A chromatic art gallery problem

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Abstract

The *art gallery problem* asks for the smallest number of guards required to see every point of the interior of a polygon P. We introduce and study a similar problem called the *chromatic art gallery problem*. Suppose that two members of a finite point guard set $S \subset P$ must be given different colors if their visible regions overlap. What is the minimum number of colors required to color any guard set (not necessarily a minimal guard set) of a polygon P? We call this number, $\chi_G(P)$, the *chromatic guard number* of P. We believe this problem has never been examined before, and it has potential applications to robotics, surveillance, sensor networks, and other areas. We show that for any spiral polygon P_{spi}, $\chi_G(P_{spi}) \leq 2$, and for any staircase polygon (strictly monotone orthogonal polygon) P_{sta}, $\chi_G(P_{sta}) \leq 3$. We also show that for any positive integer k, there exists a polygon P_k with $3k^2 + 2$ vertices such that $\chi_G(P_k) \geq k$.

Figure 1: [left] Two strictly monotone orthogonal polygons. They take the form of two vertices connected by two different staircase-shaped paths. This is the definition of "staircase polygon" that we will use. [right] Two orthogonal convex fans. This family of polygons is a subset of the strictly monotone orthogonal polygons. Some other papers use the term "staircase polygon" to refer exclusively to the orthogonal convex fans, but we will not.

1 Introduction

Suppose a robot is navigating a region populated with radio beacons. The robot is equipped with the following primitives: drive toward the beacon, drive away from the beacon, and drive along the level sets of the beacon's intensity (similar to the model in [24]). If this robot were to be in an area where two different beacons were broadcasting on the same frequency, it may become confused and the action that it takes when ordered to perform a certain primitive could become unpredictable. The same phenomenon could happen with other sensing and actuation models. A robot navigating visually and being told "drive toward the red landmark" may get confused if there are two red landmarks in its visibility region. This raises a natural question: How many classes of partially distinguishable guards are required to guard a given area? In this paper, we try to answer this question for bounded simply-connected polygonal areas. We assume that a robot cannot see a given landmark if the polygon boundary is in the way.

Spiral polygons are a heavily studied area in visibility. Special results for this class of polygons are availible for the watchman route problem [18], the weakly cooperative guard problem [12], the visibility graph recognition problem [4], point visibility isomorphisms [14], and triangulation [25]. An algorithm for decomposing general polygons into a minimum number of spiral polygons was described in [11]. We choose to focus on spiral polygons because we think that they could be a useful building block in solving the chromatic guard number problem for general polygons.

There are two commonly used definitions for staircase polygons (see Figure 1). The one we use is that a staircase polygon is a strictly monotone orthogonal polygon. This definition was also used in [8], which found an asymptotically tight bound on the number of guards required to solve the prison yard problem for these polygons. The problem of placing a maximum area staircase of this kind in a planar point set was studied in [16]. They have also been examined in the context of self-avoiding walks for physics modelling in [22] and [23]. These polygons always take the form of two convex right angle vertices joined by two subchains of alternating convex and reflex vertices (described in greater detail in Section 4.2). The other definition, which is a special case of the one we use, is that a staircase polygon is an orthogonal convex fan. This definition is used in [1]. These polygons trivially have a chromatic guard number of one (they are star-shaped). From this point forward, "staircase polygons" will refer exclusively to strictly monotone orthogonal polygons. We choose to focus on these staircase polygons because of their potential as building blocks for a bound on the chromatic guard number of general orthogonal polygons.

Section 2 contains the formal definition of the problem. Section 3 contains a proof of a lower bound on the chromatic guard number for general polygons. Section 4 contains upper bounds on the chromatic guard number for spiral polygons and staircase polygons. Section 5 discusses directions of future research.

2 Problem definition

Let a *polygon* P be a closed, simply connected, polygonal subset of $\mathbb{R}^2$ with boundary ∂P. A point $p \in P$ is *visible* from point $q \in P$ if the closed segment $\overline{pq}$ is a subset of P. The *visibility polygon* $V(p)$ of a point $p \in P$ is defined as $V(p) = \{q \in P \mid q \text{ is visible from } p\}$. Let a *guard set* S be a finite set of points in P such that $\bigcup_{s \in S} V(s) = P$. The members of a guard set are referred to as *guards.* A pair of guards $s, t \in S$ is called *conflicting* if $V(s) \cap V(t) \neq \emptyset$. Let $C(S)$ be the minimum number of colors required to color a guard set S such that no two conflicting guards are assigned the same color. Let $T(P)$ be the set of all guard sets of P. Let $\chi_G(P) = \min_{S \in T(P)} C(S)$. We will call this value $\chi_G(P)$ the *chromatic guard number* of the polygon P. Note that the number of guards used can be as high or low as is convenient. We want to minimize the number of colors used, not the number of guards.

The notion of conflict can be phrased in terms of *link distance.* The link distance between two points $p, q \in P$ (denoted $LD(p, q)$) is the minimum number of line segments required to connect p and q via a polygonal path. Each line segment must be a subset of P.

Theorem 1. *Two guards $s_1, s_2 \in P$ conflict if and only if $LD(s_1, s_2) \leq 2$.*

Proof. If $LD(s_1, s_2) = 1$, then they are mutually visible, and obviously conflict.

If $LD(s_1, s_2) = 2$, then there exists a point $r \in P$, such that $\overline{s_1 r}, \overline{r s_2} \subseteq P$. Since $\overline{s_1 r} \subseteq P$, $r \in V(s_1)$. Since $\overline{r s_2} \subseteq P$, $r \in V(s_2)$. Because r is in $V(s_1)$ and $V(s_2)$, the intersection of $V(s_1)$ and $V(s_2)$ is non-empty; therefore s_1 and s_2 conflict.

If s_1 and s_2 conflict, then let r be a point in the intersection of $V(s_1)$ and $V(s_2)$. Since $r \in V(s_1)$, $\overline{s_1 r} \subseteq P$. Since $r \in V(s_2)$, $\overline{r s_2} \subseteq P$. Because $\overline{s_1 r}, \overline{r s_2} \subseteq P$, $LD(s_1, s_2) \leq 2$. ☐

3 Lower bounds on the chromatic guard number

Theorem 2. *For every positive integer k, there exists a polygon P with $3k^2 + 2$ vertices such that $\chi_G(P) \geq k$.*

Proof. The polygon P is a version of the standard "comb" used to show the occasional necessity of $\lfloor n/3 \rfloor$ guards in the standard art gallery problem [3]. The vertex list of P is $[(0,1), (1,0), (2,1), (4,1), (5,0), (6,1) \ldots (4k^2-4, 1), (4k^2-3, 0), (4k^2-2, 1), (4k^2-2, 2k-2), (0, 2k-2)]$. This polygon has $3k^2 + 2$ vertices, and it consists of a closed rectangular region (the *body region*) with corners $(0,1), (4k^2-2, 1), (4k^2-2, 2k-2), (0, 2k-2)$ that has k^2 *notches* attached to the bottom edge. Call the vertices with a y coordinate of zero *apex points.* Note that each notch has a unique apex point. A guard with coordinates (x, y) will be referred to as an *apex guard* if $y < 1$ and will be referred to as a *body guard* if $y \geq 1$ (See Figure 2).

Each body guard can guard up to k distinct notches. However, since the visibility polygon of a body guard includes the entire body region, and every guard's visibility polygon intersects the body region, a body guard will conflict with every other guard in the polygon. Let m_{body} be the number of body guards used in a guard set of P.

Each apex guard can guard only one notch. However, two apex guards will not conflict if they are placed far enough away from each other. Since the top edge of P has a y coordinate of $2k-2$, two apex guards are only forced to conflict if the distance between the apex points of their corresponding notches is $4k$ or less. Let a set of k notches be *consecutive* if the maximum distance between the apex points of any two notches in the set is $4k$. Let m_{apex} be the maximum number of apex guards in any consecutive set of k notches in P.

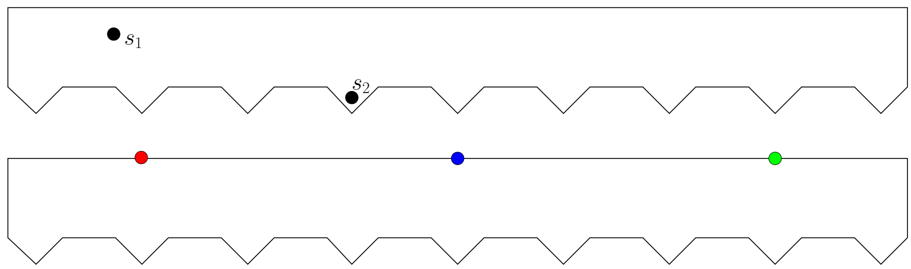

Figure 2: [top] The polygon P (Theorem 2) for $k = 3$. The guard s_1 is a body guard, and the guard s_2 is an apex guard. [bottom] A guard placement that requires three colors.

Suppose the polygon P has a guard set S assigned to it that requires only $\chi_G(P)$ colors. Consider a set of k consecutive notches in P that contains m_{apex} apex guards. All of these apex guards will conflict with each other, and all of these apex guards will conflict with all of the body guards. Therefore, $\chi_G(P) \geq m_{apex} + m_{body}$. Now, note that each body guard can guard at most k notches. Since there are k^2 notches, by the pigeonhole principle, apex guards can guard at most km_{apex} notches (see Figure 2). Since each notch must be guarded, $km_{apex} + km_{body} \geq k^2$, so $m_{apex} + m_{body} \geq k$. Therefore $\chi_G(P) \geq m_{apex} + m_{body} \geq k$.

□

4 Upper bounds on the chromatic guard number

One could just give every guard its own color. Any polygon P with n vertices can be guarded by $\lfloor n/3 \rfloor$ guards (the art gallery theorem [3]), so $\chi_G(P) \leq \lfloor n/3 \rfloor$. However, this bound is unsatisfying, because colors can often be reused. There exist polygons with an arbitrarily high number of vertices that require only two colors. We prove bounds better than $\lfloor n/3 \rfloor$ for two categories of polygons.

4.1 Spiral polygons

A *chain* is a series of points $[p_1, p_2, \ldots, p_n]$ along with line segments connecting consecutive points. A *subchain* is a chain that forms part of the boundary of a polygon. The points p_1 and p_n are called *endpoints*, and all other points are *internal vertices*. A *convex subchain* is a subchain where all the internal vertices have an internal angle of less than π radians. A *reflex subchain* is a subchain where all the internal vertices have an internal angle of greater than π radians. Note that convex and reflex subchains can trivially consist of a single line segment (if there are no internal vertices). A spiral polygon is a polygon with exactly one maximal reflex subchain (all reflex subchains of the spiral polygon must be contained within the maximal reflex subchain).

Theorem 3. *For any spiral polygon P, $\chi_G(P) \leq 2$.*

Proof. The spiral polygon consists of two subchains, a reflex subchain, and a convex subchain. Let v_s and v_t be the endpoints of the reflex subchain. Without loss of generality, assume that the path along the convex subchain from v_s to v_t runs clockwise. The guards will all be placed along the edges of the convex subchain.

Call the nth guard placed s_n. Place s_1 at v_s. Let p_n be the point most clockwise along the convex subchain that is visible from s_n. Let b_n be the most counterclockwise vertex along the reflex subchain visible from s_n. Let g_n be the vertex immediately clockwise from b_n. Let

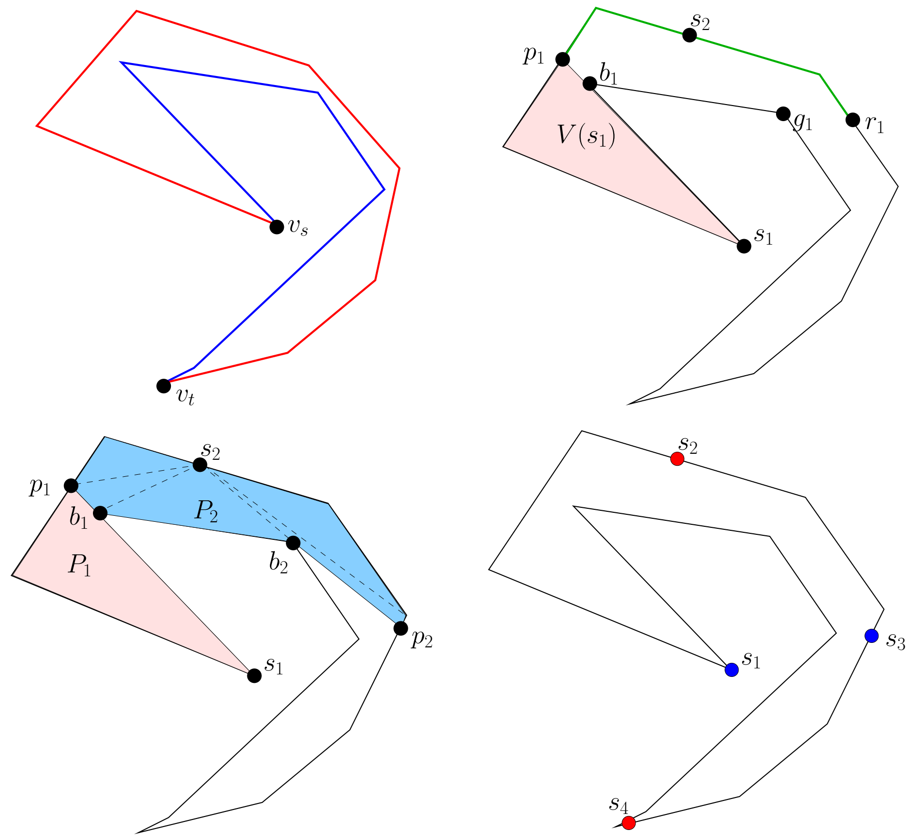

Figure 3: [top left] A spiral polygon P. The convex subchain is highlighted in red, and the reflex subchain is highlighted in blue. [top right] The first guard s_1 is placed on vertex v_s. The points p_1, b_1, g_1, and r_1 are marked and the interval that s_2 can be placed in is highlighted in green. [bottom left] Recursively showing that placed guards form a guard set. The subpolygon P_1 is assumed to be guarded by s_1. The region that s_2 is responsible for is P_2, bounded by the reflex subchain between b_1 and b_2, the edge between p_2 and b_2, the convex subchain between p_2 and p_1, and the edge between b_1 and p_1. The subpolygon P_2 has been triangulated, indicating that s_2 can guard the whole subpolygon. The triangle with endpoints p_2, b_2, and s_2 is degenerate, as those three points are colinear. [bottom right] A guard placement and 2-coloring.

r_n be the point on the convex subchain colinear with g_n and b_n and visible from both. Note that p_n and r_n define the endpoints of an interval along the convex subchain. Place s_{n+1} at a point on this interval that is not one of the endpoints. Note that this means that $s_{n+1} \notin V(s_n)$. Terminate when a guard can see v_t (see Figure 3).

We can show that this is a guard set for the polygon by triangulating the polygon using the polygon vertices, the members of S, and the points p_i and showing that each triangle has a member of S as one of its vertices. Suppose that the polygon bounded by the edges starting from p_n counterclockwise along the boundary of P until b_n and the edge between p_n and b_n has already been triangulated such that each triangle contains a vertex in the set $\{s_i | i \leq n\}$. We must show that s_{n+1} can guard the subpolygon bordered by the edges counterclockwise from p_{n+1} to p_n, the edge between p_n and b_n, the vertices counterclockwise from b_n to b_{n+1}, and the edge between b_{n+1} and p_{n+1} (call this subpolygon P_{n+1}). If each of these vertices in the subpolygon is visible from s_{n+1}, then the subpolygon can be triangulated by connecting each vertex to s_{n+1}, meaning that s_{n+1} guards the entire subpolygon (see Figure 3).

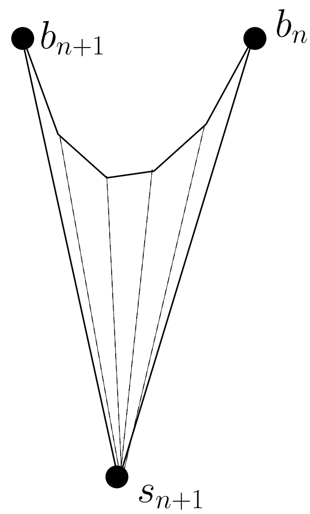

Figure 4: A polygon consisting of the edges on the reflex subchain between b_n and b_{n+1} and the edges $\overline{s_{n+1}b_n}$ and $\overline{s_{n+1}b_{n+1}}$. Since all the vertices on the reflex subchain are reflex, this polygon has only one triangulation, where all triangles have s_{n+1} as an endpoint.

Since s_{n+1} is placed on the interval in between p_n and r_n, it must be able to see the entire edge between g_n and b_n, meaning that b_n is visible from s_{n+1}. The vertex b_{n+1} is visible from s_{n+1} by definition. Examine the polygon consisting of the edges along the reflex subchain between b_n and b_{n+1}, $\overline{s_{n+1}b_n}$, and $\overline{s_{n+1}b_{n+1}}$. Since all the vertices along the reflex subchain are reflex, they cannot have edges between each other in a triangulation, so in any triangulation, they must all be connected to s_{n+1} (see Figure 4). The point p_{n+1} is visible to s_{n+1} by definition. The point p_n is visible to s_{n+1} because s_{n+1} is on the interval between p_n and r_n, and the only reflex vertex which could obstruct part of that interval's view of another part of that interval would have to lie in between b_n and g_n on the reflex subchain (by definition, there are no such vertices). Because the vertices in between p_n and p_{n+1} lie on a convex subchain, if s_{n+1} can see both p_n and p_{n+1}, then s_{n+1} can see all the vertices in between. This means that P_{n+1} can be triangulated with every triangle having s_{n+1} as an endpoint, so s_{n+1} guards P_{n+1} (the triangle with endpoints p_{n+1}, b_{n+1}, and s_{n+1} is degenerate, as those three points are colinear, but this is not a problem). This technique still works if s_{n+1} can see v_t (in this case, $p_{n+1} = b_{n+1} = v_t$). This implies inductively that S is a guard set for P.

Because all the guards are along the convex subchain, if two guards conflict, their visibility polygons must intersect somewhere along the convex subchain. Also, since $s_n \notin V(s_{n+1})$ and $s_n \notin V(s_{n-1})$, s_{n+1} cannot conflict with s_{n-1}, or there would be no room along the convex subchain to place s_n. Therefore, all evenly indexed guards can be colored red, and all oddly indexed guards can be colored blue, so $\chi_G(P) \leq 2$.

□

4.2 Staircase polygons

An *orthogonal polygon* is a polygon in which all angles are right angles. An *alternating subchain* is a subchain with at least one internal vertex, with the first and last internal vertices being convex, and with consecutive internal vertices alternating between convex and reflex. A *staircase polygon* is an orthogonal polygon consisting of two convex vertices, v_w and v_z, connected by two alternating subchains. For simplicity, we will assume without loss of generality that orthogonal

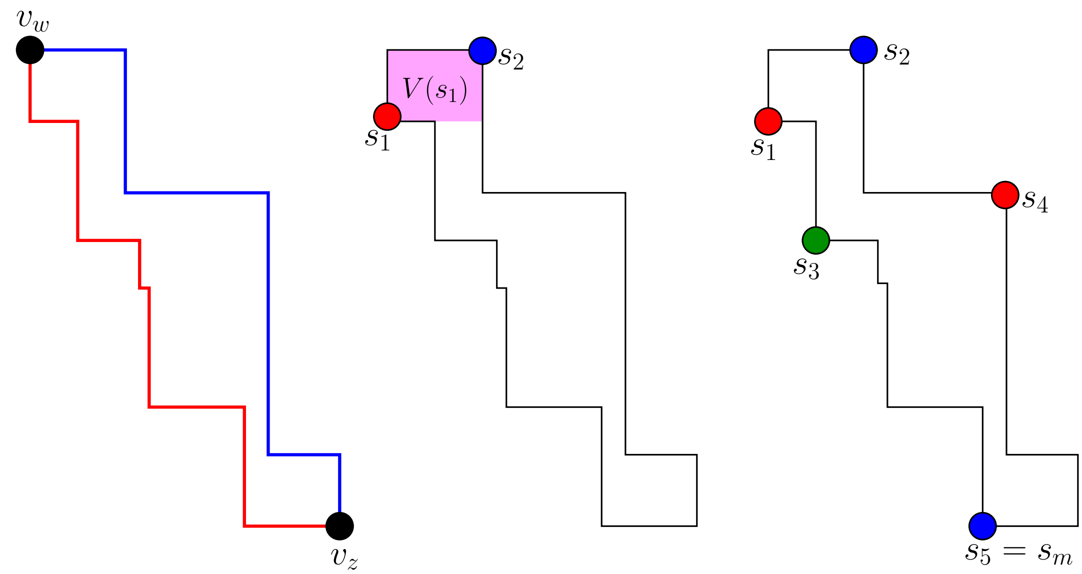

Figure 5: [left] A staircase polygon P with vertices v_w and v_z identified. The lower subchain is highlighted in red, and the upper subchain is highlighted in blue. [middle] The guard s_1 is placed on the neighbor of v_w on the lower subchain. The guard s_2 is placed on the rightmost convex vertex in $V(s_1)$. [right] A guard placement and coloring for P that uses only three colors.

polygons are always oriented such that each edge is either vertical or horizontal, and that v_w is the top left vertex, and that v_z is the bottom right vertex. Put the polygon on a coordinate plane with v_w at the $(0, 0)$ coordinate, let right be the positive x direction, and let up be the positive y direction. As mentioned earlier, "staircase polygon" is a synonym for strictly monotone orthogonal polygon.

Theorem 4. *For any staircase polygon P, $\chi_G(P) \leq 3$.*

Proof. Due to our assumptions about the orientation of the polygon P, one of the alternating subchains is going to be above the other one. Call the higher subchain the *upper subchain* and call the other subchain the *lower subchain*. Place a guard s_1 on the neighbor of v_w along the lower subchain. If guard s_i has been placed on the lower subchain, then place guard s_{i+1} on the right-most convex vertex on the upper subchain that is contained in $V(s_i)$. If guard s_i has been placed on the upper subchain, then place guard s_{i+1} on the right-most convex vertex on the lower subchain that is contained in $V(s_i)$. Stop placing guards when a guard can see v_z, and let m be the number of guards placed (See Figure 5).

First, it must be shown that s_i and s_{i+2} are not placed on the same vertex. Suppose without loss of generality that s_i is on the lower subchain. Note that the rightmost convex vertex on the lower subchain in $V(s_{i+1})$ must also be the lowest convex vertex on the lower subchain in $V(s_{i+1})$. Note also that a ray extended downward from s_{i+1} must intersect the horizontal edge incident to s_{i+2} (otherwise s_{i+2} would not be the rightmost convex vertex on the lower subchain). If this is the same horizontal edge that is incident to s_i, then the point where the ray intersects the horizontal edge incident to s_i must be a convex vertex (call it v_f). Since the convex vertex v_f neighbors the convex vertex v_i along a horizontal edge, and since v_f is to the right of v_i, v_f must be v_z. Therefore, s_{i+2} would only be placed on the same vertex as s_i if v_z is visible from s_{i+1}. Since we stop placing guards once a guard can see v_z, two guards will never be placed on the same vertex.

Next, it must be shown that this is a guard set for the staircase polygon. Suppose without loss of generality that guard s_i is placed on the lower subchain. Assume that the set $[s_1, s_2 \dots s_i]$ forms a guard set for the subpolygon that lies above the guard s_i (call this subpolygon P_i). We must show that the set $[s_1, s_2 \dots s_{i+1}]$ forms a guard set for the subpolygon that lies to the left of guard s_{i+1} (call this subpolygon P_{i+1}). Let p_{i+1} be the point where a ray extended downward from s_{i+1} intersects the lower subchain. Note that each vertex on the lower subchain between s_i and p_{i+1} is visible from s_{i+1}. We have to show that s_{i+1} guards $P_{i+1} \backslash P_i$. Let v_i^r be the reflex vertex to the right of s_i on the lower subchain. Let Q_{i+1} be the subpolygon below s_{i+1} and to the left of s_{i+1} (See Figure 6). Clearly, $Q_{i+1} \supseteq P_{i+1} \backslash P_i$ (as s_{i+1} cannot be lower than s_i). Note that every vertex of Q_{i+1} that is not connected to s_{i+1} by an edge of Q_{i+1} is on the lower subchain. For any given vertex v in Q_{i+1} that is not a connected to s_{i+1} by an edge of Q_{i+1}, all edges of Q_{i+1} not incident to s_{i+1} that lie above v must also lie to the left of v, and all edges of Q_{i+1} not incident to s_{i+1} that lie to the right of v must also lie below v. Since s_{i+1} is never lower than v, and never to the right of v, every vertex v of Q_{i+1} must be visible from s_{i+1}. This means that one could triangulate Q_{i+1} such that each triangle has s_{i+1} as one of its corners. Therefore, the guard s_{i+1} can guard Q_{i+1} by itself. Therefore, the set $[s_1, s_2 \dots s_m]$ forms a guard set for P.

Finally, it must be shown that the guard set $[s_1, s_2 \dots s_m]$ can be colored with three colors. Suppose guard s_i is placed on the lower chain. Let y_i be the y-coordinate of the lowest point visible from s_i. Note that, because s_i is on a convex right-angle vertex on the lower subchain, $V(s_i)$ is bordered on the bottom by a horizontal line at the same height as the horizontal edge incident to s_i; therefore y_i is just the y-coordinate of s_i. Let y_{i+3} be the y coordinate of the highest point in $V(s_{i+3})$. Because s_{i+3} is on a convex right-angle vertex on the upper subchain, $V(s_{i+3})$ is bordered on top by a horizontal line at the same height as the horizontal edge incident to s_{i+3}; therefore y_{i+3} is just the y-coordinate of s_{i+3}. Now, we must show that $y_i > y_{i+3}$. In the portion of the proof that showed that each guard is placed on a unique vertex, we demonstrated that the y-coordinate of s_{i+1} (call it y_{i+1}) has to be higher than the y-coordinate of s_{i+3}. If $y_i \leq y_{i+3}$, then $y_i \leq y_{i+3} < y_{i+1}$. However, this is impossible, because s_{i+1} was placed on the rightmost (and thus, lowest) vertex on the upper chain that was in $V(s_i)$. Therefore, $y_i > y_{i+3}$. Since the highest point in $V(s_{i+3})$ is lower than the lowest point in $V(s_i)$, s_i and s_{i+3} cannot conflict (see Figure 7).

Since s_i and s_{i+3} do not conflict, we can color all guards with an index of $0 \mod 3$ with green, all guards with an index of $1 \mod 3$ with red, and all guards with an index of $2 \mod 3$ with blue. Therefore $\chi_G(P) \leq 3$.

We have assumed throughout this proof that guard s_i was placed on the lower subchain. However, the arguments made above still apply if s_i was placed on the upper subchain (reflect the polygon over the $y = -x$ line).

□

5 Conclusion

One direction of future research would be to find bounds for other categories of polygons. Finding a bound better than $\chi_G(P) \leq \lfloor n/3 \rfloor$ for general polygons is the most obvious target (sources that examine visibility problems in general polygons include [3], [5], [6], [20]), but orthogonal polygons [8], [9], [13], [19], and monotone polygons [2], [17], are also heavily studied in visibility. Visibility in curvilinear bounded regions has also been researched [10]. Allowing polygons with holes is another possibility, as is placing further restrictions on the placement of

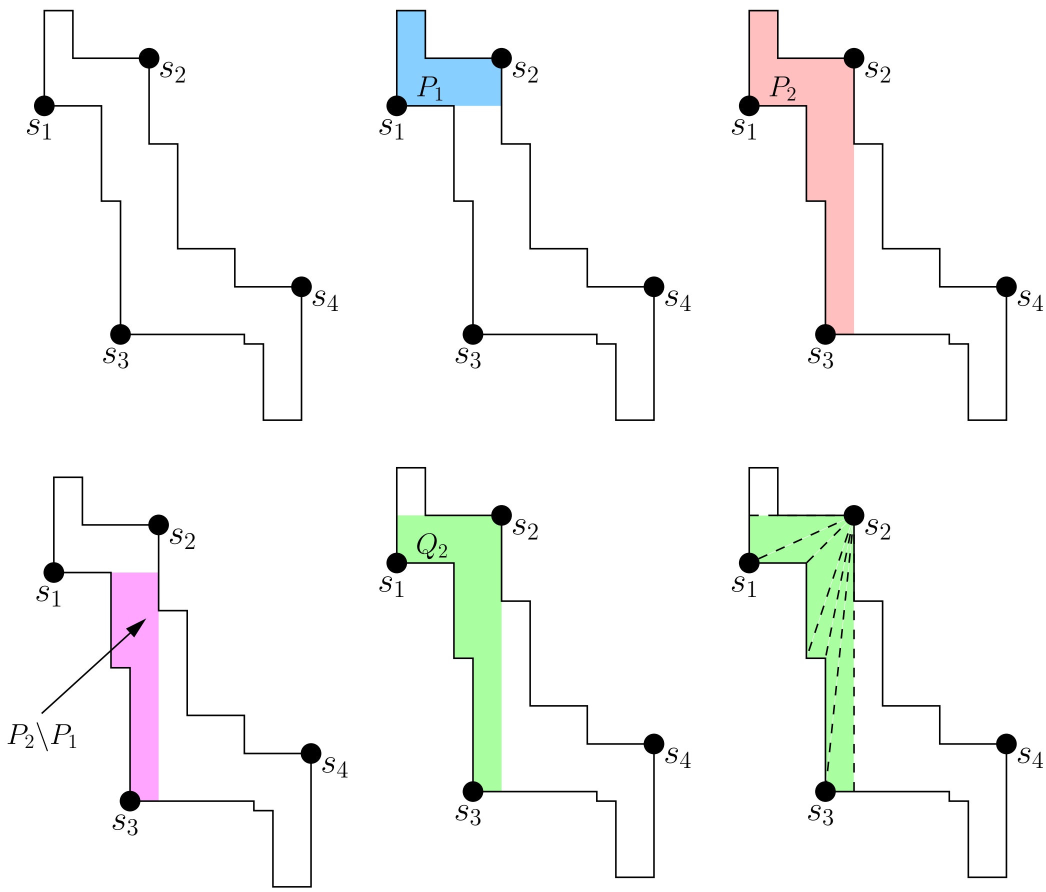

Figure 6: [top left] A polygon P with a guard placement. [top middle] The region P_1 that s_1 is responsible for guarding. [top right] The region P_2 that s_1 and s_2 are reponsible for guarding. [bottom left] The region $P_2 \backslash P_1$ that s_2 is responsible for guarding. [bottom middle] The region Q_2, which consists of the portion of P below and to the left of s_2. This region is a superset of $P_2 \backslash P_1$. [bottom right] A triangulation of Q_2 where all triangles have a vertex at the location of s_2, showing that s_2 guards Q_2.

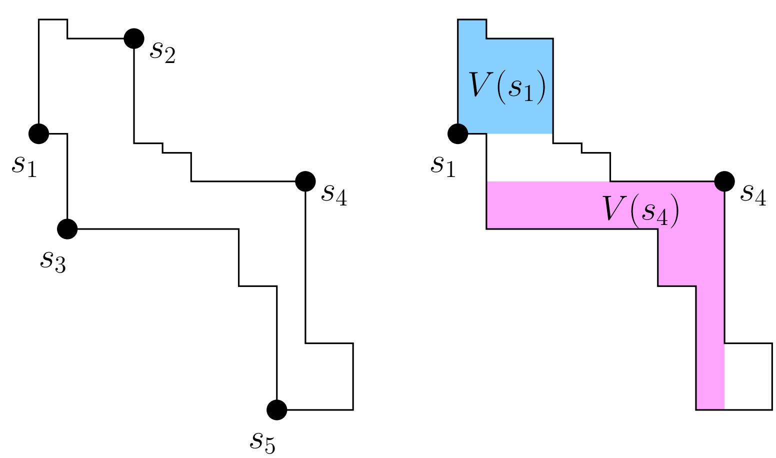

Figure 7: [left] A staircase polygon P with a guard placement. [right] The regions $V(s_1)$ and $V(s_4)$ are shown. Note that the lowest point in $V(s_1)$ is higher than the highest point in $V(s_4)$, as the horizontal line incident to s_1's vertex is higher than the horizontal line incident to s_4's vertex.

guards, perhaps forcing the guards to be strongly cooperative [21] or weakly cooperative [15].

The problem could also be attacked from a visibility graph context. The structure of standard visibility graphs for general polygons is still not completely understood, but [7] gives four necessary conditions for visibility graphs. It is likely that analogues of these four conditions could be made for “2-link” visibility.

Finally, for practical robotics purposes, it would be useful to make a more realistic model of when guards conflict. For example, using a model where a robot has limited vision, so two guards sufficiently far from each other will not conflict even if there is no obstacle between them. Alternately, it may be useful to make a model where the “signal” from a guard degrades as the robot gets further away, perhaps degrading faster if it must go through an obstacle.

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