_1}$ to share bits with source 2 and power $1 - \frac{1}{x_1}$ to send data to destination 3. Under the natural order of superposition coding, the following rates are suppported.

$$\begin{aligned}R_{1B} &= \log(1 + \frac{(1 - \frac{1}{x_1})x_1}{2}) = \log(1 + x_1) - 1\\ \frac{\mathsf{C}_{12}}{\delta} &= \log(1 + \frac{z}{x_1}) \geq (\log(\frac{z}{x_1}))^+ \geq (\beta - n_1 - 1)^+ \geq \beta - n_1 - 1.\end{aligned}$$

For the virtual channel, source 1 uses three messages W_1, U_1, V_1 and source 2 uses two messages W_2, U_2. For source 1, we allocate powers $\sigma_{W_1}^2 = \frac{1}{3}, \sigma_{U_1}^2 =$

$\frac{1}{3y_2}$, Var $(X_{V_1}) = \frac{1}{3}(1 \wedge \frac{x_2}{y_2})$, and for source 2, $\sigma^2_{W_2} = \frac{1}{3}, \sigma^2_{U_2} = \frac{1}{3y_1}$, Var $(X_{V_2}) = \frac{y_2}{x_2}$Var $(X_{V_1}) = \frac{1}{3}(1 \wedge \frac{y_2}{x_2})$. Destination 1 gets W_1, U_1, V_1, W_2, U_2 with powers $\frac{x_1}{3}, \frac{x_1}{3y_2}, \frac{\beta_1 x_1}{3}(1 \wedge \frac{x_2}{y_2}), \frac{y_1}{3}, \frac{1}{3}$, respectively, and U_2 is treated as noise. Destination 2 gets W_2, U_2, W_1, U_1 with powers $\frac{x_2}{3}, \frac{x_2}{3y_1}, \frac{y_2}{3}, \frac{1}{3}$, respectively, and U_1 is treated as noise.

Using lemma D.4, it is easy to verify that the following constraints on nonnegative rates imply all the relevant constraints in Theorem 4.1.

$$
\begin{aligned}
R_{W_1} + R_{U_1} + R_{W_2} + R_{V_1} &\leq \log(1 + \frac{x_1 + y_1}{4}) \\
R_{U_1} + R_{W_2} + R_{V_1} &\leq \log(1 + \frac{y_1 + \tilde{k}/4}{4}) \\
R_{W_1} + R_{U_1} + R_{V_1} &\leq \log(1 + \frac{x_1 + \tilde{k}/4}{4}) \\
R_{W_1} + R_{U_1} &\leq \log(1 + \frac{x_1}{4}) \\
R_{U_1} + R_{W_2} &\leq \log(1 + \frac{\frac{x_1}{y_2} + y_1}{4}) \\
R_{U_1} + R_{V_1} &\leq \log(1 + \frac{\tilde{k}/4}{4}) \\
R_{U_1} &\leq \log(1 + \frac{x_1}{4y_2}) \\
R_{V_1} &\leq \mathsf{C}_{12} \\
R_{W_1} + R_{W_2} + R_{U_2} &\leq \log(1 + \frac{x_2 + y_2}{4}) \\
R_{W_1} + R_{U_2} &\leq \log(1 + \frac{\frac{x_2}{y_1} + y_2}{4}) \\
R_{W_2} + R_{U_2} &\leq \log(1 + \frac{x_2}{4}) \\
R_{U_2} &\leq \log(1 + \frac{x_2}{4y_1}).
\end{aligned}
$$

As in region 3, it is not hard to see that $R_{1\text{A}} = (\log(1 + x_1) - R_0)^+$, $R_{1\text{B}} = 0$ satisfies these constraints when $R_0 \geq 7$.

Now, in the original constraints, eliminating V_1, U_2 with $R_{1\text{A}} = R_{W_1} + R_{U_1} + R_{V_1}$ and $R_{2\text{A}} = R_{W_2} + R_{U_2}$ and setting $R_{1\text{A}} = (\log(1 + x_1) - R_0)^+$, we get

$$
\begin{aligned}
R_{W_2} &\leq \log(1 + \frac{x_1 + y_1}{4}) - (\log(1 + x_1) - R_0)^+ \\
-R_{W_1} + R_{W_2} &\leq \log(1 + \frac{y_1 + \tilde{k}/4}{4}) - (\log(1 + x_1) - R_0)^+
\end{aligned}
$$

$$
\begin{aligned}
(\log(1+x_1)-R_0)^+ &\leq \log(1+\frac{x_1+\tilde{k}/4}{4}) \\
R_{W_1}+R_{U_1} &\leq \log(1+\frac{x_1}{4}) \\
R_{U_1}+R_{W_2} &\leq \log(1+\frac{\frac{x_1}{y_2}+y_1}{4}) \\
-R_{W_1} &\leq \log(1+\frac{\tilde{k}/4}{4})-(\log(1+x_1)-R_0)^+ \\
0\leq R_{U_1} &\leq \log(1+\frac{x_1}{4y_2}) \\
0\leq R_{W_2} &\leq \log(1+\frac{y_1}{4}) \\
R_{U_1}+R_{W_1} &\geq (\log(1+x_1)-R_0)^+-\mathsf{C}_{12} \\
R_{U_1}+R_{W_1} &\leq (\log(1+x_1)-R_0)^+ \\
R_{W_1}+R_{2\mathrm{A}} &\leq \log(1+\frac{x_2+y_2}{4}) \\
R_{W_1}+R_{2\mathrm{A}}-R_{W_2} &\leq \log(1+\frac{\frac{x_2}{y_1}+y_2}{4}) \\
R_{2\mathrm{A}} &\leq \log(1+\frac{x_2}{4}) \\
0\leq R_{W_1} &\leq \log(1+\frac{y_2}{4}) \\
0\leq R_{2\mathrm{A}}-R_{W_2} &\leq \log(1+\frac{x_2}{4y_1}).
\end{aligned}
$$

By the choice of R_0, we already have

$$
(\log(1+x_1)-R_0)^+ \leq \log(1+\frac{x_1+\tilde{\beta}_1 x_1}{4}).
$$

Again, we may simplify the constraints with linear deterministic notation to obtain the following set of constraints

$$
\begin{aligned}
R_{W_2} &\leq ([\max(\alpha_1,n_1)-2]^+-[n_1+2-R_0]^+)^+ \\
R_{W_2}-R_{W_1} &\leq (\max(\alpha_1,k)-5)^+-(n_1+2-R_0)^+ \\
R_{W_1}+R_{U_1} &\leq (n_1-2)^+ \\
R_{U_1}+R_{W_2} &\leq (\max(\alpha_1,n_1-\alpha_2)-3)^+ \\
R_{W_1} &\geq (n_1+2-R_0)^+-(k-5)^+ \\
0\leq R_{U_1} &\leq ((n_1-\alpha_2)^+-3)^+ \\
0\leq R_{W_2} &\leq (\alpha_1-2)^+ \\
R_{U_1}+R_{W_1} &\geq \min((n_1+2-R_0)^+-\mathsf{C}_{12},(n_1-R_0)^+)
\end{aligned}
$$

$$
\begin{aligned}
R_{U_1} + R_{W_1} &\leq n_1 - R_0 \\
R_{W_1} + R_{2\mathrm{A}} &\leq (\max(\alpha_2, n_2) - 2)^+ \\
R_{W_1} + R_{2\mathrm{A}} - R_{W_2} &\leq (\max(n_2 - \alpha_1, \alpha_2) - 3)^+ \\
R_{2\mathrm{A}} &\leq (n_2 - 2)^+ \\
0 \leq R_{W_1} &\leq (\alpha_2 - 2)^+ \\
0 \leq R_{2\mathrm{A}} - R_{W_2} &\leq ((n_2 - \alpha_1)^+ - 3)^+.
\end{aligned}
$$

By Fourier-Motzkin elimination, we can show that $R_{2\mathrm{A}} = \min(v_1 - 9, v_2 + \mathsf{C}_{12} - 7 + R_0, v_3 - 19, v_4 + \mathsf{C}_{12} - 16 + R_0)$ is achievable, where $v_i, i = 1, 2, 3, 4$ are defined in Proposition 6.1.1. Since we have $\mathsf{C}_{12} \geq \delta(\beta - n_1 - 1)$, we can conclude that when $R_0 \geq 7$, we may achieve $R_2 = \max_{\delta \geq 0} R_2(\delta)$, where

$$
R_2(\delta) = \frac{1}{1+\delta} R_{2\mathrm{A}} \geq \min(u_1' - 9, u_2' - 7 + R_0 - 1, u_3' - 19, u_4' - 16 + R_0 - 1).
$$

With $R_0 \geq 7$, we can see that $R_2(\delta) \geq \overline{C_{R_0}^{\mathsf{LDM}}} - 10$. Hence, $\overline{C_{R_0}^{\mathsf{LDM}}}$ can be achieved within 10 bits in this region.

APPENDIX E

PROOF OF THEOREM 6.5

For a given scheme, let $\delta \geq 0$ be the proportion of the time spent in mode B to the time spent in mode A. Note that there is no mode C in the cognitive setting. It is enough to show that for any scheme with scheduling parameter $\delta \geq 0$, $\mathscr{C}(\delta)$ is an outer bound of the achievable rate region.

Let $P_{i,t} = |X_{i,t}|^2$, $i = 1, 2$, and $t = 1, 2, \ldots, N$. We define the average power in the different modes as follows:

$$P_{1\mathrm{A}} = \frac{1+\delta}{N} \sum_{t \in \mathrm{A}} P_{1,t}, \qquad P_{1\mathrm{B}} = \frac{1+\delta}{\delta N} \sum_{t \in \mathrm{B}} P_{1,t}, \text{ and}$$
$$P_{2\mathrm{A}} = \frac{1+\delta}{N} \sum_{t \in \mathrm{A}} P_{2,t}. \qquad P_{2\mathrm{B}} = \frac{1+\delta}{\delta N} \sum_{t \in \mathrm{B}} P_{2,t} = 0.$$

By power constraint, we have $\frac{P_{i\mathrm{A}} + \delta P_{i\mathrm{B}}}{1+\delta} \leq 1, i = 1, 2$. We further define

$$V_{1\mathrm{A}}^L = h_{13} X_{1\mathrm{A}}^L + Z_{1\mathrm{A}}^L, \qquad U_{1\mathrm{A}}^L = h_{14} X_{1\mathrm{A}}^L + Z_{2\mathrm{A}}^L,$$
$$V_{2\mathrm{A}}^L = h_{24} X_{2\mathrm{A}}^L + Z_{2\mathrm{A}}^L, \qquad U_{2\mathrm{A}}^L = h_{23} X_{2\mathrm{A}}^L + Z_{1\mathrm{A}}^L.$$

1. R_2

$$\begin{aligned}
L(R_2 - \epsilon) &\leq I(W_2; Y_{4\mathrm{A}}^L) \\
&\leq I(W_2; Y_{4\mathrm{A}} | W_1) \\
&\leq I(W_2; Y_{4\mathrm{A}}^L | W_1) \\
&\leq H(Y_{4\mathrm{A}}^L | W_1) - H(Y_{4\mathrm{A}}^L | W_1, W_2, Y_{2\mathrm{B}}^L) \\
&= H(V_{2\mathrm{A}}^L | W_1) - H(Z_{4\mathrm{A}}^L) \\
&\leq H(V_{2\mathrm{A}}^L) - H(Z_{4\mathrm{A}}^L)
\end{aligned}$$

$$R_2 - \epsilon \leq \frac{1}{1+\delta}\log(1 + x_2 P_{2A}).$$

2. $R_1 + R_2$

$$L(R_1 + R_2 - \epsilon) \leq I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L) + I(W_2; , Y_{4\mathrm{A}}^L, Y_{4\mathrm{B}}^L).$$

We have

$$\begin{aligned}
&I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L) \\
\leq& I(W_1; Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{2\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2) \\
=& H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{2\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2) - H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_1, W_2, Y_{2\mathrm{B}}^L) \\
&- H(Y_{2\mathrm{B}}^L | W_1, W_2) \\
=& H(Y_{2\mathrm{B}}^L | W_2) + H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2, Y_{2\mathrm{B}}^L) \\
&- H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_1, W_2, Y_{2\mathrm{B}}^L) - H(Y_{2\mathrm{B}}^L | W_1, W_2) \\
=& H(Y_{2\mathrm{B}}^L | W_2) + H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2, Y_{2\mathrm{B}}^L, X_{2\mathrm{A}}^L) \\
&- H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_1, W_2, Y_{2\mathrm{B}}^L) - H(Y_{2\mathrm{B}}^L | W_1, W_2) \\
\leq& H(Y_{2\mathrm{B}}^L | W_2) + H(V_{1\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2, Y_{2\mathrm{B}}^L) \\
&- H(Y_{3\mathrm{A}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_1, W_2, Y_{2\mathrm{B}}^L) - H(Y_{2\mathrm{B}}^L | W_1, W_2) \\
=& H(V_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{B}}^L) - H(Z_{2\mathrm{B}}^L) \\
\leq& H(V_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2) - H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{B}}^L) - H(Z_{2\mathrm{B}}^L) \\
&+ I(U_{1\mathrm{A}}^L; X_{1\mathrm{A}}^L | V_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L, W_2) \\
=& H(V_{1\mathrm{A}}^L, U_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2) - H(U_{1\mathrm{A}}^L | X_{1\mathrm{A}}^L, V_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L, W_2) \\
&- H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{B}}^L, Z_{2\mathrm{B}}^L) \\
\leq& H(V_{1\mathrm{A}}^L | U_{1\mathrm{A}}^L) + H(U_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2) - H(Z_{4\mathrm{A}}^L) \\
&- H(Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{B}}^L, Z_{2\mathrm{B}}^L)
\end{aligned}$$

$$\begin{aligned}
&I(W_2; Y_{4\mathrm{A}}^L, Y_{4\mathrm{B}}^L) \\
\leq& I(W_2; Y_{4\mathrm{A}}^L, Y_{4\mathrm{B}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L) \\
=& H(Y_{4\mathrm{A}}^L, Y_{4\mathrm{B}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L) - H(Y_{4\mathrm{A}}^L, Y_{4\mathrm{B}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L | W_2) \\
\leq& H(Y_{4\mathrm{A}}^L) + H(Y_{4\mathrm{B}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L) - H(U_{1\mathrm{A}}^L, Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L | W_2).
\end{aligned}$$

Hence, we get

$$L(R_1 + R_2 - \epsilon) \leq H(Y_{4\mathrm{A}}^L) + H(Y_{2\mathrm{B}}^L, Y_{3\mathrm{B}}^L, Y_{4\mathrm{B}}^L) + H(V_{1\mathrm{A}}^L | U_{1\mathrm{A}}^L) \\ - H(Z_{4\mathrm{A}}^L, Z_{3\mathrm{A}}^L, Z_{3\mathrm{B}}^L, Z_{4\mathrm{B}}^L, Z_{2\mathrm{B}}^L).$$

Notice that

$$\begin{aligned}
& H(Y_{4\mathrm{A}}^L) - H(Z_{4\mathrm{A}}^L) \\
\leq & \sum_{t \in A} H(Y_{4t}) - H(Z_{4t}) \\
\leq & \sum_{A} \log(1 + (\sqrt{x_2 P_{2t}} + \sqrt{y_2 P_{1t}})^2) \\
\leq & \sum_{A} \log(1 + 2x_2 P_{2t} + 2y_2 P_{1t}) \\
\leq & \frac{L}{1+\delta} \log(1 + 2x_2 P_{2\mathrm{A}} + 2y_2 P_{1\mathrm{A}})
\end{aligned}$$

$$\begin{aligned}
& H(V_{1\mathrm{A}}^L | U_{1\mathrm{A}}^L) - H(Z_{3\mathrm{A}}^L) \\
\leq & H(V_{1\mathrm{A}}^L - cU_{1\mathrm{A}}^L) - H(Z_{3\mathrm{A}}^L) \qquad \left(c = \frac{h_{13} h_{14}^* P_{1\mathrm{A}}}{1 + y_2 P_{1\mathrm{A}}}\right) \\
\leq & \sum_{A} H(V_{1t} - cU_{1t}) - H(Z_{3t}) \\
\leq & \sum_{A} H\left(\frac{h_{13}}{1 + y_2 P_{1A}} X_{1t} + Z_{3t} - cZ_{4t}\right) - H(Z_{3t}) \\
\leq & \sum_{A} \log\left(1 + |c|^2 + \frac{x_1 P_{1t}}{1 + y_2 P_{1A}}\right) \\
\leq & \frac{L}{1+\delta} \log\left(1 + |c|^2 + \frac{x_1 P_{1A}}{(1 + y_2 P_{1A})^2}\right) \\
= & \frac{L}{1+\delta} \log\left(1 + \frac{y_2 x_1 P_{1A}^2}{(1 + y_2 P_{1A})^2} + \frac{x_1 P_{1A}}{(1 + y_2 P_{1A})^2}\right) \\
= & \frac{L}{1+\delta} \log\left(1 + \frac{x_1 P_{1A}}{1 + y_2 P_{1A}}\right).
\end{aligned}$$

Hence,

$$R_1 + R_2 - \epsilon \leq \frac{1}{1+\delta} \Big[\log(1 + 2x_2 P_{2A} + 2y_2 P_{1A}) \\ + \delta \log(1 + (x_1 + y_2 + z) P_{1B}) + \log\left(1 + \frac{x_1 P_{1A}}{1 + y_2 P_{1A}}\right) \Big].$$

3. $R_1 + R_2$

$$L(R_1 + R_2 - \epsilon) \leq I(W_1; Y_{3\text{A}}^L, Y_{3\text{B}}^L) + I(W_2; , Y_{4\text{A}}^L, Y_{4\text{B}}^L).$$

Notice that

$$\begin{aligned}
&I(W_1; Y_{3\text{A}}^L, Y_{3\text{B}}^L)\\
=&H(Y_{3\text{A}}^L, Y_{3\text{B}}^L) - H(Y_{3\text{A}}^L, Y_{3\text{B}}^L|W_1)\\
=&H(Y_{3\text{A}}^L) + H(Y_{3\text{B}}^L) - H(U_{2\text{A}}^L|W_1) - H(Z_{3\text{B}}^L)
\end{aligned}$$

$$\begin{aligned}
&I(W_2; Y_{4\text{A}}^L, Y_{4\text{B}}^L)\\
\leq&I(W_2; Y_{4\text{A}}^L, Y_{4\text{B}}^L|W_1)\\
\leq&H(Y_{4\text{A}}^L, Y_{4\text{B}}^L|W_1) - H(Y_{4\text{A}}^L, Y_{4\text{B}}^L|W_1, W_2, Y_{2\text{B}}^L)\\
=&H(V_{2\text{A}}^L|W_1) + H(Z_{4\text{B}}^L) - H(Z_{4\text{A}}^L, Z_{4\text{B}}^L)\\
\leq&H(V_{2\text{A}}^L, U_{2\text{A}}^L|W_1) - H(U_{2\text{A}}^L|V_{2\text{A}}^L, W_1, X_2^L) - H(Z_{4\text{A}}^L)\\
\leq&H(V_{2\text{A}}^L|U_{2\text{A}}^L) + H(U_{2\text{A}}^L|W_1) - H(Z_{3\text{A}}^L) - H(Z_{4\text{A}}^L).
\end{aligned}$$

We get

$$L(R_1 + R_2 - \epsilon) \leq H(Y_{3\text{A}}^L) + H(Y_{3\text{B}}^L) + H(V_{2\text{A}}^L|U_{2\text{A}}^L) - H(Z_{3\text{A}}^L, Z_{4\text{A}}^L, Z_{3\text{B}}^L).$$

Hence,

$$\begin{aligned}
R_1 + R_2 - \epsilon \leq&\frac{1}{1+\delta}\Big[\log(1 + 2x_1P_{1A} + 2y_1P_{2A}) + \delta\log(1 + x_1P_{1B})\\
&+ \log(1 + \frac{x_2P_{2A}}{1 + y_1P_{2A}})\Big].
\end{aligned}$$

4. $2R_1 + R_2$

$$L(2R_1 + R_2) \leq 2I(W_1; Y_{3\text{A}}^L, Y_{3\text{B}}^L) + I(W_2; , Y_{4\text{A}}^L, Y_{4\text{B}}^L).$$

We have

$$\begin{aligned}
&I(W_1; Y_{3\text{A}}^L, Y_{3\text{B}}^L)\\
=&H(Y_{3\text{A}}^L, Y_{3\text{B}}^L) - H(Y_{3\text{A}}^L, Y_{3\text{B}}^L|W_1)
\end{aligned}$$

$$
\begin{aligned}
&\leq H(Y_{3A}^L, Y_{3B}^L) - H(Y_{3A}^L, Y_{3B}^L | W_1, Y_{2B}^L, X_1^L) \\
&= H(Y_{3A}^L, Y_{3B}^L) - H(U_{2A}^L | W_1, Y_{2B}^L, X_1^L) - H(Z_{3B}^L) \\
&\leq H(Y_{3A}^L) + H(Y_{3B}^L) - H(U_{2A}^L | Y_{2B}^L) - H(Z_{3B}^L),
\end{aligned}
$$

where the last inequality follows from the fact that $I(U_{2A}^L; W_2, X_1^L | Y_{2B}^L) = 0$ since $U_{2A}^L - X_{2A}^L - Y_{2B}^L - (W_1, X_1^L)$ is a Markov chain. Proceeding as in case 2 (the bound for $R_1 + R_2$),

$$
\begin{aligned}
& I(W_1; Y_{3A}^L, Y_{3B}^L) \\
\leq & I(W_1; Y_{3A}^L, Y_{3B}^L, Y_{2B}^L, Y_{4B}^L | W_2) \\
= & H(Y_{3A}^L, Y_{3B}^L, Y_{2B}^L, Y_{4B}^L | W_2) - H(Y_{3A}^L, Y_{3B}^L, Y_{4B}^L | W_1, W_2, Y_{2B}^L) \\
& - H(Y_{2B}^L | W_1, W_2) \\
= & H(V_{1A}^L, Y_{2B}^L, Y_{3B}^L, Y_{4B}^L | W_2) - H(Z_{3A}^L, Z_{3B}^L, Z_{4B}^L) - H(Z_{2B}^L) \\
\leq & H(V_{1A}^L, U_{1A}^L, Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) - H(U_{1A}^L | X_1^L, V_{1A}^L, Y_{2B}^L, Y_{3B}^L, Y_{4B}^L, W_2) \\
& - H(Z_{3A}^L, Z_{3B}^L, Z_{4B}^L, Z_{2B}^L) \\
\leq & H(V_{1A}^L | U_{1A}^L) + H(U_{1A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) + H(Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) \\
& - H(Z_{4A}^L) - H(Z_{3A}^L, Z_{3B}^L, Z_{4B}^L, Z_{2B}^L)
\end{aligned}
$$

$$
\begin{aligned}
& I(W_2; Y_{4A}^L, Y_{4B}^L) \\
\leq & I(W_2; , Y_{4A}^L, Y_{4B}^L Y_{2B}^L, Y_{3B}^L, U_{2A}^L) \\
= & I(W_2; , Y_{4A}^L, U_{2A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) \\
= & H(Y_{4A}^L, U_{2A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) - H(Y_{4A}^L, U_{2A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L, W_2) \\
= & H(Y_{4A}^L, U_{2A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) - H(U_{1A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L, W_2) - H(Z_{3A}^L) \\
\leq & H(Y_{4A}^L | U_{2A}^L) + H(U_{2A}^L | Y_{2B}^L) - H(U_{1A}^L | Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) - H(Z_{3A}^L).
\end{aligned}
$$

Combining all the inequalities, we get

$$
\begin{aligned}
& L(2R_1 + R_2 - \epsilon) \\
\leq & H(Y_{3A}^L) + H(Y_{3B}^L) + H(V_{1A}^L | U_{1A}^L) + H(Y_{4A}^L | U_{2A}^L) + H(Y_{2B}^L, Y_{3B}^L, Y_{4B}^L) \\
& - H(Z_{3A}^L, Z_{3B}^L, Z_{4A}^L) - H(Z_{3A}^L, Z_{3B}^L, Z_{4B}^L, Z_{2B}^L).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&H(Y_{4\mathrm{A}}^L|U_{2\mathrm{A}}^L) - H(Z_{4\mathrm{A}}^L) \\
\le& H(Y_{4\mathrm{A}}^L - cU_{2\mathrm{A}}^L) - H(Z_{4\mathrm{A}}^L) \qquad\qquad (c = \frac{h_{24}h_{23}^* P_{2\mathrm{A}}}{1+y_1P_{2\mathrm{A}}}) \\
\le& \sum_A H(Y_{4i} - cU_{2i}) - H(Z_{4i}) \\
=& \sum_A H(\frac{h_{24}}{1+y_1P_{2A}} X_{2i} + h_{14}X_{1i} + Z_{4i} - cZ_{3i}) \\
\le& \sum_A \log\left(1 + c^2 + \frac{x_2}{(1+y_1P_{2A})^2}P_{2i} + y_2P_{1i} + \frac{2\sqrt{x_2y_2P_{1i}P_{2i}}}{1+y_1P_{2A}}\right) \\
\le& \sum_A \log\left(1 + c^2 + \frac{x_2}{(1+y_1P_{2A})^2}P_{2i} + y_2P_{1i} + \frac{x_2P_{2i} + y_2P_{1i}}{1+y_1P_{2A}}\right) \\
\le& \frac{L}{1+\delta}\log\left(1 + c^2 + \frac{x_2}{(1+y_1P_{2A})^2}P_{2A} + y_2P_{1A} + \frac{x_2P_{2A} + y_2P_{1A}}{1+y_1P_{2A}}\right) \\
=& \frac{L}{1+\delta}\log\left(1 + y_2P_{1A} + \frac{2x_2P_{2A} + y_2P_{1A}}{1+y_1P_{2A}}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&2R_1 + R_{-}\epsilon \\
\le& \frac{1}{1+\delta}\Big[\log(1 + 2x_1P_{1A} + 2y_1P_{2A}) + \delta\log(1 + x_1P_{1B}) + \log(1 + \frac{x_1P_{1A}}{1+y_2P_{1A}}) \\
&+ \log(1 + y_2P_{1A} + \frac{2x_2P_{2A} + y_2P_{1A}}{1+y_1P_{2A}}) + \delta\log(1 + (x_1 + y_2 + z)P_{1B})\Big].
\end{aligned}
$$

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