Order Number 9210893

Invariant proper holomorphic maps between balls

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University of Illinois at Urbana-Champaign, 1991

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INVARIANT PROPER HOLOMORPHIC MAPS BETWEEN BALLS

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1991

Urbana, Illinois

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

THE GRADUATE COLLEGE

AUGUST, 1991

WE HEREBY RECOMMEND THAT THE THESIS BY

DANIEL LICHTBLAU

ENTITLED INVARIANT PROPER HOLOMORPHIC MAPS BETWEEN BALLS

BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF DOCTOR OF PHILOSOPHY

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Abstract

We consider proper holomorphic maps between balls that are invariant under the action of finite groups of unitary matrices. We are primarily interested in actions of groups that are fixed-point-free; for purposes of comparison we will briefly consider matrix groups that act with fixed points (that is, groups that have at least one nontrivial element with an eigenvalue of one) in the last chapter. Forstneric showed that given any finite unitary fixed-point-free matrix group, there exists a proper holomorphic map from the ball in the appropriate dimensional complex Euclidean space to a higher dimensional ball, that is invariant under the action of that group. He showed on the other hand that if we also require the map to be smooth to the boundary, then many groups are ruled out.

One of our main results is the following theorem: if f is a proper holomorphic map between balls that is invariant under the action of some finite fixed-point-free matrix subgroup of a unitary group (acting on the domain of f), and, in addition, smooth to the boundary, then necessarily that group is cyclic and diagonally generated (with respect to some basis). We rule out some of these possibilities as well. We give corollaries concerning the nonexistence of smooth CR mappings from certain spherical space forms to spheres.

We next prove some propositions related to the theory of polynomial proper mappings between balls. As another important result, in cases where there are known finite fixed-point-free matrix group-invariant mappings we classify all such maps in terms of a group-basic map. In a subsequent chapter we investigate existence and nonexistence of various sorts of polynomial proper maps between balls, mostly invariant under some matrix group action, from a combinatorial perspective. We give a simple means of depicting monomial mappings from the ball in two-dimensional space, and show some applications. As a final theorem, we show how proper holomorphic maps between balls, invariant under the action of finite matrix groups

possibly acting with fixed points, can be "constructed". This uses a technique developed by Løw. We derive some interesting examples from this construction.

Acknowledgments

I would like to thank my advisor John D'Angelo for suggesting for consideration this topic as well as methods of attack, for giving me access to relevant preprints of his own work, and for critically reviewing the preliminary versions of the proofs, some of which were incorrect. In particular, the first (correct) proof of the main result was greatly simplified by his observation that much of it reduced to what is now a technical lemma in the first chapter. I also take this opportunity to thank him for patiently "which hunting", thereby improving the grammar. I would also like to thank Dan Grayson, Lee Rubel, and Bruce Reznick for several comments on preliminary versions that helped to clarify the exposition. Finally, I should thank my wife Eileen for not having me summarily executed for terminal ill-temper during the process of finishing this work.

Portions of the first two chapters have been submitted as [Li].

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Chapter 1 Smooth group-invariant proper holomorphic maps between balls

1.1 Introduction

We will investigate proper maps between balls in complex Euclidean space. Proper holomorphic maps in several complex variables have received a good deal of attention in recent years (see survey articles [CS3] and [Fo3], for example). When domain and range are balls, one can use powerful general results about maps between strongly pseudoconvex domains with real analytic boundaries, such as extension results that arise from polarization and related theory. One also has available techniques related to the invariance of balls under the action of the unitary group. We put such techniques to use in the main results of this thesis.

Recall that a map $f : D \to \Omega$ is proper if the preimages of compact sets in Ω are compact in D. When domain and range are (open) balls in complex Euclidean spaces this is easily seen to be equivalent to the following condition: if z_n is a sequence approaching the boundary in the domain ball, then $f(z_n)$ tends to the boundary in the target ball. Furthermore, if f is holomorphic (or just continuous) on its domain ball and continuous to the boundary, one can easily deduce that f is proper if and only if it maps boundary into boundary. It is also an easy exercise to show that a proper holomorphic map must go to a target of equal or greater dimension, and that generically such a map will look like a local embedding [Ru3]. Thus it makes sense to speak of the codimension of such a map. We use the standard notation B_n for the open unit ball centered at the origin in $n-$dimensional complex Euclidean space, and S^{2n-1} for its boundary sphere. We use the word smooth to mean $C^\infty-$smooth, unless specifically stated otherwise.

Some research ([D1], [Fo1], [Ru1], [Ru2]) has focused on proper holomorphic maps from balls that are invariant under the action of some finite unitary matrix group acting on the domain.In [Ru2] and [Fo1] it is shown that proper holomorphic maps from balls that are smooth to the boundary factor through finite unitary matrix groups (after perhaps first applying an automorphism on the domain). Rudin assumes domain and range have the same dimension, larger than one. Forstneric assumes both are of dimension larger than one, and the codimension is positive. He also assumes the range is strongly pseudoconvex. He showed that in this case the unitary matrix group must furthermore be fixed-point-free (the Rudin result, by way of contrast, always involves a reflection group). Forstneric showed that all fixed-point-free groups arise in the context of his theorem provided one removes the hypothesis of smoothness to boundary. One can even take the range to be a ball in this case.

The case where the map is smooth to boundary is much different. While little is known about the group-invariance properties of such maps to arbitrary strongly pseudoconvex domains (beyond the above mentioned factoring through a fixed-point-free group), it is noted in [Fo3] that "the problem becomes more interesting if we require the target domain...to be the unit ball...." In this case we have available more powerful techniques and also some simple examples of such group-invariant maps. Furthermore, smooth proper holomorphic maps between balls, as opposed to arbitrary domains, are in some sense quite abundant and straightforward to construct (see [D2]). Forstneric noted the seemingly curious phenomenon that in this particular case, despite the plethora of proper holomorphic maps between balls, very few unitary groups are known to arise in the context of these maps. He gave simple examples of nontrivial groups that do; a family of these appears in [Ru1]. He ruled out a large class of these groups, assuming the map to be rational in this case. This requirement is no stronger than $C^{\infty}-$smoothness; we give more details and appropriate references below. D'Angelo [D1] enlarged the set of groups for which such maps can be constructed, showing (as suggested in [Fo1]) that a second family of these groups exists for which there are particularly simple invariant maps between balls. He also gave simple proofs concerning certain uniqueness properties of the maps in [Ru1]. Most of this thesis continues the lines of enquiry pursued in [D1] and [Fo1]. We are concerned with group-invariant smooth proper holomorphic maps between balls; we contribute a few more pieces to the "solution" of the question posed by Rudin in [Ru3, chapter 15]: "What are the proper holomorphic maps between balls?"

Having introduced some of the history behind this work, it is appropriate here to note that proper holomorphic maps that extend smoothly to the boundary of a domain are of some importance in geometry and partial differential equations. The unit sphere is one of the simplest examples of a CR manifold, and the study of the Cauchy-Riemann equations on the 3–sphere led to the Lewy operator and an example of a non-locally-solvable partial differential equation. This operator, in fact, is the one that defines the CR structure on S^3, and so the CR functions that sphere are precisely the solutions to the homogeneous Lewy equation.

If a CR function on a sphere is invariant under the action of a finite unitary group, we can consider it as a function on the quotient space. These are odd dimensional spaces of constant positive curvature (spherical space forms). Thus some of our results can be interpreted as restrictions on smooth CR maps between spaces of constant positive curvature. In order that nonconstant such CR maps exist, we prove that the underlying group must be cyclic.

As shown in [Fo1], any proper holomorphic map between balls invariant under some automorphism group of the domain is, after composition with an automorphism, invariant under the action of a unitary group. Thus there is no loss of generality in restricting our attention to such groups. The main purpose of this first chapter is to show that any finite subgroup of the unitary group $U(n)$ that acts freely on $\mathbf{C}^n$, for which there is an invariant proper holomorphic map between balls, smooth on the closures, must be cyclic and generated by a diagonal matrix. We rule out some of these groups as well. We start with the relevant definitions and background theory.

We adopt the following notational conventions. By ϵ_p we denote a primitive $p^{\underline{th}}$ root of unity. When more than one of these appears in a matrix, we assume that each denotes the same primitive root. We may thence drop the subscript for the remainder of that usage. For q an integer, $\nu_2(m)$ denotes the largest power of 2 that divides m.

Recall that a subgroup Γ of $U(n)$ is said to be fixed-point-free if no member other than the identity matrix has an eigenvalue of 1. Equivalently, the only fixed point in $\mathbf{C}^n$ of any non-identity element in the group is the origin. A subgroup of $U(n)$ is said to be irreducible if it fixes no nontrivial proper subspaces of $\mathbf{C}^n$.

Throughout this paper we will work with unitary fixed-point-free matrix groups. We will also consider matrix groups acting with fixed points in the last chapter. We note that different matrix groups can be isomorphic as abstract groups and yet give completely different results in the theory we cover in this thesis. Specifically, for some finite fixed-point-free unitary groups we can show existence of certain invariant proper holomorphic maps between balls, while we can show nonexistence of such for other matrix groups isomorphic to those. In other words, suppose G is some finite abstract group and $\pi : G \mapsto \Gamma \subset U(n)$ an injective representation of G. Then whether there exists a smooth invariant proper holomorphic map between balls will depend on the particular representation π as well as on the group G.

We outline some of the relevant theory below. Forstneric showed in [Fo1] that given any finite fixed-point-free unitary group $\Gamma \subset U(n)$ there exists a proper holomorphic map $f : B_n \to B_N$ for some $N > n$ with the property that f is invariant under the action of Γ, i.e. for any $\gamma \in \Gamma$ we have $f \circ \gamma = f$. We will outline a proof of this result in chapter four. Though such functions always exist (and even can be made continuous to the boundary), the class of unitary groups for which smooth such functions are known to exist is quite small. In fact, if f is a proper holomorphic map between balls that is C^∞ to the boundary, then Forstneric has shown that f must be rational. (See [Fo2]; actually C^{N-n+1} suffices.) For a large class of finite fixed-point-free unitary groups, Forstneric [Fo1] showed that there can be no proper rational maps between balls that are invariant under the actions of those groups. In this chapter we rule out all remaining finite fixed-point-free unitary groups that are not cyclic and diagonally generated. We rule out some that are, as well. Before describing our specific results, we give a brief description of the playing field.

In [W], Wolf worked out a classification of finite fixed-point-free unitary groups in order to classify all odd-dimensional spherical space forms (complete connected odd-dimensional Riemannian manifolds of constant positive curvature; they turn out to be topological quotients of spheres modulo such groups). For our purposes a bare sketch of the details will suffice. These groups are direct sums of irreducible subgroups (so the matrix representation has block-diagonal elements, each block being an irreducible subgroup). The irreducible groups fall into two categories: those whose Sylow 2-subgroups are cyclic (type A) and those for which these subgroups are generalized quaternionic (type B). With respect to some basis, each irreducible group of type A is either 1x1 or else has only two nxn generators, one diagonal and the other of the form

$$\gamma = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & & & . & . & . & 1 \\ \delta & & & . & . & . & 0 \end{pmatrix} \tag{1.1.1}$$

where δ is a primitive $m^{\underline{th}}$ root of one. Each irreducible group of type B contains an element that, on some two dimensional subspace, is the submatrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Every finite fixed-point-free unitary group is the direct sum of irreducible representations. See [W] or [Fo1] for further details. In the latter, it is shown that there can be no proper rational map between balls that is invariant under the action of any group of type B or any group containing a $2k$x$2k$ generator of the sort in *1.1.1*. In fact such maps are ruled out when the group contains an element that, on some two dimensional subspace, is of the form

$$\begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \tag{1.1.2}$$

where ϵ is a primitive even root of unity. As we will outline in section *1.3*, this is always the case for type B groups and groups with even dimensional generators of the form *1.1.1;* hence the Forstneric results. Using a technique similar to his, we will rule out a class of generators similar to, but larger than, those of the form *1.1.2*

1.2 Statement of results

1.2.1 Theorem: There is no proper rational map $f : B_n \to B_N$ invariant under the action of any unitary group with an element γ of the form given in 1.1.1 *above. (This is true for all m, n, and N.) In particular, in order for a proper rational group-invariant map between balls to exist, the group must be cyclic with all its elements with respect to an appropriate basis diagonal.*

1.2.2 Theorem: Suppose $\Gamma \subset U(2)$ is generated by a matrix of the form $\begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^q \end{pmatrix}$ and the following condition holds: $0 < \nu_2(q-1) < \nu_2(p)$. Then there does not exist any proper rational map f from B_2 to any B_N that is invariant under the action of Γ.

Before stating our corollaries we recall some relevant definitions, versions of which can be found in [Ra, IV.2.3]. Let D be a domain in $\mathbf{C}^n$ bounded by a C^1-smooth real hypersurface M. That is, D is given (locally) by a C^1 real-valued defining equation of the form $\{r(z) < 0\}$, where r is defined in some neighborhood of a boundary point $p \in M$, and the differential dr is nonvanishing on M. The tangent space T_pM is of real dimension $2n-1$. We complexify it to form $\mathbf{C}(T_pM) = T_pM \otimes \mathbf{C}$. This gives rise to the complexified tangent bundle $\mathbf{C}(TM) = TM \otimes \mathbf{C}$. Sections of this bundle are given locally (in a neighborhood of p) by differential operators of the form $v = \sum_j a_j(z)\frac{\partial}{\partial z_j} + \sum_j b_j(z)\frac{\partial}{\partial \bar{z}_j}$, where a_j, b_j are C^1 functions near p and $v(r)|_z = 0$ for all $z \in M$ near p.

We define the subbundle of $(1,0)$ sections (locally) to be the set $T^{1,0}M = \mathbf{C}(TM) \cap \left\{\sum_j a_j \frac{\partial}{\partial z_j}\right\}$. We then define the subbundle of $(0,1)$ sections $T^{0,1}M$ to be its conjugates. It is easy to show that $T_p^{1,0}M$ and its conjugate space $T_p^{0,1}M$ each have complex dimension $n-1$.

Now let f be a complex valued function that is defined on, and C^1 in, an open subset U of the boundary M. Then f is a Cauchy-Riemann (CR) function on U if for every point $p \in U$ we have $v(f) = 0$ for all $v \in T_p^{0,1}M$. In this case we say that f satisfies the tangential Cauchy-Riemann equations.

Now suppose we have a map f, with CR coordinate functions, from M to complex Euclidean space $\mathbf{C}^N$. If the image $f(M)$ lies in some hypersurface M' then we call f a CR map from M to M'.

Some remarks: (i) One can define CR manifolds abstractly (that is, define the subbundle of $(1,0)$ sections), and then define a CR map f between CR manifolds to be one whose differential df maps $(1,0)$ sections into $(1,0)$ sections. The concrete definitions given above will serve more readily for our purposes.

(ii) If f is holomorphic on the domain D and C^1 up to the boundary hypersurface M, then it can be shown that the restriction of f to M is a CR function.

(iii) As a simple but relevant example, we work out the CR structure on the sphere S^3. The defining function is $r(z,\overline{z}) = |z_1|^2 + |z_2|^2 - 1 = z_1\overline{z}_1 + z_2\overline{z}_2 - 1$. (When the defining function is real analytic, we write it explicitly as a function of z and $\overline{z}$). The section $L = \overline{z}_2\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial z_2}$ generates $T^{1,0}S^3$. (i.e. any other section of that bundle is this one multiplied by a C^1 function). The conjugate section is the Lewy operator $\overline{L} = z_2\frac{\partial}{\partial \overline{z}_1} - z_1\frac{\partial}{\partial \overline{z}_2}$. As this is all there is of the antiholomorphic bundle $T^{0,1}S^3$ up to multiplication by functions, a function f is CR if and only if it is killed by the vector field $\overline{L}$. Finally, we note that the Lie bracket

$$\begin{aligned} T = [L,\overline{L}] &= \left[\overline{z}_2\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial z_2}, z_2\frac{\partial}{\partial \overline{z}_1} - z_1\frac{\partial}{\partial \overline{z}_2}\right] \\ &= -\overline{z}_2\frac{\partial}{\partial \overline{z}_2} - \overline{z}_1\frac{\partial}{\partial \overline{z}_1} + z_2\frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_1} \end{aligned}$$

is a vector field that is algebraically independent of L and $\overline{L}$. Hence the set $\{L,\overline{L},T\}$ spans $\mathbf{C}(TS^3)$.

We move on to the corollaries involving CR geometry.

1.2.3 Corollary: Let γ be a unitary matrix of the type in the above theorems or in the Forstneric nonexistence results mentioned above. Then there does not exist any nonconstant $\gamma-$invariant $C^\infty-$smooth CR mapping from S^{2n-1} to any sphere S^{2N-1}.

This is of course true if only a linear subspace intersected with the domain sphere is invariant under such a matrix action.

Proof of corollary: Strong pseudoconvexity of the sphere implies that such a CR mapping would extend to a holomorphic function f on the ball that is $C^\infty-$smooth on the closed ball and takes (boundary) sphere into sphere (see [Ra] II.2, IV.2). As this map is nonconstant, the maximum principle for plurisubharmonic functions ([Ra] II.4) may be applied to $||f||^2$. We see that f maps interior to interior as well as boundary to boundary; we conclude that it is proper. Note also that $f \circ \gamma - f = 0$ on the boundary and hence inside as well, also by the maximum principle. Thus f is $\gamma-$invariant. The hypothesis of $C^\infty-$smoothness to boundary then implies that f is rational ([Fo2]), which contradicts the theorems.■

A mapping from S^{2n-1} that is invariant under such a finite fixed-point-free unitary group is simply a mapping from the spherical space form that is obtained as the topological quotient of the (unit) sphere modulo the action of that group. Among the spherical space forms are the lens spaces $L(p,q)$ obtained as the quotients of S^3 modulo the cyclic groups with generators $\begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^q \end{pmatrix}$. We obtain

1.2.4 Corollary: There are no smooth nonconstant CR mappings from lens spaces $L(p,q)$ to spheres when p and q satisfy the conditions of theorem 1.2.2. There are likewise no smooth nonconstant mappings from spherical space forms S^{2n-1}/Γ to spheres when Γ is not cyclic.

The classification of lens spaces is well known; see for example [M, section 40]. In particular, there is a homeomorphism between $L(p,q)$ and $L(r,s)$ if and only if $p = r$ and $q = \pm s^{\pm 1}$ modulo p. Theorem *1.2.2* may be viewed via the corollaries as a result about maps from lens spaces; thus one is tempted to ask what role, if any, is played by homeomorphic lens spaces in finding further existence or nonexistence results concerning invariant proper maps between complex balls. We look into this in chapter four.

1.3 Remarks

We note that the nonexistence result of Forstneric concerning groups generated by $\begin{pmatrix} e^{i\pi/2k} & 0 \\ 0 & -e^{i\pi/2k} \end{pmatrix}$ also follows from theorem *1.2.2*, by taking $p = 4k$ and $q = 2k+1$. Forstneric then used this result to rule out rational maps invariant under the action of unitary groups of type B; for completeness, we outline his argument below (see [Fo1]).

The type B unitary groups have Sylow 2–subgroups isomorphic to a generalized quaternionic subgroup. These contain cyclic subgroups generated by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which, after change of basis, is just $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Eliminating groups with $2k$x$2k$ generator γ of the form *1.1.1* is similarly accomplished. This matrix has eigenvalues of $\delta^{1/2k}, \delta^{1/2k}\eta, \ldots, \delta^{1/2k}\eta^{2k-1}$ where η is a primitive $2k^{\underline{th}}$ root of one. Now $\eta^k = -1$ so on some 2–dimensional subspace, with respect to an appropriate basis, γ acts as $\begin{pmatrix} \delta^{1/n} & 0 \\ 0 & -\delta^{1/n} \end{pmatrix}$, and these likewise fall into the forbidden category.

Together, theorem *1.2.1* and the results of Forstneric show that there are no proper rational maps from B_n to B_N that are invariant under the action of non-diagonal finite fixed-point-free unitary groups. Specifically, in order to achieve such an invariant map the matrix group Γ must necessarily be cyclic with an $n \times n$ generator of the form

$$\gamma = \begin{pmatrix} \epsilon_p^{q_1} & \cdot & \cdot & \cdot & 0 \\ \cdot & \epsilon_p^{q_2} & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \epsilon_p^{q_n} \end{pmatrix}$$

(with respect to some unitary coordinate system on $\mathbf{C}^n$) where $q_1, q_2, \ldots, q_n$ are all relatively prime to p. We get an additional restriction from theorem *1.2.2*. For any pair i and j we let m be such that $mq_i = kp+1$ for some integer k (m is the representative of q_i^{-1} in the ring $\mathbf{Z}_p$).

We then let q be the remainder (representative) of mq_j modulo p. Then the pair (p, q) must not satisfy the hypotheses of theorem *1.2.2*.

Invariant proper rational (actually monomial) maps between balls are actually known for only (essentially) two subclasses of group generator of the above type. If $q_1 = q_2 = ... = q_n = 1$ then we have homogeneous monomial maps of degree p, with the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$ given by the multinomial coefficient $\sqrt{\binom{p}{\alpha}}$. Up to multiplication by a unitary matrix, these are the only invariant such maps of degree p. Note that the monomials used comprise a minimal generating set for the algebra of invariant polynomials for that group. We will call such a set a basis in this thesis. Following Rudin [Ru2] we will refer to such maps as basic polynomial maps associated to the group Γ, or more simply as $\Gamma-$basic maps. See [Ru1] or [D1] for proofs and further details concerning these maps. The second class of cyclic unitary groups for which such maps exist was conjectured by Forstneric [Fo1] and actually constructed by D'Angelo [D1]. The first few examples were known previously (see [CS2], for example). Each such group Γ is generated by a matrix of the form $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$. The corresponding map uses only basis monomials for the algebra of polynomials invariant under Γ. (That is, the map is $\Gamma-$basic). As shown in [D1], the squares of the coefficients appear in an interesting asymmetric analogue of the Pascal triangle. That is, one obtains squares of coefficients for a given map from squares of coefficients of lower degree maps. D'Angelo [D1] also gives an explicit formula for computing these coefficients. We give another algorithm (as distinct from a closed formula) for generating them in chapter three. For the first class of groups above, when $n = 2$ we get the squared coefficients from the Pascal triangle itself.

1.3.1 Example: For $p = 3$ we have the map

$$(z, w) \longmapsto \left(z^3, \sqrt{3}zw, w^3\right),$$

which is invariant under the action of the group generated by $\begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{pmatrix}$. For $p > 3$ the basis monomials are no longer symmetric.

1.3.2 Example: When $p \geq 9$ the coefficients of the basic invariant map are no longer symmetric. For the case $p = 9$ we have the map

$$(z, w) \longmapsto \left(z^9, 3z^7w, \sqrt{27}z^5w^2, \sqrt{30}z^3w^3, 3zw^4, w^9\right),$$

which is invariant under the action of the cyclic unitary group with generator $\begin{pmatrix} e^{2\pi i/9} & 0 \\ 0 & e^{4\pi i/9} \end{pmatrix}$.

In higher dimensions, if p is odd and each q_j is either 1 or 2 then the above two classes give rise to a $\Gamma-$basic map for this situation as well. In fact, since the map $z \to z^s$ properly takes the one-ball to itself, by simply replacing the condition $|z|^2 = 1$ with $||z'||^2 = |z_1|^2 + ... + |z_k|^2 = 1$ we get a homogeneous map of degree s from B_k to some higher dimensional ball. More generally, take a continuous map from B_2 invariant under the action of $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$. In the condition $|z|^2 + |w|^2 = 1$ we replace $|z|^2$ by $||z'||^2 = |z_1|^2 + ... + |z_k|^2$ and $|w|^2$ by $||w'||^2 = |w_1|^2 + ... + |w_l|^2$. We obtain a map from B_{k+l} invariant under the action of a matrix group of the sort described above, that satisfies the same condition on the sphere as before. For instance, we have

1.3.3 Example: To obtain a proper monomial map from B_4 to some ball that is invariant under the group Γ generated by

$$\gamma = \begin{pmatrix} \epsilon_5 & & & \\ & \epsilon_5 & & \\ & & \epsilon_5^2 & \\ & & & \epsilon_5^2 \end{pmatrix},$$

we modify the D'Angelo map g invariant under the action of $\begin{pmatrix} \epsilon_5 & 0 \\ 0 & \epsilon_5^2 \end{pmatrix}$. This map is given by

$$g : (z, w) \longmapsto \left(z^5, \sqrt{5}z^3w, \sqrt{5}zw^2, w^5\right).$$

Denote the variables in B_4 by $z_j, 1 \leq j \leq 4$, and let $x_j = |z_j|^2$.

A simple result of D'Angelo (proposition 1 of [D1]) states that if f is a proper monomial map between balls that takes zero to zero then the real polynomial $p = ||f||^2$ has positive coefficients and is identically 1 on the hyperplane $x_1 + ... + x_n = 1$. Specifically, if the proper monomial map has components $c_\alpha z^\alpha$ ($\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index) then we take $p(x) = \sum |c_\alpha|^2 x^\alpha$. This correspondence is unique up to what D'Angelo [D1] calls essential equivalence of monomial maps: reordering of terms and multiplication by diagonal unitaries. Conversely, from such a real polynomial one can construct a proper monomial map between balls in the obvious way. We will return to this correspondence in the next chapter.

The condition that the map g above is a proper map between balls amounts to saying that

$$|z|^{10}+5|z|^6|w|^2+5|z|^2|w|^4+|w|^{10}=1$$

when $|z|^2+|w|^2=1$. When we replace $|z|^2$ by $x_1+x_2=|z_1|^2+|z_2|^2$ and $|w|^2$ by $x_3+x_4=|z_3|^2+|z_4|^2$ we obtain the real polynomial equation

$$(x_1+x_2)^5+5(x_1+x_2)^3(x_3+x_4)+5(x_1+x_2)(x_3+x_4)^2+(x_3+x_4)^5=1$$

on $x_1+x_2+x_3+x_4=1$. Expanding this into an explicit sum of monomials then gives the recipe for our proper monomial map from B_4. We take as coefficient for the monomial z^α the square root of the coefficient of x^α in the polynomial above. This simple but tedious procedure yields a $\Gamma-$basic monomial map from B_4 to B_{26}. It is

$$\begin{aligned}(z_1,z_2,z_3,z_4)\longmapsto(&z_1^5,\sqrt{5}z_1^4z_2,\sqrt{10}z_1^3z_2^2,\sqrt{10}z_1^2z_2^3,\sqrt{5}z_1z_2^4,z_2^5,\sqrt{5}z_1^3z_3,\sqrt{15}z_1^2z_2z_3,\\&\sqrt{15}z_1z_2^2z_3,\sqrt{5}z_2^3z_3,\sqrt{5}z_1^3z_4,\sqrt{15}z_1^2z_2z_4,\sqrt{15}z_1z_2^2z_4,\sqrt{5}z_2^3z_4,\\&\sqrt{5}z_2z_3^2,\sqrt{10}z_2z_3z_4,\sqrt{5}z_2z_4^2,\sqrt{5}z_1z_3^2,\sqrt{10}z_1z_3z_4,\sqrt{5}z_1z_4^2,z_3^5,\\&\sqrt{5}z_3^4z_4,\sqrt{10}z_3^3z_4^2,\sqrt{10}z_3^2z_4^3,\sqrt{5}z_3z_4^4,z_4^5).\end{aligned}$$

From any of the above invariant maps one can construct other maps invariant under the action of the same groups by the operation of tensoring on subspaces, its inverse, and linear operations. The appropriate definitions and some remarks on this technique appear in the next chapter. We will at that time have further use for the corresponding real valued polynomials mentioned above.

A less trivial cousin, the only other known example of a group for which such invariant maps exist, was given by Chiappari in [C]. It is a monomial map invariant under the action of the group Γ generated by the matrix

$$(1.3.4) \qquad \begin{pmatrix} \epsilon_7 & 0 & 0 \\ 0 & \epsilon_7^2 & 0 \\ 0 & 0 & \epsilon_7^4 \end{pmatrix}.$$

This map has 17 components and is not Γ−basic. Note that for any pair of diagonal elements, one is the square of the other. We show a way to generate this map, as well as the D'Angelo family of invariant maps from B_2, in chapter three.

It has been suggested by D'Angelo [D3, ch.5] that the previous groups are the only ones for which there are invariant proper rational maps between balls. The proof seems elusive.

As noted above, for every group Γ (except that generated by *1.3.4*) for which an invariant proper rational map between balls is known to exist, there is in fact a Γ−basic monomial invariant map. It is not hard to show that there can be no such basic maps for any of the matrix groups not yet ruled out. We will prove this in the next chapter. Thus we have in some sense a measure of the complexity of these invariant maps, or at least of the difficulty in finding them. With those groups for which the existence of such maps is unresolved, it is clear that they will be harder to find. On the heuristic that counterexamples are generally not too hard to construct or else nonexistent, this supports D'Angelo's suggestion.

1.4 Proofs of theorems

We require a classical result concerning separate invariance of numerator and denominator for invariant rational maps. Forstneric uses it in [Fo1] and cites [S, p. 73]. For completeness, we give our own elementary proof below.

1.4.1 Lemma: Suppose $p = (p_1, ..., p_N)$ is a (vector valued) polynomial on some complex Euclidean space (i.e. $p : \mathbf{C}^n \mapsto \mathbf{C}^N$), and q is a complex valued polynomial, on the same domain, that has no factor in common with every p_j. Suppose that $q(0) \neq 0$ and that $\frac{p}{q}$ is invariant under the action of some matrix γ. Then q and each p_j are separately invariant under that action.

Remark: the condition that $q(0) \neq 0$ is necessary. To see this, take $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $p = z_1, \quad q = z_2$.

To prove this lemma we first require a special case.

1.4.2 Lemma: Suppose p and q are polynomials that have no common factor. Suppose that $q(0) \neq 0$ *and that* $\frac{p}{q}$ *is invariant under the action of some matrix* γ. *Then p and q are separately invariant under that action.*

Proof of 1.4.2: By hypothesis, $\frac{p\circ\gamma}{q\circ\gamma} = \frac{p}{q}$. We must show that numerators and denominators are respectively equal. It clearly suffices to show this for just the denominator.

We have $p \cdot \frac{q\circ\gamma}{q} = p \circ \gamma$. Since p and q have no common factor, we must have $q \mid q \circ \gamma$ (in the polynomial ring $\mathbf{C}[z_1, \ldots, z_n]$). Now the action of γ is linear and so $q \circ \gamma$ has the same degree as q. Consequently $q \circ \gamma = cq$ for some constant $c \in C$. Denote by q_l the part of q that is of degree l. Since the action of γ on the constant term is trivial,

$$q \circ \gamma(z) = q(0) + q_1(\gamma(z)) + \ldots = cq(z) = c(q(0) + \ldots).$$

This proves that $c = 1$ and thus proves the lemma.■

Proof of 1.4.1: We may first write $\frac{p}{q} = \left(\frac{p_1}{q_1}, \ldots, \frac{p_N}{q_N}\right)$ where each rational function is in lowest terms in the quotient field of $\mathbf{C}[z]$. Now $q_j \mid q$ and hence $q_j(0) \neq 0$ for each j. By *1.4.1* each p_j and q_j is thus separately invariant under the action of γ.

The denominator q is the least common multiple (lcm) of $\{q_1, \ldots, q_N\}$. This may be iterated as $q = \mathrm{lcm}(\ldots \mathrm{lcm}(\mathrm{lcm}(q_1, q_2), q_3) \ldots, q_N)$. We claim that $\mathrm{lcm}(q_1, q_2)$ is invariant. To prove this, suppose that the greatest common divisor of q_1 and q_2 is r. (We are, as always, in the polynomial ring.) Then we have $q_1 = rs$ and $q_2 = rt$ where s and t have no common factors. As above, we see that $r(0)$, $s(0)$, $t(0) \neq 0$. Since q_1 and q_2 are invariant under the action of γ so is the quotient $\frac{q_1}{q_2} = \frac{rs}{rt} = \frac{s}{t}$. As s and t have no common factors and $t(0) \neq 0$ we may again apply the previous lemma to conclude that s and t are separately invariant. Hence so is $r = \frac{q_1}{s}$ and thus $rst = \mathrm{lcm}(q_1, q_2)$ is invariant, as claimed.

By induction on N we see that q is invariant. Because the quotient $\frac{p}{q}$ was assumed to be invariant, (each component of) the numerator p is thus separately invariant as well.■

Proof of theorem 1.2.2: By composing f with an appropriate automorphism of the range B_N we may assume that $f(0) = 0$. Now f is rational, so we can write it as $\frac{1}{h}(g_1, ..., g_N)$ where h and each g_j are polynomials, and there is no factor common to all of them. Furthermore we have $h(0) \neq 0$ because otherwise one sees that h has a factor in common with every g_j. (This follows as an easy exercise from unique factorization in $\mathbb{C}[z_1, \ldots, z_n]$ and the polynomial version of the Nullstellensatz, for example. The latter can be avoided at the expense of using some basic manifold and analytic variety theory. The result is more generally true of meromorphic functions that are holomorphic at the origin, also proven by invoking suitable versions of those theorems.) Thus by lemma *1.4.1* each of these polynomials is separately invariant under the action of Γ. Furthermore, as the matrix group is diagonal, invariance of these polynomials implies invariance of each monomial term.

Since f is both proper and continuous to the boundary, we see that $||f||^2 = 1$ on $||z||^2 = 1$, where $z \in C^2$. Thus we have

$$\text{(1.4.3)} \qquad \sum_{j=1}^{N} ||g_j||^2 = |h|^2 \text{ on } ||z||^2 = 1.$$

A simple but powerful technique [D2] is to expand this identity and equate Fourier coefficients. To this end, using multi-index notation we write

$$g = (g_1, ..., g_N) = \sum_{\alpha \in \mathbf{N}^2} c_\alpha z^\alpha$$

where each c_α is a vector, and also write $h = \sum_\alpha k_\alpha z^\alpha$. Now let $z_j = r_j e^{i\theta_j}$. for $j = 1, 2$ (the usual polar notation). Then *1.4.3* becomes

$$\sum_{\alpha,\beta} \langle c_\alpha, c_\beta \rangle r_1^{\alpha_1+\beta_1} r_2^{\alpha_2+\beta_2} e^{i(\alpha_1-\beta_1)\theta_1} e^{i(\alpha_2-\beta_2)\theta_2}$$
$$= \sum_{\mu,\eta} k_\mu \overline{k}_\eta r_1^{\mu_1+\eta_1} r_2^{\mu_2+\eta_2} e^{i(\mu_1-\eta_1)\theta_1} e^{i(\mu_2-\eta_2)\theta_2}$$

on $r_1^2 = 1 - r_2^2$. (This is the "expand" part.) Let d be the maximum degree of these polynomials, and let δ be the degree of h. Then Forstneric [Fo1], D'Angelo [D3, ch.5], and others have shown that $\delta < d$. This in fact follows at once from the above identity, by independence of the different powers of $e^{i\theta_1}$ and of $e^{i\theta_2}$. Next replace r_1 with r for ease of notation, and equate powers of $e^{i\theta_j}$ on each side of the equation. We do so with the θ–constant terms (that is, those independent of $\theta = (\theta_1, \theta_2)$) to obtain

$$\sum_{|\alpha|\leq d} ||c_\alpha||^2 r^{2\alpha_1} (1-r^2)^{\alpha_2} = \sum_{|\eta|\leq\delta} |k_\eta|^2 r^{2\eta_1} (1-r^2)^{\eta_2}.$$

The high degree on the left side is $2d$ while that on the right is 2δ. Hence the high degree term on the left must vanish identically, so we have

$$\text{(1.4.4)} \qquad \sum_{|\alpha|=d} ||c_\alpha||^2 (-1)^{\alpha_2} = 0.$$

We have $p \mid \alpha_1 + q\alpha_2$ for each term that appears in the sum. Since $\alpha_1 + \alpha_2 = d$ we get $p \mid \alpha_1 + \alpha_2 + (q-1)\alpha_2 = d + (q-1)\alpha_2$. Thus $(q-1)\alpha_2 = t_\alpha p - d$ for some integer t_α. Let $m = \nu_2(q-1)$. The hypotheses then imply that α_2 is even if and only if 2^{m+1} divides d. As this condition is independent of α, each term in the left side of *1.4.4* has the same sign (every α_2 has the same parity). Thus that sum must be strictly positive or negative, hence not zero. This finishes the proof of theorem *1.2.2*.■

The proof of theorem *1.2.1* will proceed in several steps. We first investigate holomorphic maps from B^n that are invariant under the action of such a matrix γ (as given in *1.1.1* above).

This action is given by

$$\gamma\begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \\ \delta z_1 \end{pmatrix}.$$

Suppose that

$$p(z_1, z_2, ..., z_n) = \sum_{(\alpha_1,...,\alpha_n)} p_{(\alpha_1,...,\alpha_n)} z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$$

is a holomorphic map invariant under the action of γ. (Here p_α is a vector in $\mathbf{C}^N$.) By invariance we have $p(z_2, ..., z_n, \delta z_1) = p(z_1, ..., z_n)$. This becomes

$$\sum_{(\alpha_1,...,\alpha_n)} p_{(\alpha_1,...,\alpha_n)} z_2^{\alpha_1} z_3^{\alpha_2} ... z_n^{\alpha_{n-1}} (\delta z_1)^{\alpha_n} = \sum_{(\alpha_1,...,\alpha_n)} p_{(\alpha_1,...,\alpha_n)} z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}.$$

Equating monomials we see that

$$p_{(\alpha_n,\alpha_1...,\alpha_{n-1})} = \delta^{\alpha_n} p_{(\alpha_1,...,\alpha_n)}.$$

Similarly, we obtain

$$p_{(\alpha_{n-1},\alpha_n,\alpha_1...,\alpha_{n-2})} = \delta^{\alpha_{n-1}+\alpha_n} p_{(\alpha_1,...,\alpha_n)}, \tag{1.4.5}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$p_{(\alpha_2,\alpha_3,...,\alpha_n,\alpha_1)} = \delta^{\alpha_2+...+\alpha_n} p_{(\alpha_1,...,\alpha_n)},$$

$$p_{(\alpha_1,...,\alpha_n)} = \delta^{\alpha_1+...+\alpha_n} p_{(\alpha_1,...,\alpha_n)}.$$

1.4.6 Lemma: $m \mid \alpha_1 + \alpha_2 + ... + \alpha_n$ (unless $p_\alpha = 0$).

Proof: This follows from the last equation in *1.4.5*, which implies that $\delta^{\alpha_1+...+\alpha_n} = 1$.■

1.4.7 Lemma: Suppose $f = p/q$ is a proper rational map from B_2 to B_N that takes zero to zero. Then p has a term of the form $p_{(\alpha,0)} z_1^\alpha$ where the coefficient is a nonzero vector in $\mathbf{C}^N$.

Moreover, if we take the largest α such that the vector coefficient is nonzero, then $p_{(\alpha,0)}$ must be orthogonal to $p_{(0,\alpha)}$. (This latter vector could of course be zero.)

Proof: The results in [CS1] and [C] show that f is holomorphic in a neighborhood of the closed ball; that is, when expressed in lowest terms $f = p/q$, we have $q \neq 0$ on the boundary. We claim that there is a term in p of the form $p_{(\alpha,0)} z_1^{\alpha}$. If not then we have

$$f(1,0) = \frac{p(1,0)}{q(1,0)} = 0.$$

This would contradict the fact that our map takes boundary into boundary.

Note that since f extends to be holomorphic in a neighborhood of the ball, it is necessarily continuous to the boundary. That it is proper and holomorphic is then equivalent to the identity $||f||^2 = 1$ on the sphere $||z||^2 = 1$. This in turn becomes $||p||^2 - |q|^2 = 0$ on the unit sphere. We have

$$\sum_{\mu,\nu} (\langle p_\mu, p_\nu \rangle - q_\mu \overline{q}_\nu) z^\mu \overline{z}^\nu = 0$$

on $||z||^2 = 1$. Again we "expand and equate." We write $z_j = r_j e^{i\theta_j}$ for $j = 1,2$ to obtain the identity

$$\sum_{\mu,\nu} (\langle p_\mu, p_\nu \rangle - q_\mu \overline{q}_\nu) r_1^{\mu_1+\nu_1} r_2^{\mu_2+\nu_2} e^{i\theta_1(\mu_1-\nu_1)} e^{i\theta_2(\mu_2-\nu_2)} = 0 \tag{1.4.8}$$

on $r_1^2 + r_2^2 = 1$. As the different powers of $e^{i\theta}$ are independent on the sphere, we can equate Fourier coefficients in *1.4.8* above to see that

$$\sum_{\mu-\nu=\text{constant}\neq(0,0)} (\langle p_\mu, p_\nu \rangle - q_\mu \overline{q}_\nu) r_1^{\mu_1+\nu_1} r_2^{\mu_2+\nu_2} = 0$$

on $r_1^2 + r_2^2 = 1$.

Now fix $\mu_1 - \nu_1 = \alpha$ and $\mu_2 - \nu_2 = -\alpha$. Then we have

$$\sum_{\nu} \left(\langle p_{(\nu_1+\alpha,\nu_2-\alpha)}, p_{(\nu_1,\nu_2)} \rangle - q_{(\nu_1+\alpha,\nu_2-\alpha)} \overline{q}_{(\nu_1,\nu_2)} \right) r_1^{2\nu_1+\alpha} r_2^{2\nu_2-\alpha} = 0$$

on $r_1^2+r_2^2=1$, where the sum is over all $\nu=(\nu_1,\nu_2)$ for which $\nu_2 \geq \alpha$ (recall that all subscripts correspond to monomial powers and hence must be nonnegative). If α is even, leave this equation alone; otherwise multiply both sides by $r_1 r_2$. We obtain (for $k=\alpha/2$ or $k=(\alpha+1)/2$ as the case may be):

$$\sum_{\nu}\Big(\big\langle p_{(\nu_1+\alpha,\nu_2-\alpha)}, p_{(\nu_1,\nu_2)}\big\rangle - q_{(\nu_1+\alpha,\nu_2-\alpha)}\overline{q}_{(\nu_1,\nu_2)}\Big) r_1^{2\nu_1+2k} r_2^{2\nu_2-2k} = 0$$

on $r_1^2+r_2^2=1$. Replacing r_2^2 by t and r_1^2 by $1-t$ yields the identity

$$\sum_{\nu}\Big(\big\langle p_{(\nu_1+\alpha,\nu_2-\alpha)}, p_{(\nu_1,\nu_2)}\big\rangle - q_{(\nu_1+\alpha,\nu_2-\alpha)}\overline{q}_{(\nu_1,\nu_2)}\Big)(1-t)^{\nu_1+k} t^{\nu_2-k} \equiv 0.$$

The lowest degree terms in t occur when $\nu_2=\alpha$. These must vanish identically, hence

$$(1.4.9) \qquad \sum_{\nu_2=\alpha}\Big(\big\langle p_{(\nu_1+\alpha,0)}, p_{(\nu_1,\alpha)}\big\rangle - q_{(\nu_1+\alpha,0)}\overline{q}_{(\nu_1,\alpha)}\Big) = 0.$$

By the hypothesis on α the only nonzero vector of the form $p_{(\nu_1+\alpha,0)}$ is $p_{(\alpha,0)}$. Furthermore, by restriction of our map to the set $\{z_2=0\}$ we have a proper rational map between balls that takes zero to zero; hence the degree of the denominator is strictly less than that of the numerator. Thus all coefficients of the form $q_{(\nu_1+\alpha,0)}$ are zero. The only surviving term in equation *1.4.9* is thus $\langle p_{(\alpha,0)}, p_{(0,\alpha)}\rangle = 0.$ ■

With this technical lemma we now prove

Theorem 1.2.1: There is no proper rational map $f: B_n \to B_N$ invariant under the action of any matrix of the form γ.

Proof: Assume $f=p/q$ is such a map. By composing it with an automorphism of the target ball we may assume that it preserves the origin. We write $p=\sum_{\alpha} p_\alpha z^\alpha$ in the usual multi-index notation. By restricting to the set $z_3=\ldots=z_n=0$ we have a proper rational map from B_2 to B_N. Lemma *1.4.7* then implies that p has (nonzero) terms of the form $p_{(\alpha_1,0,\ldots,0)}z_1^{\alpha_1}$ and $p_{(0,\alpha_2,\ldots,0)}z_2^{\alpha_2}$. By lemma *1.4.1*, invariance of f implies separate invariance of numerator and denominator. Lemma *1.4.6* and the equations that precede it then show that terms of the above form come in pairs with exponents and vector coefficients equal. (That is, for any nonzero vector coefficient $p_{(\alpha_1,0,\ldots,0)}$ there is an equal coefficient $p_{(0,\alpha_1,\ldots,0)}$). For some pair this exponent must be maximal, as p is a polynomial. But lemma *1.4.7* states that the coefficients for that pair must be orthogonal, a contradiction to the fact that they are equal and nonzero. This proves theorem *1.2.1.*■

1.4.10 Corollary to theorem 1.2.1: There is no proper rational map from B_3 to B_N that is invariant under the action of

$$\gamma = \begin{pmatrix} \epsilon_9 & 0 & 0 \\ 0 & \epsilon_9^4 & 0 \\ 0 & 0 & \epsilon_9^7 \end{pmatrix}.$$

Proof: Take $\delta = e^{2\pi i/3}$. The matrix

$$\gamma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{pmatrix}$$

is unitarily similar to γ. That is, they have the same characteristic polynomial and thus represent the same transformation relative to different bases. Thus by a unitary change of variables, a proper rational function invariant under the action of γ becomes one invariant under the action of γ_1, contradicting the theorem.■

This result has a combinatorial interpretation given in chapter three. We can rule out some other diagonal matrix groups in this manner. Specifically, one can show that exactly those diagonal groups generated by matrices of the form

$$\gamma = \begin{pmatrix} \epsilon_9 & 0 & 0 \\ 0 & e^{2\pi i/3}\epsilon_9 & 0 \\ 0 & 0 & e^{4\pi i/3}\epsilon_9 \end{pmatrix}$$

are eliminated by this method. This generalizes to higher dimensions in a straightforward way. In the even dimensional cases we get nothing that was not already eliminated by Forstneric's results.

Chapter 2 Invariant polynomial maps: algebraic results

2.1 Preliminary material

As in the previous chapter, we may associate to a proper monomial map f between balls that takes zero to zero a real valued polynomial with nonnegative coefficients. This polynomial is $p = ||f||^2$, and it will play an important role in this and the next chapter. In more detail, if the components of f are the monomials $c_{\alpha_1,\ldots,\alpha_n} z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ then $p(x_1, \ldots, x_n) = \sum_{\alpha} |c_\alpha|^2 x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $x_j = |z_j|^2$. Note that if f is invariant under the action of some diagonal unitary matrix group, then all monomials that occur in p are squared absolute values of invariant complex monomials. Since f is a proper map between balls, we see that $p = 1$ on the hyperplane $x_1 + \ldots + x_n = 1$. Equivalently, $x_1 + \ldots + x_n - 1 \mid p(x_1, \ldots, x_n) - 1$ where the division is performed in the polynomial ring $\mathbf{R}[x_1, \ldots, x_n]$. We will henceforth refer to p as the canonical (real) polynomial associated to the map f. When our domain space is two dimensional, we will use z, w for our complex variables, and x, y for real variables, for ease of notation.

An operation of fundamental importance in D'Angelo's factorization results is that of tensoring one proper map between balls (or a part thereof) by another. Specifically, if $h = (h_1, \ldots, h_r)$ is one proper holomorphic map between balls, and $k = (k_1, \ldots, k_s)$ is another, then the map

$$h \otimes k \equiv (h_1 k_1, h_1 k_2, \ldots, h_1 k_s, h_2 k_1, \ldots, h_r k_s)$$

is easily shown to be a proper holomorphic map between balls as well (see [D3, ch.5]). We remark that the particular order in which we list the components is irrelevant; it merely amounts to a choice of basis for the range space. It is natural, in this setting, to consider maps that differ by a unitary matrix to be equivalent. Note also that the dimension of the tensor product map is the product of the dimensions of the component maps. This accounts for a crucial difference between maps to one versus multiple dimensions; tensoring pairs of the former amounts to simple multiplication and does not increase the dimension of the range.

Suppose now that A is a linear subspace of $\mathbf{C}^r$. Denote by $h_A, h_{A^\perp}$ those components of h that map into $A, A^\perp$ respectively. That is, h_A is the projection of h onto the subspace A. We write $h = h_A \oplus h_{A^\perp}$ and define $E_{(A,k)}h \equiv (h_A \otimes k) \oplus h_{A^\perp}$. ($E$ is for "extend"). This "tensor product on a subspace" is similarly seen to be a proper holomorphic map between balls. For notational convenience we generally just write E and suppress the rest. If we have a map in the form $f = (h_A \otimes k) \oplus h_{A^\perp}$ we similarly write (again suppressing some information) $h = E^{-1}f$. (This we naturally call "untensoring".)

Suppose p is a proper polynomial map between balls that takes zero to zero, and the degree of p is n. Then D'Angelo in [D2] shows that p can be obtained as $(E^{-1})^m LE^n(Id)$ where L is linear. That is, we start with the identity map, tensor it with itself n times (taking the full tensor product each time) to obtain a homogeneous map, perform some linear operation (that is unitary on the range of the map constructed thus far), and then perform some untensoring operations (these last might be done on proper subspaces). The proof proceeds as follows (see [D2] for full details). If we write $p = p_1 + ... + p_n$ where each p_j is homogeneous of degree j then it is not hard to show that the range of p_n is orthogonal to that of p_1. More generally, for nonhomogeneous polynomial maps, the range of the highest degree terms is orthogonal to that of the lowest degree nonvanishing terms. (Just set $\left\langle \sum_i p_i, \sum_j p_j \right\rangle = 1$ on $||z||^2 = 1$, let $z = re^{i\theta}$, and look at the highest powers of $e^{i\theta}$ in the resulting expression.) Letting A denote the range of p_1, we form $E_{(A,Id)}p$ and still have a proper polynomial map between balls of degree n. Moreover, now we have (the new) $p_1 = 0$ as well, and the range of p_n is thus orthogonal to that of (the new) p_2. We continue this procedure to produce a homogeneous map of degree n. Since this map is just the full tensor product of the identity with itself n times (up to multiplication by a unitary), we have our result.

Along these lines we require another basic fact.

2.1.1 Proposition (D'Angelo): Suppose f is proper polynomial map between balls that takes zero to zero. Then there is a linear map L and a proper monomial map g between balls such that $f = Lg$. Furthermore, the linear map has unit column vectors, and the coefficients in the monomial map are all positive reals.

The monomials used in g are precisely those that appear in the polynomial map. In fact, if our polynomial map is given (in multi-index notation) as $f = \sum c_\alpha z^\alpha$ where the coefficients are vector valued, then the monomial map g, which we call the monomialization of f, has components given by $g_\alpha = ||c_\alpha||\ z^\alpha$.

2.2 Results concerning polynomial invariant maps

There is a result analogous to D'Angelo's factorization, for polynomial maps between balls that take zero to zero and are invariant under the action of some unitary group. We state and prove the relevant theorem in this section.

In the D'Angelo factorization, the allowable operations are linear operations that monomialize or split or collect like monomials (these preserve length from domain to its image) and tensoring and untensoring operations on linear subspaces, by the identity map. With group-invariant maps for given unitary group Γ, we will instead do tensoring and untensoring by the $\Gamma-$basic invariant monomial map. For polynomial maps invariant under the action of a matrix generated by ϵI we will give a simple proof that essentially mimics a more general factorization of proper polynomial maps between balls given by D'Angelo in [D1]. For maps invariant under the action of a matrix group generated by some $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$ the proof we give will use different techniques.

2.2.1 Theorem: (i) Let f be a proper polynomial map that takes zero to zero and is invariant under the action of the group Γ generated by $\epsilon_k I$. Then f can be obtained from the $\Gamma-$basic *monomial map g described in section 1.3 by the type of linear operations described above and the operations of tensoring and untensoring on subspaces by that map g. That is, we can write $f = \left(E^{-1}_{(-,g)}\right)^m LE^n_{(-,g)}(g)$.*

(ii) Let f be as above but now invariant under the action of a group Γ generated by $\begin{pmatrix} \epsilon_k & 0 \\ 0 & \epsilon_k^2 \end{pmatrix}$. Then f can likewise be obtained from the $\Gamma-$basic monomial map g described in section 1.3 by linear operations as above and the operations of tensoring and untensoring on subspaces by that map g. In other words, once again we can write $f = \left(E^{-1}_{(-,g)}\right)^m LE^n_{(-,g)}(g)$.

A conjecture of D'Angelo's is that all rational maps can be obtained by a factorization similar to that for polynomial maps, if one also permits tensor products and inverses thereof with automorphisms that move the origin [D3, ch.5]. If this conjecture is verified, one expects there to be an analogous factorization result for rational group-invariant maps.

Proof of (i): By *2.1 1* we may write $f = L \circ g$ where L is such a linear operation and g is a proper monomial map between balls; it too is invariant since the matrix group is diagonal. It suffices now to obtain g as in the theorem. To do so we will "homogenize". Let d be the degree of g. Then each monomial in g is of the form $c_\alpha z_1^{\alpha_1}...z_n^{\alpha_n}$ where $k \mid \sum_{j=1}^{n} \alpha_j$ and hence $d = kr$. Since the $\Gamma-$basic monomial map b is of degree k, we can iteratively tensor every entry of g of degree less than d by the basic map b until we obtain a homogeneous map of degree d to a higher dimensional ball. It is still invariant and proper. After performing a linear operation that collects monomials (which is of the above type), we obtain a map that is unique up to multiplication by a unitary ([R1] or [D1]). This map is therefore $U \circ h$, where h is the degree d homogeneous map that is invariant under the action of $\epsilon^{1/r} I$. As this map is obtained by tensoring the $\Gamma-$basic monomial map with itself, we have $g = \left(E^{-1}\right)^s \circ L \circ (E)^t(b)$ as desired.■

It should be noted that a proof can also be modelled on the factorization result in [D2] as outlined in the previous section. The one in [D1] was used here because a similar approach will be required to prove *2.2.1 (ii)*.

2.2.2 Example: Let Γ be generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. One checks that

$$f(z,w) = \left(\frac{z^2+w^2}{\sqrt{2}}, \frac{z^2-w^2}{\sqrt{2}}, \sqrt{2}z^3w, 2z^2w^2, \sqrt{2}zw^3 \right)$$

is a proper map from B_2 to B_5 that is $\Gamma-$invariant. Then

$$g(z,w) = \left(z^2, w^2, \sqrt{2}z^3w, 2z^2w^2, \sqrt{2}zw^3 \right)$$

where the linear operation is in this instance a rotation by $\pi/4$ in the first two slots and the identity in the last three, hence is unitary. To get a homogeneous map of degree 4 we now tensor the first two slots of g with the $\Gamma-$basic map $(z,w) \longmapsto (z^2, \sqrt{2}zw, w^2)$ to obtain

$$(z,w) \longmapsto \left(z^4, \sqrt{2}z^3w, z^2w^2, z^2w^2, \sqrt{2}zw^3, w^4, \sqrt{2}z^3w, 2z^2w^2, \sqrt{2}zw^3 \right).$$

A linear operation takes this to

$$g^*(z,w) = \left(z^4, 2z^3w, \sqrt{6}z^2w^2, 2zw^3, w^4 \right)$$

which is (up to multiplication by a permutation or diagonal unitary matrix) the unique monomial map with linearly independent terms that is invariant under the action of $\gamma = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. Note that $k = 2$ in this example, and that $\gamma^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. As g^* is obtained by tensoring $(z,w) \longmapsto (z^2, \sqrt{2}zw, w^2)$ with itself and then collecting like terms via linear operation, we have f factored as specified in the theorem.

For polynomial maps that take zero to zero and are invariant under the action of a unitary group Γ generated by

$$\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix} \tag{2.2.3}$$

we first require a weaker result; it is of some interest in its own right.

2.2.4 Proposition: Suppose f is a proper polynomial map between balls that takes zero to zero and is $\gamma-$invariant with γ as in 2.2.3. Suppose g is the $\Gamma-$basic such map. Then $||g||^2 - 1$ divides $||f||^2 - 1$ in the ring $\mathbf{R}[x,y]$.

Proof: Let p, q be the canonical real polynomials associated to f, g respectively. As each is Γ-invariant, so is their quotient. Now suppose that

$$\frac{p-1}{q-1} = \frac{p^*}{q^*}$$

expressed in lowest terms. Each factor of $q - 1$ has nonzero constant term that can (dividing through by that constant) be taken to be 1. By the introductory remarks to this chapter, $x + y - 1$ divides both $p(x, y) - 1$ and $q(x, y) - 1$ as polynomials. Thus q^* is a proper divisor of $q - 1$, and we have $q^* = 1 + \ldots$. If $q^* \neq 1$ then it has terms in only one variable; otherwise the high degree terms in the product that yields $q - 1$ would have xy as a factor. This is not the case since the high degree terms in q are each in only one variable. (Recall that, for ϵ a primitive $k^{\underline{th}}$ root of 1, we have $q = ||g||^2 = x^k + kx^{k-2}y + \ldots + y^k$. The coefficients are the squares of the Γ-basic monomial map coefficients.) Now since q^* has nonzero constant term, it is separately Γ-invariant by lemma *1.4.1*; as Γ is diagonal, each term is then invariant. But the degree of q^* is strictly less than that of q, and there are no invariant terms in one variable that are of smaller degree. Hence $q^* = 1$, as required.■

Note that the quotient polynomial is invariant, for if not, then any noninvariant monomial term of minimal degree would give rise to a noninvariant term in the product $p(x, y) - 1$. We also remark that the result of this proposition also holds for maps that are invariant under the action of a group generated by ϵI; the proof is unchanged.

We now prove theorem *2.2.1 (ii)*. Let $f, g, p,\ q$ be as above. We assume ϵ is a primitive $k^{\underline{th}}$ root of 1, and so $q(x, y) = ||g||^2 = x^k + kx^{k-2}y + \ldots + kxy^{\frac{k-1}{2}} + y^k$. (As we assume Γ to be fixed-point-free, k must be odd.) By *2.2.4* we know $q - 1 \mid p - 1$ in the real polynomial ring. We will denote the quotient by Q_f; this notation is similar to that used in [D4]. Choose n large enough so that the $(n-1)$-fold tensor product of g with itself, which we will call h, contains no monomial that appears in f. (For example, take $n = 1 + \deg\ f$.) Let $r(x, y) = ||h||^2$ be the canonical real polynomial associated with h. By construction, $r = q^n$.

Thus $r-1=(q-1)(1+q+\ldots+q^{n-1})=(q-1)Q_h$. Note that our choice of n is also sufficiently large that every monomial in Q_f also appears in Q_h.

As in *2.1.1* we first apply a linear operation to f to obtain a monomial map (which we will still call f). We now claim that by applying linear operations that split monomials into two components, and then tensoring one of these components with g, and repeating this process finitely many times, we can obtain h. This suffices to prove the theorem, for it tells us that f can be obtained as $L(LE)^{-m}LE^n(g)$ where all tensoring and untensoring is by the map g. We will show that appropriate linear operations and tensoring by g will transform f into $\hat{h}$, where $\left|\left|\hat{h}\right|\right|^2=||h||^2=r$; hence $\hat{h}$ differs from h by a linear operation.

2.2.5 Lemma: Suppose $c_1x^ay^b$ is a monomial term in $p(x,y)$, and $0\le c_2\le c_1$. Then we may perform a linear operation to split the z^aw^b monomial, followed by a tensoring operation of one of the new monomial components by the basic invariant map, in such a way that the resulting monomial map, f^*, satisfies $Q_{f^*}=Q_f+c_2x^ay^b$.

We conclude that tensoring a part of f with g to obtain f^* corresponds to taking part of a term in $p=||f||^2$ and adding it to Q_f to form Q_{f^*}. A similar result is stated in [D4], though instead of working with the basic invariant map g one works with the identity map.

Proof: $f=(\ldots,\sqrt{c_1}z^aw^b,\ldots)$. A linear operation as described above takes this to $(\ldots,\sqrt{c_2}z^aw^b,\sqrt{c_1-c_2}z^aw^b,\ldots)$. We tensor the $\sqrt{c_2}z^aw^b$ entry with g to obtain

$$f^*=\left(\ldots,\sqrt{c_2}z^{a+k}w^b,\sqrt{kc_2}z^{a+k-2}w^{b+1},\ldots,\sqrt{c_2}z^aw^{b+k},\sqrt{c_1-c_2}z^aw^b,\ldots\right).$$

This map has canonical real polynomial

$$\begin{aligned}||f^*||^2&=\ldots+c_2x^ay^b(q(x,y))+(c_1-c_2)x^ay^b+\ldots\\&=\ldots+c_2x^ay^b(q(x,y)-1)+c_1x^ay^b+\ldots\\&=||f||^2+c_1x^ay^b(q-1).\end{aligned}$$

Thus $Q_{f^*}=\frac{||f^*||^2-1}{q-1}=\frac{||f||^2-1}{q-1}+c_1x^ay^b=Q_f+c_1x^ay^b.$■

To prove the claim, we have $Q_h = q^{n-1} + q^{n-2} + \ldots + q + 1 = \sum_{a,b} c_{a,b} x^a y^b$ and $Q_f = \sum_{a,b} d_{a,b} x^a y^b$ where $c_{0,0} = d_{0,0} = 1$. Suppose inductively that for some pair (a,b) we have $c_{a,b} \neq d_{a,b}$ but that for all lower degree terms the respective coefficients agree.

2.2.6 Lemma: As in lemma 2.2.5, we can perform a linear operation followed by a tensoring operation on our map f to obtain a new monomial map f^*, in such a way that $Q_{f^*} = Q_f + (c_{a,b} - d_{a,b}) x^a y^b$.

Hence Q_{f^*} agrees with Q_h in every term that Q_f did, as well as in the $x^a y^b$ term. Note that this lemma suffices to prove the claim, and hence theorem *2.2.1 (ii)*.

Proof:

$$p - 1 = (q-1)Q_f = (q-1)\Big(1 + \ldots + d_{a,b} x^a y^b + \ldots\Big)$$

$$= -1 + \ldots + [c_{a-k,b} + k c_{a-k+2,b-1} + \ldots + k c_{a-1,b-(k-1)/2} + c_{a,k-k} - d_{a,b}] x^a y^b + \ldots$$

Call the bracketed expression t. We adopt the convention that all coefficients with either subscript negative are zero. All terms except the last come from multiplying the squares of coefficients of the $\Gamma-$basic map g with appropriate $c'_{\alpha,\beta}s$ in Q_f. As p is the canonical real polynomial corresponding to the map f, we have $t \geq 0$.

Now f has an entry $\sqrt{t} z^a w^b$. We want to perform a linear operation and tensoring operation to obtain f^*, so that $Q_{f^*} = Q_f + (c_{a,b} - d_{a,b}) x^a y^b$. By the previous lemma this is possible provided $0 \leq c_{a,b} - d_{a,b} \leq t$.

Case (i): The monomial $z^a w^b$ does not appear in h. Then we have

$$c_{a-k,b} + k c_{a-k+2,b-1} + \ldots + c_{a,b-k} - c_{a,b} = 0.$$

Hence $c_{a,b} - d_{a,b} = t \geq 0$ so in this case the desired operations can be performed.

Case (ii): The monomial $z^a w^b$ appears in h. Then since

$$r = ||h||^2 = \ldots + [c_{a-k,b} + k c_{a-k+2,b-1} + \ldots + c_{a,b-k} - c_{a,b}] x^a y^b + \ldots$$

has positive coefficients, we must have $c_{a-k,b} + \ldots + c_{a,b-k} > c_{a,b}$ and hence $c_{a,b} - d_{a,b} < t$.

Now by choice of (large) n, we are in case (ii) only for pairs (a,b) for which $d_{a,b} = 0$. On the other hand, as $Q_h = 1+q+q^2+\ldots+q^{n-1}$, we see that every coefficient $c_{a,b} > 0$. Hence we also have $c_{a,b} - d_{a,b} = c_{a,b} > 0$. Thus the operations may be performed in this case as well. This concludes the proof of lemma *2.2.5*, and hence finishes the claim that proves theorem *2.2.1 (ii)*.■

The invariant map from B_3 to B_{17} found by Chiappari (and derived in chapter three), though not a basic invariant map, nevertheless has minimal possible degree and is therefore seen to satisfy proposition *2.2.4*. Hence it satisfies theorem *2.2.1 (ii)* as well. The proofs are the same as above.

The results above all hypothesize origin-preserving maps. If a proper polynomial map between balls has a nonzero constant term, then we may still perform the monomialization of *2.1.1*. We then we take the square of the norm to form the associated real polynomial, as before. We obtain the equation

$$p(x) = \sum |c_\alpha|^2 x^\alpha + |c_0|^2 = 1$$

on $x_1 + \ldots + x_n = 1$. In an obvious manner we get an origin-preserving group-invariant proper monomial map from B_n to $\left(1 - |c_0|^2\right) B_N$ for some N. We rescale to get a map between unit balls, and all above results will then apply to this map.

2.3 Nonexistence of other basic invariant maps

We now prove a result mentioned in chapter one.

2.3.1 Proposition: Let Γ be generated by $\gamma = \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^q \end{pmatrix}$, with p relatively prime to q. We assume that $2 < q < p$, and $p \neq 2q - 1$ (this last being clearly equivalent to the case $q = 2$.) Then there is no $\Gamma-$basic invariant proper map from B_2 to any B_N.

We remark that this proposition suffices to prove a corresponding result in the case of a higher dimensional domain; we simply restrict our attention to a two dimensional subspace.

Proof: The proof given here may be a bit longer than necessary; it is completely elementary and makes no use of results in the literature concerning bases of the algebra of $\Gamma-$invariant polynomials.

Since Γ is diagonal it is easy to see that we may take our basis to consist of monomials. The invariant monomials are of the form $z^a w^b$ where $p \mid a + qb$. Now z^p and w^p are basis monomials; they are invariant and clearly no lower power in either variable alone is invariant, so they are not products of other invariants.

Suppose we have such a monomial map between balls. If it has elements $c_{a,b}z^a w^b$ then as before we let $r(x,y) = \sum |c_{a,b}|^2 x^a y^b$ and note that $r(x, 1-x) \equiv 1$. Thus

$$1 = |c_{p,0}|^2 x^p + |c_{0,p}|^2 (1-x)^p + \sum_{a,b \neq 0} |c_{a,b}|^2 x^a (1-x)^b.$$

Equating terms of like degree on each side, we immediately see that $|c_{0,p}|^2 = 1$. By reversing the roles of x and y (i.e. noting that $r(1-y, y) \equiv 1$) we obtain $|c_{p,0}|^2 = 1$ as well. We see that if there are no $\Gamma-$basic monomials of degree larger than $p-2$ then we are done because in that case the only term in x^{p-1} is $(-1)^{p-1} p |c_{0,p}|^2$, which forces the contradiction $|c_{0,p}|^2 = 0$. We will show that there are indeed no such basis monomials. To this end, for any $1 \leq k \leq p-1$ we define α_k to be the smallest positive integer such that $z^k w^{\alpha_k}$ is $\Gamma-$invariant. In other words, α_k is the representative of $-q^{-1}k$ in the ring $\mathbf{Z}_p$.

2.3.2 Lemma: (i) $1 \leq \alpha_k \leq p-1$

(ii) If $k \neq l$ *then* $\alpha_k \neq \alpha_l$.

Proof of 2.3.2: Since p and q are relatively prime there exist integers s, t such that $sp + tq = 1$. We take the set of integers $\{k, k+q, k+2q, ..., k+(p-1)q\}$. Upon rewriting this as $\{ksp + ktq, ksp + (kt+1)q, ..., ksp + (kt+p-1)q\}$ it becomes clear that p divides exactly one of these integers, and not the first one, k. Hence there for some unique $1 \leq m \leq p-1$

we have $p \mid k + mq$ and so $\alpha_k = m$. Furthermore, if $\alpha_k = \alpha_l$ then we have $p \mid k + \alpha_k q$ and $p \mid l + \alpha_k q$ and thus $p \mid k - l$. As $1 \leq k, l < p$ we must then have $k = l$.■

2.3.3 Lemma: Under the hypotheses of proposition 2.3.1, *we have* $\alpha_1 + 1 < p - 1$.

Proof of 2.3.3: By lemma *2.3.2* this inequality can only be violated if either $\alpha_1 = p - 1$ or $\alpha_1 = p - 2$. If $\alpha_1 = p - 1$ then $p \mid 1 + (p-1)q$ and so $p \mid q - 1$. This implies $q = 1$ which contradicts our hypotheses. If on the other hand we have $\alpha_1 = p - 2$ then $p \mid 1 + (p-2)q$ and so $p \mid 2q - 1$. Since $q < p$ we have $2q - 1 < 2p$ and so $p = 2q - 1$ which again violates our hypotheses.■

2.3.4 Lemma: For any $1 \leq j \leq p - 1$ *either* $j + \alpha_j < p - 1$ *or else there is some* $k < j$ *such that* $\alpha_k < \alpha_j$.

We note that this lemma suffices to finish the proof of proposition *2.3.1* because if $j + \alpha_j \geq p - 1$ then $z^j w^{\alpha_j} = (z^k w^{\alpha_k})(z^{j-k} w^{\alpha_j - \alpha_k})$ where all exponents are positive. Since $z^j w^{\alpha_j}$ and $z^k w^{\alpha_k}$ are each $\Gamma-$invariant, so is $z^{j-k} w^{\alpha_j - \alpha_k}$. Hence $\alpha_{j-k} = \alpha_j - \alpha_k$ and so $z^j w^{\alpha_j}$ is not a basis monomial. Thus all basis monomials are of degree strictly less than $p-1$, and this suffices.

Proof of lemma 2.3.4: If the lemma is false, then take j to be the smallest violator. Then for every $k < j$ we have $\alpha_k > \alpha_j$. Thus we obtain $p - j - 1 \leq \alpha_j < \alpha_1, ..., \alpha_{j-1} \leq p - 1$. We now have j distinct integers occupying $j + 1$ consecutive slots; the inequalities then force either $\alpha_j = p - j - 1$ or $\alpha_j = p - j$. (We remark that from this point, it is trivial to prove by the pigeonhole principle that no basis monomial has degree strictly larger than p.)

Claim: for some $1 < k < j$ we must have $k + \alpha_k > p - 1$. Indeed, the second lemma implies that $\alpha_1 = p - 3$. Thus for some $1 < k < j$ we have $\alpha_k \in \{p-2, p-1\}$ and this proves the claim.

By the assumption of minimality of j there must be some $1 \leq l < k$ with $\alpha_l < \alpha_k$. We then have $z^k w^{\alpha_k} = (z^l w^{\alpha_l})(z^{k-l} w^{\alpha_k - \alpha_l})$ with all exponents positive; it is then clear that $\alpha_{k-l} = \alpha_k - \alpha_l$. Now $p - j < \alpha_l < \alpha_k \leq p - 1$ implies that $\alpha_{k-l} \leq (p-1) - (p-j) = j - 1$. Since $k - l < j$ we also know that $\alpha_{k-l} \geq p - j$. Thus we have $p - j \leq j - 1$ and so $j \geq \frac{1}{2}(p+1)$.

We call this condition S (for Sarah, the author's daughter) and note that it is independent of our parameter q, subject to the hypotheses that $q \neq 1$ and $2q \neq p+1$. We now show that unless $q = 2$, condition S will be violated for another matrix group $\widehat{\Gamma}$ generated by $\mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^r \end{pmatrix}$ for some r relatively prime to p.

Specifically, take $r < p$ such that $rq = 1$ modulo p. We have $r = 1$ if and only if $q = 1$ and similarly $2r = p + 1$ if and only if $q = 2$; those cases are excluded by our hypotheses. Thus we are in a position to apply condition S to the $\widehat{\Gamma}-$invariant monomials.

To this end, we now reverse the roles of our variables; that is, we take monomials invariant under the group generated by $\widehat{\gamma} = \begin{pmatrix} \epsilon^q & 0 \\ 0 & \epsilon \end{pmatrix}$. This group is just $\widehat{\Gamma}$ because $\widehat{\gamma}^r = \mu$. For this matrix group, every invariant monomial is of the form $z^{\alpha_l} w^l$ where $z^l w^{\alpha_l}$ is $\Gamma-$invariant. Given $1 \leq t \leq p-1$ we define β_t to be minimal such that $z^t w^{\beta_t}$ is $\widehat{\Gamma}-$invariant. Observe that the pairs (t, β_t) are identical to the pairs (α_l, l) where $z^l w^{\alpha_l}$ is $\Gamma-$invariant.

Now let $m = \alpha_j$. Then $\beta_m = j$. Our assumptions about j imply that $\alpha_l < \alpha_j$ only if $l > j$. That is, for a pair (t, β_t) we have $t < m$ only if $\beta_t > \beta_m$.

In summary, $m + \beta_m \geq p - 1$ and for any $t < m$ we have $\beta_t > \beta_m$. We see that m violates the lemma. But $m = \alpha_j < \frac{p}{2}$ and this contradicts condition S, which was shown to hold for the minimum (and hence any) such value violating the lemma.■

This finishes the proof of proposition *2.3.1*.■

2.4 Further remarks about polynomial maps from lens spaces

We use the first theorem of this chapter to give necessary conditions on proper polynomial maps between balls that take zero to zero and are invariant under the action on the domain of certain cyclic diagonally generated finite matrix groups. We will work with some of the groups for which we have not ruled out the existence of such maps.

2.4.1 Example: Suppose that $\epsilon = \epsilon_{12}$ and Γ is generated by $\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^5 \end{pmatrix}$. Then $\gamma^3 = \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^3 \end{pmatrix} = \pm\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. Now any γ–invariant map is γ^3–invariant as well; thus such a map may be factorized as in theorem *2.2.1 (i)* by tensoring the γ^3–basic monomial map $f = (z^4, 2z^3w, \sqrt{6}z^2w^2, 2zw^3, w^4)$ with itself, applying a linear map that is unitary on its range, and untensoring on subspaces by this map.

This sort of reasoning applies to any cyclic matrix group that contains a subgroup generated by a nontrivial element of the form $\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$ or $\begin{pmatrix} \delta & 0 \\ 0 & \delta^2 \end{pmatrix}$. An important class of such groups are those that give rise, upon taking topological quotients, to the lens spaces $L(2k, 2k-1)$. Letting $\epsilon = \epsilon_{2k}$,these are generated by $\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{2k-1} \end{pmatrix}$. (Note that if $2 \mid k$ then the existence of such maps is precluded by theorem *1.2.2.*) For these groups we have $\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{2k-1} \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon} \end{pmatrix}$ and so $\gamma^k = \begin{pmatrix} \epsilon^k & 0 \\ 0 & \epsilon^k \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus any γ–invariant map must be obtainable by a linear operation and tensoring/untensoring the γ^k–basic invariant map, which is $g(z,w) = (z^2, \sqrt{2}zw, w^2)$, with itself.

2.4.2 Example: As a related remark in the specific case of $L(6,5)$, we recall that an invariant proper map of minimal degree would exist if and only if there were a real polynomial of the form $p(x,y) = x^6 + ax^3y^3 + bx^2y^2 + cxy + y^6$ with nonnegative coefficients such that $p = 1$ on the hyperplane $x + y = 1$. There is no such polynomial. Indeed, the only polynomial of degree 6 which is identically one on the hyperplane is $p(x,y) = x^6 + 2x^3y^3 - 9x^2y^2 + 6xy + y^6$. The same proof as in proposition *2.2.4* then gives the following: any proper polynomial map f from $L(6,5)$ to some ball must satisfy $p - 1 \mid \left(||f||^2 - 1\right)$ in the polynomial ring. Thus the existence of such a map is equivalent to the existence of a real polynomial $q(x,y) = 1 + ...$ invariant under the action of $\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^5 \end{pmatrix}$, such that the product $(p-1)q$ has only nonnegative coefficients. It is not hard to show that similar results hold for other lens spaces of the form $L(2k, 2k-1)$. For example, when $k = 2$ (a case ruled out by theorem *1.2.2*) we have $p(x,y) = x^4 - 2x^2y^2 + 4xy + y^4$. It is then a straightforward exercise to see that no such quotient polynomial q can exist (one of the high degree terms in the product must have a negative coefficient.)

There is another curious but noteworthy fact about the lens spaces $L(p, p-1)$ (where p can now be odd). We may represent the quaternions over C^2 as matrices of the form $\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$ with $z, w \in C$. The three-sphere S^3 is then the group of unit quaternions, that is, those for which $|z|^2 + |w|^2 = 1$. (This is also the classical group $SU(2)$.) Now $\gamma = \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^{p-1} \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \overline{\epsilon} \end{pmatrix}$ generates a finite subgroup $\Gamma \subset S^3$. Then the lens space $L(p, p-1)$, a topological quotient of the sphere, has an algebraic structure as well; it is the homogeneous space of cosets S^3/Γ.

Chapter 3 Combinatorial aspects of this work

3.1 Symmetric and asymmetric coefficient triangles

In this section we will look into proper monomial maps from B_2/Γ to B_N where Γ is a matrix group generated by $\gamma_1 = \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p \end{pmatrix}$ or $\gamma_2 = \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^2 \end{pmatrix}$.

The γ_1-basic invariant map is $(z, w) \longmapsto \left(z^p, \sqrt{p}z^{p-1}w, \sqrt{\binom{p}{2}}z^{p-2}w^2, ..., w^p\right)$. As in the last chapter, we associate to it the real polynomial

$$g_p(x,y) = x^p + px^{p-1}y + \binom{p}{2}x^{p-2}y^2 + ... + y^p = (x+y)^p.$$

We know that $x+y-1$ divides $g_p(x,y)-1$. In fact, we have

$$\begin{aligned} \frac{g_p(x,y)-1}{x+y-1} &= (x+y)^{p-1} + (x+y)^{p-2} + \ldots + 1 \\ &= x^{p-1} + (p-1)x^{p-2}y + \binom{p-1}{2}x^{p-3}y^2 + \ldots + y^{p-1} \\ &+ x^{p-2} + (p-2)x^{p-3}y + \binom{p-2}{2}x^{p-4}y^2 + \ldots + y^{p-2} \\ &+ \ldots + x + y + 1. \end{aligned}$$

That is, the coefficients of the terms in $g_p(x,y)$ came from the $p^{\underline{th}}$ row of the the Pascal triangle, while the coefficients in the quotient are given by all the previous rows. There is nothing at all deep here; it is of some interest to compare to the case where we use the γ_2-basic invariant maps.

The first few of these have canonically associated real polynomials

$$\begin{aligned} g_3(x,y) &= x^3 + 3xy + y^3 \\ g_5(x,y) &= x^5 + 5x^3y + 5xy^2 + y^5 \\ g_7(x,y) &= x^7 + 7x^5y + 14x^3y^2 + 7xy^3 + y^7 \\ g_9(x,y) &= x^9 + 9x^7y + 27x^5y^2 + 30x^3y^3 + 9xy^4 + y^9. \end{aligned}$$

We remind the reader that the coefficients of the above polynomials form the corresponding rows of the D'Angelo triangle in [D1]. By polynomial long division one obtains the following quotients:

$$\frac{g_3(x,y)-1}{x+y-1} = x^2 + (-y+1)x + (y^2+y+1).$$

$$\begin{aligned}\frac{g_5(x,y)-1}{x+y-1} =& x^4 + (-y+1)x^3 + (y^2+3y+1)x^2 \\ &+ (-y^3-2y^2+2y+1)x + (y^4+y^3+y^2+y+1).\end{aligned}$$

$$\begin{aligned}\frac{g_7(x,y)-1}{x+y-1} =& x^6 + (-y+1)x^5 + (y^2+5y+1)x^4 + (-y^3-4y^2+4y+1)x^3 \\ &+ (y^4+3y^3+6y^2+3y+1)x^2 + (-y^5-2y^4-3y^3+3y^2+2y+1)x \\ &+ (y^6+y^5+y^4+y^3+y^2+y+1).\end{aligned}$$

Finally,

$$\begin{aligned}\frac{g_9(x,y)-1}{x+y-1} =& x^8 + (-y+1)x^7 + (y^2+7y+1)x^6 + (-y^3-6y^2+6y+1)x^5 \\ &+ (y^4+5y^3+15y^2+5y+1)x^4 + (-y^5-4y^4-10y^3+10y^2+4y+1)x^3 \\ &+ (y^6+3y^5+6y^4+10y^3+6y^2+3y+1)x^2 \\ &+ (-y^7-2y^6-3y^5-4y^4+4y^3+3y^2+2y+1)x \\ &+ (y^8+y^7+y^6+y^5+y^4+y^3+y^2+y+1).\end{aligned}$$

Each of these quotients gives rise to a triangle of coefficients in an obvious manner; we will call that triangle T_t. Let

$$\frac{g_t(x,y)-1}{(x+y-1)} = \sum c^t_{r,s} x^r y^s.$$

define the coefficients of the quotient. That is, we let $c^t_{r,s}$ denote the $r^{\underline{th}}$ entry in row s of T_t.

We write out these triangles explicitly. T_3 is

$$\begin{array}{ccccc} & & 1 & & \\ & -1 & & 1 & \\ 1 & & 1 & & 1 \end{array}$$

T_5 is

$$\begin{array}{c}
1 \\
-1 \quad 1 \\
1 \quad 3 \quad 1 \\
-1 \quad -2 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

T_7 is

$$\begin{array}{c}
1 \\
-1 \quad 1 \\
1 \quad 5 \quad 1 \\
-1 \quad -4 \quad 4 \quad 1 \\
1 \quad 3 \quad 6 \quad 3 \quad 1 \\
-1 \quad -2 \quad -3 \quad 3 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

T_9 is

$$\begin{array}{c}
1 \\
-1 \quad 1 \\
1 \quad 7 \quad 1 \\
-1 \quad -6 \quad 6 \quad 1 \\
1 \quad 5 \quad 15 \quad 5 \quad 1 \\
-1 \quad -4 \quad -10 \quad 10 \quad 4 \quad 1 \\
1 \quad 3 \quad 6 \quad 10 \quad 6 \quad 3 \quad 1 \\
-1 \quad -2 \quad -3 \quad -4 \quad 4 \quad 3 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

One now asks the obvious: (i) Does the left-vs.-right symmetry (up to sign) persist in subsequent "finite triangles"? It does. (ii) As with the Pascal and D'Angelo triangles, can the entries in one row of a given triangle be obtained in a straightforward manner from entries above? This is almost the case. In fact, both left-right symmetry and almost-recursive (in the sense of algorithmic computation) entry calculation follow from:

3.1.1 Proposition: For $r \neq \frac{s+1}{2}$ in T_t we have the $r^{\underline{th}}$ coefficient of row s given by subtracting the $r^{\underline{th}}$ coefficient from the $r-1^{\underline{th}}$ in row $s-1$. That is, $c^t_{r,s} = c^t_{r-1,s-1} - c^t_{r,s-1}$. (We consider the zeroth and $s^{\underline{th}}$ coefficients of row s to be zero.) When s is odd and $r = \frac{s+1}{2}$ this coefficient is obtained as above, but we also add to this difference the $s^{\underline{th}}$ entry of row t of D'Angelo's triangle.

Proof: For given t we show by long division that this holds for the top row of the triangle T_t (that is, the only factor of x^{t-1} in the quotient $(g_t - 1)/(x + y - 1)$ is 1.) It is simple to verify the proposition for the next two rows via long division; in doing so one sees the basic induction technique. As this is the only mathematical result I have to show for the summer of '89 (when Sarah dropped in, so to speak), I will give the cumbersome details below.

We assume the result holds for the first $2r + 1$ rows. To complete the proof we must show it for the next two rows. To this end, we multiply the factor $(x + y - 1)$ by that part of the quotient constructed thus far, and in so doing we account for all terms in the dividend in $x^t, x^{t-1}, \ldots, x^{t-2r}$. We also obtain in this product $x^{t-2r-1}(y^{2r} + \ldots + 1)(y - 1)$ where $(y^{2r} + \ldots + 1)$ is the factor of x^{t-2r} in the quotient $(g_t - 1)/(x + y - 1)$. As none of the terms in this product appears in $g_t(x, y) - 1$ (unless $t = 2r + 1$, in which case we are finished), we must utilize factors in the quotient to remove them all. Thus our factor in x^{t-2r-2} in the quotient must be $-x^{t-2r-2}(y^{2r} + ... + 1)(y - 1)$. The term in y^{2r+1} has coefficient of -1; the one in y^0 has coefficient of $+1$. For $1 \leq j \leq 2r$ the term in y^j has coefficient equal to the coefficient of the $y^j x^{t-2r-1}$ term minus that of the $y^{j-1}x^{t-2r-1}$ term (both found in the previous step). Thus the proposition holds for row $2r + 2$.

The factor $-x^{t-2r-2}(y^{2r}+...+1)(y-1)$ gives rise to terms

$$-x^{t-2r-2}(y^{2r}+...+1)(-y+1)(y-1)$$

in the expanded product. The polynomial $g_t(x,y)-1$ also has a term $k_{t,r+1}x^{t-2r-2}y^{r+1}$ where $k_{t,r+1}$ is the $r+1^{\underline{th}}$ entry of row t of D'Angelo's triangle. (Here we number the entries beginning at zero.) We therefore have in our quotient the term

$$-x^{t-2r-3}\big((y^{2r}+...+1)(-y+1)(y-1)+k_{t,r+1}y^{r+1}\big).$$

The terms in y^{2r+2} and y^0 have coefficients of $+1$. For $1 \leq j \leq 2r+1$ and $j \neq r+1$ the coefficient of the term in y^j is seen to be the difference of the coefficients in y^j and y^{j-1} from the row computed in the step just above, as required by the proposition. The term in y^{r+1} has coefficient given by adding $k_{t,r+1}$ to this difference, also as prescribed by the proposition.

This shows that the proposition holds for row $2r+3$ as well. This completes the inductive proof of the proposition.■

3.2 A depiction of monomial maps from the 2-ball

Throughout this section, unless otherwise indicated, a "map" is assumed to be a proper monomial map from the two-ball to some other ball in complex Euclidean space. We will give a precise meaning to the word "depiction". The word "representation" would probably read better, but it has been used in a much different context already in this thesis.

As previously described, D'Angelo's factorization theorem for proper polynomial maps between balls that take zero to zero begins by showing that such a map, say of degree m, can be converted to a homogeneous map between balls, also of degree m, by tensoring on certain subspaces with the identity map. As these are essentially unique ([D1], [R2]) a linear operation will convert this to the monomial map$\left\{\sqrt{\binom{\alpha}{\alpha_1,\ldots,\alpha_n}}z^\alpha\right\}$ where α ranges over all multi-indices of length m. Thus the original map may be obtained from this homogeneous map by reversing these operations.

When the domain is B_2 and we are working with monomial maps this result can be portrayed as follows. We begin with row m of the Pascal triangle. A pair of neighboring monomials in the proper homogeneous map from B_2 to B_{m+1}, say $c_{a,b}z^a w^b$ and $c_{a-1,b+1}z^{a-1}w^{b+1}$, (where $a+b=m$) are depicted by the elements $\left|c_{a,b}\right|^2 = \binom{m}{b}$ and $\left|c_{a-1,b+1}\right|^2 = \binom{m}{b+1}$ in that row of the triangle. We may untensor to form a new monomial map between balls by replacing $k\left(z^a w^b + z^{a-1}w^{b+1}\right)$ with $kz^{a-1}w^b$ where $k \leq min\left(c_{a,b}, c_{a-1,b+1}\right)$. This is depicted by placing k^2 diagonally beneath the binomial coefficient values, as in the picture below.

$$\begin{array}{ccccc} \cdots & \binom{m}{b} & & \binom{m}{b+1} & \cdots \\ & & k^2 & & \end{array} \tag{3.2.1}$$

In summary, this depicts a new map, no longer homogeneous, whose $z^a w^b$ coefficient is $\sqrt{\left|c_{a,b}\right|^2 - k^2}$, whose $z^{a-1}w^{b+1}$ coefficient is $\sqrt{\left|c_{a-1,b+1}\right|^2 - k^2}$, and with a new monomial $kz^{a-1}w^b$. We depict further untensoring in this same manner, always operating on neighboring pairs of norm-squares of coefficients of our monomial map. Our requirement is that two "children" diagonally below a coefficient never sum to more than their common parent.

3.2.2 Example: We can depict the monomial map

$$(z,w) \longmapsto \left(z^3, \sqrt{3}zw, w^3\right),$$

obtained from a homogeneous map by untensoring, as follows. We start with the third row of the Pascal triangle,

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \end{array}$$

which depicts the degree three homogeneous map

$$(z,w) \longmapsto \left(z^3, \sqrt{3}z^2w, \sqrt{3}zw^2, w^3\right).$$

We then untensor the two middle terms entirely. This is depicted by the picture

$$\begin{array}{ccccccc} 1 & & 3 & & 3 & & 1 \\ & & & 3 & & & \end{array}$$

3.2.3 Example: We similarly obtain the map

$$(z,w) \longmapsto \left(z^5, \sqrt{5}z^3w, \sqrt{5}zw^2, w^5\right)$$

from the homogeneous map

$$(z,w) \longmapsto \left(z^5, \sqrt{5}z^4w, \sqrt{10}z^3w^2, \sqrt{10}z^2w^3, \sqrt{5}zw^4, w^5\right)$$

by untensoring. This is depicted as

$$\begin{array}{ccccccccccc} 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & 5 & & 5 & & 5 & & & \\ & & & & & 5 & & & & & \end{array}$$

3.2.4 Example: The map

$$(z,w) \longmapsto \left(z^7, \sqrt{7}z^5w, \sqrt{14}z^3w^2, \sqrt{7}zw^3, w^7\right)$$

is obtained from the degree seven homogeneous map via the following depiction:

$$\begin{array}{cccccccc} 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ & 7 & 14 & 21 & 14 & 7 & & \\ & & 14 & 7 & 7 & & & \\ & & & & 7 & & & \end{array}$$

In general, we can depict the D'Angelo basic proper monomial map from B_2/Γ to B_M where Γ is generated by $\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$ with ϵ a primitive $(2M-3)^{\underline{rd}}$ root of one. We begin with row $p = 2M-3$ of the Pascal triangle and convert to a new map, as depicted in the picture below.

$$\begin{array}{cccccc} 1 & p & \binom{p}{2}\ldots & \ldots\binom{p}{2} & p & 1 \\ & p & \binom{p}{2}-p\ldots & \ldots p & & \end{array}$$

We leave the entries corresponding to the ($\gamma-$basic invariant) monomials z^p and w^p (the $1's$ at the two ends of the top row) alone, and remove all else via untensoring from the top row. By this I mean that we completely untensor all monomials that correspond to the non-end slots in the top row of the map depiction written above. (Ignore for the moment the question of whether this complete removal of all else will work out numerically. It did in the previous three examples.) Now note that the leftmost value in the newly formed row, p, is in the slot that corresponds to the $\gamma-$basic invariant monomial $z^{p-2}w$. We will leave it alone and proceed to untensor completely the remaining terms in that row. (Again, assume this can be done.) We obtain a third row in so doing. Leftmost is the value $\binom{p}{2} - p$ and it is in the slot that corresponds to the $\gamma-$basic invariant monomial $z^{p-4}w^2$. We leave it alone, untensor the rest of the row, and continue this process....The fact that each of these rows can indeed be completely untensored, upon leaving in entirety the leftmost element, is not at all trivial. Indeed, one must know D'Angelo's theorem that a $\gamma-$basic invariant proper monomial map from B_2 to some B_N actually exists. That we must leave the leftmost entry in each new row in entirety then follows from the fact that no "descendant" of a $\gamma-$basic invariant monomial is also $\gamma-$invariant.

3.2.5 Example: The procedure described above gives a technique for generating any map in D'Angelo's family of invariant maps. For instance, to find the degree eleven map invariant under the matrix group generated by $\gamma = \begin{pmatrix} e^{2\pi i/11} & 0 \\ 0 & \epsilon^{4\pi i/11} \end{pmatrix}$ we take row eleven of the Pascal triangle and apply the steps as above to obtain the map depiction

$$\begin{array}{cccccccccccc}
1 & 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 & 1 \\
 & 11 & 44 & 121 & 209 & 253 & 209 & 121 & 44 & 11 & & \\
 & & 44 & 77 & 132 & 121 & 68 & 33 & 11 & & & \\
 & & & 77 & 55 & 66 & 22 & 11 & & & & \\
 & & & & 55 & 11 & 11 & & & & & \\
 & & & & & 11 & & & & & &
\end{array}$$

The $\gamma-$basic invariant proper monomial map from B_2 is thus

$$(z, w) \mapsto \left(z^{11}, \sqrt{11}z^9w, \sqrt{44}z^7w^2, \sqrt{77}z^5w^3, \sqrt{55}z^3w^4, \sqrt{11}zw^5, w^{11}\right).$$

This mode of depicting monomial maps obtained via untensoring from homogeneous monomial maps has several nice applications. One, as just indicated, is a technique of generating the $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}-$basic invariant maps. As another, we outline a second proof of the polynomial case of theorem *1.2.2.*

3.2.5 Theorem: Suppose Γ is generated by $\gamma = \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^q \end{pmatrix}$ with p and q relatively prime, and $0 < \nu_2(q-1) < \nu_2(p)$. *Then there is no proper polynomial map from B_2/Γ to any B_N.*

First we require

3.2.7 Lemma: The conditions on p and q imply that if $z^a w^b$ and $z^{a+r}w^{b-r}$ are $\Gamma-$invariant monomials, then r is divisible by 2 (i.e. the invariant monomials are evenly spaced).

Proof: Invariance of these monomials implies that $a + qb = sp$ and $(a+r) + q(b-r) = tp$ for some integers s, t. Thus

$$p \mid (a+r) + q(b-r) - (a+qb) = r(1-q).$$

As p is divisible by a larger power of 2 than $q-1$ we see that $2 \mid r.$■

Proof of 3.2.6: Suppose f is such a map. As in *2.1.1* we may form the monomialization of f. We remove the constant term, if any, and rescale, as in the remarks at the end of section *2.2.* We are left with a proper monomial map from B_2 to some ball, and it is also $\Gamma-$invariant since Γ is a diagonal matrix group. Thus we may assume that f is a proper monomial map, say of degree m, that takes zero to zero. Hence we may start with the $m^{\underline{th}}$ row of the Pascal triangle and depict untensoring of the corresponding homogeneous map, as described above. As f is $\Gamma-$invariant, we must completely eliminate any entry in the depiction that does not correspond to an invariant monomial. (That is, the two "children" of such an entry must sum exactly to that entry.) Subsequent entries in lower rows that arise must similarly be eliminated if they do not correspond to invariant monomials, but this fact we do not need; we will show that we cannot even eliminate all noninvariant monomial entries from the top row unless we eliminate the entire row, contradicting our assumption that the degree of f is m.

We suppose, then, that there is a leftmost entry in the top $m^{\underline{th}}$ row that is not entirely removed. Say it corresponds to the invariant monomial $z^{m-k}w^k$. As in lemma *3.2.7* above, we assume the invariant monomials are spaced r apart for some even integer r. We have then removed exactly $\binom{m-1}{k-1}$ from the left side of this value, and $a < \binom{m-1}{k}$ from the right. We must then completely remove the next $r-1$ entries. We have the (partial) depiction shown below

$$\begin{array}{cccccc} & \cdots\binom{m}{k} & \binom{m}{k+1} & & & \cdots\binom{m}{k+r}\cdots \\ \binom{m-1}{k-1} & a & & \binom{m}{k+1}-a & \cdots\left[\binom{m}{k+r-1}-\ldots+a\right]\cdots & \end{array}$$

Now $a < \binom{m-1}{k}$ implies that $\binom{m}{k+r-1} - \binom{m}{k+r-2} + \ldots - a > \binom{m-1}{k+r-1}$. Thus we remove strictly more from the left side of the next invariant-monomial slot than if we were entirely removing the top row, and therefore we must remove strictly less from the right side of this entry. This is easily seen to be true at all subsequent entries corresponding to invariant monomials, by the even spacing of these entries. It is in fact clear that we are removing more (than if entirely untensoring the homogeneous map) from every subsequent entry located an odd distance from the $k^{\underline{th}}$ entry in the top row, and less from those an even distance away.

To summarize the above discussion, for any positive integer s we see that the "child" of the $m^{\underline{th}}$ row pair of entries $\binom{m}{k+2s-1}$ and $\binom{m}{k+2s}$ is strictly greater than $\binom{m-1}{k+2s-1}$, whereas the child of the next pair, $\binom{m}{k+2s}$ and $\binom{m}{k+2s+1}$, is strictly less than $\binom{m-1}{k+2s}$. If the final entry in this row is located an even distance from an entry corresponding to an invariant slot, we are forced to remove strictly more than one in untensoring with its neighbor to the left; this violates the condition of leaving nonnegative coefficients for our depiction (recall that these coefficients stand for norm-squares of monomial map coefficients). If it is located an odd distance from an invariant slot then we are forced to remove strictly less than one from its neighbor to the left and, since it has no neighbor to the right, we obtain a monomial map that has a nonzero coefficient for the monomial corresponding to this entry. But the invariant slots are evenly spaced, hence this slot does not correspond to an invariant monomial. In either case we obtain a contradiction.■

3.3 Depicting monomial maps from the 3–ball

We will return to maps from the two-ball presently. First we discuss applications similar to those of the previous section, applied to maps from the three-ball.

The trinomial coefficients can be constructed as a pyramid, just as the binomial coefficients are formed in the Pascal triangle. Thus the D'Angelo factorization theorem for proper polynomial maps between balls, applied to monomial maps from B_3 that are of degree m and take zero to zero, implies that these can be depicted in a manner similar to that for maps from B_2. Specifically, we now start out with the $m^{\underline{th}}$ level of the pyramid of trinomial coefficients; it is a triangle with $m+1$ entries on each side. Orient this triangle of values so that there is a horizontal edge on the bottom. The topmost entry will correspond to the monomial z_1^m. From this entry, or any other not on the bottom row, we may descend either to the left of the right to an adjacent entry. To obtain the monomial that corresponds to the new position we decrement the exponent of z_1 and increment that of z_2 (respectively z_3) when we descend to the left (respectively right). Thus the entry at the bottom left vertex corresponds to the monomial z_2^m, while that at the bottom right is for z_3^m.

We depict untensoring of a triple of neighboring slots by creating a new triangle of entries, corresponding to monomials of degree $m-1$, at the next level, and proceeding further down. As with the depictions of maps from the two-ball, our one requirement is that the sum of the immediate children of a given entry (there will be one, two, or three depending on whether that slot was on a vertex, interior of an edge, or interior of a triangle) in a given level of the depiction not sum to more than that parent.

We will investigate (the existence of) maps invariant under the action of particular matrix groups. Specifically, let Γ be generated by

$$\gamma = \begin{pmatrix} \epsilon_7 & 0 & 0 \\ 0 & \epsilon_7^2 & 0 \\ 0 & 0 & \epsilon_7^4 \end{pmatrix}.$$

3.3.1 Example (Chiappari): We construct a proper monomial map from B_3/Γ to B_{17}.

This is also carried out in [C]. We will use the technique above of depicting such maps to give a different derivation. If such a map were γ−basic, we would construct a depiction by starting with level 7 of the trinomial coefficient pyramid. We would then completely remove all entries (that is, each would equal the sum of its children) that correspond to non-basis invariant monomials; these would be at least equal to the sum of their children. Thus we would have a depiction of a monomial map, and all non-basis invariant monomials would have zero coefficients. When we attempt to construct this depiction we find that (i) we never have any choice in so doing if we are to entirely untensor entries corresponding to noninvariant monomials, and (ii) we will be required to not completely untensor some entries corresponding to invariant non-basis monomials.

On to the construction. We start with level 7 of the pyramid, as below.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 7 & & 7 & & & & & & \\
 & & & & & 21 & & 42 & & 21 & & & & & \\
 & & & & 35 & & 105 & & 105 & & 35 & & & & \\
 & & & 35 & & 140 & & 210 & & 140 & & 35 & & & \\
 & & 21 & & 105 & & 210 & & 210 & & 105 & & 21 & & \\
 & 7 & & 42 & & 105 & & 140 & & 105 & & 42 & & 7 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

We depict untensoring by a new level of entries that correspond to monomials of degree 6. Mentally (so as not to clutter the actual diagrams) we think of each entry in this new lower degree level as lying inside that upward pointing triangle of three entries which gives the parents of the lower degree entry. The slot that corresponds to the monomial $z_1 z_2^2 z_3^3$, for example, has three parent slots; they correspond to the monomials $z_1^2 z_2^2 z_3^3$, $z_1 z_2^3 z_3^3$, and $z_1 z_2^2 z_3^4$. In untensoring the top level of degree 7 we note that the three vertex entries must be left alone; if partly untensored they will spawn descendents of the form z_j^k for some $k < 7$ and none of these are invariant. As we proceed inwards we find that we have no choice of what values to put in level 6; the constraint of completely untensoring every noninvariant monomial of degree 7 forces every value in the depiction. Specifically, at level 6 we have the triangle of values

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 7 & & 7 & & & & & \\
 & & & & 14 & & 28 & & 14 & & & & \\
 & & & 21 & & 63 & & 56 & & 21 & & & \\
 & & 14 & & 56 & & 91 & & 63 & & 14 & & \\
 & 7 & & 28 & & 63 & & 56 & & 28 & & 7 & \\
0 & & 7 & & 14 & & 21 & & 14 & & 7 & & 0
\end{array}$$

Note that although it is not symmetric from left to right, this triangle is preserved under rotation by $2\pi/3$. Given the relations between pairs of diagonal entries in γ this is not a surprise. The invariant monomials of degree 7 are $\{z_1^7, z_2^7, z_3^7, z_1^2 z_2^4 z_3, z_1 z_2^2 z_3^4, z_1^4 z_2 z_3^2\}$. The entries in level 7 that correspond to the last three in this set are all 105. The coefficient of each of these three monomials in the map under construction is thus seen to be $\sqrt{105 - (28 + 14 + 56)} = \sqrt{7}$.

We now untensor all noninvariant degree 6 monomials created in the previous step, to form the level 5 triangle. As before, this necessity forces each value at this level upon us. We obtain the triangle

$$\begin{array}{ccccccccccc}
 & & & & & 7 & & & & & \\
 & & & & 0 & & 7 & & & & \\
 & & & 14 & & 21 & & 7 & & & \\
 & & 7 & & 28 & & 28 & & 14 & & \\
 & 7 & & 21 & & 28 & & 21 & & 0 & \\
0 & & 0 & & 14 & & 7 & & 7 & & 0
\end{array}$$

(Again note the rotational symmetry.) The invariant monomials of degree 6 are $\{z_1^5 z_2, z_2^5 z_3, z_1 z_3^5, z_1^2 z_2^2 z_3^2\}$. In the triangle depicting the degree 6 monomials, the entries that correspond to each of the first three of these invariant monomials are all 7. As is seen from level 5, we do not untensor these monomials at all. Thus the coefficients for each of these monomials in the map under construction is $\sqrt{7}$. For the last, we see that it has coefficient $\sqrt{91 - (28 + 28 + 28)} = \sqrt{7}$ as well.

We untensor again to kill off all noninvariant degree 5 monomials, and so form the triangle at level 4. Again, this forces the values of all entries. We obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 7 & & & \\
 & & 0 & & 14 & & 0 & & \\
 & 7 & & 14 & & 14 & & 0 & \\
0 & & 0 & & 0 & & 7 & & 0
\end{array}$$

The invariant monomials of degree 5 are $\{z_1^3 z_2^2, z_2^3 z_3^2, z_1^2 z_3^3\}$. Looking at the entries in level 4 that correspond to the children of these, we find that they are not at all untensored; hence the coefficient of each of these monomials in the map under construction is $\sqrt{14}$.

We untensor noninvariant monomials of degree 4 to obtain the level 3 triangle:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 14 & & 0 & \\ 0 & & 0 & & 0 & & 0 \end{array}$$

The invariant monomials of degree 4 are $\{z_1^3z_3, z_1z_2^3, z_2z_3^3\}$. Checking the entries in levels 4 and 3 of our depiction shows that these monomials each have coefficients of $\sqrt{7}$. Finally, we see that we cannot untensor at all in level three. As the single nonzero value therein, 14, is in the entry corresponding to the invariant monomial $z_1z_2z_3$, that monomial has coefficient $\sqrt{14}$.

We are done at last. Our γ−invariant proper monomial map from B_3 to B_{17} is

$$(z_1, z_2, z_3) \longmapsto (z_1^7, z_2^7, z_3^7, \sqrt{7}z_1^2z_2^4z_3, \sqrt{7}z_1z_2^2z_3^4, \sqrt{7}z_1^4z_2z_3^2, \sqrt{7}z_1^5z_2, \sqrt{7}z_2^5z_3, \sqrt{7}z_1z_3^5,$$
$$\sqrt{7}z_1^2z_2^2z_3^2, \sqrt{14}z_1^3z_2^2, \sqrt{14}z_2^3z_3^2, \sqrt{14}z_1^2z_3^3, \sqrt{7}z_1^3z_3, \sqrt{7}z_1z_2^3, \sqrt{7}z_2z_3^3, \sqrt{14}z_1z_2z_3).$$

3.3.2 Example: For another application of this technique of depicting maps from the three-ball, we let

$$\gamma = \begin{pmatrix} \epsilon_9 & 0 & 0 \\ 0 & \epsilon_9^4 & 0 \\ 0 & 0 & \epsilon_9^7 \end{pmatrix}.$$

In *1.4.10* we proved that there is no proper rational map from B_3 to any B_N invariant under the action of γ. Hence there is no such monomial map taking zero to zero. The combinatorial interpretation is as follows: we cannot start at any level of the trinomial pyramid and "undo" it (that is, depict monomial map untensoring, as above) in such a way as to leave zero in all entries corresponding to non-γ−invariant monomials, and nonnegative values in the invariant monomial slots. That is, we cannot begin with a given level of the pyramid of trinomial coefficients and form new levels such that the children of any entry corresponding to a noninvariant monomial sum exactly to that entry, while the children of an entry corresponding to an invariant monomial sum to no more than that entry, unless we kill off all invariant as well as noninvariant monomial entries.

In contrast to the combinatorial proof of theorem *3.2.6*, this fact seems to require the results of chapter one. This is because there is no "even spacing" of invariant monomials in this case, and hence we have no obvious parity argument. I believe that a direct proof of this perhaps bizarre and not very useful combinatorial result would be nontrivial. (Possibly a modulo 3 arithmetic approach would work.)

3.4 More monomial maps from the 2–ball

We return to the depiction of maps from B_2 for a final application. In [D1] it is suggested that the proper monomial map from B_2 to B_N that is of maximal degree (for fixed N) is precisely the one that is invariant under the action of the matrix group generated by $\gamma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$, with ϵ a primitive $(2N-3)^{\underline{rd}}$ root of one. This map is of degree $2N-3$. The first few of these were suggested at the end of [CS1] to maximize degree.

We do not know the validity of this assertion except when $N \leq 4$. In the case where $N = 5$ it is known that there are (modulo the usual equivalence via norm-squares of coefficients) exactly three inequivalent monomial maps of degree 7. It is believed that there are no monomial maps of higher degree in this case. One of the degree 7 maps is the one from the D'Angelo family,

$$(z, w) \longmapsto \left(z^7, \sqrt{7}z^5w, \sqrt{14}z^3w^2, \sqrt{7}zw^3, w^7\right).$$

The other two are given below. One can verify by computer that there are more than 100 inequivalent maps and families of inequivalent maps indexed by one or more real parameters, from B_2 to B_5. Seventeen of these maps/parametrized map families actually go to a proper linear subspace, that is, map properly to a ball of lower dimension.

3.4.1 Example: We find one of the degree 7 monomial maps from B_2 to B_5. Our depiction will begin with row 7 of the Pascal triangle. We leave the ends alone and completely untensor the rest of the monomials corresponding to entries in that row, as pictured below.

$$\begin{array}{cccccccccccccccc} 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\ \\ & 0 & & 7 & & 14 & & 21 & & 14 & & 7 & & 0 & \end{array}$$

We now completely untensor the end monomials, and continue to do so as we work toward the middle (which we will not completely untensor). We obtain

$$\begin{gathered} 7 \quad 14 \quad 21 \quad 14 \quad 7 \\ 7 \quad 7 \quad 7 \quad 7 \end{gathered}$$

Again we will untensor each end in entirety and work towards the middle, to obtain

$$\begin{gathered} 7 \quad 7 \quad 7 \quad 7 \\ 7 \quad 0 \quad 7 \end{gathered}$$

We can untensor no more. We have a depiction of the map

$$(z,w) \longmapsto \left(z^7, \sqrt{7}z^3w^3, \sqrt{7}z^3w, \sqrt{7}zw^3, w^7\right).$$

3.4.2 Example: We give the depiction of the third map of degree 7 from B_2 to B_5.

$$\begin{gathered} 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\ 0 \quad 7 \quad 14 \quad 21 \quad 14 \quad 7 \quad 0 \\ \frac{7}{2} \quad \frac{21}{2} \quad \frac{21}{2} \quad \frac{7}{2} \\ \frac{7}{2} \quad 7 \quad \frac{7}{2} \\ \frac{7}{2} \quad \frac{7}{2} \\ \frac{7}{2} \end{gathered}$$

This depicts the map

$$(z,w) \longmapsto \left(z^7, \sqrt{7/2}z^5w, \sqrt{7/2}zw, \sqrt{7/2}zw^5, w^7\right).$$

3.4.3 Example: We depict a map from B_2 to B_7 that has degree 11 and is inequivalent to the D'Angelo map. The strategy we employ to find it is the same as that used in *3.4.1*; we leave the ends of row 11 alone, and untensor everything else in that row. In subsequent rows we untensor the monomials corresponding to each end slot entirely, and work toward the middle by alternately doing a slot on the left and then one on the right. We find that eventually the depiction becomes asymmetric in left vs. right sides. We obtain the following:

$$\begin{array}{ccccccccccccccccccccccc}
1 & & 11 & & 55 & & 165 & & 330 & & 462 & & 462 & & 330 & & 165 & & 55 & & 11 & & 1 \\
& & & 11 & & 44 & & 121 & & 209 & & 253 & & 209 & & 121 & & 44 & & 11 & & & \\
& & & & 11 & & 33 & & 88 & & 121 & & 121 & & 88 & & 33 & & 11 & & & & \\
& & & & & 11 & & 22 & & 66 & & 55 & & 66 & & 22 & & 11 & & & & & \\
& & & & & & 11 & & 11 & & 55 & & 0 & & 11 & & 11 & & & & & & \\
& & & & & & & 11 & & 0 & & 0 & & 0 & & 11 & & & & & & &
\end{array}$$

This depicts the map

$$(z,w) \longmapsto \left(z^{11}, \sqrt{11}z^5w^5, \sqrt{11}z^5w, \sqrt{11}zw^5, \sqrt{55}z^4w^3, \sqrt{55}z^3w^5, w^{11}\right).$$

It is not clear whether there are other examples of monomial maps with the same degree as the D'Angelo family of maps. The technique used to find examples *3.4.1* and *3.4.3* seems to break down in higher degrees.

Chapter 4 Related topics and results

4.1 Introduction

In this chapter we investigate the consequences of relaxing various hypotheses throughout the preceding chapters. Specifically, we have worked with maps between balls that are (i) proper, (ii) holomorphic, (iii) smooth to the boundary, and (iv) invariant under the action of unitary matrix groups that are finite. Simple examples show where the theory changes or breaks down when hypothesis (i) or (iv) is removed. Hypothesis (ii) is of interest for the additional reason that lens space homeomorphisms are all linear but not necessarily holomorphic; thus there is a tie-in with the invariant maps from lens spaces described in chapter one. We devote a separate section to this hypothesis. Hypothesis (iii) requires nontrivial results and also warrants a separate section.

4.1.1 Proposition: Suppose that Γ is an infinite subgroup of the unitary group $U(n)$. Then there is no $\Gamma-$invariant proper holomorphic map from B_n to another ball.

Proof: Suppose f is such a map. For some z in the domain the set $\{\Gamma(z)\}$ is infinite. On the other hand, the analytic variety $W = f^{-1}(z) = \{w : f(w) = f(z)\}$ must be compact since f is assumed to be proper. Hence W, a compact variety in a complex Euclidean space, must be a finite set. (This is theorem 14.3.1 of [Ru3].) This contradicts the fact that $\{\Gamma(z)\} \subset W$.■

Another proof, that does not rely on the theory of several complex variables, is the following. We may assume that Γ is a closed subgroup, as by continuity f will be $\overline{\Gamma}-$invariant (where the overbar is our notation for closure).

Since we work with a closed group Γ, that group has some one-parameter subgroup. With respect to some basis this subgroup is diagonal, and of the form

$$(4.1.2) \qquad \gamma = \begin{pmatrix} e^{it\theta_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & & \\ 0 & & & & & e^{it\theta_n} \end{pmatrix}.$$

For some $0 < r < 1$, let $W = rB_n \cap \{(z, 0, \ldots, 0) : z \in \mathbb{C}\}$ and let g be the restriction of f to W. By $\Gamma-$invariance of f, $g(e^{i\theta}z) = g(z)$ for all $z \in \partial W$. So g is constant on the boundary circle ∂W, and hence also on W by the maximum principle. Thus $f(0, \ldots, 0) = g(0)$ lies in the boundary of the target ball, so again by the maximum principle f is a constant map, and thus not proper.■

In the case of holomorphic maps between balls that are not required to be proper, we obtain the following simple result.

4.1.3 Proposition: Let Γ be a finite subgroup of $U(n)$. Then there is a (nonproper) $\Gamma-$invariant polynomial map from B_n to some other ball.

Proof: Let $\{q_1, \ldots, q_s\}$ be a basis for the algebra of $\Gamma-$invariant polynomials. An important point is that such a finite basis exists for some $s \geq n$. This is shown in [Fl]. (This result, and one of the proofs in the reference cited, is due to Hilbert.) Each q_j is entire and hence bounded on the unit ball. Let

$$M = \max\left\{ \sqrt{\sum_{j=1}^{s} |q_j(z)|^2} : z \in \partial B_n \right\}.$$

Then $q(z) = \frac{1}{M}(q_1(z), \ldots, q_s(z))$ is such a map.

We remark that this simple technique will reappear in section *4.3* when we prove some powerful embedding theorems via proper maps. Needed are more group-invariant functions on the ball that augment the components of q, in such a way that the sum of norm squares of all components goes to one as we go to the boundary sphere. The Løw construction will provide this.

4.2 Invariant nonholomorphic proper maps between balls

We begin by describing lens space homeomorphisms. This will relate the topic of nonholomorphic proper maps between balls in a natural way to the material of chapter one.

As mentioned earlier, the lens spaces $L(p,q)$ and $L(p,s)$ are homeomorphic if and only if $q \equiv (\pm s)^{\pm 1}$ modulo p. The actual homeomorphisms are quite simple; indeed, they are $\mathbf{R}-$linear maps. We treat the four cases separately below.

Case (i): $q \equiv s$ modulo p. Then the lens spaces are in fact the same.

Case (ii): $q \equiv -s$ modulo p. Then our homeomorphism is the conjugation of one coordinate. That is, we use the map $(z,w) \longmapsto (z,\overline{w})$.

Case (iii): $q \equiv s^{-1}$ modulo p. Then the homeomorphism exchanges coordinates: $(z,w) \longmapsto (w,z)$.

Case (iv): $q \equiv (-s)^{-1}$ modulo p. Then we use the map $(z,w) \longmapsto (\overline{w},z)$.

4.2.1 Example: Let Γ_1 be generated by $\gamma = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4 \end{pmatrix}$. The basic monomial map from B_2/Γ_1 to B_5 is

$$(z,w) \longmapsto \left(z^4, 2z^3w, \sqrt{6}z^2w^2, 2zw^3, w^4\right).$$

Now $3 \equiv -1$ modulo 4. Let Γ_2 be generated by $\begin{pmatrix} \epsilon_4 & 0 \\ 0 & \overline{\epsilon}_4 \end{pmatrix} = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^3 \end{pmatrix}$. Then

$$(z,w) \longmapsto \left(z^4, 2z^3\overline{w}, \sqrt{6}z^2\overline{w}^2, 2z\overline{w}^3, \overline{w}^4\right)$$

is a nonholomorphic monomial proper map from B_2/Γ_2 to B_5.

When lens spaces are holomorphically equivalent we do not obtain any new information about existence of proper invariant maps to balls. Specifically, if $q \equiv s^{-1}$ modulo p then the pair (p,q) satisfies the hypotheses of theorem *1.2.2* if and only if (p,s) does so.

We now give a few examples to highlight some of the differences between nonholomorphic and holomorphic proper maps between balls. As before, we are primarily interested in those maps that are invariant under the action of some unitary group. We will see that virtually every important result established for such holomorphic maps can be violated by nonholomorphic ones.

First, we can no longer assert that a map that is smooth to the boundary is rational. Also, if f is a real analytic rational map proper map between balls that takes zero to zero, the degree of the numerator need not be strictly larger than that of the denominator.

4.2.2 Example: If f is any proper holomorphic map between balls, then $g_1 = \frac{2f}{1+||z||^2}$ and $g_2 = \frac{2f}{1+||f||^2}$ are nonholomorphic proper maps between the same balls, and have the same unitary group-invariance properties, as f. Thus, for instance, the map $z \longmapsto \frac{2z}{1+|z|^2}$ has denominator of degree larger than numerator.

Second, we no longer can say that the group must be fixed-point-free, or even finite, in the case where the map is smooth to the boundary. It need not have full rank on the boundary (see proof of proposition *4.3.1* below). In fact, it can even lower dimension from domain to target.

4.2.3 Example: $f(z) = ||z||^2$ takes B_n properly to B_1. This map is invariant under the action of the entire unitary group $U(n)$.

We have illustrated by way of very simple examples some of the greater diversity exhibited by nonholomorphic proper polynomial maps between balls. We have also shown how lens space homeomorphisms can give rise to (nonholomorphic) invariant proper monomial maps between balls, even in cases when it is known that no smooth holomorphic such maps can exist, as in the first example. We now give a nonexistence result, similar to theorem *1.2.2*.

4.2.4 Proposition: Suppose f is a possibly nonholomorphic proper monomial map from B_2 to some B_N and f it is invariant under the action of a unitary group containing $\gamma = \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^q \end{pmatrix}$. Then p and q cannot both be even.

Outline of proof: One proves this in a manner similar to the proof of *1.2.2*. We form the real polynomial with positive coefficients $p = ||f||^2$ that is identically 1 on the hyperplane $x + y = 1$. We look at the high degree term in $p(x, 1-x) = 1$ and employ a parity argument to show that it cannot be zero.■

4.3 Invariance under finite groups with fixed points

In this section we investigate proper holomorphic maps between balls that are invariant under the action of arbitrary finite unitary groups. In particular, these groups need not be fixed-point-free, in which case the following classical result directs us to maps that are not C^1−smooth to the boundary. (Thus we must remove hypothesis (iii) as given in section *4.1*.)

4.3.1 Proposition: Suppose $\Gamma \subset U(n)$ is a finite unitary group that acts with fixed points. That is, there is some $\gamma \in \Gamma, \gamma \neq 1$, and some $z \in \mathbf{C}^n, z \neq 0$ for which $\gamma(z) = z$. Suppose f is a proper holomorphic map from B_n to some B_N. Then f is not C^1−smooth to the boundary.

Proof: Strong pseudoconvexity of B_n and B_N implies that if f is C^1−smooth to the boundary then it must have full rank everywhere thereon ([Ru3, 15.3.8] or [CS2, lemma 1]). On the other hand, if $f(\gamma(z)) = f(z)$, then f is not locally one-to-one on some proper subspace containing the complex line spanned by z. This is so because γ by assumption does not fix all of $\mathbf{C}^n$; call the subspace it fixes S. Then as it acts linearly it will map any point close to but not on S to a different point nearby. Hence, as f is not locally one-to-one on S, it fails to have full rank there by the rank theorem of multi-variable calculus. As this subspace intersects the boundary sphere, we have a contradiction to the map f having full rank on the boundary.■

One may now ask whether for an arbitrary finite unitary group $\Gamma \subset U(n)$, there is a proper holomorphic map from B_n/Γ to some B_N, possibly not C^1−smooth to the boundary. If Γ is fixed-point-free, then Forstneric [Fo1] showed that such a map must exist. He obtained it as follows. Take $\{q_1, \ldots, q_s\}$ to be a basis for the algebra of Γ−invariant polynomials (it is finite by the Hilbert result mentioned in the proof of proposition *4.1.3*). Then the map $q(z) = (q_1(z), \ldots, q_s(z))$ maps $\mathbf{C}^n/\Gamma$ to a subvariety $V = q(\mathbf{C}^n) \subset \mathbf{C}^s$. As Γ is fixed-point-free, q is nonsingular away from the origin. Hence the image of the ball, $V_1 = q(B_n)$, is strongly pseudoconvex with nonsingular real analytic boundary in V. That this now implies V_1 imbeds into some complex ball is a deep result in [Fo4]. Again we remark that, by theorems in [Fo1] and in chapter one of this thesis, for most fixed-point-free finite unitary groups such an invariant map cannot be $C^\infty-$smooth to the boundary.

Another approach to this topic, that works for arbitrary finite unitary groups, is through the embedding theorems of Løw. I thank Franc Forstneric for suggesting this line of attack. In the remainder of this discussion, Γ is an arbitrary finite unitary group.

The following theorem and its proof are found in [Lø].

4.3.2 Theorem (Løw): Suppose Ω is a relatively compact strongly pseudoconvex domain in $\mathbf{C}^n$ with C^2-smooth boundary. For m sufficiently large (how large depends only on n), suppose ϕ is a positive real valued continuous function on the boundary $\partial\Omega$, and $f : \partial\Omega \to \mathbf{C}^m$ is continuous with $||f(z)|| < \phi(z)$ at all points. Then there exists a continuous function on the closed domain $g : \overline{\Omega} \to \mathbf{C}^m$, holomorphic in Ω, such that $||f(z) + g(z)|| = \phi(z)$ for all $z \in \partial\Omega$.

4.3.3 Proposition: Suppose Ω is the ball B_n, and f and ϕ are $\Gamma-$invariant. Then we may take g to be $\Gamma-$invariant as well.

We will outline a proof of this proposition later. First we use it to obtain the main result of this section.

4.3.4 Theorem: Suppose $\Gamma \subset U(n)$ is a finite unitary group. Then there exists a proper holomorphic map from B_n/Γ to some B_N.

Proof: Let $\{q_1, \ldots, q_s\}$ be a basis for the algebra of $\Gamma-$invariant polynomials. Define the map $\tilde{q}(z) = (q_1(z), \ldots, q_s(z))$ and let $M = 2\max\{||\tilde{q}(z)|| : z \in S^{2n-1}\}$. We now let $q = \frac{1}{M}\tilde{q}$. By construction we have $|q| < 1$ on $\overline{B}_n$. We take $f(z) = 0$ and $\phi(z) = \sqrt{1 - ||q(z)||^2}$, for z in the boundary sphere $\partial B_n = S^{2n-1}$. By *4.3.3* there is a continuous $\Gamma-$invariant function $g : \overline{B}_n \to C^m$, holomorphic on the open ball, such that $||g||^2 = 1 - ||q||^2$ on the boundary sphere. We now form the map $z \longmapsto (q(z), g(z))$. As the component polynomials in q comprise a basis for the algebra of $\Gamma-$invariant polynomials, we claim that $q(z) = q(w)$ if and only if

$z = \gamma(w)$ for some $\gamma \in \Gamma$. (We will prove this in lemma *4.3.9*.) Thus this map is an embedding of B_n/Γ into B_{s+m}.■

In light of the preceding results, this theorem is naturally of considerable interest in its own right. In addition, it allows us to derive examples with unusual properties, as below.

4.3.5 Example: Let Γ be the reflection 2–group generated by $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. A basis of the Γ–invariant polynomial algebra is $\{z^2, w\}$. Let $q(z,w) = \frac{1}{2}(z^2, w)$. Then q is Γ–invariant and maps B_2 to the nonsmooth domain $\Omega = \left\{(\zeta_1, \zeta_2) : |\zeta_1| + |\zeta_2|^2 < \frac{1}{2}\right\}$. By the proof of *4.3.4*, there exists an embedding f of Ω, a domain with non–C^2–smooth boundary, as a closed complex submanifold of some B_N. Specifically, there is a Γ–invariant function $g : \overline{B}_n \to C^m$, holomorphic on the open ball, such that $||g||^2 = 1 - ||q||^2$ on the boundary sphere. As g is invariant, it is a function of the (rescaled) invariant basis polynomials $\zeta_1 = \frac{z^2}{2}$ and $\zeta_2 = \frac{w}{2}$. The embedding of Ω is then $(\zeta_1, \zeta_2) \mapsto (\zeta_1, \zeta_2, g(\zeta_1, \zeta_2))$.

4.3.6 Example: Take Γ and f as in *4.3.5*. Then f is indeed a Γ–invariant proper map from B_2 to B_N and it is continuous on the closed ball. We write

$$f(z,w) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha w^\beta \tag{4.3.7}$$

where the coefficients are vectors in $\mathbf{C}^N$. Now we define $g : B_2 \mapsto l_2(\mathbf{C})$ a monomial map where for each pair (α, β) in *(4.3.7)* our new map g has an entry $||c_{\alpha,\beta}||z^\alpha w^\beta$. This is simply the monomialization technique in [D1]; it is shown there that such g takes B_2 properly to the open unit ball $\left\{(w_1, w_2, \ldots) : \sum_{j=1}^{\infty} |w_j|^2 < 1\right\}$ in $l_2(\mathbf{C})$. Since g is Γ–invariant (and hence a function of the basis monomials) we have rank $(g) = 1 < 2$ on the hyperplane $z = 0$ (where g is locally two-to-one). We thus obtain an example of a smooth proper holomorphic map between balls that does not have full rank at every point on the boundary, in the case where the range ball is infinite dimensional.

Note that as f is not even smooth on $\overline{B}_2$, it cannot extend to any neighborhood thereof. Thus $\left|\left|\sum c_{\alpha,\beta} z^\alpha w^\beta\right|\right|^2$ is divergent outside the closed ball, and hence $||g||^2 = \sum_{\alpha,\beta} \left|\left|c_{\alpha,\beta}\right|\right|^2 \left|z^\alpha w^\beta\right|^2$ also diverges there. Our monomial map g therefore does not converge in any neighborhood of the boundary sphere. Other examples of monomial maps that converge only on the unit ball may be found in [CS3]. They are simple to construct.

Outline of proof of proposition 4.3.3: A proof involves carefully checking the proof of theorem *4.3.2* given in [Lø] and noting that the construction of g may be averaged over the group Γ. We sketch the details below and in the next section.

Løw uses a technical lemma that is the workhorse of the proof of theorem *4.3.2*. The statement alone is a mouthful.

4.3.8 Lemma (Løw): There exist positive constants δ_0, C, D such that if $f : S^{2n-1} \to \mathbb{C}^{2N}$ is continuous, ϕ a positive real function on S^{2n-1} with $b\phi(z) \leq ||f(z)|| \leq \phi(z)$ for some $b < 1$ and all $z \in S^{2n-1}$, $0 < \epsilon \leq \delta_0$, $\epsilon \leq (1-b)^{3/4}$, then there exists an entire function $g : \mathbb{C}^n \to \mathbb{C}^{2N}$ such that for all $z \in S^{2n-1}$ we have

$$(i) \quad ||f(z) + g(z)|| \leq \left(1 + C\epsilon(1-b)^{\frac{1}{2}}\right)\phi(z)$$

$$(ii) \quad ||f(z) + g(z)|| \geq \left(b + D\epsilon^{\frac{1}{4}}(1-b)\right)\phi(z)$$

$$(iii) \quad ||g(z)|| \leq C(1-b)^{\frac{1}{2}} \; .$$

To get a group-invariant function g we must add to this lemma: (iv) We can take g to be group-invariant when f and ϕ are.

In the original statement in [Lø], g can even be made arbitrarily small on any given compact subset of the ball. We do not need this.

As it is a bit lengthy and computational, we sketch the proof of this lemma in a separate section.

Finally, Løw's proof of *4.3.2* is an iterative construction that uses *4.3.8;* it can now be carried out with this group-invariant version. (A uniform convergence on compact subset argument is used to prove that the sum of the iterations is a holomorphic map with the desired properties). We thus obtain a $\Gamma-$invariant function g for *4.3.2* and this finishes *4.3.3*.■

We now prove a result used in the proof of lemma *4.3.4*.

4.3.9 Lemma: Suppose Γ *is a finite unitary group, and* $\{q_1, \ldots, q_s\}$ *is a basis for the algebra of* $\Gamma-$*invariant polynomials in* $\mathbb{C}[z_1, \ldots, z_n]$. *Let* $q = (q_1, \ldots, q_s)$. *Then* q *is precisely* $\Gamma-$*invariant. That is,* $q(z) = q(w)$ *if and only if* $z = \gamma(w)$ *for some* $\gamma \in \Gamma$.

Proof: This result is proven by Rudin in [Ru2]. We reproduce his elegant proof below. It is clear that $z = \gamma(w)$ implies $q(z) = q(w)$. We now want to show the converse: that two points are identified by the invariant basis only when they are the same $\Gamma-$orbit. Suppose that z is not in the $\Gamma-$orbit of w. There is a polynomial g in n variables such that $g(z) = 0$ but $g(\gamma(w)) = 1$ for all $\gamma \in \Gamma$. Now let $f = \prod_{\gamma \in \Gamma} g \circ \gamma$. By construction f is a $\Gamma-$invariant polynomial; hence $f = h \circ q$ for some polynomial h. Also $f(z) = 0$ while $f(w) = 1$. Thus $q(z) \neq q(w)$.■

We make several remarks about the results in this section.

(i) We may take $m = n + 1$ in the results of Løw, under the assumption that our domain has $C^\infty-$smooth boundary, and hence in all the applications of this section. This comes from refinements on the work in [Lø] due to several authors. A good reference for this is section *4* of the survey article [Fo3].

(ii) These theorems all yield maps that are continuous (though typically no smoother) to the boundary of the domain. Using techniques presented, for example, in [CS3] and section *4* of [Fo3], one can obtain maps that are not continuous to the boundary. These techniques raise the codimension by at least n.

(iii) When we create a map invariant under the action of a unitary group acting with fixed points, then on some subspace that map has a derivative that vanishes in directions orthogonal to that subspace. This persists arbitrarily close to the boundary sphere. This is in marked contrast to lemma *1* in [CS2], where it is shown that a proper holomorphic map between balls that is

C^1-smooth to the boundary has, at any boundary point, directional derivatives of magnitude at least 1 in all directions orthogonal to the linear subspace containing that boundary point.

(iv) In light of the result of section *4.1*, we see that the group averaging technique above must fail to produce a $\Gamma-$invariant proper holomorphic map in the case where Γ is infinite. We indicate what goes astray. We may assume that we have a closed unitary subgroup, since otherwise there is no finite Haar measure with which to perform the averaging. Thus we may assume that Γ has a one parameter subgroup with a generator of the form in *4.1.2*.

In general, attempting to average any construction over a compact infinite group Γ will involve integrating some function of $\gamma(z)$ over $\gamma \in \Gamma$, and this will not yield a holomorphic function in all variables $(z_1, \ldots z_n)$, as seen in the second proof of proposition *4.1.1*.

4.4 Proof of the group-averaged Løw lemma

The proof below is essentially that found in [Lø]. We will follow exactly the notation therein. We do a slightly more general version, and also average our construction over a group.

We define a metric $\delta(z,w) = \frac{\|z-w\|}{\sqrt{2}}$. When $z, w \in S^{2n-1}$ we have $1 - \mathrm{Re}\langle z, w\rangle = \delta^2(z,w)$. Thus $|\exp(1 - \langle z, w\rangle)| = \exp(-\delta^2(z,w))$ in that case. Also, for $z \in S^{2n-1}$ we define $\widetilde{B}(z,r) = \{w \in S^{2n-1} : \delta(z,w) < r\}$. We will call this the ball of radius r centered at z; the notation and/or context will make clear that we are using the rescaling rather than the customary metric.

Løw takes the following covering lemma as known:

4.4.1 Lemma: For any positive integer n there exists a positive integer $N(n)$ such that for any $r > 0$ there are N finite families of balls of radius $3r$, $\mathcal{F}_i = \left\{\widetilde{B}(z_{i,j}, 3r) : 1 \le j \le N_i\right\}$, so that the union of balls $\widetilde{B}(z_{i,j}, r)$ of same centers and radius r covers the unit sphere S^{2n-1} while each family of balls of radius $3r$ is pairwise disjoint. (That is, $\widetilde{B}(z_{i,j}, 3r) \cap \widetilde{B}(z_{i,k}, 3r) = \emptyset$ for $j \neq k$.)

We note first that N is independent of r (though the $z_{i,j}'s$ are not). We also remark that we can take any $\alpha > 1$ in place of the factor 3 in this lemma; we then obtain N as a function of n and α.

We let $k = \min\left\{\phi(z) : z \in S^{2n-1}\right\}$. Note that $k > 0$. Next, for N as in lemma *4.3.8*, we let $e_1, \ldots e_{2N}$ denote the standard orthonormal basis on $\mathbf{C}^{2N}$. For nonzero $w \in B_{2N}$ we let $T_w = \{w + n : \langle w, n\rangle = 0\}$ denote the complex tangent space at w of (the standard metric) ball through w with center at origin.

We now define N (noncontinuous) vector fields. For $w = (w_1, \ldots w_{2N}) \in \mathbf{C}^{2N}$ and $1 \leq i \leq N$, we define the vector $n_i(w) = \overline{w_{2i}} e_{2i-1} - \overline{w_{2i-1}} e_{2i}$ unless $w_{2i-1} = w_{2i} = 0$, in which case we take $n_i(w) = e_{2i-1}$. These vector fields are orthonormal and for nonzero w we also have $n_i(w) \perp w$ (our vector field defines directions orthogonal to $w = f(z)$ in which to "push" our refinement function g).

We start out our construction by taking the case where $b > 0$. The following lemmas are straightforward.

4.4.2 Lemma: The vector $w + \sum_{i=1}^{N} \lambda_i n_i(w)$ *lies in* B_{2N} *when* $||\lambda||^2 < 1 - ||w||^2$.

4.4.3 Lemma: There exists some $\delta > 0$ *such that, if* $I \subset \{1, \ldots N\}$ *is an index set and* $w, w_i \in B_{2N}$ *(for* $i \in I$*) satisfy* $||w - w_i|| < \delta$ *and* $||w||, ||w_i|| > b$*, then there exist orthonormal vectors* $n_i \in T_w$ *with* $||n_i - n_i(w_i)|| < \epsilon$.

4.4.4 Lemma: We can pick $r > 0$ *to be sufficiently small that, whenever we have* $\delta(z_1, z_2) < 3r$*, we also have*

$$(i) \quad ||\phi(z_1) - \phi(z_2)|| < \epsilon k(1-b)^{\frac{1}{2}}$$
$$(ii) \quad \left|\left|\frac{f(z_1)}{\phi(z_1)} - \frac{f(z_2)}{\phi(z_2)}\right|\right| < \min\left(\delta, \epsilon^2 k(1-b)\right).$$

For r as found in *4.4.4*, let $\mathcal{F}_1, \ldots \mathcal{F}_N$ be the disjoint families of balls of radius $3r$ from lemma *4.4.1*, with center points denoted as in the lemma by $z_{i,j}$. That is, $\mathcal{F}_i = \left\{\widetilde{B}(z_{i,j}, 3r) : 1 \leq j \leq N_i\right\}$.

Now we define $g(z) = \sum_{i=1}^{N} g_i(z)$, where

$$g_i(z) = \frac{1}{|\Gamma|}\sum_{j=1}^{N_i}\sum_{\gamma\in\Gamma}\left[\frac{\phi^2(z_{i,j}) - ||f(z_{i,j})||^2}{N}\right]^{\frac{1}{2}} \exp\left[-m(1-\langle\gamma(z), z_{i,j}\rangle)\right] n_i(f(z_{i,j})).$$

(In [Lø] there is no averaging over the group, so the function used there is a bit simpler). The parameter m is determined below; it will be large. Note that the vector function g_i is nonzero only in the $2i-1$ and $2i$ slots. It consists of functions which peak at all the $z'_{i,j}s$ and drop off inside the ball. Also note that our function g is entire, and, as it is constructed to be group-invariant, it satisfies *4.3.8. (iv)*.

4.4.5 Lemma: There is a constant C_1 (independent of ϵ) such that if ϵ is sufficiently small, and $mr^2 = \frac{1}{9}\ln\left(\frac{C_1}{\epsilon}\right)$, then $||g_i(z)|| < \epsilon(1-b)^{\frac{1}{2}}\phi(z)$ provided that the orbit $\Gamma(z)$ does not hit any $3r$−ball in $\mathcal{F}_i$. This also holds if we sum only over those $3r$−balls in $\mathcal{F}_i$ that do not contain any points in the orbit. (Note that at most $|\Gamma|$ such balls in $\mathcal{F}_i$ can hit this orbit).

The proof is a bit subtle. An appropriate reference can be found in [Lø].

We now let $I(z) = \{i : z \in B(z_{i,j(i)}, 3r)\}$ for some (unique) $1 \leq j(i) \leq N_i$. We let $w = f(z)$, $w_i = f(z_{i,j(i)})$, and let n_i be the vector for the pair w, w_i given by lemma *4.4.3*.

On to the estimates.

$$\begin{aligned}
&||[f(z)+g(z)]- \\
&\quad [w + \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{\phi^2(z)-||w||^2}{N}\right]^{\frac{1}{2}} \exp\left(-m\left(1-\langle\gamma(z), z_{i,j(i)}\rangle\right)\right)n_i]|| \\
\leq\ & ||g(z) - \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{\phi^2(z)-||w||^2}{N}\right]^{\frac{1}{2}} \exp\left(-m\left(1-\langle\gamma(z), z_{i,j(i)}\rangle\right)\right)n_i(w_i)|| \\
&\quad + 2N^{\frac{1}{2}}\epsilon(1-b)^{\frac{1}{2}}\phi(z) \\
\leq\ & ||g(z) - \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{\phi^2(z_{i,j(i)})-||w_i||^2}{N}\right]^{\frac{1}{2}} \exp\left(-m\left(1-\langle\gamma(z), z_{i,j(i)}\rangle\right)\right)n_i(w_i)|| \\
&+ 2N^{\frac{1}{2}}\epsilon(1-b)^{\frac{1}{2}}\phi(z) \\
&+N^{-\frac{1}{2}}\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left|\left[\phi^2(z_{i,j(i)}) - ||w_i||^2\right]^{\frac{1}{2}} - \left[\phi^2(z)-||w||^2\right]^{\frac{1}{2}}\right|
\end{aligned}$$

$$\leq 2N^{\frac{1}{2}}\epsilon(1-b)^{\frac{1}{2}}\phi(z)$$
$$+N^{-\frac{1}{2}}\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left|\left[\phi^2(z_{i,j(i)})-\|w_i\|^2\right]^{\frac{1}{2}}-\left[\phi^2(z)-\|w\|^2\right]^{\frac{1}{2}}\right|$$
$$+N\epsilon(1-b)^{\frac{1}{2}}\phi(z).$$

To estimate the middle term: we have

$$N^{-\frac{1}{2}}\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left|\left[\phi^2(z_{i,j(i)})-\|w_i\|^2\right]^{\frac{1}{2}}-\left[\phi^2(z)-\|w\|^2\right]^{\frac{1}{2}}\right|$$
$$=N^{-\frac{1}{2}}\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left|\left[1-\frac{\|w_i\|^2}{\phi^2(z_{i,j(i)})}\right]^{\frac{1}{2}}\phi(z_{i,j(i)})-\left[1-\frac{\|w\|^2}{\phi^2(z)}\right]^{\frac{1}{2}}\phi(z)\right|$$
$$\leq N^{\frac{1}{2}}\epsilon(1-b)^{\frac{1}{2}}\phi(z)$$
$$+\phi(z)N^{-\frac{1}{2}}\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left|\left[1-\frac{\|w_i\|^2}{\phi^2(z_{i,j(i)})}\right]^{\frac{1}{2}}-\left[1-\frac{\|w\|^2}{\phi^2(z)}\right]^{\frac{1}{2}}\right|.$$

Exactly as in [LØ], this last term can be shown to be less than $\sqrt{10}N\epsilon(1-b)^{\frac{1}{2}}\phi(z)$.

We put these inequalities together to conclude that

$$\text{(4.4.6)}\qquad \|[f(z)+g(z)]-$$
$$[w+\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{\phi^2(z)-\|w\|^2}{N}\right]^{\frac{1}{2}}\exp\left(-m(1-\langle\gamma(z),z_{i,j(i)}\rangle)\right)n_i]\|$$
$$<12N\epsilon(1-b)^{\frac{1}{2}}\phi(z).$$

The second bracketed expression in the left hand side of *4.4.6* can be rewritten as

$$[\frac{w}{\phi(z)}+\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{1-\frac{\|w\|^2}{\phi^2(z)}}{N}\right]^{\frac{1}{2}}\exp\left(-m(1-\langle\gamma(z),z_{i,j(i)}\rangle)\right)n_i]\phi(z).$$

As the expression now in brackets lies in the unit ball by lemma *4.4.2,* we see that we have satisfied *4.3.8 (i).*

Next, $f(z)=w$, so *4.4.6* immediately yields

$$\|g(z)\|<12N\epsilon(1-b)^{\frac{1}{2}}\phi(z)+N^{\frac{1}{2}}\left[1-\frac{\|w\|^2}{\phi^2(z)}\right]^{\frac{1}{2}}\phi(z)$$
$$\leq 12N\epsilon(1-b)^{\frac{1}{2}}\phi(z)+2N^{\frac{1}{2}}(1-b)^{\frac{1}{2}}\phi(z).$$

This proves *4.3.8 (iii)*.

Inequality *4.4.6* also yields

$$(4.4.7) \qquad \|f(z)+g(z)\|$$

$$> \|w + \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{\phi^2(z)-\|w\|^2}{N}\right]^{\frac{1}{2}} \exp\left(-m\left(1-\langle\gamma(z), z_{i,j(i)}\rangle\right)\right)n_i\|$$

$$-12N\epsilon(1-b)^{\frac{1}{2}}\phi(z).$$

We now estimate the square of the norm appearing on the right hand side. We use the fact that w and each of the separate $n_i's$ are pairwise orthogonal. This norm square is thus equal to

$$\|w\|^2 + \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\frac{\phi^2(z)-\|w\|^2}{N}\exp\left(-2m\delta^2(\gamma(z), z_{i,j(i)})\right).$$

For each $\gamma \in \Gamma$ there exists some $i \in I(\gamma(z))$ such that $\delta(\gamma(z), z_{i,j(i)}) < r$. The norm square above thus is at least

$$(4.4.8) \qquad \|w\|^2 + \frac{\phi^2(z)-\|w\|^2}{N}\exp\left(-2mr^2\right)$$

$$= \|w\|^2 + \frac{\phi^2(z)-\|w\|^2}{N}\left(\frac{\epsilon}{C_1}\right)^{\frac{2}{9}}$$

$$\geq b^2\phi^2(z) + N^{-1}C_1^{-\frac{2}{9}}\epsilon^{\frac{2}{9}}(1-b^2)\phi^2(z).$$

Now use $\epsilon \leq (1-b)^{\frac{9}{4}}$ and *4.4.8* in *4.4.7* to obtain

$$\|f(z)+g(z)\|$$

$$\geq b\phi(z) + \frac{1}{2}N^{-1}C_1^{-\frac{2}{9}}\epsilon^{\frac{2}{9}}(1-b)\phi(z) - 12N\epsilon(1-b)^{\frac{1}{2}}\phi(z)$$

$$\geq \left[b + D\epsilon^{\frac{1}{4}}(1-b)\right]\phi(z)$$

for some appropriate constant D. This proves *4.3.8 (ii)*.

We still must consider the case where $b = 0$. It is similar. We replace b with ϵ^2 in lemma *4.4.3*. Then for $\|w\| \geq 2\epsilon^2$ the proof that g satisfies the properties of lemma *4.3.8* is as above. For $\|w\| \leq 2\epsilon^2$ these properties follow from the estimate

$$\|[f(z)+g(z)]-$$

$$[w + \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{i\in I(\gamma(z))}\left[\frac{\phi^2(z)-\|w\|^2}{N}\right]^{\frac{1}{2}}\exp\left(-m\left(1-\langle\gamma(z), z_{i,j(i)}\rangle\right)\right)n_i(w_i)]\|$$

$$< 2N\epsilon\phi(z)$$

This inequality is itself derived as in the previous case, but bypassing the first inequality that involved n_i where we now have $n_i(w_i)$.

This finishes our proof of lemma *4.3.8.*■

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Vita

Daniel Lichtblau was born on December 9, 1957, in New York City, and is now about 12 years old. He grew up and attended public schools in Englewood, New Jersey, graduating from Dwight Morrow High School in 1975. He attended Harvard College and received an A.B. (Latin for B.A.) degree in physics with honors in 1979. He worked at a variety of jobs requiring use of a hammer (e.g. carpentry, substitute teaching) before joining the compilers division of Intermetrics, Inc. (located in Cambridge, Massachussetts) in 1981. He worked on construction of compilers and related software at that firm until 1985, when he entered the doctoral program in mathematics at the University of Illinois at Urbana-Champaign (Champaign-Urbana). He has been a teaching assistant during academic years, with one semester on fellowship for good behavior, or perhaps because his teaching was deemed too atrocious to be allowed to continue. He has spent the past four summers as a research assistant. He has one wife, one daughter, one shepherd-husky, two mortgages, no mistresses, and no job prospects for the near future. He makes an excellent red ale and occasionally a decent bottle of grape, strawberry, raisin, or crabapple wine. He also makes decent furniture, though has found no time to engage in that hobby in recent years. He enjoys backpacking, canoeing, running, and dirty socks, uh, sex. He does not enjoy swimming but does it regularly to keep fit and save wear and tear on his knees and back. He speaks English better than a native and almost fluently. He despises all political views, particularly his own. He watches too much television, especially old movies. His wife says he reads too much. His advisor says he spends too much time gardening and working on his house. His mother says he frets too much. His daughter says "Daddy, stop it!" He says they all carp too much and should go climb trees. He is ready for a vacation.