DERIVATIONS ON O-MINIMAL FIELDS

BY

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DISSERTATION

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Abstract

Let K be an o-minimal expansion of a real closed ordered field and let T be the theory of K. In this thesis, we study derivations ∂ on K. We require that these derivations be compatible with the $\mathcal{C}^1$-functions definable in K. For example, if K defines an exponential function, then we require that $\partial \exp(a) = \exp(a)\partial a$ for all $a \in K$. We capture this compatibility with the notion of a *T-derivation*.

Let T^∂ be the theory of structures (K, ∂), where $K \models T$ and ∂ is a T-derivation on K. We show that T^∂ has a model completion $T^\partial_{\mathcal{G}}$, in which derivation behaves "generically." The theory $T^\partial_{\mathcal{G}}$ is model theoretically quite tame; it is distal, it has o-minimal open core, and it eliminates imaginaries.

Following our investigation of $T^\partial_{\mathcal{G}}$, we turn our attention to *T-convex T-differential fields*. These are models $K \models T$ equipped with a T-derivation which is continuous with respect to a T-convex valuation ring of K, as defined by van den Dries and Lewenberg. We show that if K is a T-convex T-differential field, then under certain conditions (including the necessary condition of power boundedness), K has an immediate T-convex T-differential field extension which is spherically complete.

In the penultimate chapter, we consider T-convex T-differential fields which are also H-fields, as defined by Aschenbrenner and van den Dries. We call these structures *H_T-fields*, and we show that if T is power bounded, then every H_T-field K has either exactly one or exactly two minimal Liouville closed H_T-field extensions up to K-isomorphism.

We end with two theorems when $T = T_{\mathrm{re}}$, the theory of the real field expanded by restricted elementary functions. First, we prove a model completeness result for the expansion of the ordered valued differential field $\mathbb{T}$ of logarithmic-exponential transseries by its natural restricted elementary functions. We then use this result to prove that the theory of $H_{T_{\mathrm{re}}}$-fields has a model companion.

Dedicated to Alexi, to my parents David and Veronica, and to my sister Serena.

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Contents

Conventions and Notations

We always use k, m, n, p, q, and r to denote elements of $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Citation conventions. Our primary reference for valuation theory and differential algebra is [**4**]. In this thesis, we try as much as possible to use notation consistent with the notation in [**4**]. For model theoretic background, we refer to [**4**, Appendix B] and [**66**]. Our reference for o-minimality is [**26**]. In citing these references, we do not imply that the results that we cite are originally due to the authors. We omit qualifiers in citations whenever possible. For example, we write [**4**, B.11.16] instead of [**4**, Lemma B.11.16].

Ordered set conventions. By "ordered set" we mean "totally ordered set." Let S be an ordered set, let $a \in S$, and let $A \subseteq S$. We let

$$S^{>a} \ := \ \{s \in S : s > a\};$$

similarly for $S^{\geqslant a}$, $S^{<a}$, $S^{\leqslant a}$, and $S^{\neq a}$. We write "$a > A$" (respectively "$a < A$") if a is greater (less) than each $s \in A$. For $b \in S^{>a}$, we put

$$[a, b]_A \ := \ \{s \in A : a \leqslant s \leqslant b\}.$$

If $A = S$, we drop the subscript and write $[a, b]$ instead. We say that A is *downward closed* if $s \in A$ whenever $s \in S$ is less than some $a \in A$, and we let

$$A^{\downarrow} \ := \ \bigcup_{a \in A} S^{\leqslant a}$$

denote the **downward closure of** A, so A is downward closed if and only if $A = A^{\downarrow}$. A **cut** in S is just a downward closed subset of S. If A is a cut in S and y is an element in an ordered set extending S, then we say that y *realizes the cut* A if

$$A \ < \ y \ < \ S \setminus A.$$

A **well-indexed sequence** is a sequence (a_ρ) whose terms are indexed by the ordinals ρ less than some infinite limit ordinal ν.

Algebra conventions. If Γ is an ordered abelian group, then we let $\Gamma^{>} := \Gamma^{>0}$ and we define $\Gamma^{\geqslant}$, $\Gamma^{<}$, $\Gamma^{\leqslant}$, and $\Gamma^{\neq}$ analogously. If R is a ring, then we let $R^{\times}$ denote the multiplicative group of units in R. We let $\operatorname{Mat}_{m,n}(R)$ be the collection of $m \times n$ matrices with entries in R, so $Ab \in R^m$ for $A \in \operatorname{Mat}_{m,n}(R)$ and $b \in R^n$. We identify $\operatorname{Mat}_{m,n}(R)$ with $R^{m \times n}$ in the usual way. If $m = n$, we just write $\operatorname{Mat}_n(R)$.

Model theory conventions. Let $\mathcal{L}$ be a language, let T be an $\mathcal{L}$-theory, and let K be a model of T. We regard K^0 as the one-point space $\{0\}$, and we identify each nullary map $F\colon K^0 \to K^n$ with its value $F(0) \in K^n$. Constant symbols in the language $\mathcal{L}$ are viewed as nullary function symbols.

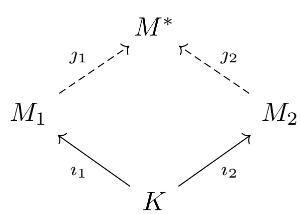
Let $A \subseteq K$ and let $D \subseteq K^n$. We say that D is **$\mathcal{L}(A)$-definable** if

$$D \ = \ \varphi(K) \ := \ \big\{y \in K^n : K \models \varphi(y)\big\}$$

for some $\mathcal{L}(A)$-formula $\varphi(y)$. Let $k \leqslant n$. We denote the projection of D onto the first k coordinates by $\pi_k(D)$ and for $y \in K^k$, we set $D_y := \{z \in K^{n-k} : (y,z) \in D\}$. Given a map $F\colon D \to K^m$, we let $\operatorname{Gr}(F) \subseteq K^{n+m}$ denote the graph of F, and we say that F is $\mathcal{L}(A)$-definable if $\operatorname{Gr}(F)$ is. Note that the domain of an $\mathcal{L}(A)$-definable map is $\mathcal{L}(A)$-definable. For $A \subseteq K$, we let $\operatorname{dcl}_{\mathcal{L}}(A)$ be the $\mathcal{L}$-definable closure of A (in K, implicitly, but this doesn't change if we pass to elementary extensions of K). If $b \in \operatorname{dcl}_{\mathcal{L}}(A)$, then $b = F(a)$ for some $\mathcal{L}(\emptyset)$-definable function F and some tuple a from A.

A **T-extension of K** is a model $M \models T$ which contains K as an $\mathcal{L}$-substructure. Let M be an elementary T-extension of K. Given an $\mathcal{L}(K)$-definable set $D \subseteq K^n$, we let D^M denote the subset of M^n defined by the same $\mathcal{L}(K)$-formula as D. We sometimes refer to D^M as the **natural extension of D to M**. Elementarity ensures that this natural extension does not depend on the choice of defining formula. If $F\colon D \to K^m$ is an $\mathcal{L}(K)$-definable map, then we let $F^M\colon D^M \to M^m$ be the $\mathcal{L}(K)$-definable map with graph $\operatorname{Gr}(F^M) = \operatorname{Gr}(F)^M$. We often drop the superscript for definable maps and just write $F\colon D^M \to M^m$.

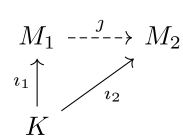
Let T^* be an $\mathcal{L}$-theory extending T. If T^* is model complete and every model of T can be extended to a model of T^*, then T^* is called the **model companion of T**. If T has a model companion, then this model companion is unique; see [**4**, B.10.11]. We say that T has the **amalgamation property** if for all models $K, M_1, M_2 \models T$ and $\mathcal{L}$-embeddings $\imath_1\colon K \to M_1$, $\imath_2\colon K \to M_2$, there exists $M^* \models T$ and $\mathcal{L}$-embeddings $\jmath_1\colon M_1 \to M^*$, $\jmath_2\colon M_2 \to M^*$ making the diagram



commute. If T^* is the model companion of T and T has the amalgamation property, then we say that T^* is the **model completion of T**. If T^* is the model completion of T and T has a universal axiomatization, then T^* has quantifier elimination [**4**, B.11.16]. We include here a useful test for whether T^* is the model completion of T. This test (a variant of Blum's criterion) is well-known, but it doesn't appear verbatim anywhere in the literature.

Model Completion Criterion. *T^* is the model completion of T if and only if the following holds:*

1. *every model of T can be extended to a model of T^*;*
2. *for any models $M_1, M_2 \models T^*$ where M_2 is $|M_1|^+$-saturated, any $K \models T$, and any $\mathcal{L}$-embeddings $\imath_1\colon K \to M_1$, $\imath_2\colon K \to M_2$, there is an $\mathcal{L}$-embedding $\jmath\colon M_1 \to M_2$ making the following diagram commute:*



Proof. Suppose conditions (1) and (2) hold. By taking K in condition (2) to be a model of T^* and using [**4**, B.10.4], we see that T^* is model complete. To see that T has the amalgamation property, let K, M_1, M_2 be models of T and let $\imath_1\colon K \to M_1$, $\imath_2\colon K \to M_2$ be $\mathcal{L}$-embeddings. Using condition (1), extend M_1, M_2 to models of $M_1^*, M_2^* \models T^*$ where M_2^* is $|M_1^*|^+$-saturated. Then condition (2) gives us an $\mathcal{L}$-embedding

$\jmath\colon M_1^* \to M_2^*$ with $\jmath \circ \imath_1 = \imath_2$. Taking $M^* := M_2^*$, $\jmath_1 := \jmath|_{M_1}$, and $\jmath_2$ to be the inclusion $M_2 \hookrightarrow M_2^*$, we see that the amalgamation property holds.

Now suppose that T^* is the model completion of T. We know that condition (1) holds, so let K, M_1, M_2 be as in (2). Viewing K as a common $\mathcal{L}$-substructure of M_1 and M_2 via the maps $\imath_1$ and $\imath_2$ and using [**4**, B.10.14], we see that M_1 and M_2 are $\mathcal{L}(K)$-elementarily equivalent. Thus, the saturation assumption on M_2 gives us an elementary $\mathcal{L}(K)$-embedding $\jmath\colon M_1 \to M_2$; see [**4**, B.9.5]. □

Suppose T is complete. Then T has a **monster model**, that is, a κ-saturated and strongly κ-homogeneous model $\mathbb{M} \models T$ for some $\kappa = \kappa(\mathbb{M}) > |T| := \max\{|\mathcal{L}|, \aleph_0\}$. When working in a monster model $\mathbb{M}$, we use *small* to mean of cardinality $< \kappa$. Let $A \subseteq \mathbb{M}$ be small, and let $(a_i)_{i\in I}$ be a sequence of elements in $\mathbb{M}^n$. The sequence (a_i) is said to be $\mathcal{L}(A)$**-indiscernible** if for all m, all indices $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$ from I, and all $\mathcal{L}(A)$-formulas $\varphi(x_1, \ldots, x_m)$, we have

$$\mathbb{M} \models \varphi(a_{i_1}, \ldots, a_{i_m}) \longleftrightarrow \varphi(a_{j_1}, \ldots, a_{j_m}).$$

Pregeometry conventions. A **pregeometry** is a set X equipped with a map $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies the following conditions, where a, b range over X and A, B range over $\mathcal{P}(X)$.

(1) Monotonicity: if $A \subseteq B$, then $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;
(2) Idempotence: $\mathrm{cl}\big(\mathrm{cl}(A)\big) = \mathrm{cl}(A)$;
(3) Finitary: if $a \in \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A_0)$ for some finite subset $A_0 \subseteq A$;
(4) Steinitz exchange: if $a \in \mathrm{cl}\big(A \cup \{b\}\big) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}\big(A \cup \{a\}\big)$.

Suppose (X, cl) is a pregeometry and let $A \subseteq X$. A set $B \subseteq X$ is said to be $\mathrm{cl}(A)$**-independent** if $b \notin \mathrm{cl}\big(A \cup (B \setminus \{b\})\big)$ for all $b \in B$. A **basis for** B **over** A is a $\mathrm{cl}(A)$-independent subset $B_0 \subseteq B$ with $B \subseteq \mathrm{cl}(A \cup B_0)$. Steinitz exchange ensures that any two bases for B over A have the same cardinality, called the **rank of** B **over** A and denoted $\mathrm{rk}(B|A)$. We just write $\mathrm{rk}(B)$ for $\mathrm{rk}(B|\emptyset)$.

Let $a = (a_i)_{i\in I}$ be a tuple from X. We say that a is $\mathrm{cl}(A)$-independent if its set of components $\{a_i : i \in I\}$ is $\mathrm{cl}(A)$-independent and no components are repeated. The rank of the tuple a over A, written $\mathrm{rk}(a|A)$, is the rank of the set $\{a_i : i \in I\}$ over A. Let $b = (b_j)_{j\in J}$ be another tuple from X. When working with pregeometries, we often abuse notation and write things like "$a \in \mathrm{cl}(Ab)$" to mean "$a_i \in \mathrm{cl}\big(A \cup \{b_j : i \in J\}\big)$ for each $i \in I$."

CHAPTER 1

Introduction

In *Orders of Infinity*, Hardy introduced the class of *Logarithmic-Exponential functions* (or *LE-functions*) [**45**]. These are the functions $f\colon (a,+\infty)\to\mathbb{R}$ with $a\in\mathbb{R}$ which are obtained from constants and the identity function x by algebraic operations and by taking exponentials and logarithms. These functions often arise in describing the asymptotic behavior of real-valued nonoscillating solutions to algebraic differential equations.

Given a function $f\colon (a,+\infty)\to\mathbb{R}$, the **germ of f at $+\infty$** is the equivalence class

$$[f] \ :=\ \big\{g\colon (b,+\infty)\to\mathbb{R} : f|_{(c,+\infty)} = g|_{(c,+\infty)} \text{ for some } c\geqslant \max\{a,b\}\big\}$$

A key fact about the class of LE-functions is that the collection

$$\mathcal{H}^{LE} \ :=\ \big\{[f] : f \text{ is an LE-function}\big\}$$

is an *ordered differential field*, that is to say, each LE-function has eventually constant sign, is eventually differentiable, and its derivative is eventually an LE-function (where a property is said to hold *eventually* if it holds for all sufficiently large x). Bourbaki took this fact as an axiom and defined a **Hardy field** to be an ordered differential field of germs at $+\infty$ of unary real-valued functions [**10**]. The study of Hardy fields was greatly advanced near the end of the 20th century by Rosenlicht, Boshernitzan, and others.

In [**2**], Aschenbrenner and van den Dries introduced the class of H-fields, which serves as an algebraic/model-theoretic generalization of Hardy fields. An **H-field** is an ordered differential field (K,∂) with constant field $C=\ker(\partial)$ such that

(1) for all $f\in K$, if $f>C$, then $\partial f>0$, and
(2) $\mathcal{O}=C+\scriptstyle\mathcal{O}$, where $\mathcal{O}$ is the convex hull of C in K and $\scriptstyle\mathcal{O}$ is the unique maximal ideal of $\mathcal{O}$.

The differential field $\mathcal{H}^{LE}$ is an H-field, as is any Hardy field containing $\mathbb{R}$, where each $r\in\mathbb{R}$ is identified with the germ of the constant function $x\mapsto r$. We always view H-fields as ordered *valued* differential fields, with valuation ring $\mathcal{O}$ as in (2). In many ways, the valuation is more important than the ordering.

In [**4**], Aschenbrenner, van den Dries, and van der Hoeven showed that the theory of H-fields has a *model companion*: the theory of ω-free newtonian Liouville closed H-fields. An H-field K is *Liouville closed* if K is real closed and for each $y\in K$, there is $f\in K$ and $g\in K^{\times}$ with $\partial f=\partial g/g=y$. The axioms of ω-freeness and newtonianity are a bit more technical, and while these axioms play a major role in [**4**], they do not appear so much in this thesis.

The theory of ω-free newtonian Liouville closed H-fields has a natural model: the field $\mathbb{T}$ of logarithmic-exponential transseries. The field $\mathbb{T}$ was introduced independently by Dahn and Göring [**18**] in their work on Tarski's problem on real exponentiation and by Écalle [**34**] in his solution to the Dulac conjecture—the best-known result on Hilbert's 16th problem. The model completeness of $\mathbb{T}$ provides a rigorous justification for Écalle's intuition that $\mathbb{T}$ is a universal domain for asymptotic differential algebra (in the same way $\mathbb{C}$ is a universal domain for algebraic geometry in characteristic 0).

Hardy fields and o-minimality. Let $\mathcal{R}$ be an expansion of the real field $\mathbb{R}$ in a language $\mathcal{L}$. Then $\mathcal{R}$ is said to be *o-minimal* if the only $\mathcal{L}(\mathbb{R})$-definable subsets of $\mathbb{R}$ are finite unions of points and intervals. Examples of specific o-minimal expansions of $\mathbb{R}$ can be found in Section 2.2, but we list a few here for the purposes of this chapter.

(1) $\mathbb{R}_{\text{re}}$ is the expansion of $\mathbb{R}$ by sine, cosine, and exponential functions, restricted to the closed interval $[-1, 1]$. These functions are collectively called *restricted elementary functions.* We let T_{re} be the elementary theory of $\mathbb{R}_{\text{re}}$.
(2) $\mathbb{R}_{\text{an}}$ is the expansion of $\mathbb{R}$ by *all* real-valued functions which are real analytic on an open neighborhood of $[-1, 1]^n$, restricted to $[-1, 1]^n$. These functions are called *restricted analytic functions.* The three restricted elementary functions are restricted analytic, so $\mathbb{R}_{\text{re}}$ is a proper reduct of $\mathbb{R}_{\text{an}}$. We let T_{an} be the elementary theory of $\mathbb{R}_{\text{an}}$.
(3) $\mathbb{R}_{\text{an,exp}}$ is the expansion of $\mathbb{R}_{\text{an}}$ by the unrestricted exponential function. We let $T_{\text{an,exp}}$ be the elementary theory of $\mathbb{R}_{\text{an,exp}}$.

O-minimality relates to Hardy fields in the following way: the expansion $\mathcal{R}$ is o-minimal if and only if the germs of all $\mathcal{L}(\mathbb{R})$-definable unary functions at $+\infty$ form a Hardy field. This equivalence was first used by van den Dries, Macintyre, and Marker to greatly simplify the proof that $T_{\text{an,exp}}$ is o-minimal [**30**]. It was subsequently exploited by Miller to prove a dichotomy theorem about the growth rates of $\mathcal{L}(\mathbb{R})$-definable functions [**53**].

Suppose that the expansion $\mathcal{R}$ is o-minimal, let T be the $\mathcal{L}$-theory of $\mathcal{R}$, and assume that T has quantifier elimination and a universal axiomatization in the language $\mathcal{L}$. This assumption on T is mostly a matter of convenience and not as unreasonable as it may first seem; see Corollary 2.5. Following [**30**], we define an $\mathcal{R}$**-Hardy field** to be a Hardy field $\mathcal{H}$ which is closed under all function symbols in $\mathcal{L}$. That is, $\mathcal{H}$ is an $\mathcal{R}$-Hardy field if for every n-ary function symbol F in $\mathcal{L}$ and all germs $[f_1], \ldots, [f_n] \in \mathcal{H}$, the germ $\big[F(f_1, \ldots, f_n)\big]$ is in $\mathcal{H}$, where $F(f_1, \ldots, f_n)$ is the composite function $x \mapsto F\big(f_1(x), \ldots, f_n(x)\big)$. If $\mathcal{H}$ is an $\mathcal{R}$-Hardy field, then we view $\mathcal{H}$ as an $\mathcal{L}$-structure as follows:

- If F is an n-ary function symbol in $\mathcal{L}$ and $[f_1], \ldots, [f_n] \in \mathcal{H}$, then
$$F\big([f_1], \ldots, [f_n]\big) \ := \ \big[F(f_1, \ldots, f_n)\big].$$
- If R is an n-ary predicate in $\mathcal{L}$ and $[f_1], \ldots, [f_n] \in \mathcal{H}$, then
$$\mathcal{H} \models R\big([f_1], \ldots, [f_n]\big) \ :\Longleftrightarrow \ \mathcal{R} \models R\big(f_1(x), \ldots, f_n(x)\big) \text{ eventually.}$$

By [**30**, 5.8], the $\mathcal{L}$-structure $\mathcal{H}$ is a model of T. If $\mathbb{R}$ is contained in $\mathcal{H}$ (under the aforementioned identification of real numbers with constant functions), then $\mathcal{H}$ may be viewed as an elementary T-extension of $\mathcal{R}$. Of course, this includes the case $\mathcal{H} = \mathcal{R}$. It also includes the $\mathcal{R}$-Hardy field

$$\mathcal{H}(\mathcal{R}) \ := \ \big\{[f] : f \colon \mathbb{R} \to \mathbb{R} \text{ is } \mathcal{L}(\mathbb{R})\text{-definable}\big\}.$$

As a consequence of our assumption on T, each $\mathcal{L}(\emptyset)$-definable function F is given piecewise by terms, so the identity

$$F\big([f_1], \ldots, [f_n]\big) \ := \ \big[F(f_1, \ldots, f_n)\big]$$

holds for arbitrary $\mathcal{L}(\emptyset)$-definable functions F, not just for function symbols in $\mathcal{L}$.

Let $\mathcal{H}$ be an $\mathcal{R}$-Hardy field. As a Hardy field, $\mathcal{H}$ admits a derivation $\partial\colon \mathcal{H} \to \mathcal{H}$ given by $\partial[f] := [f']$. How does this derivation interact with the $\mathcal{L}(\emptyset)$-definable functions on $\mathcal{H}$? To answer this, let F be an n-ary $\mathcal{L}(\emptyset)$-definable function, let $[f_1], \ldots, [f_n] \in \mathcal{H}$, and suppose that F is $\mathcal{C}^1$ at $\big([f_1], \ldots, [f_n]\big)$. The chain rule from elementary calculus gives

$$\begin{aligned}
\partial F\big([f_1], \ldots, [f_n]\big) \ &= \ \partial\big[F(f_1, \ldots, f_n)\big] \ = \ \big[F(f_1, \ldots, f_n)'\big] \\
&= \left[\frac{\partial F}{\partial X_1}(f_1, \ldots, f_n) f_1' + \cdots + \frac{\partial F}{\partial X_n}(f_1, \ldots, f_n) f_n'\right] \\
&= \ \frac{\partial F}{\partial X_1}\big([f_1], \ldots, [f_n]\big)\partial[f_1] + \cdots + \frac{\partial F}{\partial X_n}\big([f_1], \ldots, [f_n]\big)\partial[f_n].
\end{aligned}$$

We take the above identity as a definition: a *T-derivation on a model* $K \models T$ is map $\partial\colon K \to K$ such that

$$\partial F(u) \ = \ \frac{\partial F}{\partial X_1}(u)\partial u_1 + \cdots + \frac{\partial F}{\partial X_n}(u)\partial u_n$$

for all $u = (u_1, \ldots, u_n) \in K^n$ and all $\mathcal{L}(\emptyset)$-definable functions F which are $\mathcal{C}^1$ on a neighborhood of u. These T-derivations are the primary object of study in this thesis.

Let $K \models T$ and let ∂ be a T-derivation on K. If the underlying ordered differential field of $K = (K, \partial)$ is an H-field, as defined above, then we call K an *H_T-field.* Any $\mathcal{R}$-Hardy field containing $\mathbb{R}$ is an H_T-field, and the relationship between H_T-fields and $\mathcal{R}$-Hardy fields is analogous to that between H-fields and Hardy fields. We write "H_{re}-field" instead of "$H_{T_{\mathrm{re}}}$-field" for easier reading; likewise for T_{an} and $T_{\mathrm{an,exp}}$. When T is one of these theories, the differential field $\mathbb{T}$ of logarithmic-exponential transseries admits an expansion to an H_T-field:

(1) The expansion $\mathbb{T}_{\mathrm{re}}$ of $\mathbb{T}$ by restricted elementary functions is an H_{re}-field.
(2) The expansion $\mathbb{T}_{\mathrm{an}}$ of $\mathbb{T}$ by restricted analytic functions is an H_{an}-field.
(3) The expansion $\mathbb{T}_{\mathrm{an,exp}}$ of $\mathbb{T}_{\mathrm{an}}$ by the unrestricted exponential function is an $H_{\mathrm{an,exp}}$-field.

For details about these expansions, we direct the reader to Fact 2.14 and Examples 3.17 and 7.5.

Our long-term goal is to show that the theory of H_T-fields has a model companion. Another objective is to show that the expansions of $\mathbb{T}$ given above are model complete. This thesis takes first steps toward these aims. We examine various extensions of H_T-fields (often under the assumption that T is *power bounded*), we show that the theory of H_{re}-fields has a model companion, and we prove that $\mathbb{T}_{\mathrm{re}}$ is model complete.

1.1. Overview of results

Here we describe the main results in this thesis. Let T be a complete, model complete, o-minimal $\mathcal{L}$-theory extending the theory of real closed ordered fields and let $K \models T$. The theory of real closed ordered fields is formulated in the usual way in the language $\{0, 1, +, -, \cdot, <\} \subseteq \mathcal{L}$. Background on o-minimality, along with examples of o-minimal theories and o-minimal structures, is provided in Chapter 2.

Generic T-derivations. A T-derivation ∂ on K is said to be *generic* if for each $\mathcal{L}(K)$-definable function $F\colon U \to K$ with nonempty open domain $U \subseteq K^n$, there is $a \in K$ with

$$(a, a', \ldots, a^{(n-1)}) \ \in \ U, \qquad F(a, a', \ldots, a^{(n-1)}) \ = \ a^{(n)}.$$

Above, we use a' in place of ∂a and $a^{(n)}$ in place of $\partial^n a$. Generic T-derivations are studied in Chapter 4. Much of the material in Chapter 3 and nearly all the material in Chapter 4 is joint work with Antongiulio Fornasiero from [**38**]. Let $\mathcal{L}^\partial := \mathcal{L} \cup \{\partial\}$, let T^∂ be the $\mathcal{L}^\partial$-theory which extends T by axioms stating that ∂ is a T-derivation, and let $T^\partial_\mathcal{G}$ be the $\mathcal{L}^\partial$-theory which extends T^∂ by axioms asserting that ∂ is generic. The following theorem serves as a starting point for our investigation of generic T-derivations:

Theorem 4.7. *$T^\partial_\mathcal{G}$ is the model completion of T^∂. If T has a universal axiomatization, then $T^\partial_\mathcal{G}$ has quantifier elimination.*

The theory $T^\partial_\mathcal{G}$ coincides with Singer's theory of closed ordered differential fields [**65**] when T is the theory of real closed ordered fields. We show that many of the results known about closed ordered differential fields hold more generally for models of $T^\partial_\mathcal{G}$. The first of these generalizations involves distality. Distal theories are special kinds of NIP theories and were introduced by Simon [**63**].

Theorem 4.16. *$T^\partial_\mathcal{G}$ is distal.*

Any model $K \models T^\partial$ admits a closure operator, called the ∂-closure, which serves as an analog of differential algebraic closure. In Proposition 3.25, we show that this closure operator gives rise to a pregeometry on K. If $K \models T^\partial_\mathcal{G}$, then ∂-closure allows us to define a dimension function (in the sense of van den Dries [**23**]) on the algebra of $\mathcal{L}^\partial(K)$-definable sets. This dimension is a key tool in the proof of the following theorem:

Theorem 4.28. *$T^\partial_\mathcal{G}$ has T as its open core. More precisely, for $K \models T^\partial_\mathcal{G}$ and for $B \subseteq K$ with $\partial B \subseteq B$, any open $\mathcal{L}^\partial(B)$-definable subset of K^n is $\mathcal{L}(B)$-definable.*

This open core result allows us to characterize the definable closure in models $T^\partial_\mathcal{G}$, which in turn allows us to describe definable functions. Theorem 4.28 also plays a key role in the proof of the following result:

Theorem 4.39. *$T^\partial_\mathcal{G}$ eliminates imaginaries.*

In any model $K \models T^\partial$, the constant field C of K is an elementary $\mathcal{L}$-substructure of K. If K is a model of $T^\partial_\mathcal{G}$, then the constant field is also dense in K. Thus, every model of $T^\partial_\mathcal{G}$ is an expansion of the theory of dense pairs of models of T, as introduced in [**25**].

Corollary 4.41. *$T^\partial_\mathcal{G}$ is a distal theory extending the theory of dense pairs of models of T.*

This corollary is worth noting, as the theory of dense pairs itself is not distal [**46**]. Additional connections between $T^\partial_\mathcal{G}$ and the theory of dense pairs are discussed in Section 4.5.

T-convex T-differential fields. The theory $T^\partial_\mathcal{G}$ is model theoretically quite well-behaved, as indicated by the above results. However, generic derivations are highly discontinuous. In contrast, the derivation on most natural models of T^∂ is continuous. We turn our attention to continuous T-derivations in Chapter 6. Instead of working with the order topology directly, it is more convenient to work with the topology induced by a T-convex valuation ring, as introduced by van den Dries and Lewenberg [**29**]. A *T-convex valuation ring of* K is a nonempty convex set $\mathcal{O} \subseteq K$ which is closed under all $\mathcal{L}(\emptyset)$-definable continuous functions $K \to K$. We discuss T-convex valuation rings in detail in Chapter 5. For the purposes of this introduction, we just need a few definitions.

Let $\mathcal{L}^\mathcal{O} := \mathcal{L} \cup \{\mathcal{O}\}$ and let $T^\mathcal{O}$ be the $\mathcal{L}^\mathcal{O}$-theory which extends T by axioms stating that $\mathcal{O}$ is a T-convex valuation ring. Let $K = (K, \mathcal{O}) \models T^\mathcal{O}$. Then $\mathcal{O}$ has a unique maximal ideal $\smallo = \mathcal{O} \setminus \mathcal{O}^\times$, and sets of the form $a + b\smallo$ where $a \in K$ and $b \in K^\times$ form a basis for the valuation topology on K. If $\mathcal{O} \neq K$, then K is said to be *nontrivially valued*. In this case, the valuation topology and the order topology coincide. Let $\mathcal{B}$ be a collection of sets of the form $a + b\mathcal{O}$, where $a \in K$ and $b \in K^\times$. We say that $\mathcal{B}$ is *nested* if any two sets in $\mathcal{B}$ has nonempty intersection. We say that K is *spherically complete* if $\bigcap \mathcal{B} \neq \emptyset$ whenever $\mathcal{B}$ is nested.

A $T^{\mathcal{O}}$-extension $M = (M, \mathcal{O}_M)$ of K is said to be an *immediate* if for each $a \in M^\times$, there is $b \in K^\times$ with $a/b \in 1 + \smallo_M$. By a classical result of Kaplansky [**49**], every valued field of equicharacteristic zero has a spherically complete immediate extension which is unique up to isomorphism. We have the following analog for models of $T^{\mathcal{O}}$:

Corollary 5.13. *Suppose T is power bounded. Then K has a spherically complete immediate $T^{\mathcal{O}}$-extension which is unique up to $\mathcal{L}^{\mathcal{O}}(K)$-isomorphism.*

Power boundedness is the assumption that every $\mathcal{L}(K)$-definable function is eventually bounded by a power function, that is, a definable function which "behaves like" $y \mapsto y^\lambda$ for some $\lambda \in K$; see page 11 for an explicit definition. This assumption is necessary; if T is not power bounded and $\mathcal{O} \neq K$, then K has no spherically complete $T^{\mathcal{O}}$-extension by Miller's dichotomy [**54**] and a negative result of Kuhlmann, Kuhlmann, and Shelah [**51**]; see Remark 5.14 for details. Corollary 5.13 follows almost immediately from results in Tyne's thesis [**67**] and basic valuation theory, but it does not appear anywhere in the literature.

The study of immediate extensions becomes much more difficult when derivations are thrown into the mix. A relatively recent result of Aschenbrenner, van den Dries, and van der Hoeven [**5**] tells us that every equicharacteristic zero valued field with a continuous derivation has a spherically complete immediate extension. Such an extension need not be unique up to isomorphism. The main objective of Chapter 6 is to prove a similar theorem for *T-convex T-differential fields*: models of T expanded by a T-convex valuation ring and a T-derivation that is continuous with respect to the valuation topology.

Let $\mathcal{L}^{\mathcal{O},\partial} := \mathcal{L} \cup \{\mathcal{O}, \partial\}$ and let $T^{\mathcal{O},\partial}$ be the $\mathcal{L}^{\mathcal{O},\partial}$-theory of T-convex T-differential fields. The $\mathcal{R}$-Hardy fields $\mathcal{H}$ considered above are T-convex T-differential fields for $T = \mathrm{Th}(\mathcal{R})$, where

$$[f] \in \mathcal{O} \iff \lim_{x \to \infty} f(x) \in \mathbb{R}.$$

In Example 6.4, we see how certain Hahn differential fields considered by Scanlon [**61**] admit expansions to models of $T^{\mathcal{O},\partial}_{\mathrm{an}}$.

Let $K = (K, \mathcal{O}, \partial) \models T^{\mathcal{O},\partial}$. By [**4**, Lemma 4.4.7], continuity of ∂ guarantees that $\partial\smallo \subseteq \phi\smallo$ for some $\phi \in K^\times$. An element $s \in K^>$ is said to be *stabilizing* if

$$\partial\smallo \subseteq \phi\smallo \iff \partial\smallo \subseteq s\phi\smallo$$

for each $\phi \in K^\times$. The set of stabilizing elements forms a convex multiplicative subgroup of $K^>$.

Following [**5**], we say that a $T^{\mathcal{O},\partial}$-extension M of K is *strict* if

$$\partial\smallo \subseteq \phi\smallo \implies \partial_M\smallo_M \subseteq \phi\smallo_M, \qquad \partial\mathcal{O} \subseteq \phi\smallo \implies \partial_M\mathcal{O}_M \subseteq \phi\smallo_M$$

for each $\phi \in K^\times$. Strict extensions can be thought of as extensions which preserve each modulus of continuity for ∂. We can finally state the main theorem of Chapter 6:

Theorem 6.1. *Suppose that T is power bounded and that the set of stabilizing elements is closed under power functions. Then K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete.*

As discussed above, the assumption of power boundedness in Theorem 6.1 is necessary. We are unsure whether the assumption on stabilizing elements is necessary. If T is *polynomially bounded*, that is, if each $\mathcal{L}(K)$-definable function is eventually bounded by a function $y \mapsto y^n$ for some n, then the assumption on stabilizing elements is always met.

Corollary 6.28. *If T is polynomially bounded, then K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete.*

H_T-fields. Earlier, we defined an H_T-field to be a model $K \models T^{\partial}$ where the underlying ordered differential field of K is an H-field. Every H-field comes equipped with a canonical valuation ring—the convex hull of the constant field—and the derivation on any H-field is continuous with respect to the valuation topology. If K is an H_T-field, then this valuation ring is T-convex, so we view H_T-fields as $T^{\mathcal{O},\partial}$-models.

Chapter 7 is devoted to the study of H_T-fields and their simple extensions (that is, extensions generated by one element). We also study pre-H_T-fields—models of $T^{\mathcal{O},\partial}$ which arise as substructures of H_T-fields. All $\mathcal{R}$-Hardy fields are pre-H_T-fields for $T = \mathrm{Th}(\mathcal{R})$. All of our results in Chapter 7 are under the assumption that T is power bounded.

Our first result is a corollary of Theorem 6.1 above. In any pre-H_T-field, the set of stabilizing elements coincides with the set of positive elements in $\mathcal{O}^{\times}$. It follows that this set is closed under power functions, so the assumptions in Theorem 6.1 are met and every pre-H_T-field K has an immediate strict $T^{\mathcal{O},\partial}$-extension M which is spherically complete. This extension M is itself a pre-H_T-field and if K is an H_T-field, then M is as well.

Corollary 7.28. *Every pre-H_T-field has a spherically complete immediate pre-H_T-field extension. Every H_T-field has a spherically complete immediate H_T-field extension.*

Our next result shows that every pre-H_T-field has a minimal H_T-field extension with a universal property.

Theorem 7.38. *Let K be a pre-H_T-field. Then K has an H_T-field extension $H_T(K)$ such that for any H_T-field extension M of K, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $H_T(K) \to M$.*

The focus of Chapter 7 is on the existence and uniqueness of *T-Liouville closures*. An H_T-field is said to be Liouville closed if the underlying H-field is Liouville closed, that is, if for all $y \in K$, there is $f \in K$ and $g \in K^{\times}$ with $f' = g'/g = y$. A *T-Liouville closure* of an H_T-field K is a minimal Liouville closed H_T-field extension of K (this is not the definition given in Chapter 7, but it is equivalent by Corollary 7.51).

In [**2**], Aschenbrenner and van den Dries proved that every H-field has at least one and at most two Liouville closures (minimal Liouville closed H-field extensions) up to isomorphism. They used this to show that any H-field embedding of a Hardy field $\mathcal{H}$ into $\mathbb{T}$ extends to the smallest Liouville closed Hardy field extension of $\mathcal{H}$. They proved that *grounded* H-fields have exactly one Liouville closure and that certain types of ungrounded H-fields have exactly two. The precise dividing line for ungrounded H-fields was unknown until Gehret showed that λ-*freeness* is the key determining property [**41**]. We define the properties "grounded" and "λ-free" in Sections 7.1 and 7.2 respectively. The number of T-Liouville closures of an H_T-field can likewise be characterized in terms of being grounded or λ-free:

Theorem 7.57. *If K is grounded or if K is ungrounded and λ-free, then K has exactly one T-Liouville closure up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. If K is ungrounded and not λ-free, then K has exactly two T-Liouville* *closures up to* $\mathcal{L}^{\mathcal{O},\partial}(K)$*-isomorphism. For any Liouville closed* H_T*-field extension* M *of* K*, there is an* $\mathcal{L}^{\mathcal{O},\partial}(K)$*-embedding of some* T*-Liouville closure of* K *into* M*.*

As an application, we prove an analog of the embedding theorem in **[2]** for $\mathbb{R}_{\text{an}}$-Hardy fields.

Theorem 7.58. *Let* $\mathcal{H}$ *be an* $\mathbb{R}_{\text{an}}$*-Hardy field and let* $\imath\colon \mathcal{H} \to \mathbb{T}_{\text{an}}$ *be an* H_{an}*-field embedding. Then* $\imath$ *extends to an* H_{an}*-field embedding* $\mathrm{Li}_{\text{an}}(\mathcal{H}) \to \mathbb{T}_{\text{an}}$*, where* $\mathrm{Li}_{\text{an}}(\mathcal{H})$ *is the minimal Liouville closed* $\mathbb{R}_{\text{an}}$*-Hardy field extension of* $\mathcal{H}$*.*

Our final result of Chapter 7 is that every pre-H_T-field has an H_T-field extension which satisfies the "order 1 intermediate value property." This was shown for H-fields in **[3]** and for $\mathcal{R}$-Hardy fields in **[27]**.

Theorem 7.59. *Every pre-*H_T*-field* K *has an* H_T*-field extension* M *with the following property: for every* $\mathcal{L}(M)$*-definable continuous function* $F\colon M \to M$ *and every* $b_1, b_2 \in M$ *with*

$$b_1' \ < \ F(b_1), \qquad b_2' \ > \ F(b_2),$$

there is $a \in M$ *between* b_1 *and* b_2 *with* $a' = F(a)$*.*

Model completeness for $\mathbb{T}_{\text{re}}$**.** Recall that $\mathbb{T}_{\text{re}}$ is the expansion of the H-field $\mathbb{T}$ by restricted sine, cosine, and exponential functions. In the final chapter, we show how the proof given in **[4]** that $\mathbb{T}$ is model complete can be modified to show the following:

Theorem 8.17. $\mathbb{T}_{\text{re}}$ *is model complete.*

As a consequence of our proof of Theorem 8.17, we can show that H_{re}-fields have a model companion.

Theorem 8.23. *The theory of* ω*-free newtonian Liouville closed* H_{re}*-fields is the model companion of the theory of* H_{re}*-fields.*

While Theorems 8.17 and 8.23 are, of course, closely related to the rest of the thesis, the proofs of these results rely primarily on material from **[4]** and are essentially independent from the other chapters.

CHAPTER 2

O-minimality

Let $\mathcal{L}$ be a language containing the binary relation $<$ and let K be an $\mathcal{L}$-structure expanding a dense linear ordering without endpoints. If the only $\mathcal{L}(K)$-definable subsets of K are finite unions of points and intervals, then K is said to be an **o-minimal structure**. If K is o-minimal, then so is any structure which is $\mathcal{L}$-elementarily equivalent to K. An $\mathcal{L}$-theory T extending the theory of dense linear orderings without endpoints is said to be an **o-minimal theory** if every model of T is an o-minimal structure. This thesis is about **o-minimal fields**: o-minimal structures which expand ordered fields. The underlying ordered field of any o-minimal field is necessarily real closed [**26**, 1.4.6].

Assumption 2.1. *For the remainder of this thesis, T is a complete, model complete o-minimal theory which extends the theory* RCF *of real closed ordered fields in some appropriate language $\mathcal{L} \supseteq \{0, 1, +, -, \cdot, <\}$. In Chapters 2 through 7, we always use K, L, and M for models of T (or expansions thereof).*

O-minimality has strong implications for the behavior of definable sets and functions in models of T. In Section 2.1, we list some key results about o-minimal fields for use throughout the thesis. We also prove a somewhat technical result on the behavior of definable $\mathcal{C}^1$-functions. We take a look at some important o-minimal expansions of the real field in Section 2.2. Finally, in Section 2.3, we construct nonstandard Hahn field models of some of these o-minimal theories. We also discuss o-minimal expansions of the field $\mathbb{T}$ of logarithmic-exponential transseries and the field **No** of surreal numbers.

2.1. Preliminaries on o-minimality

The starting point for the study of $\mathcal{L}(K)$-definable functions and sets is the monotonicity theorem.

Monotonicity theorem ([**26**], 3.1.2)**.** *Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function. Then there are points $a_1 < \cdots < a_n$ in K such that F is continuous and either constant, strictly increasing, or strictly decreasing on each interval $(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, +\infty)$.*

Many facts from elementary calculus, such as the mean value theorem, also hold for $\mathcal{L}(K)$-definable functions.

Mean value theorem ([**26**], 7.2.3)**.** *Let $a < b \in K$, let $F\colon [a, b] \to K$ be $\mathcal{L}(K)$-definable and continuous, and suppose that F is differentiable at every point in (a, b). Then for some $c \in (a, b)$, we have*

$$F'(c) \;=\; \frac{F(b) - F(a)}{b - a}.$$

Given an $\mathcal{L}(K)$-definable $\mathcal{C}^1$-map $F = (F_1, \ldots, F_m)\colon U \to K^m$ with $U \subseteq K^n$ open, we let $\mathbf{J}_F$ denote the Jacobian matrix

$$\mathbf{J}_F \;:=\; \left(\frac{\partial F_i}{\partial Y_j}\right)_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n},$$

viewed as an $\mathcal{L}(K)$-definable map from U to $\mathrm{Mat}_{m,n}(K)$. Occasionally, we write $\frac{\partial F}{\partial Y}$ instead of $\mathbf{J}_F$ to emphasize the dependence on the variables $Y = (Y_1, \ldots, Y_n)$. If $m = n = 1$, then we write F' instead of $\mathbf{J}_F$.

Cell decompositions. A $\mathcal{C}^k$**-cell** is a special type of definable $\mathcal{C}^k$-submanifold of K^n with an associated binary sequence $(i_1, \ldots, i_n) \in \{0,1\}^n$. The cells and their sequences are defined by induction on n.

(1) A (1)-cell in K is an open interval and a (0)-cell is a singleton.

(2) Given an $(i_1, \ldots, i_n)$-cell $D \subseteq K^n$, an $\mathcal{L}(K)$-definable open set $U \supseteq D$, and an $\mathcal{L}(K)$-definable $\mathcal{C}^k$-function $F\colon U \to K$, the set $\mathrm{Gr}(F|_D) \subseteq K^{n+1}$ is an $(i_1, \ldots, i_n, 0)$-cell and the following subsets of K^{n+1} are $(i_1, \ldots, i_n, 1)$-cells:
   - $D \times K$;
   - $\big\{(x,y) \in D \times K : y < F(x)\big\}$;
   - $\big\{(x,y) \in D \times K : y > F(x)\big\}$;
   - $\big\{(x,y) \in D \times K : F(x) < y < G(x)\big\}$, where $G\colon U \to K$ is another $\mathcal{L}(K)$-definable $\mathcal{C}^k$-function with $F(x) < G(x)$ on D.

Note that a $\mathcal{C}^k$-cell is open if and only if it is a $(1, \ldots, 1)$-cell. We call the binary sequence associated to a $\mathcal{C}^k$-cell D the **type** of D. We refer to $\mathcal{C}^0$-cells just as *cells*.

One of the most useful tools in the study of o-minimal fields is the $\mathcal{C}^k$-cell decomposition theorem. A $\mathcal{C}^k$**-cell decomposition of** K^n is a finite collection $\mathcal{D}$ of disjoint $\mathcal{C}^k$-cells $D \subseteq K^n$ such that $\bigcup \mathcal{D} = K^n$ and $\big\{\pi_{n-1}(D) : D \in \mathcal{D}\big\}$ is a $\mathcal{C}^k$-cell decomposition of K^{n-1}.

$\mathcal{C}^k$**-cell decomposition theorem** ([**26**], 3.2.11, 7.3.2, and 7.3.3)**.**

(1) *For any $\mathcal{L}(K)$-definable sets $A_1, \ldots, A_p \subseteq K^n$ there is a $\mathcal{C}^k$-cell decomposition partitioning $A_1, \ldots, A_p$, that is, a $\mathcal{C}^k$-cell decomposition $\mathcal{D}$ of K^n where each $D \in \mathcal{D}$ is disjoint from or contained in each A_i.*

(2) *For every $\mathcal{L}(K)$-definable map $F\colon A \to K^m$ with $A \subseteq K^n$, there is a $\mathcal{C}^k$-cell decomposition for F, that is, a $\mathcal{C}^k$-cell decomposition $\mathcal{D}$ partitioning A where for each $D \in \mathcal{D}$ contained in A, there is an $\mathcal{L}(K)$-definable open set $U \supseteq D$ and an $\mathcal{L}(K)$-definable $\mathcal{C}^k$-map $\tilde{F}\colon U \to K^m$ with $\tilde{F}|_D = F|_D$.*

In this thesis, we only use $\mathcal{C}^0$-cell and $\mathcal{C}^1$-cell decompositions. We refer to a $\mathcal{C}^0$-cell decomposition just as a *cell decomposition*. Cell decompositions allow us to assign a *dimension* to $\mathcal{L}(K)$-definable sets as follows: For a nonempty $\mathcal{L}(K)$-definable set $A \subseteq K^n$, we let

$$\dim_{\mathcal{L}}(A) \ := \ \max\big\{i_1 + \cdots + i_n : A \text{ contains an } (i_1, \ldots, i_n)\text{-cell}\big\}.$$

We also let $\dim_{\mathcal{L}}(\emptyset) := -\infty$. This dimension is quite robust; we list some facts below.

Fact 2.2 ([**26**], Section 4.1)**.** *Let $A \subseteq K^m$ and $B \subseteq K^n$ be $\mathcal{L}(K)$-definable sets.*

(a) $\dim_{\mathcal{L}}(K^n) = n$.

(b) $\dim_{\mathcal{L}}(A \times B) = \dim_{\mathcal{L}}(A) + \dim_{\mathcal{L}}(B)$.

(c) *If $m = n$ then* $\dim_{\mathcal{L}}(A \cup B) = \max\big\{\dim_{\mathcal{L}}(A), \dim_{\mathcal{L}}(B)\big\}$.

(d) $\dim_{\mathcal{L}}(A) = 0$ *if and only if A is finite and nonempty.*

(e) *If $F\colon A \to K^n$ is an $\mathcal{L}(K)$-definable map, then for $i = 0, \ldots, m$, the set*

$$B_i \ := \ \big\{x \in K^n : \dim_{\mathcal{L}}\big(F^{-1}(x)\big) = i\big\}$$

is $\mathcal{L}(K)$-definable and $\dim_{\mathcal{L}}\big(F^{-1}(B_i)\big) = \dim_{\mathcal{L}}(B_i) + i$. *In particular,* $\dim_{\mathcal{L}}$ *is preserved under definable bijections.*

The following Lemma will be used in Section 3.6.

Lemma 2.3. *Let $A \subseteq K^n$ be $\mathcal{L}(K)$-definable and let $A_1, \ldots, A_n \subseteq K$ be infinite sets. If $A_1 \times \cdots \times A_n \subseteq A$, then $\dim_{\mathcal{L}}(A) = n$.*

Proof. This follows from (d) of Fact 2.2 for $n = 1$. Suppose it holds for a given n, let $A \subseteq K^{n+1}$ be $\mathcal{L}(K)$-definable, and let $A_1, \ldots, A_{n+1} \subseteq K$ be infinite sets with $A_1 \times \cdots \times A_{n+1} \subseteq A$. Applying (e) of Fact 2.2 to the restriction of π_n to A, we see that

$$A^* \;:=\; \big\{y \in \pi_n(A) : \dim_{\mathcal{L}}(A_y) = 1\big\}$$

is definable. If $y \in A_1 \times \cdots \times A_n$, then $A_{n+1} \subseteq A_y$, so $y \in A^*$ by the $n = 1$ case. Thus, $A_1 \times \cdots \times A_n \subseteq A^*$ and $\dim_{\mathcal{L}}(A^*) = n$, by our induction hypothesis. Set

$$B \;:=\; \big\{z \in A : \pi_n(z) \in A^*\big\}.$$

Then $\dim_{\mathcal{L}}(B) = n + 1$, again by (e) of Fact 2.2, so $\dim_{\mathcal{L}}(A) = n + 1$ as well. □

Definable closure. It is a well-known and invaluable fact that $(K, \mathrm{dcl}_{\mathcal{L}})$ is a pregeometry. Throughout this thesis, we use "$\mathcal{L}$-independence" instead of "$\mathrm{dcl}_{\mathcal{L}}$-independence" for easier reading. We let $\mathrm{rk}_{\mathcal{L}}$ be the rank function which comes from $\mathrm{dcl}_{\mathcal{L}}$. O-minimal rank and o-minimal dimension are related as follows: for each tuple $a \in K^n$ and each subset $B \subseteq K$, we have

$$\mathrm{rk}_{\mathcal{L}}(a|B) = \min\big\{\dim_{\mathcal{L}}(A) : A \subseteq K^n \text{ is } \mathcal{L}(B)\text{-definable and } a \in A\big\},$$

as can be verified using the cell decomposition theorem. As a consequence, if a is $\mathcal{L}(B)$-independent, then any $\mathcal{L}(B)$-definable set containing a also contains an open neighborhood of a. Thus, any $\mathcal{L}(B)$-definable map $F\colon K^n \to K^m$ is $\mathcal{C}^1$ at a (that is, $\mathcal{C}^1$ on an open neighborhood of a).

In addition to being a pregeometry, the $\mathcal{L}$-definable closure completely controls the substructures and extensions of K. To make this precise, we need the following fact.

Definable choice ([**26**], 6.1.2)**.** *Let $D \subseteq K^{m+n}$ be $\mathcal{L}(\emptyset)$-definable. Then there is an $\mathcal{L}(\emptyset)$-definable map $F\colon \pi_m(D) \to K^n$ such that $\mathrm{Gr}(F) \subseteq D$.*

An immediate consequence of definable choice is that T *eliminates imaginaries.* This fact will come into play in Section 4.4. Here is another consequence:

Definable Skolem functions. *For $B \subseteq K$, the set $\mathrm{dcl}_{\mathcal{L}}(B)$ is the underlying set of an elementary $\mathcal{L}$-substructure of K*

Taking B above to be the empty set, we see that $\mathrm{dcl}_{\mathcal{L}}(\emptyset) \subseteq K$ is a **prime model** for T, that is, a model of T which admits an elementary $\mathcal{L}$-embedding into any model of T. To see that this embedding property holds, we note that since T is complete, the definable closure of the empty set in any $M \models T$ is uniquely $\mathcal{L}$-isomorphic to the definable closure of the empty set in K. We denote this prime model by $\mathbb{P}$, so $\mathbb{P}$ admits a *unique* elementary $\mathcal{L}$-embedding into any model of T.

Definable Skolem functions give us control over the parameters used to define certain objects. For example, let $B \subseteq K$ and suppose that the sets $A_1, \ldots, A_p$ in part (1) of the $\mathcal{C}^k$-cell decomposition theorem are $\mathcal{L}(B)$-definable. Then by working in $\mathrm{dcl}_{\mathcal{L}}(B)$, we see that we can take a $\mathcal{C}^k$-cell decomposition partitioning $A_1, \ldots, A_p$ where each cell is $\mathcal{L}(B)$-definable. Likewise, if the map F in part (2) of the theorem is $\mathcal{L}(B)$-definable, then we may take each $\mathcal{C}^k$-map $\tilde{F}$ to be $\mathcal{L}(B)$-definable.

Corollary 2.4. *Let $B \subseteq K$, let $F\colon K^n \to K^m$ be $\mathcal{L}(B)$-definable, and let $a \in K^n$. There is an $\mathcal{L}(B)$-definable map $\tilde{F}\colon K^n \to K^m$ which is $\mathcal{C}^1$ at a with $\tilde{F}(a) = F(a)$.*

Here is one more consequence of definable Skolem functions:

Corollary 2.5. *Let $\mathcal{L}^{\mathrm{df}}$ be the extension of $\mathcal{L}$ by function symbols for all $\mathcal{L}(\emptyset)$-definable functions, and let T^{df} be the corresponding extension of T by definitions. Then T^{df} has quantifier elimination and a universal axiomatization.*

Let M be a T-extension of K and let $A \subseteq M$. We let $K\langle A\rangle$ denote the $\mathcal{L}$-substructure of M with underlying set $\mathrm{dcl}_{\mathcal{L}}(K \cup A)$, so $K\langle A\rangle$ is an elementary $\mathcal{L}$-substructure of M. If $A = \{a_1, \ldots, a_n\}$, we write $K\langle a_1, \ldots, a_n\rangle$ instead of $K\langle A\rangle$. We say that M is a **simple extension** of K if $\mathrm{rk}_{\mathcal{L}}(M|K) = 1$. Then $M = K\langle a\rangle$ for some $a \in M \setminus K$.

Power boundedness. A **power function on** K is an $\mathcal{L}(K)$-definable endomorphism of the multiplicative group $K^>$. Each power function F is $\mathcal{C}^1$ on $K^>$ and uniquely determined by $F'(1)$, and we set

$$\Lambda \;:=\; \big\{F'(1) : F \text{ is a power function on } K\big\}.$$

Then Λ is a subfield of K, and it is called the **field of exponents of** K. For $a \in K^>$ and a power function F, we suggestively write $F(a)$ as a^λ where $\lambda = F'(1)$. A straightforward computation, using that $(a+h)^\lambda = a^\lambda(1+h/a)^\lambda$ for $a, a+h > 0$, gives that the power function $y \mapsto y^\lambda$ has derivative $y \mapsto \lambda y^{\lambda-1}$. We say that K is **power bounded** if for each $\mathcal{L}(K)$-definable function $F\colon K \to K$, there is λ in the field of exponents of K with $|F(x)| < x^\lambda$ for all sufficiently large positive x.

An **exponential function on** K is an ordered group isomorphism from the additive group K to the multiplicative group $K^>$. Any $\mathcal{L}(K)$-definable exponential function E on K is $\mathcal{C}^1$ on K and uniquely determined by $E'(0)$. If there is an $\mathcal{L}(K)$-definable exponential function on K, then there is a unique $\mathcal{L}(\emptyset)$-definable exponential function on K which is equal to its own derivative. Any exponential function on K grows more quickly than every power function on K. By [**54**], either K is power bounded or K defines an exponential function. It follows that being power bounded is a property of the theory T (we say that T *is power bounded*). If T is power bounded, then each power function on K is $\mathcal{L}(\emptyset)$-definable, so we refer to the field of exponents Λ as the *field of exponents of* T, as Λ does not depend on K. If T is power bounded with archimedean field of exponents, then T is said to be **polynomially bounded**.

A fiberwise property. In this subsection, we prove a somewhat technical fiberwise property for definable $\mathcal{C}^1$-functions for use in the proof of Proposition 3.29. Let $F\colon K^{m+n} \to K$ be an $\mathcal{L}(\emptyset)$-definable function of the variables $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ and let $a \in K^m$.

Lemma 2.6. *Let $u \in K^n$ and suppose that the function $y \mapsto F(a, y)$ is $\mathcal{C}^1$ at u. Then there is an $\mathcal{L}(\emptyset)$-definable function $\tilde{F}\colon K^{m+n} \to K$ which is $\mathcal{C}^1$ at (a, u) such that $\tilde{F}(a, u) = F(a, u)$ and*

$$\frac{\partial \tilde{F}}{\partial Y_j}(a, u) \;=\; \frac{\partial F}{\partial Y_j}(a, u)$$

for $j = 1, \ldots, n$.

Proof. We proceed by induction on n. If $n = 0$, then this is an immediate consequence of Corollary 2.4, so we assume that $n > 0$. If u is an $\mathcal{L}(a)$-independent tuple, then again using Corollary 2.4, we take an

$\mathcal{L}(\emptyset)$-definable function $\tilde{F}\colon K^{m+n} \to K$ which is $\mathcal{C}^1$ at (a,u) with $\tilde{F}(a,u) = F(a,u)$. By $\mathcal{L}(a)$-independence, we have $\tilde{F}(a,y) = F(a,y)$ for all y in an open neighborhood of u, which gives

$$\frac{\partial \tilde{F}}{\partial Y_j}(a,u) \;=\; \frac{\partial F}{\partial Y_j}(a,u)$$

for $j = 1,\ldots,n$, as desired. Assume now that u is not $\mathcal{L}(a)$-independent. By permuting coordinates, we arrange that $u_n \in \operatorname{dcl}_{\mathcal{L}}(a,u_1,\ldots,u_{n-1})$. We set

$$\hat{u} \;:=\; (u_1,\ldots,u_{n-1}), \qquad \hat{Y} \;:=\; (Y_1,\ldots,Y_{n-1}).$$

Let $\theta\colon K^{m+n-1} \to K$ be an $\mathcal{L}(\emptyset)$-definable function with $u_n = \theta(a,\hat{u})$. By Corollary 2.4, we may assume that θ is $\mathcal{C}^1$ at $(a,\hat{u})$. Set

$$G(X,\hat{Y}) \;:=\; F\big(X,\hat{Y},\theta(X,\hat{Y})\big),$$

so G is an $\mathcal{L}(\emptyset)$-definable function with $G(a,\hat{u}) = F(a,u)$. Since θ is $\mathcal{C}^1$ at $(a,\hat{u})$, the function $G(a,\hat{Y})$ is $\mathcal{C}^1$ at $\hat{u}$. By applying our induction hypothesis to G, we get a function $\tilde{G}\colon K^{m+n-1} \to K$ which is $\mathcal{C}^1$ at $(a,\hat{u})$ such that $\tilde{G}(a,\hat{u}) = G(a,\hat{u}) = F(a,u)$ and

$$\frac{\partial \tilde{G}}{\partial Y_j}(a,\hat{u}) \;=\; \frac{\partial G}{\partial Y_j}(a,\hat{u}) \;=\; \frac{\partial F}{\partial Y_j}(a,u) + \frac{\partial F}{\partial Y_n}(a,u)\frac{\partial \theta}{\partial Y_j}(a,\hat{u}) \tag{2.1}$$

for $j = 1,\ldots,n-1$. Applying Corollary 2.4 to the function $\frac{\partial F}{\partial Y_n}\big(X,\hat{Y},\theta(X,\hat{Y})\big)$, we take an $\mathcal{L}(\emptyset)$-definable function $H\colon K^{m+n-1} \to K$ which is $\mathcal{C}^1$ at $(a,\hat{u})$ with

$$H(a,\hat{u}) \;=\; \frac{\partial F}{\partial Y_n}\big(a,\hat{u},\theta(a,\hat{u})\big) \;=\; \frac{\partial F}{\partial Y_n}(a,u). \tag{2.2}$$

Now define $\tilde{F}$ by

$$\tilde{F}(X,Y) \;=\; \tilde{G}(X,\hat{Y}) + H(X,\hat{Y})\big(Y_n - \theta(X,\hat{Y})\big).$$

Then $\tilde{F}$ is $\mathcal{C}^1$ at (a,u) and, since $u_n = \theta(a,\hat{u})$, we have

$$\tilde{F}(a,u) \;=\; \tilde{G}(a,\hat{u}) + H(a,\hat{u})\big(u_n - \theta(a,\hat{u})\big) \;=\; \tilde{G}(a,\hat{u}) \;=\; F(a,u).$$

As for the partial derivatives, let $j \in \{1,\ldots,n-1\}$. We have

$$\frac{\partial \tilde{F}}{\partial Y_j}(a,u) \;=\; \frac{\partial \tilde{G}}{\partial Y_j}(a,\hat{u}) + \frac{\partial H}{\partial Y_j}(a,\hat{u})\big(u_n - \theta(a,\hat{u})\big) - H(a,\hat{u})\frac{\partial \theta}{\partial Y_j}(a,\hat{u}) \;=\; \frac{\partial \tilde{G}}{\partial Y_j}(a,\hat{u}) - H(a,\hat{u})\frac{\partial \theta}{\partial Y_j}(a,\hat{u}).$$

The above equation and the identities (2.1) and (2.2) give

$$\frac{\partial \tilde{F}}{\partial Y_j}(a,u) \;=\; \frac{\partial F}{\partial Y_j}(a,u) + \frac{\partial F}{\partial Y_n}(a,u)\frac{\partial \theta}{\partial Y_j}(a,\hat{u}) - \frac{\partial F}{\partial Y_n}(a,u)\frac{\partial \theta}{\partial Y_j}(a,\hat{u}) \;=\; \frac{\partial F}{\partial Y_j}(a,u).$$

As for the derivative with respect to Y_n, we have

$$\frac{\partial \tilde{F}}{\partial Y_n}(a,u) \;=\; H(a,\hat{u}) \;=\; \frac{\partial F}{\partial Y_n}(a,u).$$

□

Corollary 2.7. *Suppose that $y \mapsto F(a,y)$ is $\mathcal{C}^1$ on an $\mathcal{L}(a)$-definable open set $U \subseteq K^m$. Then there are $\mathcal{L}(\emptyset)$-definable functions $F_1,\ldots,F_N\colon K^{m+n} \to K$ where for each $u \in U$, there is $i \in \{1,\ldots,N\}$ such that*

- *F_i is $\mathcal{C}^1$ at (a,u),*
- *$F_i(a,u) = F(a,u)$,*
- *$\frac{\partial F_i}{\partial Y_j}(a,u) = \frac{\partial F}{\partial Y_j}(a,u)$ for $j = 1,\ldots,n$.*

Proof. Let M be a $|T|^+$-saturated T-extension of K and let $u \in U^M$. By Lemma 2.6, there is an $\mathcal{L}(\emptyset)$-definable function $\tilde{F}\colon M^{m+n} \to M$ which is $\mathcal{C}^1$ at (a,u) such that $\tilde{F}(a,u) = F(a,u)$ and

$$\frac{\partial \tilde{F}}{\partial Y_j}(a,u) \;=\; \frac{\partial F}{\partial Y_j}(a,u)$$

for $j = 1,\ldots,n$. The corollary follows by a standard compactness argument. □

Remark 2.8. In [**38**, Appendix A], we prove a stronger version of Corollary 2.7. The proof of this stronger version requires the use of Verdier stratifications, and the weaker, more elementary version we prove here is enough for the purposes of this thesis.

2.2. O-minimal expansions of the real field

The most fundamental o-minimal structure is the real field $\mathbb{R} = (\mathbb{R}, 0, 1, +, -, \cdot, <)$. Let RCF be the theory of real closed ordered fields, axiomatized in the language $\mathcal{L}_{\mathrm{ring}} := \{0, 1, +, -, \cdot, <\}$. Tarski showed that RCF completely axiomatizes the $\mathcal{L}_{\mathrm{ring}}$-theory of $\mathbb{R}$, and the Tarski-Seidenberg theorem tells us that RCF eliminates quantifiers. As a consequence, RCF is polynomially bounded with field of exponents $\mathbb{Q}$. In any real closed ordered field, the definable closure coincides with the (field-theoretic) algebraic closure:

Fact 2.9. *Let $K \models \mathrm{RCF}$, let $A \subseteq K$, and let $b \in K$. Then b is $\mathcal{L}_{\mathrm{ring}}(A)$-definable if and only if there is $P \in \mathbb{Z}[A][X]$ with $P(b) = 0$ and $P'(b) \neq 0$.*

In [**21**], van den Dries noted that the expansion of $\mathbb{R}$ by all finitely subanalytic sets is o-minimal, as a consequence of Gabrielov's theorem on the complements of subanalytic sets [**39**]. Van den Dries went on to show that this expansion is also polynomially bounded with field of exponents $\mathbb{Q}$. This expansion, which we denote $\mathbb{R}_{\mathrm{an}}$, is generally axiomatized in the language $\mathcal{L}_{\mathrm{an}}$ which extends $\mathcal{L}_{\mathrm{ring}}$ by a function symbol for each n-ary function which is real analytic on a neighborhood of the closed unit cube $[-1,1]^n$. These function symbols are interpreted as usual on this unit cube, and interpreted to be identically zero outside of this cube.

Let T_{an} be the complete $\mathcal{L}_{\mathrm{an}}$-theory of $\mathbb{R}_{\mathrm{an}}$. Model completeness for T_{an} is a consequence of Gabrielov's theorem, but later work of van den Dries and Denef [**19**] showed that model completeness could be improved to quantifier elimination if one adds a function symbol for multiplicative inversion (away from zero) to the language $\mathcal{L}_{\mathrm{an}}$. In [**30**], van den Dries, Macintyre, and Marker gave an explicit axiomatization of T_{an}. They also showed that T_{an} has quantifier elimination and a universal axiomatization in the language $\mathcal{L}^*_{\mathrm{an}}$, which extends $\mathcal{L}_{\mathrm{an}}$ by function symbols for multiplicative inversion and n^{th} roots, where $n > 0$ and the function symbols for n^{th} roots are defined to be 0 for negative inputs. This allows for a very explicit description of definable functions in models of T_{an}:

Fact 2.10 ([**30**], 2.15). *Each $\mathcal{L}_{\mathrm{an}}(A)$-definable function is given piecewise by $\mathcal{L}^*_{\mathrm{an}}(A)$-terms.*

The quintessential o-minimal structure which is *not* power bounded is $\mathbb{R}_{\exp}$; the expansion of the real field by the (unrestricted) exponential function. This structure was shown to model complete by Wilkie [**68**]. O-minimality follows from model completeness and a theorem of Hovanskiĭ [**48**]. Let $T_{\exp}$ be the complete theory of $\mathbb{R}_{\exp}$ in the language $\mathcal{L}_{\exp} := \mathcal{L}_{\mathrm{ring}} \cup \{\exp\}$. In [**33**], van den Dries and Miller provided a description of the definable closure in models of $T_{\exp}$:

Fact 2.11 ([**33**], 7.1)**.** *Let $K \models T_{\text{exp}}$, let $A \subseteq K$, and let $b \in K$. Then b is $\mathcal{L}_{\text{exp}}(A)$-definable if and only if b is a coordinate of a regular solution to a system of equations*

$$t_1(Y_1, \ldots, Y_n) \;=\; \cdots \;=\; t_n(Y_1, \ldots, Y_n) \;=\; 0,$$

where $t_1, \ldots, t_n$ are $\mathcal{L}_{\text{exp}}(A)$-terms. By a regular solution to the above system, we mean a solution $a \in K^n$ at which the matrix $\big(\frac{\partial t_i}{\partial Y_j}(a)\big)_{1 \leqslant i,j \leqslant n}$ is invertible.

The expansion of the real field by both restricted analytic functions *and* the total exponential function was shown to be model complete and o-minimal by van den Dries and Miller [**33**]. This expansion, denoted $\mathbb{R}_{\text{an,exp}}$, is one of the most important structures in o-minimality, with applications to diophantine geometry, complex analysis, and Hodge theory.

Let $T_{\text{an,exp}}$ be the complete theory of $\mathbb{R}_{\text{an,exp}}$ in the language $\mathcal{L}_{\text{an,exp}} := \mathcal{L}_{\text{an}} \cup \mathcal{L}_{\text{exp}}$. A technical matter—we assume that the function symbol in $\mathcal{L}_{\text{an}}$ for restricted exponentiation differs from the function symbol exp in $\mathcal{L}_{\text{exp}}$. In [**30**], van den Dries, Macintyre, and Marker gave an explicit axiomatization of $T_{\text{an,exp}}$. They also showed that model completeness can be improved to quantifier elimination and a universal axiomatization if one extends $\mathcal{L}_{\text{an,exp}}$ by a function symbol for the logarithm function (defined to be 0 for nonpositive inputs). We denote this extended language by $\mathcal{L}^*_{\text{an,exp}}$. As with Fact 2.10, we have an explicit description of definable functions in models of $T_{\text{an,exp}}$:

Fact 2.12 ([**30**], 4.7)**.** *Each $\mathcal{L}_{\text{an,exp}}(A)$-definable function is given piecewise by $\mathcal{L}^*_{\text{an,exp}}(A)$-terms.*

One o-minimal structure which will be important to us in Chapter 8 is the expansion of $\mathbb{R}$ by the restriction of the sine, cosine, and exponential functions to the closed interval $[-1, 1]$. We denote this expansion by $\mathbb{R}_{\text{re}}$, where the subscript stands for "restricted elementary functions." Of course, $\mathbb{R}_{\text{re}}$ is o-minimal and polynomially bounded, as it is a reduct of $\mathbb{R}_{\text{an}}$. Let T_{re} be the complete theory of $\mathbb{R}_{\text{re}}$ in the language $\mathcal{L}_{\text{re}} := \mathcal{L}_{\text{ring}} \cup \{\sin, \cos, \exp\}$.

In [**22**], van den Dries showed that the theory of $\mathbb{R}_{\text{re}}$ is model complete in the language $\mathcal{L}_{\text{re}}(\mathbb{R})$ (that is, the extension of $\mathcal{L}_{\text{re}}$ by a constant symbol for each real number). Model completeness in just the language $\mathcal{L}_{\text{re}}$ follows from work of Wilkie [**68**]. One can also show that T_{re} is model complete by using a theorem of Gabrielov, which states that any expansion of $\mathbb{R}$ by a family of restricted analytic functions which is closed under partial derivatives is model complete [**40**].

2.3. Hahn field models

Let $\boldsymbol{k}$ be a field and let $\mathfrak{M} = (\mathfrak{M}, \prec)$ be a **monomial group**, that is, an ordered group written multiplicatively with identity $1_{\mathfrak{M}}$. A subset $\mathfrak{S} \subseteq \mathfrak{M}$ is said to be **well-based** if $\mathfrak{S}$ contains no infinite strictly increasing sequence $\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \cdots$ from $\mathfrak{M}$. The **Hahn field over $\boldsymbol{k}$ with monomial group $\mathfrak{M}$**, denoted $\boldsymbol{k}[[\mathfrak{M}]]$, is the collection of formal series

$$f \;=\; \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$$

where $f_{\mathfrak{m}} \in \boldsymbol{k}$ and where the *support* $\operatorname{supp}(f) := \{\mathfrak{m} \in \mathfrak{M} : f_{\mathfrak{m}} \neq 0\}$ is a well-based subset of $\mathfrak{M}$. The field operations on $\boldsymbol{k}[[\mathfrak{M}]]$ are as follows:

$$f + g \;:=\; \sum_{\mathfrak{m}} (f_{\mathfrak{m}} + g_{\mathfrak{m}})\mathfrak{m}, \qquad f \cdot g \;:=\; \sum_{\mathfrak{m}} \Big(\sum_{\mathfrak{n}_1 \cdot \mathfrak{n}_2 = \mathfrak{m}} f_{\mathfrak{n}_1} g_{\mathfrak{n}_2} \Big)\mathfrak{m}.$$

The fact that multiplication is well-defined follows from the assumption that $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are well-based. We identify $\boldsymbol{k}$ with a subfield of $\boldsymbol{k}[[\mathfrak{M}]]$ via the map $a \mapsto a \cdot 1_{\mathfrak{M}}$. For $f \in \boldsymbol{k}[[\mathfrak{M}]]$, set $\mathfrak{d}(f) := \max \operatorname{supp}(f)$. We say that $\varepsilon \in \boldsymbol{k}[[\mathfrak{M}]]$ is **infinitesimal** if $\mathfrak{d}(\varepsilon) \prec 1_{\mathfrak{M}}$. In this case, we write $\varepsilon \prec 1$.

A family $(f_i)_{i\in I}$ of series in $\boldsymbol{k}[[\mathfrak{M}]]$ is said to be **summable** if $\bigcup_i \operatorname{supp}(f_i)$ is well-based and each monomial $\mathfrak{m} \in \mathfrak{M}$ only appears in the support of only finitely many of the f_i. If (f_i) is summable, then for any family (a_i) from $\boldsymbol{k}$, the series

$$\sum_i a_i f_i \;:=\; \sum_{\mathfrak{m}} \sum_i a_i f_{i,\mathfrak{m}} \mathfrak{m}$$

is a well-defined element of $\boldsymbol{k}[[\mathfrak{M}]]$. Let $\Phi\colon \boldsymbol{k}[[\mathfrak{M}]] \to \boldsymbol{k}[[\mathfrak{M}]]$ be a map. We call Φ **strongly additive** if for each summable family (f_i) from $\boldsymbol{k}[[\mathfrak{M}]]$, the family $\big(\Phi(f_i)\big)$ is summable and $\Phi\big(\sum_i f_i\big) = \sum_i \Phi(f_i)$.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ be a tuple of infinitesimals in $\boldsymbol{k}[[\mathfrak{M}]]$. For a multi-index $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, let $\varepsilon^{\boldsymbol{i}} = \varepsilon_1^{i_1} \cdots \varepsilon_n^{i_n} \in \boldsymbol{k}[[\mathfrak{M}]]$. The family $(\varepsilon^{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}$ is summable by a result of Neumann [**55**]; see also [**31**, 1.4]. Thus, for any sequence $(a_{\boldsymbol{i}})$ from $\boldsymbol{k}$ indexed by multi-indices, the series

$$\sum_{\boldsymbol{i}\in\mathbb{N}^n} a_{\boldsymbol{i}} \varepsilon^{\boldsymbol{i}}$$

is a well-defined element of $\boldsymbol{k}[[\mathfrak{M}]]$. This can be used to show that $\boldsymbol{k}[[\mathfrak{M}]]$ is indeed a field: if $f \neq 0$ then we may take some infinitesimal $\varepsilon \in \boldsymbol{k}[[\mathfrak{M}]]$ with $f = f_{\mathfrak{d}(f)}\mathfrak{d}(f)(1+\varepsilon)$. One easily verifies that

$$f^{-1} \;=\; f_{\mathfrak{d}(f)}^{-1} \mathfrak{d}(f)^{-1} \sum_n (-\varepsilon)^n.$$

Let $(\mathfrak{M}_i)_{i\in I}$ be an increasing chain of monomial groups. We may consider the field $\bigcup_i \boldsymbol{k}[[\mathfrak{M}_i]]$. This union is not a Hahn field in general. However, we can still make sense of summability: a family (f_j) from $\bigcup_i \boldsymbol{k}[[\mathfrak{M}_i]]$ is said to be **summable** if (f_j) is a summable family in $\boldsymbol{k}[[\mathfrak{M}_i]]$ for some $i \in I$. If $\bigcup_i \boldsymbol{k}[[\mathfrak{M}_i]]$ is itself a Hahn field, then $\bigcup_i \boldsymbol{k}[[\mathfrak{M}_i]] = \boldsymbol{k}[[\mathfrak{M}_i]]$ for some $i \in I$, so this agrees with our previous definition of summability. The notion of a strongly additive map on a Hahn field may be generalized to increasing unions of Hahn fields using this definition of summability.

If $\boldsymbol{k}$ is an ordered field, then we view $\boldsymbol{k}[[\mathfrak{M}]]$ as an ordered field extension of $\boldsymbol{k}$ by declaring $f \in \boldsymbol{k}[[\mathfrak{M}]]$ to be positive whenever $f_{\mathfrak{d}(f)} \in \boldsymbol{k}$ is positive. If $\boldsymbol{k}$ is real closed and $\mathfrak{M}$ is divisible, then $\boldsymbol{k}[[\mathfrak{M}]]$ is real closed; see the remark following [**4**, 3.5.19].

Hahn field models of T_{an}. Now let $\boldsymbol{k} \models T_{\text{an}}$ and let $\mathfrak{M}$ be a divisible monomial group. We will show that $\boldsymbol{k}[[\mathfrak{M}]]$ admits a natural expansion to a model of T_{an}. This follows the method used in [**30**], where $\boldsymbol{k} = \mathbb{R}_{\text{an}}$. It is worth noting that since each real constant function is analytic, $\boldsymbol{k}$ contains $\mathbb{R}_{\text{an}}$ as an elementary substructure.

Let $y = (y_1, \ldots, y_n) \in \boldsymbol{k}[[\mathfrak{M}]]^n$ and let F be an n-ary restricted analytic function. We will show how to define $F(y)$. Of course, we define $F(y)$ to be zero if any $|y_i|$ is greater than 1, so let us assume that $|y_1|, \ldots, |y_n| \leqslant 1$. Then there are unique tuples $a = (a_1, \ldots, a_n) \in \boldsymbol{k}^n$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \boldsymbol{k}[[\mathfrak{M}]]^n$ with

$$|a_k| \;\leqslant\; 1, \qquad \varepsilon_k \;\prec\; 1, \qquad y_k \;=\; a_k + \varepsilon_k$$

for $k = 1, \ldots, n$. We set

$$F(y) \;=\; F(a+\varepsilon) \;:=\; \sum_{\boldsymbol{i}\in\mathbb{N}^n} \frac{F^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!} \varepsilon^{\boldsymbol{i}},$$

where

$$F^{(\boldsymbol{i})} \;=\; \frac{\partial^{i_1+\cdots+i_n}F}{\partial^{i_1}Y_1\cdots\partial^{i_n}Y_n}, \qquad \boldsymbol{i}! \;=\; i_1!\cdots i_n!.$$

Since $F^{(\boldsymbol{i})}$ is itself a restricted analytic function, the expression $F^{(\boldsymbol{i})}(a)$ makes sense as an element of $\boldsymbol{k}$. Doing this for all y and all restricted analytic functions F, we get an $\mathcal{L}_{\mathrm{an}}$-expansion of the Hahn field $\boldsymbol{k}[[\mathfrak{M}]]$, which we denote by $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$.

Proposition 2.13. $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}} \models T_{\mathrm{an}}$.

Proof. We use the axiomatization of T_{an} given in [**30**, 2.14]. As mentioned above, $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ is real closed, so it remains to show that axioms (AC1)–(AC4) from [**30**] hold. For (AC1)–(AC3), we need to show that the constant functions 0 and 1, the coordinate functions, addition, multiplication, and composition behave as they should. Let y, a, and ε be as above. One easily verifies that

$$0(y) \;=\; 0, \qquad 1(y) \;=\; 1, \qquad Y_i(y) \;=\; y_i \;\text{ for } i=1,\ldots,n,$$

where 0 and 1 are constant functions and Y_i is the i^{th} coordinate function. For n-ary restricted analytic functions F and G, the identities $F(y)+G(y) = (F+G)(y)$ and $F(y)G(y) = (FG)(y)$ follow from the identities

$$(F+G)^{(\boldsymbol{i})} \;=\; F^{(\boldsymbol{i})}+G^{(\boldsymbol{i})}, \qquad (FG)^{(\boldsymbol{i})} \;=\; \sum_{\boldsymbol{j}_1+\boldsymbol{j}_2=\boldsymbol{i}} \frac{\boldsymbol{i}!}{\boldsymbol{j}_1!\boldsymbol{j}_2!}F^{(\boldsymbol{j}_1)}G^{(\boldsymbol{j}_2)}$$

respectively, for $\boldsymbol{i} \in \mathbb{N}^n$. Let $G = (G_1,\ldots,G_m)$ be a tuple of n-ary restricted analytic functions with $|G_k(y)| \leqslant 1$ for $k = 1,\ldots,m$ and let F be an m-ary restricted analytic function. We need to show that $(F\circ G)(y) = F\big(G(y)\big)$. Using that $G_k(a)\in\boldsymbol{k}$ and $G_k(y)-G_k(a)\prec 1$ for each k, we have

$$F\big(G(y)\big) \;=\; \sum_{\boldsymbol{j}} \frac{F^{(\boldsymbol{j})}\big(G(a)\big)}{\boldsymbol{j}!}\big(G(y)-G(a)\big)^{\boldsymbol{j}} \;=\; \sum_{\boldsymbol{j}} \frac{F^{(\boldsymbol{j})}\big(G(a)\big)}{\boldsymbol{j}!}\prod_{k=1}^{n}\Big(\sum_{\boldsymbol{i}}\frac{G_k^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}}-G_k(a)\Big)^{j_k},$$

where $\boldsymbol{i}$ ranges over $\mathbb{N}^n$ and $\boldsymbol{j}$ ranges over $\mathbb{N}^m$. Let $d = (d_1,\ldots,d_n)\in\mathbb{R}^n$ with $|d_1|,\ldots,|d_n| \leqslant 1$. By composing Taylor expansions, we see that the following identity holds in $\mathbb{R}[[Y]]$:

$$\sum_{\boldsymbol{i}}\frac{(F\circ G)^{(\boldsymbol{i})}(d)}{\boldsymbol{i}!}Y^{\boldsymbol{i}} \;=\; \sum_{\boldsymbol{j}} \frac{F^{(\boldsymbol{j})}\big(G(d)\big)}{\boldsymbol{j}!}\prod_{k=1}^{n}\Big(\sum_{\boldsymbol{i}}\frac{G_k^{(\boldsymbol{i})}(d)}{\boldsymbol{i}!}Y^{\boldsymbol{i}}-G_k(d)\Big)^{j_k}. \tag{2.3}$$

On both sides, each coefficient of $Y^{\boldsymbol{i}}$ may be expressed as $\mathcal{L}_{\mathrm{an}}(\emptyset)$-definable function of d. Since $\boldsymbol{k}$ is an elementary extension of $\mathbb{R}_{\mathrm{an}}$, we may work coefficient by coefficient to see that (2.3) holds when d is replaced by a:

$$\sum_{\boldsymbol{i}}\frac{(F\circ G)^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}Y^{\boldsymbol{i}} \;=\; \sum_{\boldsymbol{j}} \frac{F^{(\boldsymbol{j})}\big(G(a)\big)}{\boldsymbol{j}!}\prod_{k=1}^{n}\Big(\sum_{\boldsymbol{i}}\frac{G_k^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}Y^{\boldsymbol{i}}-G_k(a)\Big)^{j_k}.$$

This gives the desired identity

$$(F\circ G)(y) \;=\; \sum_{\boldsymbol{i}}\frac{(F\circ G)^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}} \;=\; \sum_{\boldsymbol{j}} \frac{F^{(\boldsymbol{j})}\big(G(a)\big)}{\boldsymbol{j}!}\prod_{k=1}^{n}\Big(\sum_{\boldsymbol{i}}\frac{G_k^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}}-G_k(a)\Big)^{j_k} \;=\; F\big(G(y)\big).$$

Finally, for (AC4), let F be an n-ary restricted analytic function, let $d = (d_1,\ldots,d_n)\in\mathbb{R}^n$ with $|d_1|,\ldots,|d_n| \leqslant 1$, let $\delta\in\mathbb{R}^{>}$, and let G be the restricted analytic function

$$G(Y_1,\ldots,Y_n) \;=\; F(d_1+\delta Y_1,\ldots,d_n+\delta Y_n).$$

Suppose that $|d_k + \delta y_k| \leqslant 1$ for $k = 1, \ldots, n$. We need to show that $G(y) = F(d + \delta y)$, where $\delta y = (\delta y_1, \ldots, \delta y_n)$. We have

$$d + \delta y \; = \; d + \delta(a + \varepsilon) \; = \; (d + \delta a) + \delta\varepsilon$$

where $d + \delta a \in \boldsymbol{k}^n$, each $|d_k + \delta a_k| \leqslant 1$, and each $\delta\varepsilon_k \prec 1$. Thus

$$F(d + \delta y) \; = \; \sum_{\boldsymbol{i}} \frac{F^{(\boldsymbol{i})}(d + \delta a)}{\boldsymbol{i}!}(\delta\varepsilon)^{\boldsymbol{i}} \; = \; \sum_{\boldsymbol{i}} \frac{F^{(\boldsymbol{i})}(d + \delta a)\delta^{i_1 + \cdots + i_n}}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}} \; = \; \sum_{\boldsymbol{i}} \frac{G^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}(\varepsilon)^{\boldsymbol{i}} \; = \; G(y). \qquad \square$$

The method of expanding Hahn fields to models of T_{an} generalizes to increasing unions of Hahn fields. Let $(\mathfrak{M}_i)_{i \in I}$ be an increasing chain of divisible monomial groups. By Proposition 2.13 and model completeness for T_{an}, the structure $\boldsymbol{k}[[\mathfrak{M}_i]]_{\text{an}}$ is an elementary $\mathcal{L}_{\text{an}}$-substructure of $\boldsymbol{k}[[\mathfrak{M}_j]]_{\text{an}}$ whenever $i < j \in I$. Therefore, the union $\bigcup_{i \in I} \boldsymbol{k}[[\mathfrak{M}_i]]_{\text{an}}$ is a model of T_{an}.

Transseries and surreal numbers. Let us now consider two specific examples, both of which we revisit throughout the remainder of the thesis. Our first example is the field $\mathbb{T}$ of *logarithmic-exponential transseries*, mentioned in Chapter 1. In this subsection, we will not discuss the derivation on $\mathbb{T}$. Instead, we will focus on (expansions of) the underlying ordered field. We refer the reader to Examples 3.17, 6.5, and 7.5 for information about the derivation, valuation, and H-field structure on $\mathbb{T}$, and how these interact with the expansions considered here.

The field $\mathbb{T}$ consists of elements like

$$e^{e^{e^x}} + 12xe^{x^2 + \pi x} + x^{1/2} + x^{1/3} + \ldots + \log(\log x) + e^{-x} + e^{-x^2} + e^{-x^3} + \cdots,$$

that is, transfinite series of *transmonomials* such as $xe^{x^2+\pi x}$, $\log(\log x)$, and e^{-x} with real coefficients. These transmonomials are ordered as one would expect: we think of each transmonomial $\mathfrak{m}$ as a function of x, and we set $\mathfrak{m} \prec 1$ if this function approaches zero as x approaches $+\infty$. The construction of $\mathbb{T}$ is somewhat complex; for a detailed reference, see Appendix A of [**4**]. For our purposes, it is enough to know that $\mathbb{T}$ is an increasing union of Hahn fields over $\mathbb{R}$ with divisible monomial groups. As such, $\mathbb{T}$ may be expanded to an elementary extension of $\mathbb{R}_{\text{an}}$ as detailed above; we use $\mathbb{T}_{\text{an}}$ to denote this expansion.

In addition to restricted analytic functions, the field $\mathbb{T}$ admits an unrestricted exponential function which extends the exponential function on $\mathbb{R}$. The expansion of $\mathbb{T}$ by restricted analytic functions and this exponential function is denoted $\mathbb{T}_{\text{an,exp}}$.

Fact 2.14 ([**31**], 2.8)**.** $\mathbb{T}_{\text{an,exp}} \models T_{\text{an,exp}}$.

Though $\mathbb{T}$ is an increasing union of Hahn fields, it is not a Hahn field itself. In fact, no Hahn field with nontrivial monomial group admits a total exponential function by a negative result, established independently by van der Hoeven [**47**, Proposition 2.2] and by Kuhlmann, Kuhlmann, and Shelah [**51**]. This result comes up later in this thesis; see Remark 5.14. Thus, the fact that $\mathbb{T}$ is not a Hahn field is essential.

Another important structure which may be viewed as an increasing union of Hahn fields is Conway's field **No** of *surreal numbers* [**16**]. This field is a proper class, as axiomatized in Neumann–Gödel–Bernays set theory with global choice. Though elements of **No** are generally defined as maps from an ordinal to the set $\{-, +\}$, any surreal number may be represented as a transfinite series $\sum_{y \in \mathbf{No}} r_y \omega^y$, where $r_y \in \mathbb{R}$ and where $\{y : r_y \neq 0\}$ is a well-based sub*set* of **No**.

The field $\mathbf{No}$ is also increasing union of Hahn fields over $\mathbb{R}$ with divisible monomial groups, where each Hahn field is a set and the union is indexed over the proper class of ordinals. This allows us to expand $\mathbf{No}$ to an elementary extension of $\mathbb{R}_{\mathrm{an}}$, denoted $\mathbf{No}_{\mathrm{an}}$. Moreover, the exponential on $\mathbb{R}$ naturally extends to a total exponential on $\mathbf{No}$, as shown by Gonshor [**43**]. This gives us a further expansion of $\mathbf{No}_{\mathrm{an}}$, denoted $\mathbf{No}_{\mathrm{an,exp}}$.

Fact 2.15 ([**28**], 2.1)**.** $\mathbf{No}_{\mathrm{an,exp}} \models T_{\mathrm{an,exp}}$.

CHAPTER 3

T-derivations

In this chapter, we fix a map $\partial\colon K \to K$. For $a \in K$, we use a' or ∂a in place of $\partial(a)$, and we use $a^{(r)}$ in place of $\partial^r(a)$. If $a \neq 0$, then we set $a^\dagger := a'/a$. Given a tuple $b = (b_1, \ldots, b_n) \in K^n$, we denote by ∂b or b' the tuple $(b_1', \ldots, b_n')$. We let $\mathcal{L}^\partial$ be the language $\mathcal{L} \cup \{\partial\}$, and we view $K = (K, \partial)$ as an $\mathcal{L}^\partial$-structure.

Definition 3.1. Given an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function $F\colon U \to K$ with $U \subseteq K^n$ open, we say that ∂ is **compatible with** F if

$$F(u)' \;=\; \mathbf{J}_F(u)u'$$

for each $u \in U$. We say that ∂ is a **T-derivation on** K if ∂ is compatible with every $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function with open domain. Let T^∂ be the $\mathcal{L}^\partial$-theory which extends T by axioms stating that ∂ is a T-derivation.

T-derivations are the main objects of study in this thesis, and the purpose of this chapter is to provide all the necessary background on T-derivations for Chapters 4, 6, and 7. We begin with some basic facts and useful tests in Section 3.1. In Section 3.2, we examine natural T-derivations on our Hahn field models from Section 2.3. Section 3.3 is devoted to constructing a pregeometry on models of T^∂ which serves as an analog of differential algebraic closure. In Section 3.4, we prove a handful of important results on building T^∂-extensions of K. Section 3.5 is focused on various ways of manipulating definable functions for use in Chapter 6. Finally, in Section 3.6, we introduce *thin sets*, which are subsets of models of T^∂ that may be thought of as small. Almost all the material in Sections 3.1, 3.3, and 3.4 is joint work with Antongiulio Fornasiero from [**38**].

3.1. Basic properties of T-derivations

Let us begin this section by justifying the use of the name *T-derivation*. Recall that ∂ is a *derivation on K* if $(x+y)' = x' + y'$ and $(xy)' = x'y + y'x$ for all $x, y \in K$. To put it another way:

Fact 3.2. *The map ∂ is a derivation on K if and only if ∂ is compatible with addition and multiplication. In particular, any T-derivation is a derivation.*

If ∂ is a T-derivation on K, then we let $C_K := \ker(\partial) = \{a \in K : a' = 0\}$ denote the **constant field of** K. When K is clear from context, we drop the subscript and denote the constant field by C.

Lemma 3.3. *Let $K \models T^\partial$. Then C is the underlying set of an elementary $\mathcal{L}$-substructure of K.*

Proof. As T has definable Skolem functions, it suffices to show that C is $\mathcal{L}$-definably closed in K. Let $F\colon K^n \to K$ be an $\mathcal{L}(\emptyset)$-definable function and let $c \in C^n$ be $\mathcal{L}(\emptyset)$-independent, so F is $\mathcal{C}^1$ at c. Then $F(c)' = \mathbf{J}_F(c)c' = 0$, so $F(c) \in C$. $\square$

Lemma 3.3 may be viewed as an analog of the well-known fact that the constant field of a differential field is relatively algebraically closed. The **trivial derivation on** K is the map $K \to K$ which takes constant value 0. The trivial derivation is a T-derivation on K.

Corollary 3.4. *The expansion of $\mathbb{P}$, the prime model of T, by the trivial derivation admits a unique $\mathcal{L}^{\partial}$-embedding into any model of T^{∂}.*

PROOF. Let $K \models T^{\partial}$. As C is (the underlying set of) an elementary $\mathcal{L}$-substructure of K, the unique $\mathcal{L}$-embedding $\mathbb{P} \to K$ has image contained in C. Thus, this embedding is an $\mathcal{L}^{\partial}$-embedding, where the derivation on $\mathbb{P}$ is trivial. □

By our model completeness assumption for T, the union of an increasing chain of T-models is a T-model. It follows that the union of an increasing chain of T^{∂}-models is a T^{∂}-model. Additionally, if $K \models T^{\partial}$ and $E \models T$ is an $\mathcal{L}^{\partial}$-substructure of K, then $E \models T^{\partial}$. These facts, when paired with [**66**, 3.1.5 and 3.1.9], allow us to say something about the axiomatization of T^{∂}.

Corollary 3.5. *T^{∂} has an $\forall\exists$-axiomatization. If T has a universal axiomatization, then so does T^{∂}.*

Tests and preservation lemmas. Below, we provide a useful test to see if ∂ is a T-derivation. Then we prove a couple of preservation results. First, let us extend our notion of *compatibility* to definable $\mathcal{C}^1$-maps. Let $G\colon U \to K^m$ be an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-map with $U \subseteq K^n$ open. Then ∂ is said to be compatible with G if ∂ is compatible with each component function G_i. Equivalently, ∂ is compatible with G if

$$G(u)' \;=\; \mathbf{J}_G(u)u'$$

for each $u \in U$.

Lemma 3.6. *The following are equivalent:*

(1) *$K \models T^{\partial}$;*
(2) *$c' = 0$ for all $c \in \operatorname{dcl}_{\mathcal{L}}(\emptyset)$ and $F(u)' = \mathbf{J}_F(u)u'$ for all $\mathcal{L}(\emptyset)$-independent tuples u and all $\mathcal{L}(\emptyset)$-definable functions F defined at u.*

PROOF. Clearly (1) implies (2), as the constant function $x \mapsto c$ is $\mathcal{L}(\emptyset)$-definable for each $c \in \operatorname{dcl}_{\mathcal{L}}(\emptyset)$. Now suppose (2) holds, fix an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function $F\colon U \to K$ with U open, and fix a tuple $u \in U$. We need to show that $F(u)' = \mathbf{J}_F(u)u'$. If each component of u is in $\operatorname{dcl}_{\mathcal{L}}(\emptyset)$, then $F(u) \in \operatorname{dcl}_{\mathcal{L}}(\emptyset)$, so $F(u)' = \mathbf{J}_F(u)u' = 0$ by (2). If there is some component of u which is not in $\operatorname{dcl}_{\mathcal{L}}(\emptyset)$, then let v be a maximal $\mathcal{L}(\emptyset)$-independent subtuple of u and fix an $\mathcal{L}(\emptyset)$-definable map G with $u = G(v)$. As v is $\mathcal{L}(\emptyset)$-independent, we have

$$F(u)' \;=\; (F\circ G)(v)' \;=\; \mathbf{J}_{F\circ G}(v)v' \;=\; \mathbf{J}_F\big(G(v)\big)\mathbf{J}_G(v)v' \;=\; \mathbf{J}_F(u)G(v)' \;=\; \mathbf{J}_F(u)u',$$

where the second and fourth equality use (2). □

Lemma 3.7. *Let $U \subseteq K^n$ and $V \subseteq K^m$ be open, let $F\colon U \to K$ and $G\colon V \to U$ be $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-maps, and suppose ∂ is compatible with F and G. Then ∂ is compatible with the composition $F\circ G$.*

PROOF. For $u \in V$, we have

$$F\big(G(u)\big)' \;=\; \mathbf{J}_F\big(G(u)\big)G(u)' \;=\; \mathbf{J}_F\big(G(u)\big)\mathbf{J}_G(u)u' \;=\; \mathbf{J}_{F\circ G}(u)u'. \qquad \square$$

Lemma 3.8. *Let $U \subseteq K^m$ and $V \subseteq K^{m+n}$ be open, let $F\colon V \to K^n$ be an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-map in variables (X,Y), and let $G\colon U \to K^n$ be an $\mathcal{L}(\emptyset)$-definable map with $\mathrm{Gr}(G) \subseteq V$. Suppose that ∂ is compatible with F and that*

$$F\big(u, G(u)\big) \ = \ 0, \qquad \det\Big(\frac{\partial F}{\partial Y}\big(u, G(u)\big)\Big) \ \neq \ 0$$

for all $u \in U$. Then G is $\mathcal{C}^1$ and ∂ is compatible with G.

Proof. The map G is $\mathcal{C}^1$ by the o-minimal implicit function theorem; see [**26**, 7.2.11]. Let $u \in U$ and define the map $H\colon U \to K^n$ by $H(y) = F\big(y, G(y)\big)$. Then H is identically zero on U, so we have

$$\frac{\partial F}{\partial X}\big(u, G(u)\big) + \frac{\partial F}{\partial Y}\big(u, G(u)\big)\mathbf{J}_G(u) \ = \ \mathbf{J}_H(u) \ = \ 0.$$

Multiplying the left side by u' and subtracting gives

$$\frac{\partial F}{\partial X}\big(u, G(u)\big)u' \ = \ -\frac{\partial F}{\partial Y}\big(u, G(u)\big)\mathbf{J}_G(u)u'. \tag{3.1}$$

Since ∂ is compatible with F, we also have

$$\frac{\partial F}{\partial X}\big(u, G(u)\big)u' + \frac{\partial F}{\partial Y}\big(u, G(u)\big)G(u)' \ = \ H(u)' \ = \ 0.$$

Together with (3.1), this gives

$$\frac{\partial F}{\partial Y}\big(u, G(u)\big)G(u)' \ = \ \frac{\partial F}{\partial Y}\big(u, G(u)\big)\mathbf{J}_G(u)u'.$$

It remains to use the invertibility of $\frac{\partial F}{\partial Y}\big(u, G(u)\big)$. □

For an arbitrary o-minimal theory T, it may be difficult to tell whether a map on a model of T is a T-derivation. However, for many of the o-minimal theories in Section 2.2, we can use the test in Lemma 3.6 and the above preservation results to give a simple criterion for when a map is a T-derivation.

Lemma 3.9. *Let $K \models \mathrm{RCF}$ and let ∂ be a derivation on K. Then ∂ is an RCF-derivation on K.*

Proof. We use Lemma 3.6. First, let $c \in K$ be $\mathcal{L}_{\mathrm{ring}}$-definable over the empty set. Using Fact 2.9, take a polynomial $P \in \mathbb{Z}[Y]$ with $P(c) = 0$ and $P'(c) \neq 0$. Our assumption that ∂ is a derivation gives that ∂ is compatible with addition, multiplication, and the constant maps $x \mapsto 0$ and $x \mapsto 1$. Lemma 3.7 gives that ∂ is compatible with each polynomial in $\mathbb{Z}[Y]$, so

$$0 \ = \ P(c)' \ = \ P'(c)c'.$$

Thus, $c' = 0$. Now let u be an $\mathcal{L}_{\mathrm{ring}}(\emptyset)$-independent tuple and let F be an $\mathcal{L}_{\mathrm{ring}}(\emptyset)$-definable function defined at u. Fact 2.9 gives $P \in \mathbb{Z}[X,Y]$ with

$$P\big(u, F(u)\big) \ = \ 0, \qquad \frac{\partial P}{\partial Y}\big(u, F(u)\big) \ \neq \ 0.$$

As u is $\mathcal{L}_{\mathrm{ring}}(\emptyset)$-independent, there is an open neighborhood U of u such that

$$P\big(y, F(y)\big) \ = \ 0, \qquad \frac{\partial P}{\partial Y}\big(y, F(y)\big) \ \neq \ 0$$

for all $y \in U$. Then Lemma 3.8, along with the aforementioned fact that ∂ is compatible with P, gives that $F(u)' = \mathbf{J}_F(u)u'$ as desired. □

Lemma 3.10. *Let $K \models T_{\exp}$ and let ∂ be a derivation on K which is compatible with the exponential on K. Then ∂ is a $T_{\exp}$-derivation.*

Proof. We use Lemma 3.6. First, if $c \in K$ is $\mathcal{L}_{\exp}$-definable over the empty set, then by Fact 2.11 there are $c_2, \ldots, c_n \in K$ and $\mathcal{L}_{\exp}(\emptyset)$-terms $t_1, \ldots, t_n$ such that $(c, c_2, \ldots, c_n)$ is a regular solution to the system

$$t_1(Y_1, \ldots, Y_n) \;=\; \cdots \;=\; t_n(Y_1, \ldots, Y_n) \;=\; 0.$$

Lemma 3.7 and our assumption that ∂ is a derivation which is compatible with exp gives that ∂ is compatible with each $\mathcal{L}_{\exp}(\emptyset)$-term. Let $G\colon K^n \to K^n$ be the $\mathcal{L}_{\exp}(\emptyset)$-definable map $G(Y) = \big(t_1(Y), \ldots, t_n(Y)\big)$. Since $G(c, c_2, \ldots, c_n) = 0$ and ∂ is compatible with G, we have

$$G(c, c_2, \ldots, c_n)' \;=\; \mathbf{J}_G(c, c_2, \ldots, c_n)(c', c_2', \ldots, c_n') \;=\; 0.$$

Since $\mathbf{J}_G(c, c_2, \ldots, c_n)$ is invertible, we have $c' = 0$.

Now, let $u \in K^m$ be an $\mathcal{L}_{\exp}(\emptyset)$-independent tuple and let $F\colon K^m \to K$ be an $\mathcal{L}_{\exp}(\emptyset)$-definable function. Applying Fact 2.11 with u in place of A and $F(u)$ in place of b, we get $b_2, \ldots, b_n \in K$ and $\mathcal{L}_{\exp}(u)$-terms $t_1, \ldots, t_n$ such that $\big(F(u), b_2, \ldots, b_n\big)$ is a regular solution to the system

$$t_1(Y_1, \ldots, Y_n) \;=\; \cdots \;=\; t_n(Y_1, \ldots, Y_n) \;=\; 0.$$

For $i = 2, \ldots, n$, Fact 2.11 gives that b_i is also $\mathcal{L}_{\exp}(u)$-definable, so take an $\mathcal{L}_{\exp}(\emptyset)$-definable function F_i with $F_i(u) = b_i$. For $j = 1, \ldots, n$, take an $\mathcal{L}_{\exp}(\emptyset)$-term G_j with $G_j(u, Y) = t_j(Y)$. Let $G\colon K^{m+n} \to K^n$ be the $\mathcal{L}_{\exp}(\emptyset)$-definable map $G(X, Y) = \big(G_1(X, Y), \ldots, G_n(X, Y)\big)$. Then

$$G\big(u, F(u), F_2(u), \ldots, F_n(u)\big) \;=\; 0.$$

Since u is $\mathcal{L}_{\exp}(\emptyset)$-independent, we can take an $\mathcal{L}_{\exp}(\emptyset)$-definable open neighborhood U of u such that for each $y \in U$, the tuple $\big(F(y), F_2(y), \ldots, F_n(y)\big)$ is a regular solution to

$$G(y, Y_1, \ldots, Y_n) \;=\; 0.$$

Since ∂ is compatible with G, Lemma 3.8 tells us that ∂ is compatible with F. □

Remark 3.11. Lemma 3.10 provides a positive answer to Question 2.11 in [**38**].

Lemma 3.12. *Let $K \models T_{\mathrm{an}}$ and let ∂ be a derivation on K which is compatible with every restricted analytic function (restricted to the open unit cube). Then ∂ is a T_{an}-derivation.*

Proof. Let $\mathcal{L}_{\mathrm{an}}^*$ extend $\mathcal{L}_{\mathrm{an}}$ by function symbols for multiplicative inversion and n^{th} roots. As ∂ is a derivation, we know that it is compatible with addition and multiplication, and a simple application of Lemma 3.8 gives that ∂ is compatible with multiplicative inversion and n^{th} roots. Thus, our assumption and repeated applications of Lemma 3.7 gives that ∂ compatible with every $\mathcal{L}_{\mathrm{an}}^*(\emptyset)$-term. By Fact 2.10, each $\mathcal{L}_{\mathrm{an}}(\emptyset)$-definable function is given piecewise by $\mathcal{L}_{\mathrm{an}}^*(\emptyset)$-terms, so ∂ is a T_{an}-derivation. □

Lemma 3.13. *Let $K \models T_{\mathrm{an,exp}}$ and let ∂ be a derivation on K which is compatible with every restricted analytic function and with the exponential function. Then ∂ is a $T_{\mathrm{an,exp}}$-derivation.*

Proof. Let $\mathcal{L}_{\mathrm{an,exp}}^*$ extend $\mathcal{L}_{\mathrm{an,exp}}$ by a function symbol for the logarithm function. As ∂ is compatible with exp, a simple application of Lemma 3.8 gives that ∂ is compatible with log. It follows from our other assumptions on ∂ that ∂ is compatible with every $\mathcal{L}_{\mathrm{an,exp}}^*(\emptyset)$-term. By Fact 2.12, each $\mathcal{L}_{\mathrm{an,exp}}(\emptyset)$-definable function is given piecewise by $\mathcal{L}_{\mathrm{an,exp}}^*(\emptyset)$-terms, so ∂ is a $T_{\mathrm{an,exp}}$-derivation. □

3.2. Examples of T-derivations

In this section, we look at some natural models of T^∂ and construct some new models. As in Section 2.3, most of our focus is on Hahn field models of $T_{\rm an}$ and the expansions of transseries and surreal numbers to models of $T_{\rm an,exp}$. First, it's worth remarking that $T_{\rm exp}$-derivations show up frequently in work on exponential algebraicity in the real field; see **[7]** and **[50]**.

$T_{\rm an}$-derivations on Hahn field models. Let $\boldsymbol{k} \models T_{\rm an}$ and let $\mathfrak{M}$ be a divisible monomial group. Recall from Proposition 2.13 that the Hahn field $\boldsymbol{k}[[\mathfrak{M}]]$ admits a natural expansion to a model $\boldsymbol{k}[[\mathfrak{M}]]_{\rm an} \models T_{\rm an}$.

Proposition 3.14. *Let ∂ be a strongly additive derivation on $\boldsymbol{k}[[\mathfrak{M}]]_{\rm an}$ such that $\partial|_{\boldsymbol{k}}$ is a $T_{\rm an}$-derivation on $\boldsymbol{k}$. Then ∂ is a $T_{\rm an}$-derivation on $\boldsymbol{k}[[\mathfrak{M}]]_{\rm an}$.*

Proof. By Lemma 3.12, it suffices to check that $F(y)' = \mathbf{J}_F(y)y'$ for each n-ary restricted analytic function F and each $y = (y_1,\ldots,y_n) \in \boldsymbol{k}[[\mathfrak{M}]]^n$ with $|y_1|,\ldots,|y_n| < 1$. Take $a = (a_1,\ldots,a_n) \in \boldsymbol{k}^n$ and $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \in \boldsymbol{k}[[\mathfrak{M}]]^n$ with

$$|a_k| \ \leqslant\ 1, \qquad \varepsilon_k \ \prec\ 1, \qquad y_k \ =\ a_k+\varepsilon_k$$

for $k=1,\ldots,n$. By definition, we have

$$F(y) \ =\ F(a+\varepsilon) \ :=\ \sum_{\boldsymbol{i}} \frac{F^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}}, \qquad \mathbf{J}_F(y)y' \ =\ \sum_{\boldsymbol{i}} \frac{\mathbf{J}_{F^{(\boldsymbol{i})}}(a)(a+\varepsilon)'}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}}.$$

By strong additivity, we have

$$F(y)' \ =\ \sum_{\boldsymbol{i}} \left(\frac{F^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}}\right)' \ =\ \sum_{\boldsymbol{i}} \frac{F^{(\boldsymbol{i})}(a)'}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}} + \frac{F^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}(\varepsilon^{\boldsymbol{i}})'.$$

Since $\partial|_{\boldsymbol{k}}$ is a $T_{\rm an}$-derivation on $\boldsymbol{k}$, we have $F^{(\boldsymbol{i})}(a)' = \mathbf{J}_{F^{(\boldsymbol{i})}}(a)a'$ for each $\boldsymbol{i}$. We also have

$$\sum_{\boldsymbol{i}} \frac{F^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!}(\varepsilon^{\boldsymbol{i}})' \ =\ \sum_{k=1}^n \sum_{\{\boldsymbol{i}:i_k\neq 0\}} \frac{F^{(\boldsymbol{i})}(a)}{i_1!\cdots(i_k-1)!\cdots i_n!}\varepsilon_1^{i_1}\cdots\varepsilon_k^{i_k-1}\cdots\varepsilon_n^{i_n}\varepsilon_k' \ =\ \sum_{\boldsymbol{i}} \frac{\mathbf{J}_{F^{(\boldsymbol{i})}}(a)\varepsilon'}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}},$$

where the last equality comes from reindexing. Thus

$$F(y)' \ =\ \sum_{\boldsymbol{i}} \frac{\mathbf{J}_{F^{(\boldsymbol{i})}}(a)a'}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}} + \frac{\mathbf{J}_{F^{(\boldsymbol{i})}}(a)\varepsilon'}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}} \ =\ \sum_{\boldsymbol{i}} \frac{\mathbf{J}_{F^{(\boldsymbol{i})}}(a)(a+\varepsilon)'}{\boldsymbol{i}!}\varepsilon^{\boldsymbol{i}} \ =\ \mathbf{J}_F(y)y'. \qquad \square$$

The following is a variation on a structure considered by Scanlon **[61]**.

Example 3.15 (Hahn differential fields)**.** Let $\partial_{\boldsymbol{k}}$ be a $T_{\rm an}$-derivation on $\boldsymbol{k}$ and define the map ∂ on $\boldsymbol{k}[[\mathfrak{M}]]_{\rm an}$ by setting

$$\partial\big(\sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m}\big) \ :=\ \sum_{\mathfrak{m}} \partial_{\boldsymbol{k}}(f_{\mathfrak{m}})\mathfrak{m}.$$

To see that ∂ is strongly additive, we note that $\operatorname{supp}(\partial f) \subseteq \operatorname{supp}(f)$ for $f \in K$, so (∂f_i) is summable with $\sum_i \partial f_i = \partial\big(\sum_i f_i\big)$ for any summable family (f_i). Thus, ∂ is a $T_{\rm an}$-derivation on $\boldsymbol{k}[[\mathfrak{M}]]_{\rm an}$ with constant field $C_{\boldsymbol{k}}[[\mathfrak{M}]]_{\rm an}$.

Proposition 3.14 generalizes to increasing unions of Hahn fields:

Corollary 3.16. *Let $(\mathfrak{M}_i)$ be an increasing chain of divisible monomial groups, let K be the union of $\mathcal{L}_{\rm an}$-structures $\bigcup_i \boldsymbol{k}[[\mathfrak{M}_i]]_{\rm an}$, and let ∂ be a strongly additive derivation on K such that $\partial|_{\boldsymbol{k}}$ is a $T_{\rm an}$-derivation on $\boldsymbol{k}$. Then ∂ is a $T_{\rm an}$-derivation on K.*

Example 3.17 (Transseries)**.** Recall that $\mathbb{T}$, the field of logarithmic-exponential transseries, may be viewed as an increasing union of Hahn fields over $\mathbb{R}$ with divisible monomial groups. As such, it admits a natural expansion to a model $\mathbb{T}_{\text{an}} \models T_{\text{an}}$. In [**32**], a strongly additive derivation ∂ is constructed on $\mathbb{T}$ with constant field $C_{\mathbb{T}} = \mathbb{R}$. This derivation is essentially termwise differentiation with respect to x. For example, the transseries

$$f \;=\; e^{e^{e^x}} + 12xe^{x^2+\pi x} + x^{1/2} + x^{1/3} + \cdots$$

has derivative

$$f' \;=\; e^{e^{e^x}} e^{e^x} e^x + 12e^{x^2+\pi x} + 24x^2e^{x^2+\pi x} + 12\pi xe^{x^2+\pi x} + \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} + \cdots .$$

Since the trivial derivation on $\mathbb{R}$ is a T_{an}-derivation on $\mathbb{R}$, we have by Corollary 3.16 that ∂ is a T_{an}-derivation on $\mathbb{T}_{\text{an}}$; see also [**32**, 3.3]. As one may infer from the example above, ∂ is compatible with the natural exponential function on $\mathbb{T}$ [**32**, 3.4], so it is even a $T_{\text{an,exp}}$-derivation on $\mathbb{T}_{\text{an,exp}}$ by Lemma 3.13

Example 3.18 (Surreal numbers)**.** As with $\mathbb{T}$, the field **No** of surreal numbers may be viewed as an increasing union of Hahn fields over $\mathbb{R}$ with divisible monomial groups. In [**8**], a strongly additive derivation is constructed on **No** with constant field $C_{\mathbf{No}} = \mathbb{R}$. Those familiar with surreal numbers may think of this derivation as "termwise differentiation with respect to ω," though the precise construction involves a good deal of work. This derivation is compatible with Gonshor's exponential function on **No** [**8**, 6.29], so it is a $T_{\text{an,exp}}$-derivation on $\mathbf{No}_{\text{an,exp}}$ by Lemma 3.13 and Corollary 3.16.

3.3. The ∂-closure operator

Assumption 3.19. *For the remainder of this chapter, $K = (K, \partial) \models T^{\partial}$ and C is the constant field of K.*

In this section, we introduce the ∂-closure operator on K. First, some notation: given $a \in K$, we define the **jets** of a:

$$\mathcal{J}^r_\partial(a) \;:=\; (a, a', \ldots, a^{(r)}), \qquad \mathcal{J}^\infty_\partial(a) := (a^{(n)})_{n\in\mathbb{N}}.$$

Given $b \in K^m$, $B \subseteq K^m$ and $\alpha \in \mathbb{N} \cup \{\infty\}$, we set

$$\mathcal{J}^\alpha_\partial(b) \;:=\; \big(\mathcal{J}^\alpha_\partial(b_1), \ldots, \mathcal{J}^\alpha_\partial(b_m)\big), \qquad \mathcal{J}^\alpha_\partial(B) \;:=\; \big\{\mathcal{J}^\alpha_\partial(b) : b \in B\big\}.$$

To make some statements cleaner, we let $\mathcal{J}^{-1}_\partial(b)$ be the empty tuple and we let $\mathcal{J}^{-1}_\partial(B)$ be the empty set. Occasionally, we omit the parentheses and write $\mathcal{J}^\alpha_\partial b$.

Definition 3.20. Given $a \in K$ and $B \subseteq K$, we say that a **is in the ∂-closure of** B, written $a \in \text{cl}^\partial(B)$, if $\text{rk}_{\mathcal{L}}\big(\mathcal{J}^\infty_\partial(a)\big|\mathcal{J}^\infty_\partial(B)\big)$ is finite.

This section is devoted to showing that (K, cl^∂) is a pregeometry and exploring the corresponding rank function. Throughout this section, A, B, and D denote subsets of K and a, b, and d denote elements of K.

Lemma 3.21. *If $A \subseteq \text{dcl}_{\mathcal{L}}(B)$, then $\partial A \subseteq \text{dcl}_{\mathcal{L}}(B\partial B)$.*

Proof. Given $a \in A$, we may write $a = F(b)$ for some $\mathcal{L}(\emptyset)$-independent tuple b from B and some $\mathcal{L}(\emptyset)$-definable function F. Then $a' = \mathbf{J}_F(b)b'$, so $a' \in \text{dcl}_{\mathcal{L}}(B\partial B)$. ◻

Corollary 3.22. *We have $\text{rk}_{\mathcal{L}}(\partial A|AB\partial B) \leqslant \text{rk}_{\mathcal{L}}(A|B)$.*

Proof. Let $A_0 \subseteq A$ be a $\mathrm{dcl}_{\mathcal{L}}$-basis for A over B, so $\mathrm{rk}_{\mathcal{L}}(A|B) = |A_0|$. Then $\partial A \subseteq \mathrm{dcl}_{\mathcal{L}}(A_0 B \partial A_0 \partial B)$ by Lemma 3.21, so we have

$$\mathrm{rk}_{\mathcal{L}}(\partial A | AB\partial B) \ \leqslant\ |\partial A_0| \ \leqslant\ |A_0| \ =\ \mathrm{rk}_{\mathcal{L}}(A|B). \qquad \square$$

Lemma 3.23. *The following are equivalent:*

1. $a \in \mathrm{cl}^{\partial}(B)$;
2. $\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^n_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \leqslant n$ *for some* n;
3. $a^{(n)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{\infty}_{\partial}(B)\big)$ *for some* n;
4. *there are* n *and* m *such that* $a^{(r)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{m+r}_{\partial}(B)\big)$ *for all* $r \geqslant n$.

Proof. Suppose (1) holds and set $n := \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{\infty}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big)$. Then $\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \leqslant n$. Now suppose (2) holds and let n be least such that $\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \leqslant n$. Then

$$\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \ =\ n$$

by minimality of n, so $a^{(n)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{\infty}_{\partial}(B)\big)$. Suppose (3) holds and take n with $a^{(n)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{\infty}_{\partial}(B)\big)$. As $\mathrm{dcl}_{\mathcal{L}}$ is finitary, we may take m with

$$a^{(n)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{m+n}_{\partial}(B)\big).$$

Lemma 3.21 gives that

$$a^{(n+1)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\mathcal{J}^{m+n+1}_{\partial}(B)\big) = \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{m+n+1}_{\partial}(B)\big).$$

By induction, $a^{(r)} \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^{n-1}_{\partial}(a)\mathcal{J}^{m+r}_{\partial}(B)\big)$ for all $r \geqslant n$. The final implication, (4) implies (1), is clear. $\square$

Fact 3.24. *The following are consequences of (3) of Lemma 3.23:*

- $\mathrm{dcl}(B) \subseteq \mathrm{cl}^{\partial}(B)$;
- $\partial B \subseteq \mathrm{cl}^{\partial}(B)$;
- $a \notin \mathrm{cl}^{\partial}(B)$ *if and only if* $\mathcal{J}^{\infty}_{\partial}(a)$ *is* $\mathcal{L}\big(\mathcal{J}^{\infty}_{\partial}(B)\big)$*-independent.*

Proposition 3.25. $(K, \mathrm{cl}^{\partial})$ *is a pregeometry.*

Proof. It is clear that if $A \subseteq B$, then $A \subseteq \mathrm{cl}^{\partial}(A) \subseteq \mathrm{cl}^{\partial}(B)$. The fact that cl^{∂} is finitary follows from (3) of Lemma 3.23 and the fact that $\mathrm{dcl}_{\mathcal{L}}$ is finitary. We will show that $\mathrm{cl}^{\partial}\big(\mathrm{cl}^{\partial}(B)\big) = \mathrm{cl}^{\partial}(B)$. Fix $a \in \mathrm{cl}^{\partial}\big(\mathrm{cl}^{\partial}(B)\big)$ and take a finite set $D \subseteq \mathrm{cl}^{\partial}(B)$ with $a \in \mathrm{cl}^{\partial}(D)$. Then

$$\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{\infty}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \ \leqslant\ \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{\infty}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(D)\big) + \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{\infty}_{\partial}(D)\big|\mathcal{J}^{\infty}_{\partial}(B)\big).$$

The first summand on the right is finite since $a \in \mathrm{cl}^{\partial}(D)$. The second summand is finite since $D \subseteq \mathrm{cl}^{\partial}(B)$ and D is finite. Thus, $\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{\infty}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big)$ is finite, so $a \in \mathrm{cl}^{\partial}(B)$.

It remains to show that cl^{∂} satisfies Steinitz exchange. Fix a, b, and B with $a \in \mathrm{cl}^{\partial}(Bb) \setminus \mathrm{cl}^{\partial}(B)$. Using (2) of Lemma 3.23, take n with $\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(Bb)\big) \leqslant n$. Since $\mathrm{dcl}_{\mathcal{L}}$ is finitary, we may take m with $\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\mathcal{J}^{m}_{\partial}(b)\big) \leqslant n$. We have

$$\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{m}_{\partial}(b)\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \ =\ \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{m}_{\partial}(b)\mathcal{J}^{\infty}_{\partial}(B)\big) + \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{m}_{\partial}(b)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \ \leqslant\ n+m+1. \tag{3.2}$$

On the other hand,

$$\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{m}_{\partial}(b)\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big) \ =\ \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{m}_{\partial}(b)\big|\mathcal{J}^{n}_{\partial}(a)\mathcal{J}^{\infty}_{\partial}(B)\big) + \mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^{n}_{\partial}(a)\big|\mathcal{J}^{\infty}_{\partial}(B)\big).$$

Since $a \notin \mathrm{cl}^\partial(B)$, we have $\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^n_\partial(a)\big|\mathfrak{J}^\infty_\partial(B)\big) = n+1$. This gives us

$$\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^m_\partial(b)\mathfrak{J}^n_\partial(a)\big|\mathfrak{J}^\infty_\partial(B)\big) \ = \ \mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^m_\partial(b)\big|\mathfrak{J}^n_\partial(a)\mathfrak{J}^\infty_\partial(B)\big) + n + 1. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^m_\partial(b)\big|\mathfrak{J}^n_\partial(a)\mathfrak{J}^\infty_\partial(B)\big) \ \leqslant \ m,$$

so $b \in \mathrm{cl}^\partial(Ba)$, again by (2) of Lemma 3.23. □

Since $(K, \mathrm{cl}^\partial)$ is a pregeometry, it has an associated rank function which we call the **∂-rank** and which we denote by rk^∂. The next proposition and its corollary give us a two explicit descriptions of the ∂-rank of a finite set:

Proposition 3.26. *Let A be finite and suppose $\partial B \subseteq B$. Then*

$$\mathrm{rk}^\partial(A|B) \ = \ \lim_{r\to\infty} \frac{\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(A)\big|B\big)}{r+1}.$$

In particular, this limit exists.

Proof. We prove this proposition by induction on $|A|$. Clearly this holds if $A = \emptyset$. Suppose that the proposition holds for A and let $a \in K \setminus A$. We need to show that

$$\mathrm{rk}^\partial(Aa|B) \ = \ \lim_{r\to\infty} \frac{\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(Aa)\big|B\big)}{r+1}.$$

Using that $\mathrm{rk}^\partial(Aa|B) = \mathrm{rk}^\partial(A|B) + \mathrm{rk}^\partial(a|AB)$, that $\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(Aa)\big|B\big) = \mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(A)\big|B\big) + \mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(a)\big|\mathfrak{J}^r_\partial(A)B\big)$, and our induction hypothesis, it suffices to show that the limit

$$\lim_{r\to\infty} \frac{\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(a)\big|\mathfrak{J}^r_\partial(A)B\big)}{r+1}$$

exists and is equal to $\mathrm{rk}^\partial(a|AB)$.

If $a \notin \mathrm{cl}^\partial(AB)$, then $\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(a)\big|\mathfrak{J}^r_\partial(A)B\big) = r+1$ for each r by (2) of Lemma 3.23, so

$$\lim_{r\to\infty} \frac{\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(a)\big|\mathfrak{J}^r_\partial(A)B\big)}{r+1} \ = \ 1 \ = \ \mathrm{rk}^\partial(a|AB).$$

Suppose that $a \in \mathrm{cl}^\partial(AB)$. By (4) of Lemma 3.23, there are m and n such that $a^{(r)} \in \mathrm{dcl}_\mathcal{L}\big(\mathfrak{J}^{n-1}_\partial(a)\mathfrak{J}^{m+r}_\partial(A)B\big)$ for all $r \geqslant n$. For these r, we have

$$\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^{m+r}_\partial(a)\big|\mathfrak{J}^{m+r}_\partial(A)B\big) \ \leqslant \ \mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^{n-1}_\partial(a)\big|\mathfrak{J}^{m+r}_\partial(A)B\big) + \mathrm{rk}_\mathcal{L}\big(\{a^{(r+1)},\ldots,a^{(r+m)}\}\big|\mathfrak{J}^{m+r}_\partial(A)B\big) \ \leqslant \ n+m.$$

Therefore,

$$\lim_{r\to\infty} \frac{\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^{m+r}_\partial(a)\big|\mathfrak{J}^{m+r}_\partial(A)B\big)}{m+r+1} \ \leqslant \ \lim_{r\to\infty} \frac{n+m}{m+r+1} \ = \ 0 \ = \ \mathrm{rk}^\partial(a|AB).$$

Since $\mathrm{rk}_\mathcal{L}$ is nonnegative, this gives the desired equality. □

Corollary 3.27. *If A is finite and $\partial B \subseteq B$, then*

$$\mathrm{rk}^\partial(A|B) \ = \ \lim_{r\to\infty} \mathrm{rk}_\mathcal{L}\big(\partial^r A\big|\mathfrak{J}^{r-1}_\partial(A)B\big).$$

Proof. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be the map given by $\sigma(n) := \mathrm{rk}_\mathcal{L}\big(\partial^n A\big|\mathfrak{J}^{n-1}_\partial(A)B\big)$. By Corollary 3.22, we have $\sigma(n) \geqslant \sigma(n+1)$, so σ is decreasing, thus eventually constant. As $\mathrm{rk}_\mathcal{L}\big(\mathfrak{J}^r_\partial(A)\big|B\big) = \sum_{n=0}^r \sigma(n)$, we have by

Proposition 3.26 that

$$\mathrm{rk}^{\partial}(A|B) \;=\; \lim_{r\to\infty}\frac{\mathrm{rk}_{\mathcal{L}}\big(\mathcal{J}^r_{\partial}(A)\big|B\big)}{r+1} \;=\; \lim_{r\to\infty}\frac{\sum_{n=0}^{r}\sigma(n)}{r+1} \;=\; \lim_{r\to\infty}\sigma(r). \qquad \square$$

Remark 3.28. In [**38**], we study maps δ on arbitrary pregeometries (X,cl) which satisfy the inequality $\mathrm{rk}(\delta A|AB\delta B) \leqslant \mathrm{rk}(A|B)$ from Corollary 3.22. We call such a map δ a *quasi-endomorphism.* Given a quasi-endomorphism δ, one can define the δ-*closure* in the same way that we define ∂-closure here: $a \in X$ is in the δ-closure of $B \subseteq X$ if $\mathrm{rk}(a,\delta a,\delta^2 a,\ldots|B,\delta B,\delta^2 B,\ldots)$ is finite. All the results proven in this section hold for arbitrary quasi-endomorphisms. For example, δ-closure gives us a pregeometry on X and the corresponding rank function satisfies the equalities in Proposition 3.26 and Corollary 3.27.

3.4. Extending T-derivations

In this section, we prove some results about expanding T-extensions of K to T^{∂}-extensions of K. First, we need an important proposition about the behavior of definable functions in T^{∂}-extensions of K.

Proposition 3.29. *Let $U \subseteq K^n$ be open and let $G\colon U \to K$ be an $\mathcal{L}(K)$-definable $\mathcal{C}^1$-function. Then there is a (necessarily unique) $\mathcal{L}(K)$-definable function $G^{[\partial]}\colon U \to K$ such that for all T^{∂}-extensions M of K and all $u \in U^M$, we have*

$$G(u)' \;=\; G^{[\partial]}(u) + \mathbf{J}_G(u)u'.$$

If G is $\mathcal{L}(C)$-definable, then $G^{[\partial]} = 0$.

Proof. Take a tuple $a = (a_1,\ldots,a_m) \in K^m$ and an $\mathcal{L}(\emptyset)$-definable function $F\colon K^{m+n} \to K$ in variables $(X_1,\ldots,X_m,Y_1,\ldots,Y_n)$ such that $G(u) = F(a,u)$ for all $u \in U$. By Corollary 2.7, we can find $\mathcal{L}(\emptyset)$-definable functions $F_1,\ldots,F_N\colon K^{m+n} \to K$ where for each $u \in U$, there is $i \in \{1,\ldots,N\}$ such that

- F_i is $\mathcal{C}^1$ at (a,u),
- $F_i(a,u) = F(a,u)$,
- $\frac{\partial F_i}{\partial Y_j}(a,u) = \frac{\partial F}{\partial Y_j}(a,u)$ for $j = 1,\ldots,n$.

For $i = 1,\ldots,N$, let D_i be the set of all $u \in U$ such that F_i satisfies the above criteria. Then each D_i is $\mathcal{L}(a)$-definable, so for all T^{∂}-extensions M of K and all $u \in D_i^M$, we have

$$G(u)' \;=\; F_i(a,u)' \;=\; \mathbf{J}_{F_i}(a,u)(a_1',\ldots,a_m',u_1',\ldots,u_n') \;=\; G_i^{[\partial]}(u) + \mathbf{J}_G(u)u',$$

where

$$G_i^{[\partial]}(u) \;:=\; \frac{\partial G_i}{\partial X_1}(a,u)a_1' + \cdots + \frac{\partial G_i}{\partial X_m}(a,u)a_m'.$$

Let $G^{[\partial]}\colon U \to K$ be defined by $G^{[\partial]}(u) := G_i^{[\partial]}(u)$ whenever $u \in D_i \setminus (D_1 \cup \cdots \cup D_{i-1})$. It is immediate from the definition of $G_i^{[\partial]}$ that $G^{[\partial]} = 0$ whenever $a \in C^m$. $\square$

Lemma 3.30. *Let M be a T^{∂}-extension of K, let $A \subseteq M$, and suppose $a' \in K\langle A\rangle$ for all $a \in A$. Then $K\langle A\rangle$ is the underlying set of an $\mathcal{L}^{\partial}$-substructure of M. In particular, $K\langle \mathcal{J}^{\infty}_{\partial} a\rangle$ is the underlying set of an $\mathcal{L}^{\partial}$-substructure of M for any $a \in M$.*

Proof. Let $y \in K\langle A\rangle$ and take an $\mathcal{L}(K)$-independent tuple a from A and an $\mathcal{L}(K)$-definable function F with $y = F(a)$. Then

$$y' \;=\; F(a)' \;=\; F^{[\partial]}(a) + \mathbf{J}_F(a)a'$$

by Proposition 3.29. Since $F^{[\partial]}$ and $\mathbf{J}_F$ are $\mathcal{L}(K)$-definable and since the components of a' lie in $K\langle A\rangle$ by assumption, we have $F^{[\partial]}(a) + \mathbf{J}_F(a)a' \in K\langle A\rangle$. Thus, $y' \in K\langle A\rangle$. $\square$

As is the case for derivations, one can extend a T-derivation ∂ on K to any T-extension M of K by specifying the values of ∂ on a $\operatorname{dcl}_{\mathcal{L}}$-basis for M over K.

Lemma 3.31. *Let M be a T-extension of K, let A be a $\operatorname{dcl}_{\mathcal{L}}$-basis for M over K, and let $s\colon A \to M$ be a map. There is a unique extension of ∂ to a T-derivation on M such that $a' = s(a)$ for all $a \in A$.*

Proof. Let $y \in M \setminus K$, so there is a tuple $a = (a_1, \ldots, a_n)$ of distinct elements from A and an $\mathcal{L}(K)$-definable n-ary function F with $y = F(a)$. By Proposition 3.29, any T-derivation on M must satisfy the identity

$$F(a)' \;=\; F^{[\partial]}(a) + \mathbf{J}_F(a)a',$$

so if we want a' to equal $s(a) := \big(s(a_1), \ldots, s(a_n)\big)$, we only have one choice: we set

$$y' \;=\; F(a)' \;:=\; F^{[\partial]}(a) + \mathbf{J}_F(a)s(a).$$

Doing this for all $y \in M \setminus K$, we define a map $\partial\colon M \to M$ which extends the T-derivation on K. Clearly, $a' = s(a)$ for all $a \in A$. Using that A is $\mathcal{L}(K)$-independent, it is routine to show that this doesn't depend on our choice of a or F. If we can show that ∂ is a T-derivation, then uniqueness follows from the above considerations.

Let $U \subseteq M^m$ be open, let $G\colon U \to M$ be an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function, and let $u \in U$. We need to show that $G(u)' = \mathbf{J}_G(u)u'$. Take an $\mathcal{L}(K)$-definable $\mathcal{C}^1$-map $F\colon V \to K^m$ with V open and a tuple a of distinct elements from A with $a \in V^M$ and $u = F(a)$. By shrinking V, we may assume that $F(V) \subseteq U$. Set $H := G \circ F$ and set $s := s(a)$. By definition, we have

$$G(u)' \;=\; H(a)' \;=\; H^{[\partial]}(a) + \mathbf{J}_H(a)s, \qquad u' \;=\; F(a)' \;=\; F^{[\partial]}(a) + \mathbf{J}_F(a)s. \tag{3.4}$$

For all $v \in V$, we have

$$H^{[\partial]}(v) + \mathbf{J}_H(v)v' \;=\; H(v)' \;=\; G\big(F(v)\big)' \;=\; \mathbf{J}_G\big(F(v)\big)\big(F^{[\partial]}(v) + \mathbf{J}_F(v)v'\big).$$

Using also that $\mathbf{J}_H(v) = \mathbf{J}_{G\circ F}(v) = \mathbf{J}_G\big(F(v)\big)\mathbf{J}_F(v)$, we see that $H^{[\partial]}(v) = \mathbf{J}_G\big(F(v)\big)F^{[\partial]}(v)$ for all $v \in V$. Since $a \in V^M$ we have by $\mathcal{L}$-elementarity that

$$H^{[\partial]}(a) \;=\; \mathbf{J}_G\big(F(a)\big)F^{[\partial]}(a) \;=\; \mathbf{J}_G(u)F^{[\partial]}(a), \qquad \mathbf{J}_H(a) \;=\; \mathbf{J}_G\big(F(a)\big)\mathbf{J}_F(a) \;=\; \mathbf{J}_G(u)\mathbf{J}_F(a).$$

This, along with the identities in (3.4), gives

$$G(u)' \;=\; \mathbf{J}_G(u)F^{[\partial]}(a) + \mathbf{J}_G(u)\mathbf{J}_F(a)s \;=\; \mathbf{J}_G(u)u'. \qquad \square$$

Corollary 3.32. *Let M be a T-extension of K with $\operatorname{rk}_{\mathcal{L}}(M|K) = n$ and let $A \subseteq M^{n+1}$ be an $\mathcal{L}(K)$-definable set with $\dim_{\mathcal{L}} \pi_n(A) = n$. Then there is $b \in M$ and an extension of ∂ to a T-derivation on M such that $\mathfrak{J}^n_\partial(b) \in A$.*

Proof. We claim that there is some $\mathcal{L}(K)$-independent tuple $a \in M^n$ such that $a \in \pi_n(A)$. We construct a coordinate by coordinate. Fix $i \in \{1, \ldots, n\}$ and suppose we have already chosen an $\mathcal{L}(K)$-independent tuple $\tilde{a} = (a_1, \ldots, a_{i-1}) \in \pi_{i-1}(A)$. We need to find $a_i \in M \setminus \operatorname{dcl}_{\mathcal{L}}(K\tilde{a})$ with $(\tilde{a}, a_i) \in \pi_i(A)$. As $\pi_i(A)$ has nonempty interior, we can find $r_1 < r_2 \in \operatorname{dcl}_{\mathcal{L}}(K\tilde{a}) \cup \{\pm\infty\}$ with $(r_1, r_2) \subseteq \pi_i(A)_{\tilde{a}}$. Take $d \in M \setminus \operatorname{dcl}_{\mathcal{L}}(K\tilde{a})$.

By negating and inverting d if need be, we may assume that $0 < d < 1$. Set

$$a_i \;:=\; \begin{cases} d & \text{if } r_1 = -\infty \text{ and } r_2 = +\infty \\ r_1 + d & \text{if } r_1 \in \operatorname{dcl}_{\mathcal{L}}(K\tilde{a}) \text{ and } r_2 = +\infty \\ r_2 - d & \text{if } r_1 = -\infty \text{ and } r_2 \in \operatorname{dcl}_{\mathcal{L}}(K\tilde{a}) \\ r_1 + d(r_2 - r_1) & \text{if } r_1, r_2 \in \operatorname{dcl}_{\mathcal{L}}(K\tilde{a}). \end{cases}$$

Then $a_i \notin \operatorname{dcl}_{\mathcal{L}}(K\tilde{a})$ and $a_i \in \pi_i(A)_{\tilde{a}}$, as required.

With the claim proven, we may assume that $M = K\langle a\rangle$ for some $a \in M^n$ with $a \in \pi_n(A)$. By definable choice, there is an $\mathcal{L}(K)$-definable map $F\colon \pi_n(A) \to M$ with $\operatorname{Gr}(F) \subseteq A$. By Lemma 3.31, there is a unique extension of ∂ to a T-derivation on M such that $a_i' = a_{i+1}$ for $i = 1, \ldots, n-1$ and such that $a_n' = F(a)$. Then $\mathcal{J}_\partial^n(a_1) = \big(a, F(a)\big) \in A$, so we may take $b = a_1$. □

3.5. Affine and compositional conjugation

In this section, fix $r \geqslant 0$ and let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function. For $k = 0, \ldots, r$, we identify each variable Y_k with the k^{th} coordinate function $K^{1+r} \to K$. We let $Y = (Y_0, \ldots, Y_r)$, so $Y\colon K^{1+r} \to K^{1+r}$ is the identity map.

Definition 3.33. F is said to be in **implicit form** if

$$F \;=\; \mathfrak{m}_F\big(Y_r - I_F(Y_0, \ldots, Y_{r-1})\big)$$

for some $\mathfrak{m}_F \in K^\times$ and some $\mathcal{L}(K)$-definable function $I_F\colon K^r \to K$.

If F is in implicit form, then $F(a, b) = 0$ if and only if $b = I_F(a)$ for $a \in K^r$ and $b \in K$. Thus, I_F is an *implicit function* for F. This is the source of the name "implicit form" and the notation I_F. By our convention for nullary functions, the unary functions in implicit form are exactly the functions of the form $\mathfrak{m}(Y_0 - d)$ where $\mathfrak{m} \in K^\times$ and $d \in K$. Often, we omit the variables $Y_0, \ldots, Y_{r-1}$ and just write $F = \mathfrak{m}_F(Y_r - I_F)$ for F in implicit form.

We may associate to F the unary $\mathcal{L}^\partial(K)$-definable function $y \mapsto F(\mathcal{J}_\partial^r y)$. For $k \leqslant r$ and $y \in K$, we have $Y_k(\mathcal{J}_\partial^r y) = y^{(k)}$. As is the case with differential polynomials, these functions $y \mapsto F(\mathcal{J}_\partial^r y)$ can be additively and multiplicatively conjugated (an operation we call *affine conjugation*). They can also be compositionally conjugated. These conjugations will play a role in Section 3.6, but otherwise, the results in this section will not be used until Chapter 6.

Affine conjugation. Let $a, d \in K$. We let $Y^\partial_{+a,\times d} = \big((Y_0)^\partial_{+a,\times d}, \ldots, (Y_r)^\partial_{+a,\times d}\big)\colon K^{1+r} \to K^{1+r}$ be the map with coordinate functions

$$(Y_k)^\partial_{+a,\times d} \;:=\; a^{(k)} + \sum_{i=0}^{k} \binom{k}{i} d^{(k-i)} Y_i, \qquad k = 0, \ldots, r.$$

Then for $y \in K$ and $k \leqslant r$, we have

$$(Y_k)^\partial_{+a,\times d}(\mathcal{J}_\partial^r y) \;=\; a^{(k)} + \sum_{i=0}^{k} \binom{k}{i} d^{(k-i)} y^{(i)} \;=\; (dy + a)^{(k)},$$

so $Y^\partial_{+a,\times d}(\mathcal{J}^r_\partial y) = \mathcal{J}^r_\partial(dy+a)$. Note that $Y^\partial_{+a,\times d}$ is a K-affine map which is bijective if $d \neq 0$ and which takes constant value $\mathcal{J}^r_\partial(a)$ if $d=0$. We let $F^\partial_{+a,\times d} := F \circ Y^\partial_{+a,\times d}$, so

$$F^\partial_{+a,\times d}(\mathcal{J}^r_\partial y) \;=\; F\big(\mathcal{J}^r_\partial(dy+a)\big)$$

for each $y \in K$. When ∂ is clear from context, we drop the superscript and just write $F_{+a,\times d}$. For notational simplicity, we let

$$F_{+a} \;:= F_{+a,\times 1}, \qquad F_{\times d} \;:= F_{+0,\times d}, \qquad F_{-a,\times d} \;:=\; F_{+(-a),\times d}.$$

Lemma 3.34. *Suppose F is in implicit form and $d \neq 0$. Then $F_{+a,\times d}$ is also in implicit form with*

$$\mathfrak{m}_{F_{+a,\times d}} \;=\; d\mathfrak{m}_F, \qquad I_{F_{+a,\times d}} \;=\; d^{-1}\Big((I_F)_{+a,\times d} - a^{(r)} - \sum_{i=0}^{r-1}\binom{r}{i} d^{(r-i)} Y_i\Big).$$

Proof. Define $G\colon K^r \to K$ by

$$G \;:=\; d^{-1}\Big((I_F)_{+a,\times d} - a^{(r)} - \sum_{i=0}^{r-1}\binom{r}{i} d^{(r-i)} Y_i\Big).$$

Then

$$\begin{aligned} F_{+a,\times d} \;&=\; \mathfrak{m}_F\big((Y_r)_{+a,\times d} - (I_F)_{+a,\times d}\big) \;=\; \mathfrak{m}_F\Big(a^{(r)} + \sum_{i=0}^{r}\binom{r}{i} d^{(r-i)} Y_i - (I_F)_{+a,\times d}\Big) \\ &=\; \mathfrak{m}_F\Big(dY_r + a^{(r)} + \sum_{i=0}^{r-1}\binom{r}{i} d^{(r-i)} Y_i - (I_F)_{+a,\times d}\Big) \;=\; d\mathfrak{m}_F(Y_r - G), \end{aligned}$$

so $\mathfrak{m}_{F_{+a,\times d}} = d\mathfrak{m}_F$ and $I_{F_{+a,\times d}} = G$. □

Given an $\mathcal{L}(K)$-definable set $A \subseteq K^{1+r}$, we set

$$A^\partial_{+a,\times d} \;:=\; \big\{u \in K^{1+r} : Y^\partial_{+a,\times d}(u) \in A\big\},$$

so $A^\partial_{+a,\times d}$ is $\mathcal{L}(K)$-definable and

$$\mathcal{J}^r_\partial(y) \in A^\partial_{+a,\times d} \iff \mathcal{J}^r_\partial(dy+a) \in A$$

for $y \in K$. If $d \neq 0$, then $Y_{+a,\times d}$ is an $\mathcal{L}(K)$-definable bijection, so $\dim_{\mathcal{L}}(A^\partial_{+a,\times d}) = \dim_{\mathcal{L}}(A)$. Again, we drop the superscript if ∂ is clear from context.

Compositional conjugation. Let $\phi \in K^\times$. Then $\delta := \phi^{-1}\partial$ is also a T-derivation on K, since for each $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function $G\colon U \to K$ with $U \subseteq K^n$ open and for each $u \in U$, we have

$$\delta G(u) \;=\; \phi^{-1}\partial G(u) \;=\; \phi^{-1}\big(\mathbf{J}_G(u)\partial u\big) \;=\; \mathbf{J}_G(u)\delta u.$$

We let $K^\phi = (K, \delta)$ be the expansion of the $\mathcal{L}$-structure K by the T-derivation δ, and we refer to K^ϕ as the **compositional conjugate of** K **by** ϕ. For $\psi \in K^\times$, we have $(K^\phi)^\psi = K^{\phi\psi}$.

Let $n \geqslant 0$. Subsection 5.7 in [**4**] gives for each $k \leqslant n$ an element $\xi^n_k(\phi) \in \mathbb{Q}[\phi, \partial\phi, \ldots, \partial^n\phi]$ such that

$$\partial^n y \;=\; \xi^n_0(\phi)y + \xi^n_1(\phi)\delta y + \cdots + \xi^n_n(\phi)\delta^n y.$$

In [**4**], $\xi^n_k(\phi)$ is instead called $F^n_k(\phi)$; we use different notation here and reserve F for definable functions. The values of $\xi^n_k(\phi)$ are given by the recurrence relation:

$$\xi^n_n(\phi) \;=\; \phi^n, \qquad \xi^n_0(\phi) \;=\; 0 \text{ for } n > 0, \qquad \xi^{n+1}_k(\phi) \;=\; \partial\xi^n_k(\phi) + \phi\xi^n_{k-1}(\phi) \text{ for } 0 < k \leqslant n. \tag{3.5}$$

Let Y^ϕ_∂ be the K-linear map $K^{1+r} \to K^{1+r}$ with matrix

$$\begin{pmatrix} \xi^0_0(\phi) & 0 & 0 & \cdots & 0 \\ \xi^1_0(\phi) & \xi^1_1(\phi) & 0 & \cdots & 0 \\ \xi^2_0(\phi) & \xi^2_1(\phi) & \xi^2_2(\phi) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi^r_0(\phi) & \xi^r_1(\phi) & \xi^r_2(\phi) & \cdots & \xi^r_r(\phi) \end{pmatrix}.$$

Then Y^ϕ_∂ is bijective and $Y^\phi_\partial(\mathcal{J}^r_\delta y) = \mathcal{J}^r_\partial(y)$ for each $y \in K$. An easy induction on r using (3.5) gives that Y^1_∂ is the identity map on K^{1+r}.

Lemma 3.35. *For $\psi \in K^\times$, we have $Y^\phi_\partial \circ Y^\psi_\delta = Y^{\phi\psi}_\partial$.*

Proof. Clearly, this identity holds on a closed subset of K^{1+r}, so it is enough to show that it holds on a dense subset. Let $U \subset K^{1+r}$ be $\mathcal{L}(K)$-definable and open. Using Corollary 3.32 with $\psi^{-1}\delta$ in place of ∂, we take a T^∂-extension $M = K\langle \mathcal{J}^r_{\psi^{-1}\delta} b\rangle$ with $\mathcal{J}^r_{\psi^{-1}\delta}(b) \in U^M$. We have

$$Y^\phi_\partial\big(Y^\psi_\delta(\mathcal{J}^r_{\psi^{-1}\delta} b)\big) \;=\; Y^\phi_\partial(\mathcal{J}^r_\delta b) \;=\; \mathcal{J}^r_\partial(b) \;=\; Y^{\phi\psi}_\partial(\mathcal{J}^r_{\psi^{-1}\delta} b).$$

Since $Y^\phi_\partial \circ Y^\psi_\delta$ and $Y^{\phi\psi}_\partial$ are $\mathcal{L}(K)$-definable and K is an elementary $\mathcal{L}$-substructure of M, there is a tuple $u \in U$ with

$$(Y^\phi_\partial \circ Y^\psi_\delta)(u) \;=\; Y^{\phi\psi}_\partial(u). \qquad \square$$

Lemma 3.36. *We have*

$$Y^\partial_{+a,\times d} \circ Y^\phi_\partial \;=\; Y^\phi_\partial \circ Y^\delta_{+a,\times d}.$$

Proof. As with Lemma 3.35, this identity holds on a closed subset of K^{1+r}, so it is enough to show that it holds on a dense subset. Let $U \subset K^{1+r}$ be $\mathcal{L}(K)$-definable and open and, using Corollary 3.32 with δ in place of ∂, take a T^∂-extension $M = K\langle \mathcal{J}^r_\delta b\rangle$ with $\mathcal{J}^r_\delta(b) \in U^M$. We have

$$Y^\partial_{+a,\times d}\big(Y^\phi_\partial(\mathcal{J}^r_\delta b)\big) \;=\; Y^\partial_{+a,\times d}(\mathcal{J}^r_\partial b) \;=\; \mathcal{J}^r_\partial(db+a) \;=\; Y^\phi_\partial\big(\mathcal{J}^r_\delta(db+a)\big) \;=\; Y^\phi_\partial\big(Y^\delta_{+a,\times d}(\mathcal{J}^r_\delta b)\big).$$

Since $Y^\partial_{+a,\times d} \circ Y^\phi_\partial$ and $Y^\phi_\partial \circ Y^\delta_{+a,\times d}$ are $\mathcal{L}(K)$-definable and K is an elementary $\mathcal{L}$-substructure of M, there is a tuple $u \in U$ with

$$Y^\partial_{+a,\times d}\big(Y^\phi_\partial(u)\big) \;=\; Y^\phi_\partial\big(Y^\delta_{+a,\times d}(u)\big). \qquad \square$$

We set $F^\phi_\partial := F \circ Y^\phi_\partial$, so F^ϕ_∂ is $\mathcal{L}(K)$-definable and

$$F^\phi_\partial(\mathcal{J}^r_\delta y) \;=\; F(\mathcal{J}^r_\partial y)$$

for all $y \in K$. When ∂ is clear from context, we drop the subscript and just write F^ϕ. We have $F^1_\partial = F$ and Lemma 3.35 gives $(F^\phi_\partial)^\psi_\delta = F^{\phi\psi}_\partial$ for $\psi \in K^\times$. Lemma 3.36 gives

$$(F^\partial_{+a,\times d})^\phi_\partial \;=\; (F^\phi_\partial)^\delta_{+a,\times d},$$

This function, which we denote by $F^\phi_{+a,\times d}$, satisfies the identity

$$F^\phi_{+a,\times d}(\mathcal{J}^r_\delta y) \;=\; F\big(\mathcal{J}^r_\partial(dy+a)\big)$$

for all $y \in K$.

Lemma 3.37. *Suppose F is in implicit form. Then F^ϕ is also in implicit form with*

$$\mathfrak{m}_{F^\phi} \ = \ \phi^r \mathfrak{m}_F, \qquad I_{F^\phi} \ = \ \phi^{-r}\Big(I_F^\phi - \sum_{i=0}^{r-1} \xi_i^r(\phi) Y_i\Big).$$

Proof. Set $G := \phi^{-r}\big(I_F^\phi - \sum_{i=0}^{r-1} \xi_i^r(\phi) Y_i\big)$. Then

$$F^\phi \ = \ \mathfrak{m}_F(Y_r^\phi - I_F^\phi) \ = \ \mathfrak{m}_F\Big(\sum_{i=0}^{r} \xi_i^r(\phi) Y_i - I_F^\phi\Big) \ = \ \mathfrak{m}_F\Big(\phi^r Y_r + \sum_{i=0}^{r-1} \xi_i^r(\phi) Y_i - I_F^\phi\Big) \ = \ \phi^r \mathfrak{m}_F(Y_r - G),$$

so $\mathfrak{m}_{F^\phi} = \phi^r \mathfrak{m}_F$ and $I_{F^\phi} = G$. □

Given an $\mathcal{L}(K)$-definable set $A \subseteq K^{1+r}$, we set

$$A_\partial^\phi \ := \ \big\{u \in K^{1+r} : Y_\partial^\phi(u) \in A\big\},$$

so A_∂^ϕ is $\mathcal{L}(K)$-definable and

$$\mathscr{J}_\delta^r(y) \in A_\partial^\phi \iff \mathscr{J}_\partial^r(y) \in A$$

for $y \in K$. As with definable functions, we drop the subscript ∂ and just write A^ϕ when ∂ is clear from context. Since $Y_\partial^\phi(A^\phi) = A$ and Y_∂^ϕ is an $\mathcal{L}(K)$-definable bijection, we have $\dim_{\mathcal{L}}(A^\phi) = \dim_{\mathcal{L}}(A)$.

3.6. Thin sets

A subset $Z \subseteq K$ is said to be **thin** if $\mathscr{J}_\partial^r(Z) \subseteq A$ for some r and some $\mathcal{L}(K)$-definable set $A \subseteq K^{1+r}$ with $\dim_{\mathcal{L}}(A) \leqslant r$. The union of any two thin sets is thin, any subset of a thin set is thin, and the singleton $\{a\}$ is thin for $a \in K$. The constant field C of K is thin, since

$$\mathscr{J}_\partial^1(C) \ = \ \big\{(c,0) : c \in C\big\} \ \subseteq \ K \times \{0\},$$

and $K \times \{0\}$ is a 1-dimensional subset of K^2. Thus, K is thin if ∂ is trivial. Here is a strong converse:

Proposition 3.38. *If ∂ is nontrivial, then no open subinterval of K is thin.*

Proof. Suppose ∂ is nontrivial, let $I \subseteq K$ be an open interval, let $A \subseteq K^{1+r}$ be $\mathcal{L}(K)$-definable, and suppose $\mathscr{J}_\partial^r(y) \in A$ for each $y \in I$. We will show that $\dim_{\mathcal{L}}(A) = 1 + r$. By replacing I with an open subinterval, we may assume that $I = (a-d, a+d)$ for some $a \in K$ and some $d \in K^\times$. By replacing A with $A_{+a,\times d}$, we arrange that $I = (-1,1)$. Since ∂ is nontrivial, we have $x \in K$ with $x' \neq 0$. By inverting x if need be, we arrange that $-1 < x < 1$, so

$$c_0 + c_1 x + \frac{1}{2} c_2 x^2 + \cdots + \frac{1}{r!} c_r x^r \in I$$

for any constants $c_0, \ldots, c_r \in [0, 1/3]_C$. Set $\phi := x'$. By replacing K and A with K^ϕ and A^ϕ, we arrange that $x' = 1$. Let

$$P \ := \ \begin{pmatrix} 1 & x & \frac{1}{2}x^2 & \cdots & \frac{1}{r!}x^r \\ 0 & 1 & x & \cdots & \frac{1}{(r-1)!}x^{r-1} \\ 0 & 0 & 1 & \cdots & \frac{1}{(r-2)!}x^{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \ \in \ \mathrm{Mat}_{1+r}(K).$$

So P is invertible and for each $c = (c_0, \ldots, c_r) \in [0, 1/3]_C^{1+r}$, we have

$$Pc \;=\; \begin{pmatrix} c_0 + c_1 x + \frac{1}{2} c_2 x^2 + \cdots + \frac{1}{r!} c_r x^r \\ c_1 + c_2 x + \cdots + \frac{1}{(r-1)!} c_r x^{r-1} \\ c_2 + \cdots + \frac{1}{(r-2)!} c_r x^{r-2} \\ \vdots \\ c_r \end{pmatrix} \;=\; \mathcal{J}_\partial^r\Big(c_0 + c_1 x + \frac{1}{2} c_2 x^2 + \cdots + \frac{1}{r!} c_r x^r\Big) \;\in\; A.$$

Thus, $[0, 1/3]_C^{1+r} \subseteq P^{-1}A := \{P^{-1}a : a \in A\}$. Lemma 2.3 (with $P^{-1}A$ in place of A and $[0, 1/3]_C$ in place of each A_i) gives

$$\dim_{\mathcal{L}}(A) \;=\; \dim_{\mathcal{L}}(P^{-1}A) \;=\; 1 + r. \qquad \square$$

CHAPTER 4

Generic T-derivations

In this chapter, we fix a model $K = (K, \partial) \models T^\partial$ with constant field C.

Definition 4.1. We say that ∂ is **generic** if for each n and each $\mathcal{L}(K)$-definable set $A \subseteq K^{n+1}$ with $\dim_{\mathcal{L}}\big(\pi_n(A)\big) = n$, there is $a \in K$ with $\jmath^n_\partial(a) \in A$. Let $T^\partial_{\mathcal{G}}$ be the $\mathcal{L}^\partial$-theory which extends T^∂ by axioms asserting that ∂ is generic.

Definition 4.1 differs from the definition given in Chapter 1, but the definition given there is equivalent; see Corollary 4.9 below. In Section 4.1, we prove our main theorem—that $T^\partial_{\mathcal{G}}$ is the model completion of T^∂. We also collect some immediate consequences of this result. We prove that $T^\partial_{\mathcal{G}}$ is distal but not strongly dependent in Section 4.2. In Section 4.3, we see how the ∂-closure constructed in Section 3.3 gives rise to a well-behaved dimension function on the algebra of definable subsets of models of $T^\partial_{\mathcal{G}}$. We also adapt the cell decomposition for closed ordered differential fields developed in **[12]** to our setting. Section 4.4 is devoted to proving two results: $T^\partial_{\mathcal{G}}$ has o-minimal open core and $T^\partial_{\mathcal{G}}$ eliminates imaginaries. In Section 4.5, we conclude with some remarks about the relationship between $T^\partial_{\mathcal{G}}$ and the theory of dense pairs of models of T, as studied in **[25]**. Almost all the material in this chapter is joint work with Antongiulio Fornasiero from **[38]**.

4.1. The model completion of T^∂

Extensions and embeddings. We begin this section with the extension and embedding lemmas needed for our main result. We also investigate which models of T admit an expansion to a model of $T^\partial_{\mathcal{G}}$.

Proposition 4.2. *Let M be a T-extension of K with $\mathrm{rk}_{\mathcal{L}}(M|K) = |M|$. Then there is an extension of ∂ to a T-derivation on M making M a model of $T^\partial_{\mathcal{G}}$.*

Proof. Set $\kappa := |M|$ and let B be a $\mathrm{dcl}_{\mathcal{L}}$-basis for M over K, so $|B| = \kappa$. Let $B_1, B_2, \ldots$ be disjoint subsets of B of cardinality κ with $\bigcup_k B_k = B$. We will construct an increasing chain of T^∂-models $(M_k)_{k\in\mathbb{N}}$ such that

- as T-models, $M_{k+1} = M_k\langle B_{k+1}\rangle$ for $k \geqslant 0$ and $\bigcup_k M_k = M$;
- as T^∂-models, $M_0 = K$, $M_k \subseteq M_{k+1}$ and $\bigcup_k M_k \models T^\partial_{\mathcal{G}}$.

Suppose we have already constructed M_k. Let $(A_\rho)_{\rho<\kappa}$ be an enumeration of all $\mathcal{L}(M_k)$-definable subsets $A_\rho \subseteq M_k^{n_\rho+1}$ with $\dim_{\mathcal{L}}\big(\pi_n(A_\rho)\big) = n_\rho$. Let $(B_\rho)_{\rho<\kappa}$ be an enumeration of pairwise disjoint finite subsets of B_{k+1} such that $|B_\rho| = n_\rho$ and $\bigcup_{\rho<\kappa} B_\rho = B_{k+1}$. We construct an increasing chain of T^∂-models $(M_{k,\rho})_{\rho<\kappa}$ as follows:

- let $M_{k,0} := M_k$;
- if ρ is an infinite limit ordinal, then let $M_{k,\rho} := \bigcup_{\gamma<\rho} M_{k,\gamma}$;
- let $M_{k,\rho+1} := M_{k,\rho}\langle B_\rho\rangle$ and use Corollary 3.32 to extend the T-derivation on $M_{k,\rho}$ to a T-derivation on $M_{k,\rho+1}$ so that $\jmath^{n_\rho}_\partial(b) \in A_\rho$ for some $b \in M_{k,\rho+1}$.

Finally, set $M_{k+1} := \bigcup_{\rho<\kappa} M_{k,\rho}$.

Since $\bigcup_k M_k = M$ as T-models, may view M as a model of T^∂. We claim that $M \models T^\partial_{\mathcal{G}}$. Let $A \subseteq M^{n+1}$ be an $\mathcal{L}(M)$-definable set with $\dim_{\mathcal{L}}\big(\pi_n(A)\big) = n$. Then A is $\mathcal{L}(M_k)$-definable for some k, so there is $b \in M_{k+1}$ with $\jmath^n_\partial(b) \in A$. □

Corollary 4.3. *K has a T^∂-extension which models $T^\partial_{\mathcal{G}}$.*

Proof. Let M be a T-extension of K with $|M| > |K|$. Then $\mathrm{rk}_{\mathcal{L}}(M|K) = |M|$, so we may apply Proposition 4.2. □

Corollary 4.4. *Let $M \models T$ with $\mathrm{rk}_{\mathcal{L}}(M) = |M|$. Then M admits an expansion to a model of $T^\partial_{\mathcal{G}}$. In particular, if T is countable and has an archimedean model, then there is an expansion of $\mathbb{R}$ to a model of $T^\partial_{\mathcal{G}}$.*

Proof. Apply Proposition 4.2 to the prime model $\mathbb{P}$ with the trivial derivation in place of K. If T has an archimedean model then by [**29**, 2.17], there is a unique expansion of $\mathbb{R}$ to a model of T. If in addition T is countable, then $\mathrm{rk}_{\mathcal{L}}(\mathbb{R}) = |\mathbb{R}|$. □

Remark 4.5. We would conjecture that a partial converse to Corollary 4.4 holds as well: if a model $M \models T$ admits an expansion to a model of $T^\partial_{\mathcal{G}}$, then $\mathrm{rk}_{\mathcal{L}}(M) \geqslant \aleph_0$. This is true for $T = \mathrm{RCF}$, since by results of Rosenlicht [**58**], any sequence of distinct elements (a_n) in a differential field of characteristic 0 with $a_n' = a_n^3 - a_n^2 \neq 0$ are necessarily algebraically independent. One can easily show that infinitely many such elements must exist in any model of $T^\partial_{\mathcal{G}}$. It remains to note that the $\mathrm{dcl}_{\mathcal{L}}$-rank and the transcendence degree agree when $T = \mathrm{RCF}$.

Lemma 4.6. *Let M, L be T^∂-extensions of K and suppose that $L \models T^\partial_{\mathcal{G}}$ and that L is $|M|^+$-saturated. Then there is an $\mathcal{L}^\partial(K)$-embedding $\imath\colon M \to L$.*

Proof. It suffices to find an embedding $\mathcal{L}^\partial(K)$-embedding $\imath\colon M_0 \to L$ for some T^∂-submodel M_0 of M properly containing K. Let $a \in M \setminus K$. Then $K\langle \jmath^\infty_\partial a\rangle \models T^\partial$ by Lemma 3.30, so we assume that $M = K\langle \jmath^\infty_\partial a\rangle$. We first consider the case that $a \in \mathrm{cl}^\partial(K)$. Then by Lemma 3.23, there is some minimal n with $M = K\langle \jmath^{n-1}_\partial a\rangle$. Let $F\colon K^n \to K$ be an $\mathcal{L}(K)$-definable function with $a^{(n)} = F(\jmath^{n-1}_\partial a)$. We want to find $b \in L$ such that

(i) $b^{(n)} = F(\jmath^{n-1}_\partial b)$, and
(ii) $\jmath^{n-1}_\partial(b) \in B^L$ for every $\mathcal{L}(K)$-definable set B with $\jmath^{n-1}_\partial(a) \in B^M$.

If we can do this, then we can construct an $\mathcal{L}^\partial(K)$-embedding $\imath\colon M \to L$ by sending $\jmath^{n-1}_\partial(a)$ to $\jmath^{n-1}_\partial(b)$. Let B be an arbitrary $\mathcal{L}(K)$-definable set with $\jmath^{n-1}_\partial(a) \in B^M$. By saturation, it is enough to find $b \in L$ with $\jmath^{n-1}_\partial(b) \in B^L$ and with $b^{(n)} = F(\jmath^{n-1}_\partial b)$. Set $A := \mathrm{Gr}(F|_B)$, so we want to find $b \in L$ with $\jmath^n_\partial(b) \in A$. By minimality of n, the tuple $\jmath^{n-1}_\partial(a)$ is $\mathcal{L}(K)$-independent, so $\dim_{\mathcal{L}}\big(\pi_n(A)\big) = \dim_{\mathcal{L}}(B) = n$. Since $L \models T^\partial_{\mathcal{G}}$, there is some $b \in L$ with $\jmath^n_\partial(b) \in A$, as desired.

Now suppose $a \notin \mathrm{cl}^\partial(K)$. We need to find $b \in L$ with $\jmath^n_\partial(b) \in A^L$ for every n and every $\mathcal{L}(K)$-definable set $A \subseteq K^{n+1}$ with $\jmath^n_\partial(a) \in A^M$. If we can do this, then we can construct an $\mathcal{L}^\partial(K)$-embedding $\imath$ by sending $\jmath^\infty_\partial(a)$ to $\jmath^\infty_\partial(b)$. Again by saturation, it suffices to do this for an arbitrary n and an arbitrary $\mathcal{L}(K)$-definable set $A \subseteq K^{n+1}$ with $\jmath^n_\partial(a) \in A^M$. For such a set A, we have $\dim_{\mathcal{L}}(A) = n+1$ since $\jmath^n_\partial(a)$ is $\mathcal{L}(K)$-independent. In particular, $\dim_{\mathcal{L}}\big(\pi_n(A)\big) = n$, so there is $b \in L$ with $\jmath^n_\partial(b) \in A$ since $L \models T^\partial_{\mathcal{G}}$. □

The model completion. We can now prove our main theorem and collect some immediate consequences.

Theorem 4.7. *$T^\partial_\mathcal{G}$ is the model completion of T^∂. If T has a universal axiomatization, then $T^\partial_\mathcal{G}$ has quantifier elimination.*

PROOF. The fact that $T^\partial_\mathcal{G}$ is the model completion of T^∂ follows from Corollary 4.3, Lemma 4.6, and our Model Completion Criterion. If T has a universal axiomatization, then so does T^∂ by Corollary 3.5, so $T^\partial_\mathcal{G}$ eliminates quantifiers by [**4**, B.11.16]. □

Corollary 4.8. *$T^\partial_\mathcal{G}$ is complete.*

PROOF. By Corollary 3.4, the expansion of the prime model $\mathbb{P}$ by the trivial derivation embeds into any model of $T^\partial_\mathcal{G}$. Completeness follows from Theorem 4.7. □

Theorem 4.7 allows us to give alternate axiomatizations of $T^\partial_\mathcal{G}$:

Corollary 4.9. *The following are equivalent:*

(1) *$K \models T^\partial_\mathcal{G}$;*
(2) *for each n and each $\mathcal{L}(K)$-definable set $X \subseteq K^{2n}$ with $\dim_\mathcal{L}\big(\pi_n(X)\big) = n$, there is $a \in K^n$ with $(a, a') \in X$;*
(3) *for each n and each $\mathcal{L}(K)$-definable function $F\colon U \to K$ with $U \subseteq K^n$ nonempty and open, there is $a \in K$ with $\jmath^{n-1}_\partial(a) \in U$ and $F(\jmath^{n-1}_\partial a) = a^{(n)}$.*

PROOF. Suppose that $K \models T^\partial_\mathcal{G}$ and let $X \subseteq K^{2n}$ be an $\mathcal{L}(K)$-definable set with $\dim_\mathcal{L}\big(\pi_n(X)\big) = n$. Since $\pi_n(X)$ has nonempty interior, there is a T-extension M of K containing an $\mathcal{L}(K)$-independent tuple $b \in \pi_n(X)^M$. Using definable choice, take an $\mathcal{L}(K)$-definable map $G\colon \pi_n(X) \to K^n$ with $\mathrm{Gr}(G) \subseteq X$. Lemma 3.31 gives a unique extension of ∂ to a T-derivation on $K\langle b\rangle$ with $b' = G(b)$, and we view $K\langle b\rangle$ as a T^∂-extension of K with this derivation. Theorem 4.7 gives that K is existentially closed in $K\langle b\rangle$, so there is $a \in \pi_n(X)$ with $a' = G(a)$. Then $(a, a') \in X$. Now suppose that (2) holds and let F and U be as in (3). Define $X \subseteq K^{2n}$ by

$$X \ := \ \big\{(x, y) \in U \times K : y_i = x_{i+1} \text{ for } i = 1, \ldots, n-1 \text{ and } y_n = F(x)\big\}.$$

Then $\pi_n(X) = U$ and for $a = (a_1, \ldots, a_n) \in K^n$ with $(a, a') \in X$, we have $\jmath^{n-1}_\partial(a_1) \in U$ and $F(\jmath^{n-1}_\partial a) = a^{(n)}$. Finally, suppose (3) holds and let $A \subseteq K^{n+1}$ be an $\mathcal{L}(K)$-definable set with $\dim_\mathcal{L}\big(\pi_n(A)\big) = n$. Let $U \subseteq \pi_n(A)$ be a nonempty $\mathcal{L}(K)$-definable open set and, using definable choice, take an $\mathcal{L}(K)$-definable function $F\colon U \to K$ with $\mathrm{Gr}(F) \subseteq A$. Then for $a \in K$ as in (3), we have $\jmath^n_\partial(a) \in \mathrm{Gr}(F) \subseteq A$. □

As $T^\partial_\mathcal{G}$ is complete, we may make the following assumption:

Assumption 4.10. *For the remainder of this chapter, we let $\mathbb{M} = (\mathbb{M}, \partial)$ be a monster model of $T^\partial_\mathcal{G}$. We assume that $K = (K, \partial)$ is a small elementary $\mathcal{L}^\partial$-substructure of $\mathbb{M}$, so $K \models T^\partial_\mathcal{G}$ as well.*

We have for each $\mathcal{L}^\partial(\emptyset)$-formula $\varphi(x)$ that $T^\partial_\mathcal{G} \vdash \forall x \varphi(x)$ if and only if $\mathbb{M} \models \varphi(a)$ for all $a \in \mathbb{M}$. We usually drop the universal quantifier and write $T^\partial_\mathcal{G} \vdash \varphi(x)$ in place of $T^\partial_\mathcal{G} \vdash \forall x \varphi(x)$. We can use our quantifier elimination result to show that each $\mathcal{L}^\partial(\emptyset)$-formula is equivalent to a formula of a special form.

Lemma 4.11. *For every $\mathcal{L}^\partial(\emptyset)$-formula φ there is some r and some $\mathcal{L}(\emptyset)$-formula $\tilde{\varphi}$ such that*

$$T^\partial_\mathcal{G} \ \vdash \ \varphi(x) \longleftrightarrow \tilde{\varphi}(\jmath^r_\partial x).$$

Proof. By extending $\mathcal{L}$ with function symbols for all $\mathcal{L}(\emptyset)$-definable functions, we may assume that T has a universal axiomatization; see Corollary 2.5. Then $T^{\partial}_{\mathcal{G}}$ has quantifier elimination by Theorem 4.7, so it is enough to show that the lemma holds for quantifier-free $\mathcal{L}^{\partial}(\emptyset)$-formulas. Given a quantifier-free $\mathcal{L}^{\partial}(\emptyset)$-formula φ, let $e(\varphi)$ be the number of times in φ that ∂ is applied to a term that is not of the form $\partial^k x_i$. We proceed by induction on $e(\varphi)$. If $e(\varphi) = 0$ then we are done. If $e(\varphi) > 0$, then φ is of the form

$$\varphi(x) \;=\; \psi\big(x, \partial F(\mathcal{J}^n_{\partial} x)\big)$$

for some n, where F is an $(n+1)$-ary $\mathcal{L}(\emptyset)$-definable function and where ψ is an $\mathcal{L}^{\partial}(\emptyset)$-formula. For the remainder of the proof, we work in $\mathbb{M}$. Let $\mathcal{D}$ be an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-cell decomposition for F, so for each $D \in \mathcal{D}$, there is an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function F_D defined in an open neighborhood of D with $F_D(y) = F(y)$ for all $y \in D$. Define the map $G\colon \mathbb{M}^{n+1} \to \mathbb{M}^{n+1}$ by setting $G(y) := \mathbf{J}_{F_D}(y)$ whenever y in D. Then G is $\mathcal{L}(\emptyset)$-definable and $\partial F(y) = G(y)\partial y$ for all $y \in \mathbb{M}^{n+1}$. Set

$$\theta(x) \;:=\; \psi\big(x, G(\mathcal{J}^n_{\partial} x)\partial(\mathcal{J}^n_{\partial} x)\big) \;=\; \psi\big(x, G(\mathcal{J}^n_{\partial} x)(\partial x, \ldots, \partial^{n+1} x)\big).$$

Then $e(\theta) < e(\varphi)$ and

$$\mathbb{M} \;\models\; \varphi(a) \longleftrightarrow \theta(a)$$

for all $a \in \mathbb{M}$. □

Lemma 4.11 can be rephrased in terms of $\mathcal{L}^{\partial}(\mathbb{M})$-definable sets:

Corollary 4.12. *Let $B \subseteq \mathbb{M}$ with $\partial B \subseteq B$ and let $A \subseteq \mathbb{M}^n$ be $\mathcal{L}^{\partial}(B)$-definable. Then there is r and an $\mathcal{L}(B)$-definable set $\tilde{A} \subseteq \mathbb{M}^{n(r+1)}$ with*

$$A \;=\; \big\{a \in \mathbb{M}^n : \mathcal{J}^r_{\partial}(a) \in \tilde{A}\big\}.$$

Proof. Take a tuple b from B and an $\mathcal{L}^{\partial}(\emptyset)$-formula $\varphi(x; y)$ with $A = \varphi(\mathbb{M}; b)$. By Lemma 4.11, we have r and an $\mathcal{L}(\emptyset)$-formula $\tilde{\varphi}$ such that

$$\mathbb{M} \;\models\; \varphi(a; b) \longleftrightarrow \tilde{\varphi}\big(\mathcal{J}^r_{\partial}(a); \mathcal{J}^r_{\partial}(b)\big)$$

for all $a \in \mathbb{M}^n$. Set $\tilde{A} := \tilde{\varphi}\big(\mathbb{M}; \mathcal{J}^r_{\partial}(b)\big)$. □

We can use Lemma 4.11 to show the following:

Corollary 4.13. *Suppose T is countable. Then $T^{\partial}_{\mathcal{G}}$ does not have a prime model.*

Proof. Note that $T^{\partial}_{\mathcal{G}}$ is also countable. We use the fact that if the isolated types are not dense in the unary type space $S_1(T^{\partial}_{\mathcal{G}})$, then $T^{\partial}_{\mathcal{G}}$ does not have a prime model; see [**66**, 4.5.7]. Given a unary $\mathcal{L}^{\partial}(\emptyset)$-formula $\varphi(x)$, we let $[\varphi(x)]$ denote the clopen subset of $S_1(T^{\partial}_{\mathcal{G}})$ consisting of all unary types containing φ. We claim that $[\partial x = 1]$ contains no isolated types. Suppose toward contradiction that $[\psi(x)]$ is a basic clopen set contained in $[\partial x = 1]$ which isolates a type. By Lemma 4.11, we may assume that $\psi(x)$ is of the form $\tilde{\psi}(\mathcal{J}^r_{\partial} x)$ for some r, where $\tilde{\psi}$ is an $\mathcal{L}(\emptyset)$-formula. Since $\psi(x)$ implies that $\partial x = 1$, we may replace ∂x by 1 and $\partial^k x$ by 0 for all $k > 1$. Thus, we may assume that $\psi(x)$ is actually an $\mathcal{L}(\emptyset)$-formula, so ψ defines a finite union of points and open intervals. Since $[\psi]$ is assumed to isolate a type, the set defined by ψ is just one point. However, this point lies in the prime model of T, so $\psi(x)$ implies $\partial x = 0$, a contradiction. □

Closed ordered differential fields. Let $R \models \mathrm{RCF}$ and let ∂ be a derivation on R. In [**65**], Singer axiomatized the theory of closed ordered differential fields as follows:

Definition 4.14. R is a **closed ordered differential field** if the following holds for any $P \in R[X_0, \ldots, X_n]$, and any $Q_1, \ldots, Q_k \in R[X_0, \ldots, X_{n-1}]$: if there is $a \in R^{n+1}$ with

$$P(a) = 0, \qquad \frac{\partial P}{\partial X_n}(a) \neq 0, \qquad Q_i(a) > 0 \text{ for } i = 1, \ldots, k,$$

then there is $b \in R$ with $P(\mathcal{J}^n_\partial b) = 0$ and $Q_i(\mathcal{J}^{n-1}_\partial b) > 0$ for $i = 1, \ldots, k$.

Singer went on to show that the $\mathcal{L}^\partial_{\mathrm{ring}}$-theory CODF of closed ordered differential fields is the model completion of the theory of ordered differential fields. By Fact 3.2 and Lemma 3.9, RCF-derivations are the same as derivations on real closed ordered fields, so CODF is the model completion of RCF^∂. By the uniqueness of model completions, we have the following:

Proposition 4.15. *$R \models \mathrm{RCF}^\partial_{\mathcal{G}}$ if and only if $R \models$ CODF.*

Corollary 4.4 generalizes a result of Brouette, who showed that $\mathbb{R}$ admits a derivation making it a closed ordered differential field [**13**]. Our alternate axiomatization of $T^\partial_{\mathcal{G}}$ in (2) of Corollary 4.9 is similar to the geometric axiomatization for CODF given by Michaux and Rivière [**52**]. Our proof of Corollary 4.13 is essentially the same as Singer's proof that CODF has no prime model [**65**].

4.2. Distality and strong dependence

The class of distal theories was introduced by Simon [**63**] as a subclass of NIP theories. Strongly dependent theories form another subclass of NIP theories and were introduced by Shelah [**62**]. The goal in this section is to show that $T^\partial_{\mathcal{G}}$ is distal but not strongly dependent. At this point, it is well-known that distality implies NIP; see [**15**, 2.6] or [**42**, 2.9] for a proof. Therefore, we deduce NIP as a consequence of distality.

Theorem 4.16. *$T^\partial_{\mathcal{G}}$ is distal.*

Proof. Let $B \subseteq \mathbb{M}$, let $I = I_1 + (c) + I_2$ be an index set where I_1 and I_2 are infinite without endpoints, let $(a_i)_{i\in I}$ be an $\mathcal{L}^\partial(\emptyset)$-indiscernible sequence from $\mathbb{M}^m$, and suppose that $(a_i)_{i\in I_1+I_2}$ is $\mathcal{L}^\partial(B)$-indiscernible. If we can show that $(a_i)_{i\in I}$ is $\mathcal{L}^\partial(B)$-indiscernible, then distality follows; see [**63**, 2.7]. Let b be a tuple from B, let $\varphi(x_1, \ldots, x_n; y)$ be an $\mathcal{L}^\partial(\emptyset)$-formula, and let $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ be indices from I. We need to show that

$$\mathbb{M} \models \varphi(a_{i_1}, \ldots, a_{i_n}; b) \longleftrightarrow \varphi(a_{j_1}, \ldots, a_{j_n}; b).$$

By Lemma 4.11 there is some r and some $\mathcal{L}(\emptyset)$-formula $\tilde{\varphi}$ such that

$$T^\partial_{\mathcal{G}} \vdash \varphi(x_1, \ldots, x_n; y) \longleftrightarrow \tilde{\varphi}(\mathcal{J}^r_\partial x_1, \ldots, \mathcal{J}^r_\partial x_n; \mathcal{J}^r_\partial y).$$

Since $(a_i)_{i\in I}$ is $\mathcal{L}^\partial(\emptyset)$-indiscernible, the sequence $(\mathcal{J}^r_\partial a_i)_{i\in I}$ is also $\mathcal{L}^\partial(\emptyset)$-indiscernible. In particular, $(\mathcal{J}^r_\partial a_i)_{i\in I}$ is $\mathcal{L}(\emptyset)$-indiscernible. Likewise, $(\mathcal{J}^r_\partial a_i)_{i\in I_1+I_2}$ is $\mathcal{L}(\mathcal{J}^r_\partial b)$-indiscernible. Since o-minimal theories are distal [**63**, 2.30], we have

$$\mathbb{M} \models \tilde{\varphi}(\mathcal{J}^r_\partial a_{i_1}, \ldots, \mathcal{J}^r_\partial a_{i_n}; \mathcal{J}^r_\partial b) \longleftrightarrow \tilde{\varphi}(\mathcal{J}^r_\partial a_{j_1}, \ldots, \mathcal{J}^r_\partial a_{j_n}; \mathcal{J}^r_\partial b). \qquad \square$$

Corollary 4.17. *$T^\partial_{\mathcal{G}}$ has NIP.*

Proposition 4.18. *$T^\partial_{\mathcal{G}}$ is not strongly dependent.*

Proof. Let x and y be unary variables. We will find $\mathcal{L}^\partial(\emptyset)$-formulas $\big(\varphi_k(x;y)\big)_{k\in\mathbb{N}}$ and parameters $(b_m)_{m\in\mathbb{N}}$ from $\mathbb{M}$ such that

(i) $\varphi_k(\mathbb{M}; b_m) \cap \varphi_k(\mathbb{M}; b_n) = \emptyset$ for all k and all $m \neq n$, and

(ii) $\bigcap_k \varphi_k(\mathbb{M}; b_{\eta(k)}) \neq \emptyset$ for any function $\eta\colon \mathbb{N} \to \mathbb{N}$.

From this, we see that the dp-rank of the formula $x = x$ is at least $\aleph_0$, so $T^\partial_\mathcal{G}$ is not strongly dependent; see [**64**, 4.22 and 4.23]. For $k \in \mathbb{N}$, let $\varphi_k(x; y)$ be the formula

$$y < \partial^k x < y + 1.$$

Let (b_m) be any increasing sequence in $\mathbb{M}$ with $b_{m+1} - b_m > 1$ for all m. Then for any k and any $m \neq n$, the intervals $(b_m, b_m + 1)$ and $(b_n, b_n + 1)$ are disjoint, so $\varphi_k(\mathbb{M}; b_m) \cap \varphi_k(\mathbb{M}; b_n) = \emptyset$. Fix a function $\eta\colon \mathbb{N} \to \mathbb{N}$. By saturation, $\bigcap_k \varphi_k(\mathbb{M}; b_{\eta(k)})$ is nonempty so long as any finite intersection $\bigcap_{k \leqslant n} \varphi_k(\mathbb{M}; b_{\eta(k)})$ is nonempty. Let n be given and set

$$A \ := \ (b_{\eta(0)}, b_{\eta(0)} + 1) \times (b_{\eta(1)}, b_{\eta(1)} + 1) \times \cdots \times (b_{\eta(n)}, b_{\eta(n)} + 1).$$

Since $\pi_n(A)$ is open, there is $a \in \mathbb{M}$ with $\mathcal{J}^n_\partial(a) \in A$. Then $\varphi_k(a; b_{\eta(k)})$ holds for all $k \leqslant n$, so $a \in \bigcap_{k \leqslant n} \varphi_k(\mathbb{M}; b_{\eta(k)})$ □

Remark 4.19. Theorem 4.16 was first shown for CODF by Chernikov, and this has been employed to construct a distal extension of the theory of differentially closed fields of characteristic zero; see [**1**]. Corollary 4.17 was first shown for CODF in [**52**]. Proposition 4.18 was first shown for CODF by Brouette [**13**]. Since every model of $T^\partial_\mathcal{G}$ is an expansion of a model of CODF, Brouette's result suffices to prove Proposition 4.18. Our proof, which was suggested by Itay Kaplan, differs from Brouette's.

4.3. Dimension and cell decomposition

In this section, we establish a dimension theory for models of $T^\partial_\mathcal{G}$ and adapt the cell decomposition result in [**12**] to our setting.

Dimension in models of $T^\partial_\mathcal{G}$. In [**37**], Fornasiero introduced the notion of an *existential matroid* and showed how these matroids induce a dimension function on the algebra of definable sets in the sense of van den Dries [**23**]. In this section, we apply these results. First, let us see how the ∂-closure behaves in models of $T^\partial_\mathcal{G}$.

Lemma 4.20. *Let $B \subseteq \mathbb{M}$ and suppose $\mathrm{cl}^\partial(B) = B$. Then $(B, \partial|_B) \models T^\partial_\mathcal{G}$.*

Proof. Since $\mathrm{dcl}_\mathcal{L}(B) \subseteq \mathrm{cl}^\partial(B) = B$, we see that B is an elementary $\mathcal{L}$-substructure of $\mathbb{M}$. Since $\partial B \subseteq \mathrm{cl}^\partial(B) = B$ we also see that B is closed under ∂, so $(B, \partial|_B) \models T^\partial$. Fix n and some $\mathcal{L}(B)$-definable set $A \subseteq \mathbb{M}^{n+1}$ with $\dim_\mathcal{L}\big(\pi_n(A)\big) = n$. We need to show that there is some $a \in B$ with $\mathcal{J}^n_\partial(a) \in A$. By definable choice, there is an $\mathcal{L}(B)$-definable function $F\colon \pi_n(A) \to \mathbb{M}$ with $\mathrm{Gr}(F) \subseteq A$, so we may replace A by $\mathrm{Gr}(F)$. As $\mathbb{M} \models T^\partial_\mathcal{G}$, there is some $a \in \mathbb{M}$ with $\mathcal{J}^n_\partial(a) \in A$. Then $a^{(n)} \in \mathrm{dcl}_\mathcal{L}\big(\mathcal{J}^{n-1}_\partial(a)B\big)$, so $a \in \mathrm{cl}^\partial(B) = B$. □

It is not true that every elementary $\mathcal{L}^\partial$-substructure of $\mathbb{M}$ is cl^∂-closed. Indeed, $\mathrm{cl}^\partial(K) \neq K$ since $C_\mathbb{M} \subseteq \mathrm{cl}^\partial(\emptyset) \subseteq \mathrm{cl}^\partial(K)$ but $|C_\mathbb{M}| > |K|$ by saturation (recall that K is assumed to be small).

Proposition 4.21. $(\mathbb{M}, \mathrm{cl}^\partial)$ *is an existential matroid, as defined in* [**37**]

Proof. We begin by recalling that $(\mathbb{M}, \mathrm{cl}^\partial)$ is a pregeometry by Proposition 3.25. Moreover, cl^∂ is *nontrivial* (that is, $\mathrm{cl}^\partial(\emptyset) \neq \mathbb{M}$) since we can find $a \in \mathbb{M}$ such that $\mathcal{J}^\infty_\partial(a)$ is $\mathcal{L}(\emptyset)$-independent by saturation. Now let $a \in \mathbb{M}$ and $B \subseteq \mathbb{M}$ with $a \in \mathrm{cl}^\partial(B)$. We will find an $\mathcal{L}^\partial(B)$-definable set A with $a \in A \subseteq \mathrm{cl}^\partial(B)$, which will show that cl^∂ is *definable*. To do this, use (3) of Lemma 3.23 to find n and an $\mathcal{L}\big(\jmath_\partial^\infty(B)\big)$-definable n-ary function F with $a^{(n)} = F(\jmath_\partial^{n-1}a)$. Then the $\mathcal{L}^\partial(B)$-definable set

$$A \ := \ \big\{x \in \mathbb{M} : x^{(n)} = F(\jmath_\partial^{n-1}x)\big\}$$

contains a and is contained in $\mathrm{cl}^\partial(B)$. Finally, we show that cl^∂ satisfies *existence*. By [**37**, 3.23], it is enough to show that $\mathrm{cl}^\partial(B)$ is an elementary substructure of $\mathbb{M}$ for each subset B of $\mathbb{M}$, and this holds by Lemma 4.20. □

We now define a dimension function on the algebra of $\mathcal{L}^\partial(K)$-definable sets:

Definition 4.22. Let $A \subseteq \mathbb{M}^n$ be a nonempty $\mathcal{L}^\partial(K)$-definable set. We set

$$\dim^\partial(A) \ := \ \max\big\{\operatorname{rk}^\partial(a|K) : a \in A\big\}$$

and we call this the **∂-dimension of** A. We also set $\dim^\partial(\emptyset) := -\infty$.

By [**37**, 4.3], this ∂-dimension satisfies the following axioms in [**23**]:

(D1) $\dim^\partial(\{a\}) = 0$ for each $a \in K$ and $\dim^\partial(K) = 1$;
(D2) $\dim^\partial(A \cup B) = \max\big\{\dim^\partial(A), \dim^\partial(B)\big\}$ for $\mathcal{L}^\partial(K)$-definable sets $A, B \subseteq \mathbb{M}^n$;
(D3) $\dim^\partial$ is invariant under permutation of coordinates;
(D4) if $A \subseteq \mathbb{M}^{n+1}$ is $\mathcal{L}^\partial(K)$-definable, then the set

$$A_i \ := \ \big\{x \in \mathbb{M}^n : \dim^\partial(A_x) = i\big\}$$

is $\mathcal{L}^\partial(K)$-definable for $i = 0, 1$ and $\dim^\partial\big(\{(x,y) \in A : x \in A_i\}\big) = \dim^\partial(A_i) + i$.

By [**23**, 1.7], this dimension does not change if we pass to an elementary T^∂-extension of $\mathbb{M}$, so it does not depend on the choice of the monster model $\mathbb{M}$, only on K. We collect some consequences below, all of which are from [**23**]:

Corollary 4.23. *Let $A \subseteq \mathbb{M}^m$ and $B \subseteq \mathbb{M}^n$ be $\mathcal{L}^\partial(K)$-definable sets.*

(a) $\dim^\partial(\mathbb{M}^n) = n$.
(b) $\dim^\partial(A \times B) = \dim^\partial(A) + \dim^\partial(B)$.
(c) *If $m = n$ and $A \subseteq B$, then* $\dim^\partial(A) \leqslant \dim^\partial(B)$.
(d) *If A is finite and nonempty, then* $\dim^\partial(A) = 0$.
(e) *If $F \colon A \to \mathbb{M}^n$ is an $\mathcal{L}^\partial(K)$-definable map, then for $i = 0, \ldots, m$, the set*

$$A_i \ := \ \big\{x \in \mathbb{M}^n : \dim^\partial\big(F^{-1}(x)\big) = i\big\}$$

is $\mathcal{L}^\partial(K)$-definable and $\dim^\partial\big(F^{-1}(A_i)\big) = \dim^\partial(A_i) + i$. *In particular, ∂-dimension is preserved under definable bijections.*

Finite sets are not the only sets of ∂-dimension 0. For example, the constant field has ∂-dimension 0, since it is contained in $\mathrm{cl}^\partial(\emptyset) \subseteq \mathrm{cl}^\partial(K)$. Below, we characterize unary ∂-dimension 0 sets in terms of thin sets (as defined in Section 3.6):

Proposition 4.24. *Let $Z \subseteq \mathbb{M}$ be $\mathcal{L}^\partial(K)$-definable. Then* $\dim^\partial(Z) = 0$ *if and only if Z is thin.*

Proof. Suppose Z is thin. Take r and an $\mathcal{L}(K)$-definable subset $A \subseteq K^{1+r}$ with $\dim_{\mathcal{L}}(A) \leqslant r$ such that $\jmath_\partial^r(Z) \subseteq A$. Then for any $a \in Z$, we have $\operatorname{rk}_{\mathcal{L}}(\jmath_\partial^r a|K) \leqslant \dim_{\mathcal{L}}(A) \leqslant r$. It follows from (2) of Lemma 3.23 that

$a \in \mathrm{cl}^\partial(K)$. Conversely, suppose $\dim^\partial(Z) = 0$. Then for each $a \in Z$, we have $\mathrm{rk}_{\mathcal{L}}(\mathcal{J}^r_\partial a|K) \leqslant r$ for some r, again by (2) of Lemma 3.23. Thus, $\mathcal{J}^r_\partial(a)$ is contained in some $\mathcal{L}(K)$-definable set $A \subseteq K^{1+r}$ with $\dim_{\mathcal{L}}(A) \leqslant r$. Observe that if $\mathcal{J}^r_\partial(a) \in A$ and $\dim_{\mathcal{L}}(A) \leqslant r$, then $\mathcal{J}^{r+m}_\partial(a) \in A \times K^m$ and $\dim_{\mathcal{L}}(A \times K^m) \leqslant r + m$ for all m. Thus, by saturation, we can find n, r, and $\mathcal{L}(K)$-definable sets $A_1, \ldots, A_n \subseteq K^{1+r}$ of $\mathcal{L}$-dimension $\leqslant r$ such that for each $a \in Z$, there is $i \in \{1, \ldots, n\}$ with $\mathcal{J}^r_\partial(a) \in A_i$ (our observation ensures that we can find one r instead of a separate r_i for each A_i). Then

$$\mathcal{J}^r_\partial(Z) \subseteq \bigcup_{i=1}^n A_i, \qquad \dim_{\mathcal{L}}\Big(\bigcup_{i=1}^n A_i\Big) \leqslant r,$$

so Z is thin. □

Cell decomposition. In [**12**], Brihaye, Michaux and Rivière proved a cell decomposition result for definable sets in closed ordered differential fields. As they remarked in the final section of this paper, the only results that they used are quantifier elimination for CODF, o-minimal cell decomposition for RCF, and the density of the image of $\mathcal{J}^r_\partial$ in any model of CODF. Thus, their results also apply in our case in light of the following lemma:

Lemma 4.25. *$\mathcal{J}^r_\partial(K^n)$ is dense in $K^{n(r+1)}$.*

Proof. Let $U_1, \ldots, U_n \subseteq K^{r+1}$ be basic (hence $\mathcal{L}(K)$-definable) open sets. By the axioms of $T^\partial_{\mathcal{G}}$, there is $a_i \in K$ with $\mathcal{J}^r_\partial(a_i) \in U_i$ for $i = 1, \ldots, n$. Then $\mathcal{J}^r_\partial(a_1, \ldots, a_n) = (\mathcal{J}^r_\partial a_1, \ldots, \mathcal{J}^r_\partial a_n) \in U_1 \times \cdots \times U_n$. □

A **∂-cell** is a subset $D \subseteq K^n$ of the form

$$D = \big\{x \in K^n : \mathcal{J}^r_\partial(x) \in \tilde{D}\big\}$$

for some r and some $\mathcal{L}(K)$-definable cell $\tilde{D} \subseteq K^{n(r+1)}$. Note that any ∂-cell is $\mathcal{L}^\partial(K)$-definable. The set $\tilde{D}$ above is called a **source cell for** D.

Each ∂-cell D has an associated binary sequence $(\boldsymbol{i}_1; \ldots; \boldsymbol{i}_n) \in \{0,1\}^n$, called the **$\partial$-type of** D. This sequence is defined as follows: let $\tilde{D}$ be a source cell for D and let

$$(i_{1,0}, i_{1,1}, \ldots, i_{1,r}; i_{2,0}, \ldots, i_{2,r}; \ldots; i_{n,0}, \ldots, i_{n,r})$$

be the type of the cell $\tilde{D}$. For $k = 1, \ldots, n$, set

$$\boldsymbol{i}_k := \begin{cases} 1 & \text{if } i_{k,j} = 1 \text{ for all } j \leqslant r \\ 0 & \text{otherwise.} \end{cases}$$

In Lemma 4.5 of [**12**], it is shown that this ∂-type is independent of the choice of r and $\tilde{D}$. A **∂-cell decomposition of** K^n is a finite collection $\mathcal{D}$ of disjoint ∂-cells such that $\bigcup \mathcal{D} = K^n$ and such that $\big\{\pi_{n-1}(D) : D \in \mathcal{D}\big\}$ is a ∂-cell decomposition of K^{n-1}.

Theorem 4.26 ([**12**], 4.9)**.** *For any $\mathcal{L}^\partial(K)$-definable sets $A_1, \ldots, A_m \subseteq K^n$, there is a ∂-cell decomposition $\mathcal{D}$ of K^n partitioning $A_1, \ldots, A_m$.*

Brihaye, Michaux and Rivière used their cell decomposition theorem to assign a dimension (which they also call the ∂-dimension) to the definable sets in any closed ordered differential field. They showed that the ∂-dimension of a set A is the same as the maximum differential transcendence degree of a tuple contained in the natural extension of A to a sufficiently saturated elementary extension. Their argument can be adapted with virtually no change in proof to show that this dimension is equal to our ∂-dimension:

Proposition 4.27 ([**12**], 5.23)**.** *Let $A \subseteq K^n$ be an $\mathcal{L}^\partial(K)$-definable set and let $\dim^\partial(A)$ be as in Definition 4.22. Then*

$$\dim^\partial(A) \ = \ \max\big\{\boldsymbol{i}_1 + \cdots + \boldsymbol{i}_n : A \text{ contains a } \partial\text{-cell of } \partial\text{-type } (\boldsymbol{i}_1; \ldots; \boldsymbol{i}_n)\big\}.$$

As in the o-minimal setting, this maximum is always realized in any ∂-cell decomposition partitioning A. This correspondence gives us another way to compute the ∂-dimension of certain sets. For example, the constant field C is of the form

$$C \ = \ \big\{x \in K : (x, x') \in K \times \{0\}\big\}.$$

Thus C is a ∂-cell since $K \times \{0\}$ is a cell. The binary sequence associated to $K \times \{0\}$ is $(1, 0)$, so the ∂-type of C is (0) and $\dim^\partial(C) = 0$.

4.4. Open core and elimination of imaginaries

In this section, we show that $T^\partial_{\mathcal{G}}$ has T as its open core, and we use this result to analyze the definable closure in models of $T^\partial_{\mathcal{G}}$. We also show that $T^\partial_{\mathcal{G}}$ eliminates imaginaries.

Open core. Using a theorem of Dolich, Miller and Steinhorn [**20**], Point proved that CODF has o-minimal open core [**56**]. While Point's proof works in our case, we can gather more information about definable open sets by using a criterion developed by Boxall and Hieronymi [**11**]. In this subsection, we verify that the conditions for this criterion are met, which will allow us to prove the following:

Theorem 4.28. *$T^\partial_{\mathcal{G}}$ has T as its open core. More precisely, for $B \subseteq \mathbb{M}$ with $\partial B \subseteq B$, any open $\mathcal{L}^\partial(B)$-definable set is $\mathcal{L}(B)$-definable.*

For the remainder of this subsection let n be fixed, let $a \in \mathbb{M}^n$, and let $B \subseteq \mathbb{M}$ be a small set with $\partial B \subseteq B$. Set

$$\Xi_{\mathcal{L}}(a|B) \ := \ \big\{b \in \mathbb{M}^n : \operatorname{tp}_{\mathcal{L}}(b|B) = \operatorname{tp}_{\mathcal{L}}(a|B)\big\}, \qquad \Xi_{\mathcal{L}^\partial}(a|B) \ := \ \big\{b \in \mathbb{M}^n : \operatorname{tp}_{\mathcal{L}^\partial}(b|B) = \operatorname{tp}_{\mathcal{L}^\partial}(a|B)\big\}.$$

The next few results are dedicated to studying these sets.

Lemma 4.29. $\operatorname{rk}_{\mathcal{L}}(a|B) = n$ *if and only if* $\Xi_{\mathcal{L}}(a|B)$ *is somewhere dense.*

Proof. If $\operatorname{rk}_{\mathcal{L}}(a|B) = n$, then any $\mathcal{L}(B)$-definable set X containing a contains an open neighborhood of a. Let $(X_i)_{i\in I}$ be a list of all $\mathcal{L}(B)$-definable sets containing a, so $\Xi_{\mathcal{L}}(a|B) = \bigcap_{i\in I} X_i$. Since I is small and $\bigcap_{i\in I_0} X_i$ contains an open neighborhood of a for each finite $I_0 \subseteq I$, we can use saturation to find an open neighborhood U of a contained in $\bigcap_{i\in I} X_i$. Thus, a is in the interior of $\Xi_{\mathcal{L}}(a|B)$. In particular, $\Xi_{\mathcal{L}}(a|B)$ is somewhere dense.

Now suppose $\operatorname{rk}_{\mathcal{L}}(a|B) < n$ and take an $\mathcal{L}(B)$-definable set X containing a with $\dim_{\mathcal{L}}(X) < n$. Then X is nowhere dense and $\Xi_{\mathcal{L}}(a|B) \subseteq X$, so $\Xi_{\mathcal{L}}(a|B)$ is nowhere dense. ☐

Lemma 4.30. *Suppose $\operatorname{rk}^\partial(a|B) = n$ and let $X \subseteq \mathbb{M}^n$ be an $\mathcal{L}^\partial(B)$-definable set containing a. Then there is an $\mathcal{L}(B)$-definable open set $A \subseteq \mathbb{M}^n$ containing a such that $X \cap A$ is dense in A.*

Proof. Using Corollary 4.12, take r and an $\mathcal{L}(B)$-definable set $\tilde{X} \subseteq \mathbb{M}^{n(r+1)}$ with

$$X \;=\; \{x \in \mathbb{M}^n : \mathfrak{J}^r_\partial(x) \in \tilde{X}\}.$$

Let $\tilde{A} \subseteq \tilde{X}$ be an $\mathcal{L}(B)$-definable cell containing $\mathfrak{J}^r_\partial(a)$. Then $\tilde{A}$ must be open, since $\operatorname{rk}^\partial(a|B) = n$. Let $\pi\colon \mathbb{M}^{n(r+1)} \to \mathbb{M}^n$ be the projection map

$$(x_{1,0}, x_{1,1}, \ldots, x_{1,r}; x_{2,0}, \ldots, x_{2,r}; \ldots; x_{n,0}, \ldots, x_{n,r}) \;\mapsto\; (x_{1,0}; x_{2,0}; \ldots; x_{n,0}).$$

Then $\pi(\mathfrak{J}^r_\partial x) = x$ for all $x \in \mathbb{M}^n$, so

$$X \;=\; \pi\big(\mathfrak{J}^r_\partial(\mathbb{M}^n) \cap \tilde{X}\big) \;\supseteq\; \pi\big(\mathfrak{J}^r_\partial(\mathbb{M}^n) \cap \tilde{A}\big).$$

Since $\mathfrak{J}^r_\partial(\mathbb{M}^n)$ is dense in $\mathbb{M}^{n(r+1)}$ by Lemma 4.25, the intersection $\mathfrak{J}^r_\partial(\mathbb{M}^n) \cap \tilde{A}$ is dense in $\tilde{A}$. Therefore, $X \cap \pi(\tilde{A})$ is dense in $\pi(\tilde{A})$, so we may set $A := \pi(\tilde{A})$. ☐

Corollary 4.31. *Suppose* $\operatorname{rk}^\partial(a|B) = n$. *Then* $\Xi_{\mathcal{L}^\partial}(a|B)$ *is dense in* $\Xi_{\mathcal{L}}(a|B)$.

Proof. Fix $b \in \Xi_{\mathcal{L}}(a|B)$. We need to show that if $U \subseteq \mathbb{M}^n$ is an open set containing b, then $\Xi_{\mathcal{L}^\partial}(a|B) \cap U$ is nonempty. By saturation, it suffices to show that $U \cap X \neq \emptyset$ for any $\mathcal{L}^\partial(B)$-definable set $X \subseteq \mathbb{M}^n$ containing a. By Lemma 4.30, there is an $\mathcal{L}(B)$-definable open set $A \subseteq \mathbb{M}^n$ with $a \in A$ such that $X \cap A$ is dense in A. Since $b \in \Xi_{\mathcal{L}}(a|B) \subseteq A$, the intersection $U \cap A$ is nonempty and open, so $U \cap X$ is nonempty by density of $X \cap A$ in A. ☐

Lemma 4.32. $\operatorname{rk}^\partial(a|B) < n$ *if and only if* a *is contained in some* $\mathcal{L}^\partial(B)$*-definable set* A *with* $\dim^\partial(A) < n$.

Proof. If $\operatorname{rk}^\partial(a|B) = n$, then any $\mathcal{L}^\partial(B)$-definable set containing a has ∂-dimension n by Definition 4.22. Suppose $\operatorname{rk}^\partial(a|B) < n$. Proposition 3.26 gives that

$$\lim_{r\to\infty} \frac{\operatorname{rk}_{\mathcal{L}}(\mathfrak{J}^r_\partial a|B)}{r+1} \;<\; n.$$

Take r large enough that $\operatorname{rk}_{\mathcal{L}}(\mathfrak{J}^r_\partial a|B) < n(r+1)$ and take an $\mathcal{L}(B)$-definable set $\tilde{A} \subseteq \mathbb{M}^{n(r+1)}$ with $\mathfrak{J}^r_\partial(a) \in \tilde{A}$ and $\dim_{\mathcal{L}}(\tilde{A}) < n(r+1)$. Then a is contained in the set

$$A \;:=\; \{x \in \mathbb{M}^n : \mathfrak{J}^r_\partial(x) \in \tilde{A}\}.$$

It remains to note that $\dim^\partial(A) < n$. ☐

Proof of Theorem 4.28. Let A_n be the set of all $a \in \mathbb{M}^n$ such that $\Xi_{\mathcal{L}}(a|B)$ is somewhere dense and let A^*_n be the set of all $a \in A_n$ such that $\Xi_{\mathcal{L}^\partial}(a|B)$ is dense in $\Xi_{\mathcal{L}}(a|B)$. If we can show that A^*_n is dense in $\mathbb{M}^n$, then the theorem follows by [**11**, 2.2], Set

$$D \;:=\; \{a \in \mathbb{M}^n : \operatorname{rk}^\partial(a|B) = n\}.$$

If $a \in D$, then $\operatorname{rk}_{\mathcal{L}}(a|B) = n$ and so $D \subseteq A_n$ by Lemma 4.29. By Corollary 4.31 we even have $D \subseteq A^*_n$, so it is enough to show that D is dense in $\mathbb{M}^n$. Lemma 4.32 gives that

$$D \;=\; \mathbb{M}^n \setminus \bigcup\{X \subseteq \mathbb{M}^n : X \text{ is } \mathcal{L}^\partial(B)\text{-definable and } \dim^\partial(X) < n\}.$$

Let $U \subseteq \mathbb{M}^n$ be a basic open set. We need to show that D intersects U and by saturation, it suffices to show that $U \setminus X \neq \emptyset$ for an arbitrary $\mathcal{L}^\partial(B)$-definable set X of ∂-dimension $< n$. This follows immediately from the fact that $\dim^\partial(U) = n$. ☐

We list below two standard consequences of having o-minimal open core; see [**20**] for proofs.

Corollary 4.33. *$T^\partial_{\mathcal{G}}$ eliminates the quantifier $\exists^\infty$ and every model of $T^\partial_{\mathcal{G}}$ is definably complete.*

We can use Theorem 4.28 to analyze the definable closure in models of $T^\partial_{\mathcal{G}}$:

Corollary 4.34. *Let $A \subseteq \mathbb{M}$. Then*

$$\mathrm{dcl}_{\mathcal{L}^\partial}(A) \ = \ \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^\infty_\partial(A)\big).$$

Thus A is $\mathrm{dcl}_{\mathcal{L}^\partial}$-closed if and only if $(A, \partial|_A) \models T^\partial$.

Proof. Since $\mathcal{J}^\infty_\partial(A) \subseteq \mathrm{dcl}_{\mathcal{L}^\partial}(A)$ and since $\mathrm{dcl}_{\mathcal{L}^\partial}(A)$ is $\mathrm{dcl}_{\mathcal{L}}$-closed, we have $\mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^\infty_\partial(A)\big) \subseteq \mathrm{dcl}_{\mathcal{L}^\partial}(A)$. For the other inclusion, fix $b \in \mathrm{dcl}_{\mathcal{L}^\partial}(A)$. Since $\{b\}$ is closed and $\mathcal{L}^\partial(A)$-definable, Theorem 4.28 gives that $\{b\}$ is $\mathcal{L}\big(\mathcal{J}^\infty_\partial(A)\big)$-definable. □

As usual, this allows us to understand the $\mathcal{L}^\partial(B)$-definable functions:

Corollary 4.35. *Let $F\colon \mathbb{M}^n \to \mathbb{M}$ be an $\mathcal{L}^\partial(B)$-definable function. Then there is r and an $\mathcal{L}(B)$-definable function $\tilde{F}\colon \mathbb{M}^{n(r+1)} \to \mathbb{M}$ such that*

$$F(y) \ = \ \tilde{F}(\mathcal{J}^r_\partial y)$$

for all $y \in \mathbb{M}^n$.

Proof. For $a \in \mathbb{M}^n$, we have $F(a) \in \mathrm{dcl}_{\mathcal{L}}\big(\mathcal{J}^\infty_\partial(a)B\big)$ by Corollary 4.34. Saturation yields m, r, and $\mathcal{L}(B)$-definable functions $\tilde{F}_1, \ldots, \tilde{F}_m\colon \mathbb{M}^{n(r+1)} \to \mathbb{M}$ such that for each $y \in \mathbb{M}^n$, there is $i \in \{1, \ldots, m\}$ with $F(y) = \tilde{F}_i(\mathcal{J}^r_\partial y)$. For $i = 1, \ldots, m$, set

$$A_i \ := \ \big\{y \in \mathbb{M}^n : F(y) = \tilde{F}_i(\mathcal{J}^r_\partial y)\big\}.$$

Then each A_i is $\mathcal{L}^\partial(B)$-definable, so by possibly increasing r, Corollary 4.12 gives an $\mathcal{L}(B)$-definable set $\tilde{A}_i \subseteq \mathbb{M}^{n(r+1)}$ with

$$A_i \ = \ \big\{y \in \mathbb{M}^n : \mathcal{J}^r_\partial(y) \in \tilde{A}_i\big\}.$$

Now define $\tilde{F}\colon \mathbb{M}^{n(r+1)} \to \mathbb{M}$ by setting $\tilde{F}(x) := \tilde{F}_i(x)$ whenever $x \in \tilde{A}_i \setminus (\tilde{A}_1 \cup \cdots \cup \tilde{A}_{i-1})$. □

Elimination of imaginaries. Recall that T eliminates imaginaries as a consequence of definable choice. In this subsection, we combine this fact with Theorem 4.28 to show that $T^\partial_{\mathcal{G}}$ eliminates imaginaries. This proof was communicated to us by Marcus Tressl. In [**56**], Point used that CODF has o-minimal open core to prove that CODF eliminates imaginaries. Our method differs, but Point's method also works in our case. Yet another proof of elimination of imaginaries for CODF can be found in [**14**].

Fix an $\mathcal{L}^\partial(\mathbb{M})$-definable set $A \subseteq \mathbb{M}^n$. A **canonical parameter** for A is a tuple a such that each $\mathcal{L}^\partial$-automorphism $\sigma\colon \mathbb{M} \to \mathbb{M}$ fixes A setwise if and only if σ fixes a componentwise. In order to show that $T^\partial_{\mathcal{G}}$ eliminates imaginaries, we need to find a canonical parameter for A; see [**66**, 8.4.2 and 8.4.3]. We do this over the next few lemmas by writing A as the difference of two $\mathcal{L}^\partial(\mathbb{M})$-definable sets and finding a canonical parameter for each of these two sets. Let σ be an arbitrary $\mathcal{L}^\partial$-automorphism of $\mathbb{M}$. Note that $\mathcal{J}^r_\partial\big(\sigma(b)\big) = \sigma\big(\mathcal{J}^r_\partial(b)\big)$ for all r and all $b \in \mathbb{M}^n$.

Using Corollary 4.12, take r and an $\mathcal{L}(\mathbb{M})$-definable set $B \subseteq \mathbb{M}^{n(r+1)}$ with

$$A \ = \ \big\{x \in \mathbb{M}^n : \mathcal{J}^r_\partial(x) \in B\big\}.$$

The topological closure $\overline{\mathfrak{J}^r_\partial(A)}$ of $\mathfrak{J}^r_\partial(A)$ is $\mathcal{L}(\mathbb{M})$-definable by Theorem 4.28, so by replacing B with $B\cap\overline{\mathfrak{J}^r_\partial(A)}$, we arrange that $\mathfrak{J}^r_\partial(A)\subseteq B\subseteq\overline{\mathfrak{J}^r_\partial(A)}$. We associate to A two other $\mathcal{L}^\partial(\mathbb{M})$-definable sets:

$$A^{\mathrm{cl}} \;:=\; \big\{x\in\mathbb{M}^n : \mathfrak{J}^r_\partial(x)\in\overline{\mathfrak{J}^r_\partial(A)}\big\},\qquad A^{\mathrm{fr}} \;:=\big\{x\in\mathbb{M}^n : \mathfrak{J}^r_\partial(x)\in\overline{\mathfrak{J}^r_\partial(A)}\setminus B\big\}.$$

Note that $A\cup A^{\mathrm{fr}}=A^{\mathrm{cl}}$ and that $A\cap A^{\mathrm{fr}}=\emptyset$.

Lemma 4.36. *$\sigma(A)=A$ if and only if $\sigma(A^{\mathrm{cl}})=A^{\mathrm{cl}}$ and $\sigma(A^{\mathrm{fr}})=A^{\mathrm{fr}}$.*

Proof. Suppose $\sigma(A)=A$. Then $\sigma\big(\mathfrak{J}^r_\partial(A)\big)=\mathfrak{J}^r_\partial(A)$ and so $\sigma\big(\overline{\mathfrak{J}^r_\partial(A)}\big)=\overline{\mathfrak{J}^r_\partial(A)}$. We have

$$b\in A^{\mathrm{cl}}\iff \mathfrak{J}^r_\partial(b)\in\overline{\mathfrak{J}^r_\partial(A)}\iff\sigma(\mathfrak{J}^r_\partial b)\in\sigma\big(\overline{\mathfrak{J}^r_\partial(A)}\big)\iff\sigma(b)\in A^{\mathrm{cl}},$$

since $\overline{\mathfrak{J}^r_\partial(A)}$ is σ-invariant. Thus, $\sigma(A^{\mathrm{cl}})=A^{\mathrm{cl}}$ and so $\sigma(A^{\mathrm{fr}})=\sigma(A^{\mathrm{cl}}\setminus A)=A^{\mathrm{cl}}\setminus A=A^{\mathrm{fr}}$. For the other direction, we use that $\sigma(A)=\sigma(A^{\mathrm{cl}}\setminus A^{\mathrm{fr}})=\sigma(A^{\mathrm{cl}})\setminus\sigma(A^{\mathrm{fr}})$. □

Lemma 4.37. *If $A=A^{\mathrm{cl}}$, then A has a canonical parameter.*

Proof. We first note that since $\overline{\mathfrak{J}^r_\partial(A)}$ is $\mathcal{L}(\mathbb{M})$-definable and since T eliminates imaginaries, there is a canonical parameter a for $\overline{\mathfrak{J}^r_\partial(A)}$. We claim that a is also a canonical parameter for A. We need to show that

$$\sigma(A)=A\iff\sigma\big(\overline{\mathfrak{J}^r_\partial(A)}\big)=\overline{\mathfrak{J}^r_\partial(A)}.$$

First, if $\sigma(A)=A$ then $\sigma\big(\mathfrak{J}^r_\partial(A)\big)=\mathfrak{J}^r_\partial(A)$ and so $\sigma\big(\overline{\mathfrak{J}^r_\partial(A)}\big)=\overline{\mathfrak{J}^r_\partial(A)}$. Now, suppose $\sigma\big(\overline{\mathfrak{J}^r_\partial(A)}\big)=\overline{\mathfrak{J}^r_\partial(A)}$ and fix $b\in A$. Then $\mathfrak{J}^r_\partial(b)\in\overline{\mathfrak{J}^r_\partial(A)}$ and so $\sigma(\mathfrak{J}^r_\partial b)\in\sigma\big(\overline{\mathfrak{J}^r_\partial(A)}\big)=\overline{\mathfrak{J}^r_\partial(A)}$, so $\sigma(b)\in A^{\mathrm{cl}}=A$. □

Lemma 4.38. $\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A^{\mathrm{fr}})}\big)<\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A)}\big)$.

Proof. Set $B_0:=\overline{\mathfrak{J}^r_\partial(A)}\setminus B$, so $\mathfrak{J}^r_\partial(A^{\mathrm{fr}})\subseteq B_0$. Since $\overline{B}=\overline{\mathfrak{J}^r_\partial(A)}$, we have $\dim_{\mathcal{L}}(B_0)<\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A)}\big)$ by [**26**, 4.1.8]. Since the dimension of an $\mathcal{L}(\mathbb{M})$-definable set doesn't increase when we take its closure (again by [**26**, 4.1.8]), we get

$$\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A^{\mathrm{fr}})}\big)\;\leqslant\;\dim_{\mathcal{L}}(\overline{B_0})\;=\;\dim_{\mathcal{L}}(B_0)\;<\;\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A)}\big).$$ □

Theorem 4.39. *$T^\partial_{\mathcal{G}}$ eliminates imaginaries.*

Proof. By Lemma 4.36, it is enough to find canonical parameters for A^{cl} and for A^{fr}. By Lemma 4.37, there is a canonical parameter for A^{cl}. By Lemma 4.38, we have $\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A^{\mathrm{fr}})}\big)<\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A)}\big)$, so by induction on $\dim_{\mathcal{L}}\big(\overline{\mathfrak{J}^r_\partial(A)}\big)$, we may assume that there is a canonical parameter for A^{fr} as well. □

4.5. Dense pairs

In [**25**], van den Dries introduced the theory of dense pairs of o-minimal structures:

Definition 4.40. Let P be a unary predicate and let $\mathcal{L}^P:=\mathcal{L}\cup\{P\}$. An $\mathcal{L}^P$-structure (M,P) is said to be a **dense pair of models of** T if $M\models T$ and $P(M)$ is the underlying set of a proper dense elementary $\mathcal{L}$-substructure of M. Let T^P be the $\mathcal{L}^P$-theory axiomatizing dense pairs of models of T.

Recall our assumption that $K\models T^\partial_{\mathcal{G}}$. For each open interval $I\subseteq K$, the axioms of $T^\partial_{\mathcal{G}}$ give $c\in I$ with $c'=0$, so C is dense in K. Moreover, C is the underlying set of an elementary $\mathcal{L}$-substructure of K by Lemma 3.3. Let (K,P) be the $\mathcal{L}^P$-structure where $P(K)=C$. Then $(K,P)\models T^P$. Note that (K,P) is a reduct of (K,∂) in the sense of definability. In [**46**], Hieronymi and Nell showed that T^P is not distal. However, since distality is not preserved under reducts, the question remained open whether models of T^P have distal expansions. In light of Theorem 4.16, we are able to give a partial answer:

Corollary 4.41. *$T^\partial_{\mathcal{G}}$ is a distal theory extending T^P.*

We say that this is only a partial answer because dense pairs are defined for o-minimal theories extending the theory of divisible ordered abelian groups, whereas Corollary 4.41 is only a statement about o-minimal theories extending RCF. In the case that $T = \mathrm{RCF}$, Corollary 4.41 was first observed by Cubides Kovacsics and Point [**17**].

Remark 4.42. It is worth noting that we do not have a method of expanding a given dense pair to a model of $T^\partial_{\mathcal{G}}$. Indeed, there is a dense pair of real closed ordered fields which does not admit an expansion to a model of CODF. To see this, let R be the real closure of $\mathbb{Q}(\pi)$ and expand R to a model $(R, P) \models \mathrm{RCF}^P$ by letting $P(R)$ be the real closure of $\mathbb{Q}$. Since the $\mathcal{L}_{\mathrm{ring}}$-rank of R (which is the same as the transcendence degree of R) is equal to 1, Remark 4.5 tells us that R does not admit an expansion to a closed ordered differential field.

Internality to the constants. A subset $A \subseteq K^n$ is said to be **$\mathcal{L}^\partial(K)$-internal to** C if $A \subseteq F(C^m)$ for some $\mathcal{L}^\partial(K)$-definable map $F\colon K^m \to K^n$. If this map F can be taken to be $\mathcal{L}(K)$-definable, then A is said to be **$\mathcal{L}(K)$-internal to** C.

Lemma 4.43. *Let $A \subseteq K^n$ be $\mathcal{L}^\partial(K)$-definable. Then A is $\mathcal{L}^\partial(K)$-internal to C if and only if A is $\mathcal{L}^P(K)$-definable and $\mathcal{L}(K)$-internal to C.*

Proof. This is clear if A is empty, so we may assume that A is nonempty. Suppose that A is $\mathcal{L}^\partial(K)$-internal to C and let $F\colon K^m \to K^n$ be an $\mathcal{L}^\partial(K)$-definable map with $A \subseteq F(C^m)$. Choose some arbitrary $a \in A$ and let $G\colon K^m \to K^n$ be the function

$$G(y) \ := \ \begin{cases} F(y) & \text{if } F(y) \in A \\ a & \text{otherwise.} \end{cases}$$

Then G is $\mathcal{L}^\partial(K)$-definable (since A is) and $G(C^m) = A$. Using Corollary 4.35, take r and an $\mathcal{L}(K)$-definable map $\tilde{G}\colon K^{m(r+1)} \to K^n$ such that $G(y) = \tilde{G}(\jmath^r_\partial y)$ for all $y \in \mathbb{M}^m$. Then

$$A \ = \ G(C^m) \ = \ \big\{\tilde{G}(c_1, 0, 0, \ldots; c_2, 0, 0, \ldots; \ldots; c_m, 0, 0, \ldots) : (c_1, \ldots, c_m) \in C^m\big\},$$

so A is $\mathcal{L}^P(K)$-definable and $\mathcal{L}(K)$-internal to C. The other direction is immediate. □

If A is an $\mathcal{L}^\partial(K)$-definable subset of C^n, then A is $\mathcal{L}^\partial(K)$-internal to C, so A is $\mathcal{L}^P(K)$-definable. In any dense pair (M, P), the $\mathcal{L}^P(M)$-definable subsets of $P(M)^n$ are just traces of $\mathcal{L}(M)$-definable sets [**25**], so it follows that $A = B \cap C^n$ for some $\mathcal{L}(K)$-definable set $B \subseteq K^n$. We include a short, direct proof of this corollary below:

Corollary 4.44. *Let $A \subseteq C^n$ be $\mathcal{L}^\partial(K)$-definable. Then $A = B \cap C^n$ for some $\mathcal{L}(K)$-definable set $B \subseteq K^n$.*

Proof. Using Corollary 4.12, take r and an $\mathcal{L}(K)$-definable set $\tilde{A} \subseteq K^{n(r+1)}$ with

$$A \ = \ \big\{x \in K^n : \jmath^r_\partial(x) \in \tilde{A}\big\}.$$

For any $a \in A$ and any $k > 0$, we have $a^{(k)} = 0$ since $A \subseteq C^n$. Thus,

$$A \ = \ C^n \cap \big\{x \in K^n : (x_1, 0, 0, \ldots; x_2, 0, 0, \ldots; \ldots; x_n, 0, 0, \ldots) \in \tilde{A}\big\}.$$ □

In [**35**], the following dichotomy was established for definable sets in dense pairs:

Fact 4.45 ([**35**], 3.11)**.** *Let (M, P) be a dense pair and let $A \subseteq M^n$ be $\mathcal{L}^P(M)$-definable. Exactly one of the following holds:*

(1) *A is $\mathcal{L}(M)$-internal to $P(M)$, or*
(2) *there is m and an $\mathcal{L}(M)$-definable function $F\colon M^{nm} \to M$ such that $F(A^m)$ contains an open interval.*

This dichotomy can be used to characterize $\mathcal{L}(K)$-internal $\mathcal{L}^P(K)$-definable sets in terms of their ∂-dimension:

Lemma 4.46. *Let $A \subseteq K^n$ be $\mathcal{L}^P(K)$-definable. Then A is $\mathcal{L}(K)$-internal to C if and only if $\dim^\partial(A) = 0$.*

Proof. Suppose A is $\mathcal{L}(K)$-internal to C and let $F\colon K^m \to K^n$ be an $\mathcal{L}(K)$-definable map with $A \subseteq F(C^m)$. As $\dim^\partial(C) = 0$, we have $\dim^\partial(C^m) = 0$, so $\dim^\partial(A) \leqslant \dim^\partial(C^m) = 0$ by (e) of Corollary 4.23. Now suppose A is not $\mathcal{L}(K)$-internal to C. Using Fact 4.45, take m and an $\mathcal{L}(K)$-definable function $F\colon K^{nm} \to K$ such that $F(A^m)$ contains an open interval. Then $F(A^m)$ has ∂-dimension 1, so $\dim^\partial(A^m) > 0$ by (e) of Corollary 4.23. It follows that $\dim^\partial(A) > 0$. □

Combining Lemmas 4.43 and 4.46, we get the following corollary:

Corollary 4.47. *Let $A \subseteq K^n$ be $\mathcal{L}^\partial(K)$-definable. The following are equivalent:*

(1) *A is $\mathcal{L}^\partial(K)$-internal to C;*
(2) *A is $\mathcal{L}^P(K)$-definable and $\mathcal{L}(K)$-internal to C;*
(3) *A is $\mathcal{L}^P(K)$-definable and $\dim^\partial(A) = 0$.*

Remark 4.48. Corollary 4.47 was first shown for CODF by Eleftheriou, León Sánchez, and Regnault [**36**]. Our method of proof is essentially the same as theirs.

CHAPTER 5

T-convex valuation rings

The fundamentals of valuation theory for o-minimal fields were established by van den Dries and Lewenberg in [**29**]. In this chapter, we set up valuation theoretic notation, recall some important results, and establish a number of lemmas for use in the next two chapters. We try as much as possible to use the same notation as in [**4**]. Much of Sections 5.1 and 5.2 is spent verifying that well-known facts and constructions in valuation theory also hold in the o-minimal setting (often, under the assumption of power boundedness). In Section 5.3, we discuss the Wilkie inequality and collect some useful consequences.

Following [**29**], we say that a nonempty convex set $\mathcal{O} \subseteq K$ is a **T-convex valuation ring of K** if $F(\mathcal{O}) \subseteq \mathcal{O}$ for all $\mathcal{L}(\emptyset)$-definable continuous functions $F\colon K \to K$. Any T-convex valuation ring is a valuation ring, that is, a subring of K which contains either a or a^{-1} for each $a \in K$. Let $\mathcal{L}^{\mathcal{O}} := \mathcal{L} \cup \{\mathcal{O}\}$ be the extension of $\mathcal{L}$ by a unary predicate $\mathcal{O}$ and let $T^{\mathcal{O}}$ be the $\mathcal{L}^{\mathcal{O}}$-theory which extends T by axioms asserting that $\mathcal{O}$ is a T-convex valuation ring. For the rest of this chapter, let $K = (K, \mathcal{O}) \models T^{\mathcal{O}}$ (so K is now an $\mathcal{L}^{\mathcal{O}}$-structure). Unlike in [**29**], we allow $\mathcal{O} = K$, in which case K is said to be **trivially valued**. The theory $T^{\mathcal{O}}$ is *weakly o-minimal*—every $\mathcal{L}^{\mathcal{O}}(K)$-definable subset of K is a finite union of convex subsets of K [**29**, 3.14]. The following is an easy consequence of the o-minimal monotonicity theorem:

Fact 5.1. *The convex hull of an elementary $\mathcal{L}$-substructure of K is a T-convex valuation ring of K.*

We let $\smallo$ denote the unique maximal ideal of $\mathcal{O}$ and for $a, b \in K$, we set

$$a \preccurlyeq b \;:\Longleftrightarrow\; a \in b\mathcal{O}, \qquad a \prec b \;:\Longleftrightarrow\; b \neq 0 \text{ and } a \in b\smallo, \qquad a \asymp b \;:\Longleftrightarrow\; a \preccurlyeq b \text{ and } b \preccurlyeq a$$

$$a \succcurlyeq b \;:\Longleftrightarrow\; b \preccurlyeq a, \qquad a \succ b \;:\Longleftrightarrow\; b \prec a, \qquad a \sim b \;:\Longleftrightarrow\; a - b \prec a.$$

Note that if $a \sim b$, then a, b are nonzero and that $\sim$ is an equivalence relation on $K^\times$. Moreover, if $a \sim b$, then $a \asymp b$ and a is positive if and only if b is. We let $v\colon K^\times \to \Gamma$ be the (surjective) Krull valuation corresponding to $\mathcal{O}$, so $\ker(v) = \mathcal{O}^\times$. The **value group** Γ (or Γ_K if K is not clear from context) is written additively and ordered as follows:

$$va \geqslant 0 \iff a \in \mathcal{O}, \qquad va > 0 \iff a \in \smallo.$$

We set $\Gamma_\infty := \Gamma \cup \{\infty\}$ where $\infty > \Gamma$, and we extend v to all of K by setting $v(0) := \infty$. For $a, b \in K$, we have

$$v(a+b) \;\geqslant\; \min\{va, vb\}, \qquad v(ab) \;=\; va + vb.$$

Subsets of K of the form

$$\big\{y \in K : v(y-a) > \gamma\big\}, \qquad \big\{y \in K : v(y-a) \geqslant \gamma\big\}$$

for $a \in K$ and $\gamma \in \Gamma$ are called **open v-balls** and **closed v-balls**, respectively. The open v-balls form a basis for a field topology on K, called the **valuation topology on K**. Both open and closed v-balls are clopen with respect to this topology. If $\mathcal{O} \neq K$, then the valuation topology on K agrees with the order topology. If $\mathcal{O} = K$, then the valuation topology is the discrete topology.

If T is power bounded with field of exponents Λ, then the value group Γ naturally admits the structure of an ordered Λ-vector space by

$$\lambda v(a) \ := \ v(a^{\lambda})$$

for $a \in K^{>}$ (this does not depend on the choice of a).

The **residue field of** K is the quotient $\operatorname{res}(K) = \mathcal{O}/\smallo$. For $a \in \mathcal{O}$, we let $\bar{a} := a + \smallo$ denote the image of a under the quotient map $\mathcal{O} \to \operatorname{res}(K)$. Under this map, $\operatorname{res}(K)$ admits a natural expansion to a T-model; see [**29**, 2.16] for details. A **lift of** $\operatorname{res}(K)$ is an elementary $\mathcal{L}$-substructure $\boldsymbol{k}$ of K contained in $\mathcal{O}$ such that the map $a \mapsto \bar{a} \colon \boldsymbol{k} \to \operatorname{res}(K)$ is an $\mathcal{L}$-isomorphism. By [**29**, 2.12], we can always find a lift of $\operatorname{res}(K)$. This gives a converse to Fact 5.1 every T-convex valuation ring of K is the convex hull of some elementary $\mathcal{L}$-substructure of K (namely, a lift of the residue field). For $a = (a_1, \ldots, a_n) \in \mathcal{O}^n$, we let $\bar{a} := (\bar{a}_1, \ldots, \bar{a}_n) \in \operatorname{res}(K)^n$. For $D \subseteq K^n$, we let

$$\overline{D} \ := \ \{\bar{a} : a \in D \cap \mathcal{O}^n\} \ \subseteq \ \operatorname{res}(K)^n.$$

Fact 5.2 ([**24**], 1.10)**.** *If $D \subseteq K^n$ is $\mathcal{L}(K)$-definable, then $\overline{D}$ is $\mathcal{L}(\operatorname{res} K)$-definable and*

$$\dim_{\mathcal{L}} \overline{D} \ \leqslant \ \dim_{\mathcal{L}} D.$$

Let M be a $T^{\mathcal{O}}$-extension of K with T-convex valuation ring $\mathcal{O}_M$ and maximal ideal $\smallo_M$. We view Γ as a subgroup of Γ_M and $\operatorname{res}(K)$ as an $\mathcal{L}$-substructure of $\operatorname{res}(M)$ in the obvious way. We let v and $x \mapsto \bar{x}$ denote their extensions to functions $M^{\times} \to \Gamma_M$ and $\mathcal{O}_M \to \operatorname{res}(M)$. If $\mathcal{O} \neq K$, then $\mathcal{O}_M \neq M$ and M is an elementary $T^{\mathcal{O}}$-extension of K by [**29**, 3.13]. If $\mathcal{O}_M = M$, then $\mathcal{O} = K$ so M is again an elementary $T^{\mathcal{O}}$-extension of K.

Fact 5.3 ([**29**], Section 3)**.** *Let $K\langle a \rangle$ be a simple T-extension of K. There are at most two T-convex valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ of $K\langle a \rangle$ which make $K\langle a \rangle$ a $T^{\mathcal{O}}$-extension of K:*

$$\mathcal{O}_1 \ := \ \big\{y \in K\langle a \rangle : |y| < u \text{ for some } u \in \mathcal{O}\big\}, \qquad \mathcal{O}_2 \ := \ \big\{y \in K\langle a \rangle : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

If there is $b \in K\langle a \rangle$ which realizes the cut $\mathcal{O}^{\downarrow}$, then b is contained in $\mathcal{O}_2$ but not $\mathcal{O}_1$, so $\mathcal{O}_1 \subsetneq \mathcal{O}_2$. If there is no such b, then $\mathcal{O}_1 = \mathcal{O}_2$.

5.1. Immediate extensions

In this section, let M be a $T^{\mathcal{O}}$-extension of K. If $\Gamma_M = \Gamma$ and $\operatorname{res}(M) = \operatorname{res}(K)$, then M is said to be an **immediate extension of** K. If M is an immediate extension of K, then M is an elementary $T^{\mathcal{O}}$-extension of K. Note that M is an immediate extension of K if and only if for all $a \in M^{\times}$ there is $b \in K^{\times}$ with $a \sim b$. The next lemma shows that in an immediate extension of K, we can approximate $\mathcal{L}^{\mathcal{O}}(K)$-definable sets by $\mathcal{L}(K)$-definable sets.

Lemma 5.4. *Suppose M is an immediate extension of K, let $A \subseteq K^n$ be an $\mathcal{L}^{\mathcal{O}}(K)$-definable set, and let $a \in A^M$. Then there is an $\mathcal{L}(K)$-definable cell $D \subseteq A$ with $a \in D^M$.*

Proof. It suffices to find an $\mathcal{L}(K)$-definable set $B \subseteq A$ with $a \in B^M$, for then we can replace B with a subcell D in a cell decomposition of B. If K is trivially valued, then A is already $\mathcal{L}(K)$-definable, so we may assume $\mathcal{O} \neq K$. By extending $\mathcal{L}$ with function symbols for all $\mathcal{L}(\emptyset)$-definable functions, we may assume that T has quantifier elimination and a universal axiomatization; see Corollary 2.5. Let T^* be the extension of $T^{\mathcal{O}}$ by an axiom asserting that $\mathcal{O} \neq K$. Then T^* eliminates quantifiers by **[29]**, so

$$A \;=\; \bigcup_{i \leqslant m} \bigcap_{j \leqslant k} A_{i,j}$$

where either $A_{i,j}$ is $\mathcal{L}(K)$-definable or

$$A_{i,j} \;=\; \big\{y \in K^n : F(y) \in \mathcal{O}\big\} \quad \text{or} \quad A_{i,j} \;=\; \big\{y \in K^n : F(y) \notin \mathcal{O}\big\}$$

for some $\mathcal{L}(K)$-definable function $F\colon K^n \to K$. For each $i \leqslant m$ and each $j \leqslant k$, we take an $\mathcal{L}(K)$-definable set $B_{i,j} \subseteq A_{i,j}$ such that if $a \in A_{i,j}^M$, then $a \in B_{i,j}^M$. We do this as follows.

(i) If $A_{i,j}$ is $\mathcal{L}(K)$-definable, then we set $B_{i,j} := A_{i,j}$.

(ii) Suppose $A_{i,j} = \big\{y \in K^n : F(y) \in \mathcal{O}\big\}$ for some $\mathcal{L}(K)$-definable F. If $F(a) \notin \mathcal{O}_M$, then we set $B_{i,j} := \emptyset$. If $F(a) \in \mathcal{O}_M$, then since $\operatorname{res}(M) = \operatorname{res}(K)$, we may take $u \in K^>$ with $u \asymp 1$ and $|F(a)| < u$. We set

$$B_{i,j} \;:=\; \big\{y \in K^n : |F(y)| < u\big\}.$$

(iii) Suppose $A_{i,j} = \big\{y \in K^n : F(y) \notin \mathcal{O}\big\}$ for some $\mathcal{L}(K)$-definable F. If $F(a) \in \mathcal{O}_M$, then we set $B_{i,j} := \emptyset$. If $F(a) \notin \mathcal{O}_M$, then since $\Gamma_M = \Gamma$, we may take $d \in K^>$ with $d \succ 1$ and $|F(a)| > d$. We set

$$B_{i,j} \;:=\; \big\{y \in K^n : |F(y)| > d\big\}.$$

Now set

$$B \;:=\; \bigcup_{i \leqslant m} \bigcap_{j \leqslant k} B_{i,j}. \qquad \square$$

Corollary 5.5. *Suppose M is an immediate extension of K, let $F\colon A \to K$ be an $\mathcal{L}^{\mathcal{O}}(K)$-definable function, and let $a \in A^M$. Then there is an $\mathcal{L}(K)$-definable cell $D \subseteq A$ with $a \in D^M$ such that either $F(y) = 0$ for all $y \in D^M$ or $F(y) \sim F(a)$ for all $y \in D^M$.*

Proof. If $F(a) = 0$, then apply Lemma 5.4 to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set $\big\{y \in A : F(y) = 0\big\}$. If $F(a) \neq 0$, then take $b \in K^\times$ with $F(a) \sim b$ and apply Lemma 5.4 to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set $\big\{y \in A : F(y) \sim b\big\}$. $\square$

Let ℓ be an element in an immediate $T^{\mathcal{O}}$-extension of K. Then the set

$$v(\ell - K) \;:=\; \big\{v(\ell - y) : y \in K\big\}$$

is contained in Γ and has no largest element. To see this, let $y \in K$ be given and take $b \in K$ with $\ell - y \sim b$. Then $v(\ell - y - b) > v(\ell - y) = v(b) \in \Gamma$. These values $v(\ell - y)$ completely determine the extension $K\langle \ell \rangle$ up to $\mathcal{L}^{\mathcal{O}}(K)$-isomorphism:

Corollary 5.6. *Let $K\langle \ell \rangle$ be a simple immediate $T^{\mathcal{O}}$-extension of K and let $a \in M$ with $v(a-y) = v(\ell-y) \in \Gamma$ for each $y \in K$. Then there is a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle \ell \rangle \to M$ sending ℓ to a.*

Proof. First, we will show that a and ℓ realize the same cut in K. Let $y \in K$ with $y < \ell$ and take $f \in K^>$ with $\ell - y \sim f$. Then $\ell - y - f \prec f$, so $a - y - f \asymp \ell - y - f \prec f$. Thus $a - y \sim f > 0$. Likewise, if $y \in K$ and $y > \ell$, then $y > a$. This gives us a unique $\mathcal{L}(K)$-embedding $\iota\colon K\langle \ell \rangle \to M$ sending ℓ to a. To get that ι is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding, let $F\colon K \to K$ be $\mathcal{L}(K)$-definable. We need to show that $F(\ell) \in \mathcal{O}_{K\langle \ell \rangle}$ if and only if $F(a) \in \mathcal{O}_M$. We assume that $F(\ell) \neq 0$, and we will show that $F(\ell) \sim F(a)$. Using Corollary 5.5, take an interval $I \subseteq K$ with $\ell \in I^{K\langle \ell \rangle}$ such that $F(y) \sim F(\ell)$ for all $y \in I^{K\langle \ell \rangle}$. Since K has a proper immediate extension, it is not trivially valued, so M is an *elementary* $T^{\mathcal{O}}$-extension of K. Thus, $F(a) \sim F(y) \sim F(\ell)$, since $a \in I^M$. □

If a is an element in a simple immediate $T^{\mathcal{O}}$-extension $K\langle \ell \rangle$ of K, the set $v(a-K)$ can be expressed as a translate of $v(\ell - K)$ as follows:

Lemma 5.7. *Let $K\langle \ell \rangle$ be a simple immediate $T^{\mathcal{O}}$-extension of K and let $a \in K\langle \ell \rangle \setminus K$. Then*

$$v(a-K) \;=\; \gamma + v(\ell - K)$$

for some $\gamma \in \Gamma$.

Proof. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(\ell) = a$. Since $a \notin K$, we have $F'(\ell) \neq 0$. Using Lemma 5.4 and Corollary 5.5, take an open interval $I \subseteq K$ with $\ell \in I^{K\langle \ell \rangle}$ such that F is $\mathcal{C}^1$ on I and such that $F'(y) \sim F'(\ell)$ for all $y \in I^{K\langle \ell \rangle}$. Set $\gamma := vF'(\ell) \in \Gamma$ and let $u \in I$. By the o-minimal mean value theorem, we have

$$F(\ell) - F(u) \;=\; F'(b)(\ell - u)$$

for some $b \in K\langle \ell \rangle$ between ℓ and u. Then $vF'(b) = \gamma$ since $b \in I^{K\langle \ell \rangle}$, so

$$v\big(a - F(u)\big) \;=\; v\big(F(\ell) - F(u)\big) \;=\; \gamma + v(\ell - u).$$

The set $\big\{v(\ell - u) : u \in I\big\}$ is cofinal in $v(\ell - K)$ and, since $F(I)$ contains an interval around a, the set $\big\{v\big(a - F(u)\big) : u \in I\big\}$ is cofinal in $v(a - K)$. This gives $v(a-K) = \gamma + v(\ell - K)$, as desired. □

Spherical completeness and pseudocauchy sequences. Let $\mathcal{B}$ be a collection of closed v-balls in K. Then $\mathcal{B}$ is said to be **nested** if $B_1 \cap B_2 \neq \emptyset$ for any $B_1, B_2 \in \mathcal{B}$. If $\mathcal{B}$ is nested, then it is totally ordered by inclusion. If every nested collection of closed v-balls in K has nonempty intersection in K, then K is said to be **spherically complete**. If M is an elementary $T^{\mathcal{O}}$-extension of K, then we let $\mathcal{B}^M$ denote the collection $\{B^M : B \in \mathcal{B}\}$. Below we list some facts about spherical completeness. These are all standard facts from valuation theory, but we include brief proofs.

Lemma 5.8.

1. *Let $\mathcal{B}$ be a nested collection of closed v-balls in K with empty intersection. Then K has a simple elementary $T^{\mathcal{O}}$-extension $K\langle a \rangle$ with $a \in \bigcap \mathcal{B}^{K\langle a \rangle}$.*
2. *Let $\mathcal{B}$ and a be as in (1). Then the set $v(a-K)$ has no largest element and for $y \in K$, the value $v(a-y)$ does not depend on the choice of a, just on the assumption $a \in \bigcap \mathcal{B}^{K\langle a \rangle}$.*
3. *Suppose M is an immediate extension of K and let $a \in M \setminus K$. Then the collection of all closed v-balls B in K with $a \in B^M$ is nested and has empty intersection in K.*
4. *If K is spherically complete, then K has no proper immediate $T^{\mathcal{O}}$-extensions.*

Proof. For (1), the assumption that $\mathcal{B}$ is nested gives that $\bigcap \mathcal{B}_0$ is nonempty for any finite subcollection $\mathcal{B}_0 \subseteq \mathcal{B}$. By model theoretic compactness, we can find a in an elementary $T^{\mathcal{O}}$-extension of K with $a \in \bigcap \mathcal{B}^{K\langle a \rangle}$. For (2), let $y \in K$, take a v-ball $B \in \mathcal{B}$ which doesn't contain y, and let $b \in B$. Then $v(a-b) > v(a-y)$, so $v(a-K)$ has no largest element. We also have $v(a-y) = v(b-y)$, and $v(b-y)$ clearly does not depend on a. For (3), let B_1, B_2 be closed v-balls with $a \in B_1^M \cap B_2^M$. As M is an elementary $T^{\mathcal{O}}$-extension of K and B_1, B_2 are $\mathcal{L}^{\mathcal{O}}(K)$-definable, there is $b \in K$ with $b \in B_1 \cap B_2$. This shows that the collection of all closed

v-balls B in K with $a \in B^M$ is nested. To see that this collection has empty intersection in K, let $b \in K$ be given and take $d \in K$ with $v(a-d) > v(a-b)$. Let

$$\gamma \ := \ v(a-d), \qquad B \ = \ \{y \in K : v(y-d) \geqslant \gamma\}.$$

Then $a \in B^M$ but $b \notin B$. Finally, (4) follows immediately from (3). □

Instead of working with nested collections of closed v-balls, it is sometimes more useful to work with sequences of elements from K. A **pseudocauchy sequence (pc-sequence) in** K is a well-indexed sequence (a_ρ) in K such that

$$a_\tau - a_\sigma \ \prec \ a_\sigma - a_\rho$$

for all $\tau > \sigma > \rho$ greater than some index ρ_0. Let (a_ρ) be a pc-sequence. An element a in a $T^{\mathcal{O}}$-extension of K is said to be a **pseudolimit of** (a_ρ) if for some index ρ_0, we have

$$a - a_\sigma \ \prec \ a - a_\rho$$

for all $\sigma > \rho > \rho_0$. In this case, we say that (a_ρ) **pseudoconverges to** a and we write $a_\rho \rightsquigarrow a$. The pc-sequence (a_ρ) is said to be **divergent** if it has no pseudolimit in K. Suppose (a_ρ) is divergent with pseudolimit a in some $T^{\mathcal{O}}$-extension of K. Given $y \in K$, we have $a - a_\rho \prec a - y$ for all sufficiently large ρ; otherwise, we would have $a_\rho \rightsquigarrow y$. If (b_σ) is another pc-sequence in K, then (b_σ) is said to be *equivalent* to (a_ρ) if (a_ρ) and (b_σ) have the same pseudolimits in every $T^{\mathcal{O}}$-extension of K.

By removing some initial terms of the pc-sequence (a_ρ), we arrange that $a_\tau - a_\sigma \prec a_\sigma - a_\rho$ for all $\tau > \sigma > \rho$. We may then associate to (a_ρ) a family of closed v-balls (B_ρ) as follows: let

$$\gamma_\rho \ := \ v(a_{\rho+1} - a_\rho), \qquad B_\rho \ = \ \{y \in K : v(y - a_\rho) \geqslant \gamma_\rho\}. \tag{5.1}$$

The family (B_ρ) is nested: if $\sigma \geqslant \rho$, then $a_\sigma \in B_\sigma \cap B_\rho$. Moreover, for $a \in M$, we have $a \in \bigcap_\rho B_\rho^M$ if and only if $a_\rho \rightsquigarrow a$. Thus, (a_ρ) is divergent if and only if $\bigcap_\rho B_\rho = \emptyset$, so if K is spherically complete, then every pc-sequence in K has a pseudolimit in K. The converse also holds; see [**4**, 2.2.10]. This correspondence, when paired with (1) of Lemma 5.8, also tells us that every divergent pc-sequence pseudoconverges in an elementary $T^{\mathcal{O}}$-extension of K.

Immediate extensions of power bounded models. If T is power bounded, then we have a better understanding of the immediate $T^{\mathcal{O}}$-extensions of K. This is due to the following result of Tyne:

Fact 5.9 ([**67**], 12.10 and 13.4)**.** *Let T be power bounded and let $K\langle \ell \rangle$ be a simple $T^{\mathcal{O}}$-extension of K.*

1. *If $\Gamma_{K\langle \ell \rangle} \neq \Gamma$, then there is $a \in K$ with $v(\ell - a) \notin \Gamma$.*
2. *If $\operatorname{res} K\langle \ell \rangle \neq \operatorname{res} K$, then there are $a, b \in K$ with $b(\ell - a) \preccurlyeq 1$ and $\overline{b(\ell - a)} \notin \operatorname{res} K$.*

Item (1) above is often referred to as the **valuation property**. Item (2) is called the **residue property**. These properties allow us to characterize the simple immediate $T^{\mathcal{O}}$-extensions of K as follows:

Lemma 5.10. *Let $K\langle \ell \rangle$ be a simple $T^{\mathcal{O}}$-extension of K. The following are equivalent:*

1. *$v(\ell - K)$ has no largest element;*
2. *there is a divergent pc-sequence in K which pseudoconverges to ℓ;*
3. *for each $a \in K$ there is $d \in K$ with $\ell - a \sim d$.*

If in addition, T is power bounded, then the above three conditions are all equivalent to:

(4) $K\langle\ell\rangle$ *is an immediate extension of* K.

PROOF. Assume (1) holds and let (a_ρ) be a well-indexed sequence in K such that $v(\ell - a_\rho)$ is strictly increasing and cofinal in $v(\ell - K)$. One easily verifies that (a_ρ) is a divergent pc-sequence in K which pseudoconverges to ℓ. Now, assume (2) holds and let (a_ρ) be a pc-sequence witnessing this. Given $a \in K$, take ρ with $\ell - a_\rho \prec \ell - a$. Then $\ell - a \sim a_\rho - a \in K$. Finally, suppose (3) holds, let $a \in K$, and take $d \in K$ with $\ell - a \sim d$. Then $v(\ell - a - d) > v(\ell - a)$, so $v(\ell - K)$ has no largest element.

Clearly, (4) implies (3), even without the assumption of power boundedness. Suppose now that T is power bounded and that (3) holds. Then $v(\ell - a) \in \Gamma$ for each $a \in K$, so $\Gamma_{K\langle\ell\rangle} = \Gamma$ by the valuation property. Let $a, b \in K$ with $b(\ell - a) \preccurlyeq 1$ and take $d \in K$ with $\ell - a \sim d$. If $b = 0$, then $\overline{b(\ell - a)} = 0 \in \operatorname{res}(K)$ and if $b \neq 0$, then $b(\ell - a) \sim bd$ so $\overline{b(\ell - a)} = \overline{bd} \in \operatorname{res}(K)$. Thus, $\operatorname{res} K\langle\ell\rangle = \operatorname{res} K$ by the residue property. □

We can use this equivalence together with Lemma 5.8 to show that if T is power bounded, then any nested collection of closed v-balls has nonempty intersection in an *immediate* $T^{\mathcal{O}}$-extension of K.

Corollary 5.11. *Let T be power bounded and let $\mathcal{B}$ be a nested collection of closed v-balls in K with empty intersection. Then there is a simple immediate $T^{\mathcal{O}}$-extension $K\langle\ell\rangle$ of K with $\ell \in \bigcap \mathcal{B}^{K\langle\ell\rangle}$. Given $a \in \bigcap \mathcal{B}^M$, there is a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a.*

PROOF. Using (1) of Lemma 5.8, let $K\langle\ell\rangle$ be a simple elementary $T^{\mathcal{O}}$-extension of K with $\ell \in \bigcap \mathcal{B}^{K\langle\ell\rangle}$. By (2) of Lemma 5.8, the set $v(\ell - K)$ has no largest element, so $K\langle\ell\rangle$ is an immediate extension of K by Lemma 5.10. For $a \in \bigcap \mathcal{B}^M$, we have $v(\ell - y) = v(a - y)$ for all $y \in K$ by (2) of Lemma 5.8, so Corollary 5.6 gives us a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a. □

Corollary 5.12. *Let T be power bounded and let (a_ρ) be a divergent pc-sequence in K. Then there is an immediate $T^{\mathcal{O}}$-extension $K\langle\ell\rangle$ of K with $a_\rho \rightsquigarrow \ell$. If $a_\rho \rightsquigarrow a \in M$, then there is a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a.*

PROOF. Apply Corollary 5.11 to the family (B_ρ) given in (5.1). □

Corollary 5.13. *Suppose T is power bounded. Then K has a spherically complete immediate $T^{\mathcal{O}}$-extension which is unique up to $\mathcal{L}^{\mathcal{O}}(K)$-isomorphism.*

PROOF. If K is not itself spherically complete, then K has a proper immediate $T^{\mathcal{O}}$-extension by Corollary 5.11. It follows by Zorn's lemma that K has a spherically complete immediate $T^{\mathcal{O}}$-extension. For uniqueness, let L and M be two spherically complete immediate $T^{\mathcal{O}}$-extensions of K. We first show that there is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $L \to M$. For this, we assume $K \neq L$, and we let $\ell \in L \setminus K$. Let $\mathcal{B}$ be the collection of all closed v-balls B in K with $\ell \in B^L$. This collection is nested and has empty intersection in K by (3) of Lemma 5.8. Let $a \in \bigcap \mathcal{B}^M$ and, again using Corollary 5.11, take an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a. Continuing in this manner, we construct an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $L \to M$, and we identify L with an $\mathcal{L}^{\mathcal{O}}$-substructure of M via this embedding. Then M is an immediate extension of L, so $L = M$ by (4) of Lemma 5.8, as L is spherically complete. □

The assumption of power boundedness in Corollary 5.13 is necessary.

Remark 5.14. If T is not power bounded and $\mathcal{O} \neq K$, then K has no spherically complete $T^{\mathcal{O}}$-extension. To see this, we use Miller's dichotomy and a theorem of Kuhlmann, Kuhlmann, and Shelah. Suppose toward contradiction that K is itself spherically complete, $\mathcal{O} \neq K$, and T is not power bounded. By [**54**], K

admits an $\mathcal{L}(\emptyset)$-definable exponential function exp. Let $\boldsymbol{k} \subseteq \mathcal{O}$ be a lift of $\operatorname{res}(K)$, so $\exp|_{\boldsymbol{k}}$ is an exponential function on $\boldsymbol{k}$. Using [**4**, 3.3.32], we take a subgroup $\mathfrak{M} \subseteq K^{>}$ such that $v|_{\mathfrak{M}} \colon \mathfrak{M} \to \Gamma$ is a group isomorphism. Using [**4**, 3.3.42], we get an $\mathcal{L}^{\mathcal{O}}_{\mathrm{ring}}$-isomorphism from K to the Hahn field $\boldsymbol{k}[[\mathfrak{M}]]$ which is the identity on $\mathfrak{M}$ and $\boldsymbol{k}$. The exponential on K induces an exponential on $\boldsymbol{k}[[\mathfrak{M}]]$ which restricts to the exponential $\exp|_{\boldsymbol{k}}$ on $\boldsymbol{k}$, contradicting the main theorem in [**51**].

5.2. Coarsening and specialization

In this section, we set up notation and prove some basic lemmas about coarsening and specialization. For the remainder of this section, we assume that T is power bounded with field of exponents Λ, and we let Δ be a nontrivial convex Λ-subspace of Γ. We set $\dot{\Gamma} := \Gamma/\Delta$, and we let $\dot{v} \colon K^{\times} \to \dot{\Gamma}$ be the map $a \mapsto va + \Delta \in \dot{\Gamma}$. Then $\dot{v}$ is a Krull valuation on K with valuation ring and maximal ideal

$$\dot{\mathcal{O}} \ := \ \{y \in K : vy \geqslant \delta \text{ for some } \delta \in \Delta\}, \qquad \dot{o} \ := \ \{y \in K : vy > \Delta\}.$$

Lemma 5.15. *$\dot{\mathcal{O}}$ is a T-convex valuation ring of K.*

Proof. Let $F \colon K \to K$ be a continuous $\mathcal{L}(\emptyset)$-definable function and let $a \in \dot{\mathcal{O}}$. We need to show that $F(a) \in \dot{\mathcal{O}}$. If $va \geqslant 0$, then $a \in \mathcal{O}$, so $F(a) \in \mathcal{O} \subseteq \dot{\mathcal{O}}$. Suppose $va < 0$, so $va \in \Delta^{<}$. Take $\lambda \in \Lambda$ and $b \in K^{>}$ such that $|F(y)| < |y|^{\lambda}$ for all $y \in K$ with $|y| > b$. As F is $\mathcal{L}(\emptyset)$-definable, we may assume $\{b\}$ is $\mathcal{L}(\emptyset)$-definable as well, so $vb = 0$ and $|a| > b$. Then $|F(a)| < |a|^{\lambda}$, so $vF(a) \geqslant \lambda va \in \Delta$. □

We let K_{Δ} denote the $\mathcal{L}^{\mathcal{O}}$-structure $(K, \dot{\mathcal{O}})$, so $K_{\Delta} \models T^{\mathcal{O}}$ by the above lemma. We refer to K_{Δ} as the **Δ-coarsening of** K. The residue field $\operatorname{res}(K_{\Delta}) = \dot{\mathcal{O}}/\dot{o}$ is itself a $T^{\mathcal{O}}$-model with valuation $v \colon \operatorname{res}(K_{\Delta}) \to \Delta$ given by

$$v(a + \dot{o}) \ := \ va$$

for $a \in \dot{\mathcal{O}} \setminus \dot{o}$ and with T-convex valuation ring $\mathcal{O}_{\operatorname{res}(K_{\Delta})} = \{a + \dot{o} : a \in \mathcal{O}\}$. We refer to $\operatorname{res}(K_{\Delta})$ with this valuation ring as the **Δ-specialization of** K. Note that $\mathcal{O}_{\operatorname{res}(K_{\Delta})}/o_{\operatorname{res}(K_{\Delta})}$ is naturally $\mathcal{L}$-isomorphic to $\operatorname{res}(K)$.

Fact 5.16 ([**4**], 3.4.6)**.** *K is spherically complete if and only if K_{Δ} and $\operatorname{res}(K_{\Delta})$ are both spherically complete.*

Let M be a $T^{\mathcal{O}}$-extension of K_{Δ} with $\Gamma_M = \dot{\Gamma}$. Let $\mathcal{O}_{\operatorname{res}(M)}$ be a T-convex valuation ring of $\operatorname{res}(M)$ and suppose that the expansion of $\operatorname{res}(M)$ by $\mathcal{O}_{\operatorname{res}(M)}$ is a $T^{\mathcal{O}}$-extension of $\operatorname{res}(K_{\Delta})$ with $\Gamma_{\operatorname{res}(M)} = \Delta$. Let $\mathcal{O}^*_M \subseteq M$ be the convex subring

$$\mathcal{O}^*_M \ := \ \{a \in M : a \in \mathcal{O}_M \text{ and } \bar{a} \in \mathcal{O}_{\operatorname{res}(M)}\}.$$

Lemma 5.17. *$\mathcal{O}^*_M$ is a T-convex valuation ring of M and $\mathcal{O}^*_M \cap K = \mathcal{O}$.*

Proof. Let $a \in \mathcal{O}^*_M$ and let $F \colon M \to M$ be an $\mathcal{L}(\emptyset)$-definable continuous function. Since $\bar{a} \in \mathcal{O}_{\operatorname{res}(M)}$ and $\mathcal{O}_{\operatorname{res}(M)}$ is T-convex, we have $\overline{F(a)} = F(\bar{a}) \in \mathcal{O}_{\operatorname{res}(M)}$ by [**29**, 1.13]. Thus, $F(a) \in \mathcal{O}^*_M$, so $\mathcal{O}^*_M$ is T-convex. The equality $\mathcal{O}^*_M \cap K = \mathcal{O}$ follows from the equivalence

$$y \in \mathcal{O} \iff y \in \dot{\mathcal{O}} \text{ and } y + \dot{o} \in \mathcal{O}_{\operatorname{res}(K_{\Delta})}$$

for $y \in K$. □

Let M^* be the $T^{\mathcal{O}}$-model with underlying T-model M and T-convex valuation ring $\mathcal{O}_{M^*} = \mathcal{O}^*_M$, as defined above. Then M^* is a $T^{\mathcal{O}}$-extension of K, and we have $M^*_{\Delta} = M$ (as $T^{\mathcal{O}}$-models).

Lemma 5.18. $\Gamma_{M^*} = \Gamma$ *and* $\operatorname{res}(M^*)$ *is naturally* $\mathcal{L}$*-isomorphic to* $\mathcal{O}_{\operatorname{res}(M)}/\smallo_{\operatorname{res}(M)}$.

Proof. As M^* is a $T^{\mathcal{O}}$-extension of K, we have $\Gamma \subseteq \Gamma_{M^*}$. For the other inclusion, let $a \in (M^*)^\times$. We need to find $b \in K^\times$ with $ab^{-1} \in \mathcal{O}_{M^*}^\times$. First, take $f \in K_\Delta^\times$ with $af^{-1} \in \mathcal{O}_M^\times$ and set $u := \overline{af^{-1}} \in \operatorname{res}(M)^\times$. Next, take $g \in \dot{\mathcal{O}}^\times$ with $u\bar{g}^{-1} \in \mathcal{O}_{\operatorname{res}(M)}^\times$. Then for $b := fg$, we have $ab^{-1} \in \mathcal{O}_{M^*}^\times$, as desired. As for the residue field, note that $\{\bar{a} : a \in \mathcal{O}_{M^*}\} = \mathcal{O}_{\operatorname{res}(M)}$, so we have a surjection

$$a \mapsto \bar{a} + \smallo_{\operatorname{res}(M)} : \mathcal{O}_{M^*} \to \mathcal{O}_{\operatorname{res}(M)}/\smallo_{\operatorname{res}(M)}$$

with kernel

$$\{a \in \mathcal{O}_{M^*} : \bar{a} \in \smallo_{\operatorname{res}(M)}\} \;=\; \smallo_{M^*}.$$

This induces a natural isomorphism $\operatorname{res}(M^*) \to \mathcal{O}_{\operatorname{res}(M)}/\smallo_{\operatorname{res}(M)}$, as required. $\square$

By the above lemma, we see that M^* is an immediate extension of K if and only if $\operatorname{res}(M)$ is an immediate extension of $\operatorname{res}(K_\Delta)$. We summarize the discussion above with a diagram:

$$\begin{array}{ccc}
K_\Delta & \xrightarrow[\mathcal{L}^{\mathcal{O}}]{} & M = M_\Delta^* \\
\downarrow & & \downarrow \\
\operatorname{res}(K_\Delta) & \xrightarrow[\mathcal{L}^{\mathcal{O}}]{} & \operatorname{res}(M) \\
\downarrow & & \downarrow \\
\operatorname{res}(K) & \xrightarrow[\mathcal{L}]{} & \operatorname{res}(M^*)
\end{array}$$

Horizontal arrows are all embeddings in the indicated language. Downward arrows are projections and are only defined on the T-convex valuation ring of their source. Every square commutes.

5.3. The Wilkie inequality

In this section, we assume that T is power bounded with field of exponents Λ. Many classical results about valued fields and their extensions have useful analogs which hold in the power bounded setting. We list one such result below.

Fact 5.19 ([**24**], Section 5)**.** *Let* M *be a* $T^{\mathcal{O}}$*-extension of* K *and suppose* $\operatorname{rk}_{\mathcal{L}}(M|K)$ *is finite. Then*

$$\operatorname{rk}_{\mathcal{L}}(M|K) \geqslant \operatorname{rk}_{\mathcal{L}}(\operatorname{res} M|\operatorname{res} K) + \dim_\Lambda(\Gamma_M/\Gamma).$$

Fact 5.19 is an analog of the Abhyankar-Zariski inequality, and it is referred to in the literature as the **Wilkie inequality**. We most frequently use the Wilkie inequality when M is a simple extension of K. Here is a consequence of the Wilkie inequality:

Lemma 5.20. *Let* S *be a cut in* Γ*. Then there is a simple* $T^{\mathcal{O}}$*-extension* $K\langle f\rangle$ *of* K *where* $f > 0$ *and where* vf *realizes the cut* S*. This extension is unique up to* $\mathcal{L}^{\mathcal{O}}(K)$*-isomorphism and is completely described as follows:* f *realizes the cut*

$$\{y \in K : y \leqslant 0 \text{ or } vy > S\}$$

and $\mathcal{O}_{K\langle f\rangle}$ *is the convex hull of* $\mathcal{O}$ *in* $K\langle f\rangle$*.*

Proof. Let $K\langle f\rangle$ be a simple extension of K where f realizes the cut

$$\{y \in K : y \leqslant 0 \text{ or } vy > S\}.$$

Let $\mathcal{O}_{K\langle f\rangle}$ be the convex hull of $\mathcal{O}$ in $K\langle f\rangle$, so $\mathcal{O}_{K\langle f\rangle}$ is indeed a T-convex valuation ring of K by Fact 5.3. One easily verifies that vf realizes the cut S, where v is the valuation induced by $\mathcal{O}_{K\langle f\rangle}$, so it remains to show uniqueness. Let $\mathcal{O}^*$ be another T-convex valuation ring of $K\langle f\rangle$ with $\mathcal{O}^* \cap K = \mathcal{O}$. If $\mathcal{O}^* \neq \mathcal{O}_{K\langle f\rangle}$, then by Fact 5.3, there is $g \in \mathcal{O}^*$ with $g > \mathcal{O}$. Then the residue field of $K\langle f\rangle$ with respect to $\mathcal{O}^*$ is strictly bigger than $\operatorname{res}(K)$, as it contains the image of g. By the Wilkie inequality, the value group of $K\langle f\rangle$ with respect to $\mathcal{O}^*$ is equal to Γ, so $v^*(f) \in \Gamma$, where v^* is the valuation induced by $\mathcal{O}^*$. Thus $v^*(f)$ doesn't realize the cut S. □

In Proposition 5.23 below, we use the Wilkie inequality to bound the derivative of a unary $\mathcal{L}(K)$-definable function. This proposition will be used a number of times in this thesis. First, we need two lemmas.

Lemma 5.21. *Let $M = K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \succ 1$ and $va \notin \Gamma$. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \preccurlyeq 1$. Then $F'(a) \prec a^{-1}$.*

Proof. By replacing a with $-a$, we may assume that $a > 0$. The Wilkie inequality gives that $\operatorname{res}(M) = \operatorname{res}(K)$, so we may take $u \in \mathcal{O}^\times$ with $F(a) - u \prec 1$. Replacing F with $F - u$, we may assume that $F(a) \prec 1$. Note that this does not change F'.

We first handle the case that $\mathcal{O} = K$, so $a > K$. Phrased in terms of limits, we have

$$\lim_{x\to\infty} |F(x)| \;=\; 0,$$

and we want to show that

$$\lim_{x\to\infty} x|F'(x)| \;=\; 0.$$

Let $\delta, g \in K^>$ be given. We need to find $d > g$ with $d|F'(d)| < \delta$. By increasing g, we may assume that $|F(g)| < \delta/4$, that F is continuous and $|F|$ is decreasing on $[g, +\infty)$, and that F is $\mathcal{C}^1$ on $(g, +\infty)$. The o-minimal mean value theorem gives $d \in (g, 2g)$ with

$$|F'(d)| \;=\; \left|\frac{F(2g) - F(g)}{g}\right| \;\leqslant\; \frac{2|F(g)|}{g}.$$

Since $d < 2g$ and $|F(g)| < \delta/4$ we have

$$d|F'(d)| \;\leqslant\; d\frac{2|F(g)|}{g} \;<\; 4|F(g)| \;<\; \delta.$$

Now suppose $\mathcal{O} \neq K$. Then M is an elementary $T^{\mathcal{O}}$-extension of K, so it suffices to show that for any $\mathcal{L}^{\mathcal{O}}(K)$-definable set $A \subseteq K^>$ with $a \in A^M$, there is $y \in A$ with $F'(y) \prec y^{-1}$. Let A be such a set. By shrinking A, we arrange that A is convex, that F is $\mathcal{C}^1$ on A, and that $F(y) \prec 1$ for all $y \in A$. Since $va \notin \Gamma$ and Γ is densely ordered, the set A contains elements $y_1 \prec y_2$. The o-minimal mean value theorem gives

$$F'(y) \;=\; \frac{F(y_2) - F(y_1)}{y_2 - y_1}.$$

for some $y \in A$ between y_1 and y_2. Since $F(y_2) - F(y_1) \prec 1$ and $y_2 - y_1 \asymp y_2 \succcurlyeq y$, we have $F'(y) \prec y^{-1}$, as desired. □

Lemma 5.22. *Let $M = K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \asymp 1$ and $\bar{a} \notin \operatorname{res}(K)$. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \preccurlyeq 1$. Then $F'(a) \preccurlyeq 1$.*

Proof. Let $\boldsymbol{k} \subseteq \mathcal{O}^\times$ be a lift of $\operatorname{res}(K)$, so $\boldsymbol{k}\langle a\rangle$ is a lift of $\operatorname{res}(M)$ by [**29**, 5.1]. The Wilkie inequality gives that $\Gamma_M = \Gamma$, so since $\Gamma^<$ has no largest element, it suffices to show that $|F'(a)| < d$ for each $d \in K^>$ with

$d \succ 1$. Let d be given. We will show that for any subinterval $I \subseteq K^>$ with $a \in I^M$, there is $y \in I$ with $|F'(y)| < d$. Let I be such an interval and take an $\mathcal{L}(\boldsymbol{k})$-definable function $G\colon K \to K$ with $|F(a)| < G(a)$. By shrinking I, we arrange that F is $\mathcal{C}^1$ on I and that $|F(y)| < G(y)$ for all $y \in I$. As $\bar{a} \in \bar{I}^{\mathrm{res}(M)}$, we see that $\bar{I}$ must be infinite, so $I \cap \boldsymbol{k}$ is infinite. Take $y_1 < y_2 \in I \cap \boldsymbol{k}$, so $y_2 - y_1 \asymp 1$. Note that $G(y_i) \in \boldsymbol{k}$, so $|F(y_i)| < G(y_i) \prec d$ for $i = 1, 2$. The o-minimal mean value theorem gives

$$F'(y) \;=\; \frac{F(y_2) - F(y_1)}{y_2 - y_1} \;\prec\; d$$

for some $y \in I$ between y_1 and y_2. In particular, $|F'(y)| < d$. □

Proposition 5.23. *Let $M = K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \not\sim f$ for all $f \in K$ and let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function. Then $F'(a) \preccurlyeq a^{-1}F(a)$.*

Proof. First, suppose $a \succ 1$ and $va \notin \Gamma$. The Wilkie inequality gives $\Gamma_M = \Gamma \oplus \Lambda va$, so take $d \in K^>$ and $\lambda \in \Lambda$ with $F(a) \asymp da^{\lambda}$. Then $d^{-1}a^{-\lambda}F(a) \preccurlyeq 1$ and, applying Lemma 5.21 to the function $y \mapsto d^{-1}y^{-\lambda}F(y)$, we get

$$d^{-1}a^{-\lambda}F'(a) - \lambda d^{-1}a^{-\lambda-1}F(a) \;\prec\; a^{-1}.$$

Since $-\lambda d^{-1}a^{-\lambda-1}F(a) \preccurlyeq a^{-1}$, we see that $d^{-1}a^{-\lambda}F'(a) \preccurlyeq a^{-1}$, so

$$F'(a) \;\preccurlyeq\; a^{-1}da^{\lambda} \;\asymp\; a^{-1}F(a).$$

Now, suppose $a \prec 1$ and $va \notin \Gamma$. Let $G\colon K \to K$ be the function given by

$$G(y) \;=\; \begin{cases} F(y^{-1}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Then $F(a) = G(a^{-1})$. By applying the previous case to G and $a^{-1} \succ 1$, we get

$$F'(a) \;=\; -G'(a^{-1})a^{-2} \;\preccurlyeq\; aG(a^{-1})a^{-2} \;=\; a^{-1}F(a).$$

Finally, suppose $va \in \Gamma$ and take $b \in K$ with $b \asymp a$, so $b^{-1}a \asymp 1$. Note that $\overline{b^{-1}a} \notin \mathrm{res}(K)$, for otherwise we would have $a \sim bu$ for some $u \in \mathcal{O}^{\times}$, contradicting our assumption on a. The Wilkie inequality gives $\Gamma_M = \Gamma$, so take $d \in K^>$ with $F(a) \asymp d$. Applying Lemma 5.22 with $b^{-1}a$ in place of a and with the function $y \mapsto d^{-1}F(by)$ in place of F, we see that

$$d^{-1}bF'(a) \;\preccurlyeq\; 1,$$

so $F'(a) \preccurlyeq b^{-1}d \asymp a^{-1}F(a)$. □

Note that our standing assumption of power boundedness is necessary for Proposition 5.23, as the proposition clearly fails when a is infinite and F is an exponential function with $F' = F$. Our assumption that $a \not\sim f$ for all $f \in K$ is also necessary. To see this, suppose $a \sim f \in K$ and let $F(Y) = Y - f$. Then $F(a) \prec a$ so $a^{-1}F(a) \prec 1$, but $F'(a) = 1$. Here is an application of Proposition 5.23 for use in the proof of Lemma 7.32.

Corollary 5.24. *Suppose $\mathcal{O} = K$, let $b \in K^n$ be an $\mathcal{L}(\emptyset)$-independent tuple, and let $K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \prec 1$. Let $G\colon K^{1+n} \to K$ be an $\mathcal{L}(\emptyset)$-definable function with $G(a, b) \prec 1$ and let $d = (d_0, \ldots, d_n) \in K^{1+n}$. Then $\mathbf{J}_G(a, b)d \prec a^{-1}$.*

Proof. Viewing G as a function of the variables $Y_0, \ldots, Y_n$, we have

$$\mathbf{J}_G(a, b)d \;=\; \frac{\partial G}{\partial Y_0}(a, b)d_0 + \frac{\partial G}{\partial Y_1}(a, b)d_1 + \cdots + \frac{\partial G}{\partial Y_n}(a, b)d_n.$$

Since $d_i \preccurlyeq 1$ for $i = 0, \ldots n$, it suffices to show that $\frac{\partial G}{\partial Y_i} \prec a^{-1}$ for each i. For $i = 0$, we apply Proposition 5.23 to the function $y \mapsto G(y, b)$ to get $\frac{\partial G}{\partial Y_0}(a, b) \preccurlyeq a^{-1}G(a, b) \prec a^{-1}$. For $i > 0$, we will again use Proposition 5.23, but doing so requires a bit of an argument. By symmetry, it suffices to show that $\frac{\partial G}{\partial Y_1}(a, b) \prec a^{-1}$. Let $E := \mathrm{dcl}(b_2, \ldots, b_n)$ and view E as an elementary $\mathcal{L}^{\mathcal{O}}$-substructure of K with $\mathcal{O}_E = \mathcal{O} \cap E = E$. Then $b_1 \notin E$, so $b_1 \not\sim f$ for any $f \in E$, since E is trivially valued. Viewing $E\langle a\rangle$ as a $T^{\mathcal{O}}$-extension of E with $va \notin \Gamma_E = \{0\}$, the Wilkie inequality gives $\operatorname{res} E\langle a\rangle = \operatorname{res} E$, so $b_1 \not\sim f$ for any $f \in E\langle a\rangle$. Thus, we may apply Proposition 5.23 with $E\langle a\rangle$ in place of K, with b_1 in place of a, and with the function $y \mapsto G(a, y, b_2, \ldots, b_n)$ in place of F to get $\frac{\partial G}{\partial Y_1}(a, b) \preccurlyeq b_1^{-1}G(a, b)$. Since $b_1^{-1} \in K$ and $G(a, b) \prec 1$, this gives $\frac{\partial G}{\partial Y_1}(a, b) \prec 1 \prec a^{-1}$. $\square$

CHAPTER 6

T-convex T-differential Fields

In this chapter, let $K \models T$, let $\mathcal{O}$ be a T-convex valuation ring of K with maximal ideal $\mathcal{o}$, and let ∂ be a T-derivation on K. If ∂ is continuous with respect to the valuation topology on K, then we call K a **T-convex T-differential field**. By [**4**, Lemma 4.4.7], ∂ is continuous if and only if $\partial\mathcal{o} \subseteq \phi\mathcal{o}$ for some $\phi \in K^\times$. If $\partial\mathcal{o} \subseteq \mathcal{o}$, then we say that K has **small derivation**.

Let $\mathcal{L}^{\mathcal{O},\partial} := \mathcal{L}^{\mathcal{O}} \cup \mathcal{L}^{\partial} = \mathcal{L} \cup \{\mathcal{O}, \partial\}$ and let $T^{\mathcal{O},\partial}$ be the $\mathcal{L}^{\mathcal{O},\partial}$-theory of T-convex T-differential fields. For the remainder of this chapter, let $K = (K, \mathcal{O}, \partial) \models T^{\mathcal{O},\partial}$. Following [**5**], we say that a $T^{\mathcal{O},\partial}$-extension M of K is **strict** if

$$\partial\mathcal{o} \subseteq \phi\mathcal{o} \implies \partial_M\mathcal{o}_M \subseteq \phi\mathcal{o}_M, \qquad \partial\mathcal{O} \subseteq \phi\mathcal{o} \implies \partial_M\mathcal{O}_M \subseteq \phi\mathcal{o}_M$$

for each $\phi \in K^\times$. Let Γ be the value group of K and consider the following subsets of Γ, introduced in [**5**]:

$$\Gamma_K(\partial) \ := \ \big\{v(\phi) : \phi \in K^\times \text{ and } \partial\mathcal{o} \subseteq \phi\mathcal{o}\big\}, \qquad S_K(\partial) \ := \ \big\{\gamma \in \Gamma : \Gamma(\partial) + \gamma = \Gamma(\partial)\big\}.$$

We omit the subscript and write $\Gamma(\partial)$ and $S(\partial)$ when K is clear from context. Note that $\Gamma(\partial) < v(\partial\mathcal{o})$ is a downward closed subset of Γ and that $S(\partial)$, the stabilizer of $\Gamma(\partial)$, is a convex subgroup of Γ. The purpose of this chapter is to prove the following:

Theorem 6.1. *Suppose that T is power bounded with field of exponents Λ and that $S(\partial)$ is a Λ-subspace of Γ. Then K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete.*

The condition that $S(\partial)$ is a Λ-subspace of Γ is equivalent to the condition that the set of stabilizing elements is closed under power functions given in Chapter 1, since $S(\partial) = \{vf : f \text{ is a stabilizing element}\}$. This condition is satisfied when T is polynomially bounded. It is also satisfied when K is H_T-asymptotic, a case we study in Chapter 7.

The assumption of power boundedness in Theorem 6.1 is necessary. If T is not power bounded and $\mathcal{O} \neq K$, then K has no T-convex extension which is spherically complete by Remark 5.14. We are unsure whether $S(\partial)$ is always a Λ-subspace of Γ, or whether this assumption is necessary. Note that if $\mathcal{O} = K$, then K is itself spherically complete. Moreover, if ∂ is trivial and T is power bounded, then K has a spherically complete immediate $T^{\mathcal{O}}$-extension M by Corollary 5.13. Viewed as a $T^{\mathcal{O},\partial}$-model with trivial derivation, M is a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension of K. Accordingly, we make the following assumption:

Assumption 6.2. *For the remainder of this chapter, T is power bounded with field of exponents Λ and the derivation and valuation on K are nontrivial.*

A consequence which we use freely is that any $T^{\mathcal{O}}$-extension of K is an elementary $T^{\mathcal{O}}$-extension.

In [**5**], Aschenbrenner, van den Dries, and van der Hoeven proved that every equicharacteristic zero valued field with a continuous derivation has an immediate strict valued differential field extension which is spherically complete. The case $T = \mathrm{RCF}$ of Theorem 6.1 is a special case of the main result in [**5**]. Our proof of Theorem 6.1 is similar in structure to the proof of this result. As in [**5**], we first handle the case $S(\partial) = \{0\}$, and then we reduce to this case using a coarsening argument. Our definition of the set $Z(K, \ell)$ is also quite similar to that in [**5**]. However, since we work with $\mathcal{L}(K)$-definable functions instead of differential polynomials over K, many of the key tools from [**5**] are not available to us. For example, the property of *having positive Newton degree*, which plays a starring role in [**5**], is here replaced with the property of *eventual smallness*.

In Section 6.1, we prove some basic lemmas about T-convex T-differential fields. In Section 6.2, we assign a valuation to definable functions in implicit form, which acts as an analog of the Gaussian valuation for polynomials. We use this to study the behavior of these functions. The aforementioned property of eventual smallness is introduced in Section 6.3. In Section 6.4, we introduce the sets $Z(K, \ell)$. These sets tell us when a cut in K can be filled by an element contained in the ∂-closure of K. In Section 6.5 we consider the special case $S(\partial) = \{0\}$, and in Section 6.6, we prove Theorem 6.1. First, let us start with some examples.

Examples of T-convex T-differential fields. As mentioned above, the following example fits comfortably within the framework of [**5**]:

Example 6.3 (Real closed valued fields)**.** Let $(R, \mathcal{O})$ be a real closed valued field. Then $\mathcal{O}$ is RCF-convex by [**29**, 4.2]. Let ∂ be a derivation on R which is continuous with respect to the valuation topology. By Lemma 3.9, ∂ is an RCF-derivation, so $(R, \mathcal{O}, \partial) \models \mathrm{RCF}^{\mathcal{O},\partial}$. Conversely, every model of $\mathrm{RCF}^{\mathcal{O},\partial}$ is a real closed valued field with a continuous derivation.

For examples which are new to our setting, we look to the models considered in Section 3.2. Let us begin with Hahn fields.

Example 6.4 (Hahn differential fields)**.** Let $(\boldsymbol{k}, \partial_{\boldsymbol{k}}) \models T_{\mathrm{an}}^{\partial}$, let $\mathfrak{M}$ be a divisible monomial group, and extend $\partial_{\boldsymbol{k}}$ to a derivation ∂ on $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ by setting $\partial\big(\sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m}\big) := \sum_{\mathfrak{m}} \partial_{\boldsymbol{k}}(f_{\mathfrak{m}})\mathfrak{m}$. Then ∂ is a T_{an}-derivation on $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$, as shown in Example 3.15. Let $\mathcal{O}$ be the convex hull of $\boldsymbol{k}$ in $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$, so $\mathcal{O}$ is T_{an}-convex by Fact 5.1. With respect to $\mathcal{O}$, the derivation ∂ is **monotone**, that is, $\partial f \preccurlyeq f$ for all $f \in \boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$. In particular, ∂ is small, so $\big(\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}, \mathcal{O}, \partial\big) \models T_{\mathrm{an}}^{\mathcal{O},\partial}$. This model is spherically complete and has *many constants*: for each $f \in \boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$, there is a constant c in the constant field $C_{\boldsymbol{k}}[[\mathfrak{M}]]_{\mathrm{an}}$ with $f \asymp c$.

In [**61**], Scanlon proved an Ax-Kochen-Eršov (AKE) result for Hahn differential fields $\boldsymbol{k}[[\mathfrak{M}]]$ where $\boldsymbol{k}$ is an (unordered) differential field of characteristic zero and where the derivation on $\boldsymbol{k}$ is extended to $\boldsymbol{k}[[\mathfrak{M}]]$ as in the example above. As a consequence, he showed that if $\mathfrak{M}$ is divisible and $\boldsymbol{k}$ is differentially closed, then the valued differential field $\boldsymbol{k}[[\mathfrak{M}]]$ is model complete. Scanlon's AKE result can also be used to show that if $\mathfrak{M}$ is divisible and $\boldsymbol{k}$ is a closed ordered differential field (or equivalently, a model of $\mathrm{RCF}_{\mathcal{G}}^{\partial}$; see Definition 4.14 and Proposition 4.15), then the *ordered* valued differential field $\boldsymbol{k}[[\mathfrak{M}]]$ is model complete in the language $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$. This raises the question: if $\boldsymbol{k} \models (T_{\mathrm{an}})_{\mathcal{G}}^{\partial}$ and $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}} \models T_{\mathrm{an}}^{\mathcal{O},\partial}$ is as in Example 6.4, is $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ model complete in the language $\mathcal{L}_{\mathrm{an}}^{\mathcal{O},\partial}$? As with the uniqueness questions discussed at the end of the chapter, answering this question likely requires some analog of *differential henselianity*. For a generalization of Scanlon's AKE result, see [**44**].

Now let us turn our attention to transseries and surreal numbers:

Example 6.5 (Transseries)**.** Recall from Example 3.17 that the field $\mathbb{T}$ of logarithmic-exponential transseries admits a natural expansion to model $\mathbb{T}_{\mathrm{an,exp}} \models T^{\partial}_{\mathrm{an,exp}}$. Let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in $\mathbb{T}$, so $\mathcal{O}$ is $T_{\mathrm{an,exp}}$-convex by Fact 5.1. Since ∂ is small by [**32**, 4.1], it is continuous, so we may view $\mathbb{T}_{\mathrm{an,exp}}$ as a model of $T^{\mathcal{O},\partial}_{\mathrm{an,exp}}$.

Example 6.6 (Surreal numbers)**.** As discussed in Example 3.18, we have a natural expansion of the field **No** of surreal numbers to a model $\mathbf{No}_{\mathrm{an,exp}} \models T^{\partial}_{\mathrm{an,exp}}$. As with $\mathbb{T}$, we let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in **No**, so $\mathcal{O}$ is $T_{\mathrm{an,exp}}$-convex. The Berarducci-Mantova derivation is small [**8**, 7.7], so $\mathbf{No}_{\mathrm{an,exp}}$ can also be viewed as a model of $T^{\mathcal{O},\partial}_{\mathrm{an,exp}}$.

6.1. Additional background

Here, we say a bit more about small derivation, coarsening, strict extensions, and the sets $\Gamma(\partial)$ and $S(\partial)$.

The differential residue field. The following fact demonstrates how $\partial\mathcal{O}$ is controlled by $\partial\smallo$.

Fact 6.7 ([**4**], 4.4.2)**.** *If K has small derivation, then $\partial\mathcal{O} \subseteq \mathcal{O}$. Consequently, $\partial\smallo \subseteq \phi\smallo \Longrightarrow \partial\mathcal{O} \subseteq \phi\mathcal{O}$ for each $\phi \in K^{\times}$.*

Suppose K has small derivation. By the above fact, ∂ induces a map $\bar{a} \mapsto \overline{\partial a}\colon \mathrm{res}(K) \to \mathrm{res}(K)$. We denote this map also by ∂, and we call it the **induced derivation on** $\mathrm{res}(K)$. This induced derivation is even a T-derivation. To see this, let F be an n-ary $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function with open domain and let φ be the $\mathcal{L}(\emptyset)$-formula defining the domain of F. Set

$$U \ :=\ \varphi(K)\ \subseteq\ K^n, \qquad V\ :=\ \varphi(\mathrm{res}\,K)\ \subseteq\ \mathrm{res}(K)^n,$$

so $V \subseteq \overline{U}$. Let F denote both its interpretation as a function $U \to K$ and its interpretation as a function $V \to \mathrm{res}(K)$ and let $u \in U$ with $\bar{u} \in V$. By [**29**, 1.13], we have $\overline{F(u)} = F(\bar{u})$, so

$$\partial F(\bar{u})\ =\ \partial\overline{F(u)}\ =\ \overline{\partial F(u)}\ =\ \overline{\mathbf{J}_F(u)\partial u}\ =\ \mathbf{J}_F(\bar{u})\partial\bar{u}.$$

Accordingly, we view $\mathrm{res}(K)$ as a model of T^{∂}. Note that the induced derivation on $\mathrm{res}(K)$ is trivial if and only if $\partial\mathcal{O} \subseteq \smallo$.

Coarsening and specialization. Let Δ be a nontrivial convex Λ-subspace of Γ. Recall the $T^{\mathcal{O}}$-models K_Δ and $\mathrm{res}(K_\Delta)$ associated to K and Δ from Section 5.2. We may view K_Δ as a $T^{\mathcal{O},\partial}$-model with the same derivation as K; the valuation topology induced by $\dot{\mathcal{O}}$ is either discrete or the same as the topology induced by $\mathcal{O}$, so ∂ is still continuous in K_Δ.

Suppose K has small derivation. Then K_Δ does as well by [**4**, 4.4.4], so we may consider the induced derivation on $\mathrm{res}(K_\Delta)$. This induced derivation is a T-derivation by the remarks above, and it is also small, hence continuous. Thus, we may view $\mathrm{res}(K_\Delta)$ as a $T^{\mathcal{O},\partial}$-model as well.

Strict extensions. Let M be a $T^{\mathcal{O},\partial}$-extension of K. Below we list some basic but important facts about strict extensions.

(1) M is a strict $T^{\mathcal{O},\partial}$-extension of K if and only if M^{ϕ} is a strict $T^{\mathcal{O},\partial}$-extension of K^{ϕ} for $\phi \in K^{\times}$.
(2) If M is a $T^{\mathcal{O},\partial}$-extension of K and M is contained in a strict $T^{\mathcal{O},\partial}$-extension of K, then M is itself a strict extension of K.

(3) If M is a strict $T^{\mathcal{O},\partial}$-extension of L and L is a strict $T^{\mathcal{O},\partial}$-extension of K, then M is a strict extension of K.
(4) If M is an elementary $T^{\mathcal{O},\partial}$-extension of K, then M is a strict extension of K.

We have a useful test for determining whether an immediate $T^{\mathcal{O},\partial}$-extension is strict:

Fact 6.8 ([**5**], 1.5)**.** *Suppose that M is an immediate $T^{\mathcal{O}}$-extension of K and let ∂_M be a T-derivation on M which extends ∂. If*

$$\partial\smallo \subseteq \phi\smallo \implies \partial_M\smallo_M \subseteq \phi\smallo_M$$

for each $\phi \in K^\times$, then M is a strict $T^{\mathcal{O},\partial}$-extension of K.

More on $\Gamma(\partial)$ and $S(\partial)$. For $\phi \in K^\times$, we have

$$\Gamma(\phi^{-1}\partial) \;=\; \Gamma(\partial) - v\phi, \qquad S(\phi^{-1}\partial) \;=\; S(\partial),$$

so $S(\partial)$ is invariant under compositional conjugation. If M is a strict $T^{\mathcal{O},\partial}$-extension of K with $\Gamma_M = \Gamma$, then $\Gamma_M(\partial) = \Gamma(\partial)$ and $S_M(\partial) = S(\partial)$. Using that K is real closed, we can show that $\Gamma(\partial)$ is closed in Γ (with respect to the order topology).

Lemma 6.9. *Let $\alpha \in \Gamma$. If $\alpha - \varepsilon \in \Gamma(\partial)$ for each $\varepsilon \in \Gamma^>$, then $\alpha \in \Gamma(\partial)$.*

Proof. Suppose $\alpha \notin \Gamma(\partial)$. Take $a \in \smallo$ with $a > 0$ and $v(a') \leqslant \alpha$. Then

$$v\big((a^{1/2})'\big) \;=\; v(a') - \frac{1}{2}va \;\leqslant\; \alpha - \frac{1}{2}va,$$

so $\alpha - \frac{1}{2}va \notin \Gamma(\partial)$. □

6.2. Valuations of definable functions

In this section, let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function in implicit form. We set $vF := v(\mathfrak{m}_F) \in \Gamma$, and we call vF the **valuation of** F. Lemmas 3.34 and 3.37 give

$$vF_{+a,\times d} \;=\; vF + vd, \qquad vF^{\phi} \;=\; vF + rv\phi$$

for $a \in K$ and $d, \phi \in K^\times$. The valuation of F acts as a sort of crude replacement for the Gaussian valuation associated to differential polynomials, used frequently throughout [**4**]. Below, we use vF to find points $u \in K$ where $F(\jmath^r_\partial u)$ is "not too small."

Lemma 6.10. *Suppose that K has small derivation and that the induced derivation on* $\mathrm{res}(K)$ *is nontrivial. Then*

$$vF(\jmath^r_\partial u) \;\leqslant\; vF$$

for some $u \in \mathcal{O}^\times$.

Proof. Fact 5.2 gives that $\overline{\mathrm{Gr}(I_F)} \subseteq \mathrm{res}(K)^{1+r}$ is $\mathcal{L}(\mathrm{res}\,K)$-definable and $\dim_{\mathcal{L}}\big(\overline{\mathrm{Gr}(I_F)}\big) \leqslant r$. Thus, the set

$$\big\{y \in \mathrm{res}(K) : \jmath^r_\partial(y) \in \overline{\mathrm{Gr}(I_F)}\big\} \cup \{0\}$$

is a thin subset of $\mathrm{res}(K)$. Proposition 3.38 applied to $\mathrm{res}(K)$ gives $u \in \mathcal{O}^\times$ with $\jmath^r_\partial(\bar{u}) = \overline{\jmath^r_\partial(u)} \notin \overline{\mathrm{Gr}(I_F)}$. Then either $I_F(\jmath^{r-1}_\partial u) \succ 1$ or $I_F(\jmath^{r-1}_\partial u) \preccurlyeq 1$ and

$$\overline{u^{(r)}} \;\neq\; \overline{I_F(\jmath^{r-1}_\partial u)}.$$

In either case, $u^{(r)} - I_F(\jmath_\partial^{r-1}u) \succcurlyeq 1$, so

$$F(\jmath_\partial^r u) \;=\; \mathfrak{m}_F\big(u^{(r)} - I_F(\jmath_\partial^{r-1}u)\big) \;\succcurlyeq\; \mathfrak{m}_F. \qquad \square$$

Lemma 6.11. *Suppose $S(\partial) = \{0\}$ and let $\beta \in \Gamma^>$. Then there is $\gamma \in \Gamma(\partial)$ and $u \in K$ with $|vu| < \beta$ such that*

$$vF(\jmath_\partial^r u) \;\leqslant\; vF + r\gamma + \beta.$$

Proof. We claim that for any $\varepsilon \in \Gamma^>$, we can find $\gamma \in \Gamma(\partial)$ and $a \in \mathcal{O}$ such that $v(a') - \gamma \leqslant \varepsilon$. Let $\varepsilon \in \Gamma^>$ be given and, using that $\varepsilon \notin S(\partial) = \{0\}$, take $\gamma \in \Gamma(\partial)$ with $\gamma + \varepsilon \notin \Gamma(\partial)$. Then there is $a \in \mathcal{O}$ with $v(a') \leqslant \gamma + \varepsilon$, as desired. This claim yields an elementary $T^{\mathcal{O},\partial}$-extension M of K with $\gamma \in \Gamma_M(\partial)$ and $a \in \mathcal{O}_M$ such that

$$v(a') - \gamma \;<\; \Gamma^>.$$

Let Δ be the convex Λ-subspace of Γ_M consisting of all $\varepsilon \in \Gamma_M$ with $|\varepsilon| < \Gamma^>$ and let $\phi \in M^\times$ with $v\phi = \gamma$. Then M^ϕ has small derivation, so M^ϕ_Δ does as well by [**4**, 4.4.4]. Since $v(\phi^{-1}a') < \Gamma^>$ and $v(a') > v\phi$, we have $\dot{v}(\phi^{-1}a') = 0$, so the derivation on $\operatorname{res}(M^\phi_\Delta)$ is nontrivial. Applying Lemma 6.10 to M^ϕ_Δ and F^ϕ, we get $u \in M$ with $\dot{v}u = 0$ and

$$\dot{v}F^\phi(\jmath^r_{\phi^{-1}\partial_M}u) \;\leqslant\; \dot{v}F^\phi.$$

Then $|vu| < \Gamma^>$ and

$$vF^\phi(\jmath^r_{\phi^{-1}\partial_M}u) - vF^\phi \;=\; vF(\jmath^r_{\partial_M}u) - (vF + rv\phi) \;=\; vF(\jmath^r_{\partial_M}u) - vF - r\gamma \;<\; \Gamma^>.$$

In particular, $|vu| < \beta$ and $vF(\jmath^r_{\partial_M}u) < vF + r\gamma + \beta$. As M is an elementary $T^{\mathcal{O},\partial}$-extension of K, the lemma follows. $\square$

Corollary 6.12. *Suppose $S(\partial) = \{0\}$, let $\beta \in \Gamma^>$, and suppose that $vF(\jmath_\partial^r a) = vF(\jmath_\partial^r b)$ for all $a, b \in K$ with $|va|, |vb| < \beta$. Then there is $\gamma \in \Gamma(\partial)$ such that $vF(\jmath_\partial^r u) \leqslant vF + r\gamma$ for all $u \in K$ with $|vu| < \beta$.*

Proof. We first handle the case $r = 0$, so $I_F \in K$. Take $a \in \mathcal{O}^\times$ with $a \not\sim I_F$. Then

$$F(a) \;=\; \mathfrak{m}_F(a - I_F) \;\succcurlyeq\; \mathfrak{m}_F,$$

so $vF(a) \leqslant vF$. For $u \in K$ with $|vu| < \beta$, we have $vF(u) = vF(a) \leqslant vF$, as desired. Now assume $r > 0$. Let $a \in K$ with $|va| < \beta$ and set

$$\alpha \;:=\; r^{-1}\big(vF(\jmath_\partial^r a) - vF\big).$$

For $u \in K$ with $|vu| < \beta$, we have $vF(\jmath_\partial^r u) = vF(\jmath_\partial^r a) = vF + r\alpha$, so it suffices to show that $\alpha \in \Gamma(\partial)$. Let $\varepsilon \in \Gamma^>$ with $r\varepsilon < \beta$. Using Lemma 6.11, take $b \in K^\times$ and $\gamma \in \Gamma(\partial)$ with $|vb| < r\varepsilon < \beta$ and

$$vF(\jmath_\partial^r b) \;\leqslant\; vF + r\gamma + r\varepsilon.$$

By assumption, $vF(\jmath_\partial^r a) = vF(\jmath_\partial^r b)$, so

$$\alpha - \varepsilon \;=\; r^{-1}\big(vF(\jmath_\partial^r a) - vF\big) - \varepsilon \;=\; r^{-1}\big(vF(\jmath_\partial^r b) - vF\big) - \varepsilon \;\leqslant\; \gamma \;\in\; \Gamma(\partial).$$

As ε can be taken to be arbitrarily small, we have $\alpha \in \Gamma(\partial)$ by Lemma 6.9. $\square$

6.3. Eventual smallness

In this section, let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function in implicit form, let ϕ range over $K^\times$, let $\ell \prec 1$ be an element in a $T^{\mathcal{O},\partial}$-extension of K, and suppose $v(\ell - K)$ has no largest element.

A property is said to hold *for all $y \in K$ sufficiently close to ℓ* if there is $\eta \in v(\ell - K)$ such that the property holds for all $y \in K$ with $v(\ell - y) > \eta$. We say that F **is small near** (K, ℓ) if $I_F(\jmath_\partial^{r-1} y) \prec 1$ for all $y \in K$ sufficiently close to ℓ.

Lemma 6.13. *Let $\phi_0 \in K^\times$ with $v\phi_0 \in \Gamma(\partial)$ and suppose $v\phi \leqslant v\phi_0$. If F^{ϕ_0} is small near (K^{ϕ_0}, ℓ), then F^ϕ is small near (K^ϕ, ℓ).*

PROOF. By replacing K, F, and ϕ with K^{ϕ_0}, F^{ϕ_0}, and $\phi_0^{-1}\phi$, we may assume $\phi_0 = 1$ (so K has small derivation) and $\phi \succcurlyeq 1$. Set $\delta := \phi^{-1}\partial$. Suppose F is small near (K, ℓ) and let $y \in \mathcal{O}$ be close enough to ℓ so that $I_F(\jmath_\partial^{r-1} y) \prec 1$. We claim that $I_{F^\phi}(\jmath_\delta^{r-1} y) \prec 1$. By Lemma 3.37, we have

$$I_{F^\phi}(\jmath_\delta^{r-1} y) \ = \ \phi^{-r}\Big(I_F^\phi(\jmath_\delta^{r-1} y) - \sum_{i=0}^{r-1} \xi_i^r(\phi)\delta^i y\Big) \ = \ \phi^{-r} I_F(\jmath_\partial^{r-1} y) - \sum_{i=0}^{r-1} \phi^{-r}\xi_i^r(\phi)\delta^i y.$$

As $I_F(\jmath_\partial^{r-1} y) \prec 1$ and $\phi \succcurlyeq 1$, we have $\phi^{-r} I_F(\jmath_\partial^{r-1} y) \prec 1$, so it remains to show

$$\sum_{i=0}^{r-1} \phi^{-r}\xi_i^r(\phi)\delta^i y \ \prec \ 1.$$

We claim that $\phi^\dagger \preccurlyeq \phi$. This is clear in the case that $\phi' \preccurlyeq \phi$, for then $\phi^\dagger \preccurlyeq 1 \preccurlyeq \phi$ (note that if $\phi \asymp 1$, then we are in this case by Fact 6.7). On the other hand, if $\phi' \succ \phi \succ 1$, then $\phi^\dagger \preccurlyeq \phi$ by [**4**, 6.4.1]. Now [**5**, 2.2] gives $\phi^{-r}\xi_i^r(\phi) \preccurlyeq 1$ for each $i < r$. Since K^ϕ has small derivation and $y \prec 1$, we have $\phi^{-r}\xi_i^r(\phi)\delta^i y \prec 1$ for each $i < r$ as desired. □

We say that F **is eventually small near** (K, ℓ) if F^ϕ is small near (K^ϕ, ℓ) whenever $v\phi \in \Gamma(\partial)$. Unlike smallness, eventual smallness is invariant under compositional conjugation: F is eventually small near (K, ℓ) if and only if F^ϕ is eventually small near (K^ϕ, ℓ). By the above lemma, the set

$$\big\{v\phi \in \Gamma(\partial) : F^\phi \text{ is small near } (K^\phi, \ell)\big\}$$

is a downward closed subset of $\Gamma(\partial)$. Thus, F is eventually small near (K, ℓ) if and only if F^ϕ is small near (K^ϕ, ℓ) for all sufficiently large $v\phi \in \Gamma(\partial)$, and F is not eventually small near (K, ℓ) if and only if F^ϕ is not small near (K^ϕ, ℓ) for all sufficiently large $v\phi \in \Gamma(\partial)$. Eventual smallness serves as a crude analog of the Newton degree in [**4**] and [**5**]; one should think of F being eventually small as analogous to F having positive Newton degree. Of course, Newton degree makes no sense for arbitrary definable functions.

Lemma 6.14. *Suppose that $S(\partial) = \{0\}$ and that F is eventually small near (K, ℓ). For each $\beta \in \Gamma^>$, we have*

$$F(\jmath_\partial^r a) \ \not\asymp \ F(\jmath_\partial^r b)$$

for some $a, b \in K$ with $va, vb > -\beta$.

PROOF. Let $\beta \in \Gamma^>$ and let $a \in K$ with $|va| < \beta$. If $F(\jmath_\partial^r a) \not\asymp F(\jmath_\partial^r b)$ for some $b \in K$ with $|vb| < \beta$, then we are done, so we may assume $F(\jmath_\partial^r a) \asymp F(\jmath_\partial^r b)$ for all $b \in K$ with $|vb| < \beta$. Then Corollary 6.12 gives $\gamma \in \Gamma(\partial)$ with $vF(\jmath_\partial^r a) \leqslant vF + r\gamma$. Take ϕ with $v\phi = \gamma$. Then F^ϕ is small near (K^ϕ, ℓ), so we may take $y \in \mathcal{O}$ close enough to ℓ so that $I_{F^\phi}(\jmath_\delta^{r-1} y) \prec 1$. Since $\delta^r y \prec 1$ as well, we have

$$F(\jmath_\partial^r y) \ = \ F^\phi(\jmath_\delta^r y) \ = \ \phi^r \mathfrak{m}_F\big(\delta^r y - I_{F^\phi}(\jmath_\delta^{r-1} y)\big) \ \prec \ \phi^r \mathfrak{m}_F,$$

so $vF(\jmath_\partial^r y) > vF + r\gamma \geqslant vF(\jmath_\partial^r a)$. □

6.4. Vanishing

In this section, let ϕ range over $K^\times$, let ℓ be an element in a strict $T^{\mathcal{O},\partial}$-extension L of K, and suppose $v(\ell - K)$ has no largest element. Unlike in Section 6.3, we do not assume $\ell \prec 1$.

Definition 6.15. Let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function in implicit form. We say F **vanishes at** (K,ℓ) if $F_{+a,\times d}$ is eventually small near $\big(K, d^{-1}(\ell - a)\big)$ for all $a \in K$ and $d \in K^\times$ with $\ell - a \prec d$.

Let $Z(K,\ell)$ be the set of all $\mathcal{L}(K)$-definable functions in implicit form which vanish at (K,ℓ). For each r, we let $Z_r(K,\ell)$ be the functions in $Z(K,\ell)$ of arity $1+r$, so $Z(K,\ell)$ is equal to the disjoint union $\bigcup_r Z_r(K,\ell)$. The set $Z(K,\ell)$ serves a similar purpose to the set in **[4]** and **[5]** with the same name. Note that $Z(K,\ell)$ does not depend on ℓ, only on the $\mathcal{L}^{\mathcal{O}}$-type of ℓ over K. we will show in Proposition 6.18 below that if $Z(K,\ell) = \emptyset$ then $F(\jmath^r_\partial \ell) \neq 0$ for all $\mathcal{L}(K)$-definable functions F in implicit form.

Lemma 6.16. $Z_0(K,\ell) = \emptyset$.

Proof. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function in implicit form, so $I_F \in K$. Let $d \in K$ with $d \asymp \ell - I_F$ and let $a \in K$ with $\ell - a \prec d$. Then $I_F - a \asymp d$, so

$$I_{F_{+a,\times d}} \;=\; d^{-1}(I_F - a) \;\asymp\; 1$$

and $F_{+a,\times d}$ is not small near $\big(K, d^{-1}(\ell - a)\big)$. As $F^\phi_{+a,\times d} = F_{+a,\times d}$ for all ϕ, we see that $F_{+a,\times d}$ is not eventually small near $\big(K, d^{-1}(\ell-a)\big)$, so $F \notin Z(K,\ell)$. □

Lemma 6.17. *Suppose $S(\partial) = \{0\}$, let $F \in Z_r(K,\ell)$, and let $y \in K$. Then there is $z \in K$ with $\ell - z \prec \ell - y$ and $F(\jmath^r_\partial y) \not\asymp F(\jmath^r_\partial z)$.*

Proof. Let $a \in K$ and $d \in K^\times$ with $\ell - a \prec d \prec \ell - y$. Set $\beta := vd - v(\ell - y) > 0$. Since $F_{+a,\times d}$ is eventually small near $\big(K, d^{-1}(\ell - a)\big)$, Lemma 6.14 gives $b_1, b_2 \in K$ with $vb_1, vb_2 > -\beta$ and

$$vF_{+a,\times d}(\jmath^r_\partial b_1) \;\neq\; vF_{+a,\times d}(\jmath^r_\partial b_2).$$

Either $vF_{+a,\times d}(\jmath^r_\partial b_1)$ or $vF_{+a,\times d}(\jmath^r_\partial b_2)$ is different from $vF(\jmath^r_\partial y)$, so suppose $vF_{+a,\times d}(\jmath^r_\partial b_1) \neq vF(\jmath^r_\partial y)$ and set $z := db_1 + a$. Then $F(\jmath^r_\partial z) = F_{+a,\times d}(\jmath^r_\partial b_1) \not\asymp F(\jmath^r_\partial y)$ and, since $v(db_1) > vd - \beta = v(\ell - y)$, we have

$$v(\ell - z) \;=\; v\big((\ell - a) - db_1\big) \;\geqslant\; \min\big\{v(\ell - a), v(db_1)\big\} \;>\; v(\ell - y). \qquad \square$$

Behavior of nonvanishing functions. Fix r and suppose $Z_q(K,\ell) = \emptyset$ for all $q < r$. Our goal is to prove the following result:

Proposition 6.18. *Let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function in implicit form with $F \notin Z_r(K,\ell)$. Then $F(\jmath^r_\partial \ell) \neq 0$ and*

$$F(\jmath^r_\partial \ell) \;\sim\; F(\jmath^r_\partial y)$$

for all $y \in K$ sufficiently close to ℓ.

This proposition requires a somewhat lengthy proof by induction, so we make the following hypothesis.

Induction Hypothesis (IH)**.** *We assume that for all $q < r$ and all $\mathcal{L}(K)$-definable functions $F\colon K^{1+q} \to K$ in implicit form, we have $F(\jmath^q_\partial \ell) \neq 0$ and*

$$F(\jmath^q_\partial \ell) \;\sim\; F(\jmath^q_\partial y)$$

for all $y \in K$ sufficiently close to ℓ.

Lemma 6.19. *Suppose (IH) holds. Then $K\langle \jmath_\partial^{r-1}\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K.*

Proof. Let $q<r$ be given and assume $K\langle \jmath_\partial^{q-1}\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K (this holds vacuously when $q=0$). We will show that $K\langle \jmath_\partial^{q}\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of $K\langle \jmath_\partial^{q-1}\ell\rangle$, from which the lemma follows by induction. Let $G\colon K^q\to K$ be an $\mathcal{L}(K)$-definable function. For all $u\in K$ sufficiently close to ℓ, we have

$$\ell^{(q)}-G(\jmath_\partial^{q-1}\ell)\ \sim\ y^{(q)}-G(\jmath_\partial^{q-1}y)\ \in\ K$$

by (IH). Since G is arbitrary, we may apply Lemma 5.10 with $K\langle \jmath_\partial^{q-1}\ell\rangle$ and $\ell^{(q)}$ in place of K and ℓ to get that $K\langle \jmath_\partial^{q}\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of $K\langle \jmath_\partial^{q-1}\ell\rangle$. □

Lemma 6.20. *Suppose (IH) holds, let $A\subseteq K^r$ be $\mathcal{L}^{\mathcal{O}}(K)$-definable, and suppose $\jmath_\partial^{r-1}(\ell)\in A^L$. Then A has nonempty interior and $\jmath_\partial^{r-1}(y)\in A$ for all $y\in K$ sufficiently close to ℓ.*

Proof. Using Lemmas 5.4 and 6.19, we take an $\mathcal{L}(K)$-definable cell D contained in A with $\jmath_\partial^{r-1}(\ell)\in D^L$. Let $q<r$ be given and assume $\pi_q(D)$ is open and $\jmath_\partial^{q-1}(y)\in\pi_q(D)$ for all $y\in K$ sufficiently close to ℓ (this holds vacuously when $q=0$). We will show that $\pi_{q+1}(D)$ is open and $\jmath_\partial^{q}(y)\in\pi_{q+1}(D)$ for all $y\in K$ sufficiently close to ℓ, from which the lemma follows by induction. If $\pi_{q+1}(D)$ is not open, then $\pi_{q+1}(D)=\operatorname{Gr}\big(G|_{\pi_q(D)}\big)$ for some $\mathcal{L}(K)$-definable function $G\colon K^q\to K$, so $\ell^{(q)}=G(\jmath_\partial^{q-1}\ell)$, contradicting (IH). Therefore, $\pi_{q+1}(D)$ is open. Suppose $\pi_{q+1}(D)$ is of the form

$$\big\{(a,b): a\in\pi_q(D)\text{ and }G(a)<b<H(a)\big\}$$

for some $\mathcal{L}(K)$-definable functions $G,H\colon K^q\to K$. Then (IH) gives

$$y^{(q)}-G(\jmath_\partial^{q-1}y)\ \sim\ \ell^{(q)}-G(\jmath_\partial^{q-1}\ell)\ >\ 0\ >\ \ell^{(q)}-H(\jmath_\partial^{q-1}\ell)\ \sim\ y^{(q)}-H(\jmath_\partial^{q-1}y)$$

for all $y\in K$ sufficiently close to ℓ, so $G(\jmath_\partial^{q-1}y)<y^{(q)}<H(\jmath_\partial^{q-1}y)$ for these y. This gives $\jmath_\partial^{q}(y)\in\pi_{q+1}(D)$ for all $y\in K$ sufficiently close to ℓ as desired. If $\pi_{q+1}(D)$ is of the form

$$\big\{(a,b): a\in\pi_q(D)\text{ and }b>G(a)\big\}\ \text{ or }\ \big\{(a,b): a\in\pi_q(D)\text{ and }b<H(a)\big\},$$

then a simpler version of the above argument works. If $\pi_{q+1}(D)=\pi_q(D)\times K$, then the result follows immediately from the inductive assumption. □

Corollary 6.21. *Suppose (IH) holds and let $G\colon K^r\to K$ be an $\mathcal{L}(K)$-definable function. If $G(\jmath_\partial^{r-1}\ell)=0$, then $G(\jmath_\partial^{r-1}y)=0$ for all $y\in K$ sufficiently close to ℓ. If $G(\jmath_\partial^{r-1}\ell)\neq 0$, then*

$$G(\jmath_\partial^{r-1}\ell)\ \sim\ G(\jmath_\partial^{r-1}y)$$

for all $y\in K$ sufficiently close to ℓ.

Proof. If $G(\jmath_\partial^{r-1}\ell)=0$, then apply Lemma 6.20 to the $\mathcal{L}(K)$-definable set

$$\big\{a\in K^r: G(a)=0\big\}.$$

If $G(\jmath_\partial^{r-1}\ell)\neq 0$, then since $K\langle \jmath_\partial^{r-1}\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K by Lemma 6.19, we may take $g\in K^\times$ with $G(\jmath_\partial^{r-1}\ell)\sim g$. Now apply Lemma 6.20 to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set

$$\big\{a\in K^r: G(a)\sim g\big\}.$$ □

We are now ready to prove Proposition 6.18.

Proof of Proposition 6.18. Suppose Proposition 6.18 holds with q in place of r for all $q < r$ (this is vacuous if $r = 0$). Then (IH) holds, since we are assuming that $Z_q(K, \ell) = \emptyset$ for all $q < r$. Let $F\colon K^{1+r} \to K$ be as in the statement of the proposition. Since $F \notin Z_r(K, \ell)$, we may take $a \in K$ and $d \in K^\times$ with $\ell - a \prec d$ such that $F_{+a,\times d}$ is not eventually small near $\big(K, d^{-1}(\ell - a)\big)$. Set $e := d^{-1}(\ell - a) \prec 1$ and take ϕ with $v\phi \in \Gamma(\partial)$ such that $F^\phi_{+a,\times d}$ is not small near (K^ϕ, e). Set $\delta := \phi^{-1}\partial$ and set

$$\mathfrak{m} \ := \ \mathfrak{m}_{F^\phi_{+a,\times d}}, \qquad G \ := \ \big(I_{F^\phi_{+a,\times d}}\big)^{\phi^{-1}}_{-d^{-1}a,\times d^{-1}}.$$

We have

$$F(\jmath^r_\partial \ell) \ = \ F^\phi_{+a,\times d}(\jmath^r_\delta e) \ = \ \mathfrak{m}_{F^\phi_{+a,\times d}}\big(\delta^r e - I_{F^\phi_{+a,\times d}}(\jmath^{r-1}_\delta e)\big) \ = \ \mathfrak{m}\big(\delta^r e - G(\jmath^{r-1}_\partial \ell)\big).$$

Using Corollary 6.21, we take $\eta \in v(\ell - K)$ such that for all $y \in K$ with $v(\ell - y) > \eta$, either

$$G(\jmath^{r-1}_\partial y) \ = \ G(\jmath^{r-1}_\partial \ell) \ = \ 0 \ \text{ or } \ G(\jmath^{r-1}_\partial y) \ \sim \ G(\jmath^{r-1}_\partial \ell) \ \neq \ 0.$$

Then $\eta - vd \in v(e - K)$ and, since $e \prec 1$, we may increase η and arrange $\eta - vd > 0$. Since $F^\phi_{+a,\times d}$ is not small near (K^ϕ, e), we may take $z_0 \in K$ with

$$v(e - z_0) \ > \ \eta - vd, \qquad I_{F^\phi_{+a,\times d}}(\jmath^{r-1}_\delta z_0) \ \succcurlyeq \ 1.$$

Then $v\big(\ell - (dz_0 + a)\big) > \eta$, so we have

$$G(\jmath^{r-1}_\partial \ell) \ \sim \ G\big(\jmath^{r-1}_\partial (dz_0 + a)\big) \ = \ I_{F^\phi_{+a,\times d}}(\jmath^{r-1}_\delta z_0) \ \succcurlyeq \ 1.$$

Since δ is small, L strictly extends K, and $e \prec 1$, we have $\delta^r e \prec 1$, so

$$F(\jmath^r_\partial \ell) \ = \ \mathfrak{m}\big(\delta^r e - G(\jmath^{r-1}_\partial \ell)\big) \ \sim \ -\mathfrak{m}G(\jmath^{r-1}_\partial \ell) \ \neq \ 0.$$

Now, let $y \in K$ with $v(\ell - y) > \eta$ and set $z := d^{-1}(y - a)$, so

$$F(\jmath^r_\partial y) \ = \ F^\phi_{+a,\times d}(\jmath^r_\delta z) \ = \ \mathfrak{m}_{F^\phi_{+a,\times d}}\big(\delta^r z - I_{F^\phi_{+a,\times d}}(\jmath^{r-1}_\delta z)\big) \ = \ \mathfrak{m}\big(\delta^r z - G(\jmath^{r-1}_\partial y)\big).$$

Since $v(e - z) > \eta - vd > 0$, we have $z \prec 1$, so $\delta^r z \prec 1$. Since $v(\ell - y) > \eta$, we also have $G(\jmath^{r-1}_\partial y) \sim G(\jmath^{r-1}_\partial \ell) \succcurlyeq 1$. Thus,

$$F(\jmath^r_\partial y) \ = \ \mathfrak{m}\big(\delta^r z - G(\jmath^{r-1}_\partial y)\big) \ \sim -\mathfrak{m}G(\jmath^{r-1}_\partial y) \ \sim \ -\mathfrak{m}G(\jmath^{r-1}_\partial \ell) \ \sim \ F(\jmath^r_\partial \ell). \qquad \square$$

6.5. Constructing immediate extensions when $S(\partial) = \{0\}$

As in the previous section, let ℓ be an element in a strict $T^{\mathcal{O},\partial}$-extension L of K and suppose $v(\ell - K)$ has no largest element.

Proposition 6.22. *Suppose $Z(K, \ell) = \emptyset$. Then $K\langle \jmath^\infty_\partial \ell\rangle$ is an immediate strict $T^{\mathcal{O},\partial}$-extension of K. Let b be an element in a strict $T^{\mathcal{O},\partial}$-extension M of K with $v(b - y) = v(\ell - y)$ for each $y \in K$. Then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle \jmath^\infty_\partial \ell\rangle \to M$ sending ℓ to b.*

Proof. By Lemma 6.19, $K\langle \jmath^\infty_\partial \ell\rangle$ is an increasing union of immediate $T^{\mathcal{O}}$-extensions of K, so it is itself an immediate $T^{\mathcal{O}}$-extension of K. It is also strict, as L is strict. As for the existence of an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle \jmath^\infty_\partial \ell\rangle \to M$, we proceed by induction. Let $r \geqslant 0$ and suppose we have an $\mathcal{L}^{\mathcal{O}}(K)$-embedding

$$\imath\colon K\langle \jmath^{r-1}_\partial \ell\rangle \to M$$

which sends the tuple $\jmath_\partial^{r-1}(\ell)$ to $\jmath_\partial^{r-1}(b)$ (this holds vacuously when $r = 0$). Let $G\colon K^r \to K$ be an $\mathcal{L}(K)$-definable function. As $Z(K, b) = Z(K, \ell) = \emptyset$, Proposition 6.18 (applied to both ℓ and b) gives $\eta \in v(\ell - K)$ such that

$$\ell^{(r)} - G(\jmath_\partial^{r-1}\ell) \;\sim\; y^{(r)} - G(\jmath_\partial^{r-1}y) \;\sim\; b^{(r)} - G(\jmath_\partial^{r-1}b)$$

for $y \in K$ with $v(\ell - y) = v(b - y) > \eta$. Since G is arbitrary and $\imath\big(G(\jmath_\partial^{r-1}\ell)\big) = G(\jmath_\partial^{r-1}b)$, we may apply Corollary 5.6 with $K\langle \jmath_\partial^{r-1}\ell\rangle$, $\ell^{(r)}$, and $b^{(r)}$ in place of K, ℓ, and b to extend $\imath$ to an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle \jmath_\partial^{r}\ell\rangle \to M$ sending $\ell^{(r)}$ to $b^{(r)}$. The union of these embeddings is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle \jmath_\partial^{\infty}\ell\rangle \to M$ which sends $\jmath_\partial^{\infty}(\ell)$ to $\jmath_\partial^{\infty}(b)$. This is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Lemma 3.31. As an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, it is uniquely determined by the condition that ℓ be sent to b. □

Proposition 6.23. *Suppose $S(\partial) = \{0\}$, let $F \in Z_{r+1}(K, \ell)$, and suppose that $Z_q(K, \ell) = \emptyset$ for all $q \leqslant r$. Then K has an immediate strict $T^{\mathcal{O},\partial}$-extension $K\langle \jmath_\partial^{r}a\rangle$ with $F(\jmath_\partial^{r+1}a) = 0$ and $v(a - y) = v(\ell - y)$ for each $y \in K$. Let b be an element in a strict $T^{\mathcal{O},\partial}$-extension M of K with $F(\jmath_\partial^{r+1}b) = 0$ and $v(b - y) = v(\ell - y)$ for each $y \in K$. Then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle \jmath_\partial^{r}a\rangle \to M$ sending a to b.*

Proof. Let $(a_0, \ldots, a_r)$ realize the $\mathcal{L}^{\mathcal{O}}(K)$-type of $\jmath_\partial^{r}(\ell)$ in some $T^{\mathcal{O}}$-extension of K. Then $K\langle a_0, \ldots, a_r\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K by Lemma 6.19. The tuple $\jmath_\partial^{r}(\ell)$ is $\mathcal{L}(K)$-independent by Proposition 6.18, so $(a_0, \ldots, a_r)$ is $\mathcal{L}(K)$-independent as well. Using Lemma 3.31, we extend ∂ to a T-derivation on $K\langle a_0, \ldots, a_r\rangle$ with $a_i' = a_{i+1}$ for $i < r$ and $a_r' = I_F(a_0, \ldots, a_r)$. Set $a := a_0$ so $K\langle a_0, \ldots, a_r\rangle = K\langle \jmath_\partial^{r}a\rangle$ and $a^{(r+1)} = I_F(\jmath_\partial^{r}a)$.

We need to show that $K\langle \jmath_\partial^{r}a\rangle$ is a strict extension of K. Let $\phi \in K^\times$ with $v\phi \in \Gamma(\partial)$ and let $G\colon K^{r+1} \to K$ be an $\mathcal{L}(K)$-definable function with $G(\jmath_\partial^{r}a) \prec 1$. By Fact 6.8, it suffices to show that $G(\jmath_\partial^{r}a)' \prec \phi$. We assume that $G(\jmath_\partial^{r}a) \neq 0$, and we take an $\mathcal{L}(K)$-definable open set $U \subseteq K^{1+r}$ on which G is $\mathcal{C}^1$ and which contains $\jmath_\partial^{r}(a)$ in its natural extension. By Proposition 3.29, we have

$$G(\jmath_\partial^{r}a)' \;=\; G^{[\partial]}(\jmath_\partial^{r}a) + \mathbf{J}_G(\jmath_\partial^{r}a)\big(a', a'', \ldots, a^{(r)}, I_F(\jmath_\partial^{r}a)\big).$$

Let $Y = (Y_0, \ldots, Y_r)$ and let $H\colon U \to K$ be the function

$$H(Y) \;:=\; G^{[\partial]}(Y) + \mathbf{J}_G(Y)\big(Y_1, \ldots, Y_r, I_F(Y)\big),$$

so $H(\jmath_\partial^{r}a) = G(\jmath_\partial^{r}a)'$. Suppose toward contradiction that $H(\jmath_\partial^{r}a) \succcurlyeq \phi$. Since $\jmath_\partial^{r}(\ell)$ and $\jmath_\partial^{r}(a)$ have the same $\mathcal{L}^{\mathcal{O}}(K)$-type and $Z_r(K, \ell) = \emptyset$, Lemma 6.20 and Corollary 6.21 give $\eta \in v(\ell - K)$ with

$$\jmath_\partial^{r}(y) \;\in\; U, \qquad G(\jmath_\partial^{r}y) \;\sim\; G(\jmath_\partial^{r}a) \;\prec\; 1, \qquad H(\jmath_\partial^{r}y) \;\sim\; H(\jmath_\partial^{r}a) \;\succcurlyeq\; \phi$$

for all $y \in K$ with $v(\ell - y) > \eta$. For the remainder of this proof, we let $y \in K$ with $v(\ell - y) > \eta$. Since $G(\jmath_\partial^{r}y) \prec 1$ and $v\phi \in \Gamma(\partial)$, we have

$$G(\jmath_\partial^{r}y)' \;=\; G^{[\partial]}(\jmath_\partial^{r}y) + \mathbf{J}_G(\jmath_\partial^{r}y)(y', \ldots, y^{(r)}, y^{(r+1)}) \;\prec\; \phi \;\preccurlyeq\; H(\jmath_\partial^{r}y).$$

Thus

$$G(\jmath_\partial^{r}y)' - H(\jmath_\partial^{r}y) \;=\; \frac{\partial G}{\partial Y_r}(\jmath_\partial^{r}y)\big(y^{(r+1)} - I_F(\jmath_\partial^{r}y)\big) \;\sim\; -H(\jmath_\partial^{r}y).$$

Since $H(\jmath_\partial^{r}y) \neq 0$, we have $\frac{\partial G}{\partial Y_r}(\jmath_\partial^{r}y) \neq 0$, so

$$y^{(r+1)} - I_F(\jmath_\partial^{r}y) \;\sim\; -H(\jmath_\partial^{r}y)\left(\frac{\partial G}{\partial Y_r}(\jmath_\partial^{r}y)\right)^{-1}.$$

We have $H(\jmath_\partial^r y) \sim H(\jmath_\partial^r a)$ and, by increasing η, we may assume $\frac{\partial G}{\partial Y_r}(\jmath_\partial^r y) \sim \frac{\partial G}{\partial Y_r}(\jmath_\partial^r a)$. Thus

$$F(\jmath_\partial^{r+1} y) \;=\; \mathfrak{m}_F\big(y^{(r+1)} - I_F(\jmath_\partial^r y)\big) \sim\; -\mathfrak{m}_F H(\jmath_\partial^r a)\left(\frac{\partial G}{\partial Y_r}(\jmath_\partial^r a)\right)^{-1}.$$

In particular, $F(\jmath_\partial^{r+1} y) \sim F(\jmath_\partial^{r+1} z)$ for all $y, z \in K$ with $v(\ell - y), v(\ell - z) > \eta$, contradicting Lemma 6.17.

Now let M and b be as in the statement of the proposition. Since $\jmath_\partial^r(\ell)$ and $\jmath_\partial^r(a)$ have the same $\mathcal{L}^{\mathcal{O}}(K)$-type and $Z_r(K,\ell) = \emptyset$, we may construct an $\mathcal{L}^{\mathcal{O}}(K)$-embedding

$$K\langle \jmath_\partial^r a\rangle \;\to\; M$$

which sends $\jmath_\partial^r(a)$ to $\jmath_\partial^r(b)$ as in the proof of the previous proposition. This is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Lemma 3.31. As an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, it is uniquely determined by the condition that a be sent to b. □

Theorem 6.24. *Suppose $S(\partial) = \{0\}$. Then K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension.*

Proof. We may assume that K is not itself spherically complete. It suffices to show that K has a proper immediate strict $T^{\mathcal{O},\partial}$-extension, as the property $S(\partial) = \{0\}$ is preserved by immediate strict extensions. Let $\mathcal{B}$ be a nested collection of closed v-balls in K with empty intersection in K and let ℓ be an element in an elementary $T^{\mathcal{O},\partial}$-extension of L of K with $\ell \in \bigcap \mathcal{B}^L$. Then $v(\ell - K)$ has no largest element by Lemma 5.8. If $Z(K,\ell) = \emptyset$, then $K\langle \jmath_\partial^\infty \ell\rangle$ is a proper immediate strict $T^{\mathcal{O},\partial}$-extension of K by Proposition 6.22. Suppose $Z(K,\ell) \neq \emptyset$. Lemma 6.16 gives $Z_0(K,\ell) = \emptyset$, so take r maximal such that $Z_q(K,\ell) = \emptyset$ for all $q \leqslant r$. Then Proposition 6.23 provides a proper immediate strict $T^{\mathcal{O},\partial}$-extension $K\langle \jmath_\partial^r a\rangle$ of K where a is in the natural extension of each $B \in \mathcal{B}$. □

Before moving on, let us consider a case that can be handled by Theorem 6.24. Suppose that K has small derivation and that the induced derivation on $\operatorname{res}(K)$ is nontrivial. Then $\Gamma(\partial) = \Gamma^{\leqslant}$, so $S(\partial) = \{0\}$; see [**5**, 1.7 and 1.15]. Given a $T^{\mathcal{O},\partial}$-extension M of K, it follows from Fact 6.7 that M is a strict extension of K if and only if M has small derivation. Thus, we have the following:

Corollary 6.25. *If K has small derivation and the induced derivation on $\operatorname{res}(K)$ is nontrivial, then K has a spherically complete immediate $T^{\mathcal{O},\partial}$-extension with small derivation.*

6.6. Coarsening by $S(\partial)$

In this section, we prove our main theorem. First, we establish some results on residue field extensions.

Lemma 6.26. *Suppose K has small derivation and let $L = K\langle a\rangle$ be a simple $T^{\mathcal{O},\partial}$-extension of K with $a \asymp 1$, $\bar{a} \notin \operatorname{res}(K)$, and $a' \preccurlyeq 1$. Then L has small derivation. Moreover, if $\partial\mathcal{O} \subseteq \smallo$ and $a' \prec 1$, then $\partial_L \mathcal{O}_L \subseteq \smallo_L$.*

Proof. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \prec 1$. We need to show that

$$F(a)' \;=\; F^{[\partial]}(a) + F'(a)a' \;\prec\; 1.$$

Proposition 5.23 gives that $F'(a) \preccurlyeq a^{-1}F(a) \prec 1$, so $F'(a)a' \prec 1$ and it remains to show that $F^{[\partial]}(a) \prec 1$. The Wilkie inequality gives $\Gamma_L = \Gamma$, so L is an elementary $T^{\mathcal{O}}$-extension of K and it suffices to show that for any $\mathcal{L}^{\mathcal{O}}(K)$-definable set $A \subseteq \mathcal{O}$ with $a \in A^L$, there is $y \in A$ with $F^{[\partial]}(y) \prec 1$. Let A be such a set and, by shrinking A if need be, assume that F is $\mathcal{C}^1$ on A and that $F(y) \prec 1$ for all $y \in A$. Since $F'(a) \prec 1$, we can use $\mathcal{L}^{\mathcal{O}}$-elementarity to take $y \in A$ with $F'(y) \prec 1$. Since $y' \preccurlyeq 1$ by Fact 6.7, we have $F'(y)y' \prec 1$. Since $F(y)' \prec 1$ as well, this gives

$$F^{[\partial]}(y) \ = \ F(y)' - F'(y)y' \ \prec \ 1.$$

This takes care of the first part of the lemma.

For the second part, assume that $\partial\mathcal{O} \subseteq \smallo$ and that $a' \prec 1$. We need to show that $F(a)' \prec 1$ for each $\mathcal{L}(K)$-definable function $F\colon K \to K$ with $F(a) \preccurlyeq 1$. The proof is essentially the same as the proof of the first part, but now Proposition 5.23 only gives that $F'(a) \preccurlyeq 1$. We make up for this by using our assumption that $\partial\mathcal{O} \subseteq \smallo$ and that $a' \prec 1$. $\square$

The following corollary serves as an analog of [**5**, 6.7].

Corollary 6.27. *Suppose $\partial\mathcal{O} \subseteq \smallo$ and let E be a T-extension of* $\operatorname{res}(K)$*. Then there is a strict $T^{\mathcal{O},\partial}$-extension L of K such that $\Gamma_L = \Gamma$, the derivation on* $\operatorname{res}(L)$ *is trivial, and* $\operatorname{res}(L)$ *is $\mathcal{L}(\operatorname{res} K)$-isomorphic to E.*

Proof. It suffices to consider the case $E = \operatorname{res}(K)\langle f\rangle$ where $f \notin \operatorname{res}(K)$. Let $L = K\langle a\rangle$ be a simple T-extension of K where a realizes the cut

$$\{y \in K : y < \mathcal{O} \text{ or } y \in \mathcal{O} \text{ and } \bar{y} < f\}.$$

We expand L to an $\mathcal{L}^{\mathcal{O}}$-structure by letting

$$\mathcal{O}_L \ := \ \big\{y \in L : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

Fact 5.3 gives that this expansion of L is a $T^{\mathcal{O}}$-extension of K. Note that $a \in \mathcal{O}_L$ and that $\operatorname{res}(L) = \operatorname{res}(K)\langle\bar{a}\rangle$ is $\mathcal{L}(\operatorname{res} K)$-isomorphic to E, since $\bar{a}$ and f realize the same cut in $\operatorname{res}(K)$. In particular, $\operatorname{res}(L) \neq \operatorname{res}(K)$, so $\Gamma_L = \Gamma$ by the Wilkie inequality. Using Lemma 3.31, we extend ∂ uniquely to a T-derivation on L with $a' = 0$. We claim that L is a strict $T^{\mathcal{O},\partial}$-extension of K. Let $\phi \in K^{\times}$ and note that $\phi^{-1}a' = a' = 0$. If $\partial\smallo \subseteq \phi\smallo$, then Lemma 6.26 applied to K^{ϕ} and L^{ϕ} in place of K and L gives $\partial_L\smallo_L \subseteq \phi\smallo_L$. Likewise, if $\partial\mathcal{O} \subseteq \phi\smallo$, then Lemma 6.26 gives $\partial_L\mathcal{O}_L \subseteq \phi\smallo_L$. The case $\phi = 1$ gives that the derivation on $\operatorname{res}(L)$ is trivial. $\square$

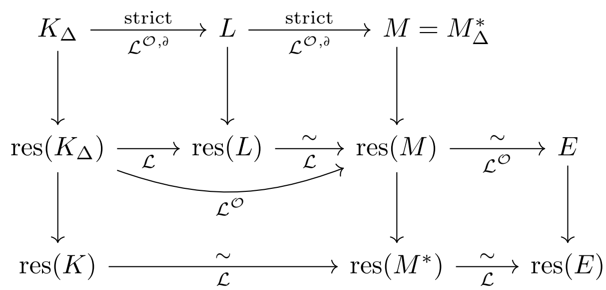
We are now ready to prove the main theorem:

Proof of Theorem 6.1. We assume that $S(\partial)$ is a Λ-subspace of K and we need to show that K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete. Set $\Delta := S(\partial)$. If $\Delta = \{0\}$, then we are done by Theorem 6.24, so we may assume that $\Delta \neq \{0\}$. We arrange by compositional conjugation that K has small derivation. The assumption that Δ is a Λ-subspace of K allows us to coarsen by Δ, which we do, yielding the $T^{\mathcal{O},\partial}$-model K_{Δ}. The derivation on $\operatorname{res}(K_{\Delta})$ is trivial by [**5**, 6.1] and $S_{K_{\Delta}}(\partial) = \{0\}$ by [**5**, 6.2]. Let E be a spherically complete immediate $T^{\mathcal{O}}$-extension of $\operatorname{res}(K_{\Delta})$; such an extension exists by Corollary 5.13. Using Corollary 6.27, we take a strict $T^{\mathcal{O},\partial}$-extension L of K_{Δ} such that $\Gamma_L = \dot{\Gamma}$, the derivation on $\operatorname{res}(L)$ is trivial, and $\operatorname{res}(L)$ is $\mathcal{L}(\operatorname{res} K_{\Delta})$-isomorphic to E. Then $S_L(\partial) = \{0\}$ as well, and we apply Theorem 6.24 to L to get a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension M of L. We have $\operatorname{res}(M) = \operatorname{res}(L)$ as T-models, so $\operatorname{res}(M)$ is $\mathcal{L}(\operatorname{res} K_{\Delta})$-isomorphic to E. We equip $\operatorname{res}(M)$ with a T-convex valuation ring $\mathcal{O}_{\operatorname{res}(M)}$ so that $\operatorname{res}(M)$ is $\mathcal{L}^{\mathcal{O}}(\operatorname{res} K_{\Delta})$-isomorphic to E; then $\operatorname{res}(M)$ is a spherically complete immediate $T^{\mathcal{O}}$-extension of $\operatorname{res}(K_{\Delta})$. Now let M^* be the $T^{\mathcal{O},\partial}$-model with underlying T^{∂}-model M and T-convex valuation ring

$$\mathcal{O}_{M^*} \ := \ \{a \in \mathcal{O}_M : \bar{a} \in \mathcal{O}_{\operatorname{res}(M)}\}.$$

Then M^* is an immediate $T^{\mathcal{O},\partial}$-extension of K with $M^*_\Delta = M$; see Section 5.2. By [**5**, 6.4], M^* is a strict $T^{\mathcal{O},\partial}$-extension of K. As $M^*_\Delta = M$ and $\mathrm{res}(M^*_\Delta) = \mathrm{res}(M)$ are both spherically complete, M^* is spherically complete by Fact 5.16. $\square$

The diagram below catalogs the objects and maps involved in the proof of Theorem 6.1.



As with the diagram in Section 5.2, each horizontal arrow is an embedding in the indicated language and every downward arrow is a partially defined projection. Isomorphisms are labeled as such and every square commutes.

If Λ is archimedean, then $S(\partial)$ is always a Λ-subspace of Γ. Thus, we have the following:

Corollary 6.28. *If T is polynomially bounded, then K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete.*

Uniqueness. Suppose K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension M. It is natural to ask: under which circumstances is M the *unique* spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension of K up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism? Uniqueness holds if K is itself spherically complete, for then $M = K$. In particular, it holds if $\mathcal{O} = K$. If ∂ is trivial, then any immediate strict $T^{\mathcal{O},\partial}$-extension of K has trivial derivation as well, so K has a unique spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism by Corollary 5.13.

Suppose $T = \mathrm{RCF}$. If ∂ is small, $\mathrm{res}(K)$ is linearly surjective, and K is monotone, then M is unique up to $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}(K)$-isomorphism by [**4**, Section 7.4]. If ∂ is small, $\mathrm{res}(K)$ is linearly surjective, and K is *asymptotic* then again M is unique up to $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}(K)$-isomorphism by [**57**]. See Section 7.1 for more information about asymptotic fields. An example of a real closed H-field R which does not have a unique spherically complete immediate strict $\mathrm{RCF}^{\mathcal{O},\partial}$-extension up to $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}(R)$-isomorphism is given in [**5**].

When $T \neq \mathrm{RCF}$, nothing is known about uniqueness outside of the trivial cases. All the results in the case $T = \mathrm{RCF}$ depend crucially on *differential henselianity*, a differential-algebraic property which we have yet to generalize to our setting.

CHAPTER 7

H_T-fields

In this chapter, let $K = (K, \mathcal{O}, \partial)$ be a model of $T^{\mathcal{O},\partial}$ with constant field C, value group Γ, and maximal ideal $\smallo \subseteq \mathcal{O}$.

Definition 7.1. K is an H_T**-field** if the following conditions hold:

(H1) for all $f \in K$, if $f > \mathcal{O}$, then $f' > 0$;
(H2) $\mathcal{O} = C + \smallo$.

The purpose of this chapter is to study H_T-fields and their simple extensions. We also consider pre-H_T-fields, which arise as $T^{\mathcal{O},\partial}$-submodels of H_T-fields, and H_T-asymptotic fields, which arise as coarsenings of H_T-fields. Our investigation of these structures is based on the study of H-fields, pre-H-fields, and H-asymptotic fields conducted in [**4**]. Indeed, (pre)-H_T-fields are just $T^{\mathcal{O},\partial}$-models which are also (pre)-H-fields in the sense of [**4**]. This is not quite true for H_T-asymptotic fields, as we will see in Section 7.1. All of our results require power boundedness, so we make the following assumption.

Assumption 7.2. *For the remainder of this chapter, T is power bounded with field of exponents Λ.*

Of course, the definitions made in this chapter make sense even when T is not power bounded. Our main result is that every H_T-field has a *Liouville closed* H_T-field extension, that is, an H_T-field extension where every element has an integral and an exponential integral. In [**2**], Aschenbrenner and van den Dries showed that every H-field has at least one and at most two Liouville closures (minimal Liouville closed H-field extensions) up to isomorphism. Though some H-fields were known to have exactly one Liouville closure up to isomorphism and others were known to have exactly two, the precise dividing line was unknown until Gehret showed that the number of Liouville closures is determined by the property of $\boldsymbol{\lambda}$*-freeness* [**41**]. The same is true in our setting; we give a precise characterization of the number of nonisomorphic minimal Liouville closed H_T-field extensions of a given H_T-field in Theorem 7.57. As an application, we show in Theorem 7.58 that any $\mathcal{L}_{\mathrm{an}}^{\mathcal{O},\partial}$-embedding of an $\mathbb{R}_{\mathrm{an}}$-Hardy field $\mathcal{H}$ into $\mathbb{T}_{\mathrm{an}}$ extends to an $\mathcal{L}_{\mathrm{an}}^{\mathcal{O},\partial}$-embedding of a Liouville closed $\mathbb{R}_{\mathrm{an}}$-Hardy field extension of $\mathcal{H}$.

We consider more general simple extensions of H_T-fields in Section 7.7. In Theorem 7.59, we show that every H_T-field has an H_T-field extension which satisfies an "order 1 intermediate value property." This was first shown for H-fields in [**3**] and for $\mathcal{R}$-Hardy fields in [**27**].

After giving some examples of H_T-fields, we introduce the class of H_T-asymptotic fields in Section 7.1. There, we recall a number of important results from [**4**] and apply some of our results from Chapter 6. In Section 7.2, we discuss the aforementioned property of $\boldsymbol{\lambda}$-freeness and adapt some arguments from [**41**] to our setting. Pre-H_T-fields are introduced in Section 7.3, and the relationship between pre-H_T-fields, H_T-fields, and H_T-asymptotic fields is discussed in more detail.

Extensions of pre-H_T-fields by integrals are studied in Section 7.4, and we show that every pre-H_T-field has a minimal H_T-field extension (called the H_T-field hull) which is unique up to unique isomorphism. In Section 7.5, we study extensions of pre-H_T-fields by exponential integrals and extensions of H_T-fields by constants. Section 7.6 is devoted to the proof of Theorem 7.57 on the existence and uniqueness of minimal Liouville closed H_T-field extensions. Finally, in Section 7.7, we prove Theorem 7.59.

Examples of H_T-fields.

Example 7.3 (Real closed H-fields)**.** Recall from Example 6.3 that the models of $\mathrm{RCF}^{\mathcal{O},\partial}$ are exactly the real closed valued fields with a continuous derivation. Thus, the class of H_{RCF}-fields is the same as the class of real closed H-fields.

Example 7.4 ($\mathcal{R}$-Hardy fields)**.** Let $\mathcal{R}$ be an o-minimal expansion of the real field and let $\mathcal{L}_{\mathcal{R}}$ be a language in which $T_{\mathcal{R}}$, the theory of $\mathcal{R}$, has quantifier elimination and a universal axiomatization. Recall from Chapter 1 that an *$\mathcal{R}$-Hardy field* is a Hardy field $\mathcal{H}$ which is closed under all function symbols in $\mathcal{L}_{\mathcal{R}}$. Let $\mathcal{H}$ be an $\mathcal{R}$-Hardy field and suppose that $\mathbb{R}$ is contained in $\mathcal{H}$, where each $r \in \mathbb{R}$ is identified with the germ of the constant function $x \mapsto r$. As discussed in Chapter 1, $\mathcal{H}$ admits an expansion to an elementary $T_{\mathcal{R}}$-extension of $\mathcal{R}$. Moreover, the natural derivation on $\mathcal{H}$ is a $T_{\mathcal{R}}$-derivation and the underlying ordered differential field of $\mathcal{H}$ is an H-field. Thus, $\mathcal{H}$ can be further expanded to an $H_{T_{\mathcal{R}}}$-field where the valuation ring, $\mathcal{O}_{\mathcal{H}}$, is the convex hull of $\mathbb{R}$ in $\mathcal{H}$. Arbitrary $\mathcal{R}$-Hardy fields (those which may not contain $\mathbb{R}$) are not $H_{T_{\mathcal{R}}}$-fields in general, but as we will see in Section 7.3, all $\mathcal{R}$-Hardy fields are pre-$H_{T_{\mathcal{R}}}$-fields.

Now we turn our attention to transseries and surreal numbers.

Example 7.5 (Transseries)**.** Recall from Example 6.5 that the expansion $\mathbb{T}_{\mathrm{an,exp}}$ of $\mathbb{T}$ is a model of $T^{\mathcal{O},\partial}_{\mathrm{an,exp}}$. It is well-known that $\mathbb{T}$ is an H-field; the axiom (H1) is verified in [**32**, 4.3] and the axiom (H2) follows easily since the valuation ring $\mathcal{O}$ is the convex hull of the (Dedekind complete) constant field $\mathbb{R}$. Therefore, $\mathbb{T}_{\mathrm{an,exp}}$ is an $H_{\mathrm{an,exp}}$-field.

Example 7.6 (Surreal numbers)**.** The surreal numbers with derivation and valuation as in Example 6.6 is also an H-field; see [**8**, 6.24 and 6.25]. Thus, the expansion $\mathbf{No}_{\mathrm{an,exp}}$ is an $H_{\mathrm{an,exp}}$-field.

A major result in [**4**] is that $\mathbb{T}$ is model complete in the language $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$. It is natural to ask: is $\mathbb{T}_{\mathrm{an,exp}}$ model complete in the language $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$? In [**6**], it is shown that $\mathbb{T}$ admits a canonical elementary $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding into $\mathbf{No}$. In particular, $\mathbb{T}$ and $\mathbf{No}$ are elementarily equivalent as $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-structures. It is easy to see that this embedding is even an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding (it is strongly $\mathbb{R}$-linear and is compatible with the exponential on both $\mathbb{T}$ and $\mathbf{No}$). One may wonder: is this embedding $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-elementary? Answers to these questions are likely some ways off even for the reduct $\mathbb{T}_{\mathrm{an}}$, though this thesis is a first step. Of course, $T_{\mathrm{an,exp}}$ is not power bounded, so the results this chapter and the previous chapter can not be applied directly to $H_{\mathrm{an,exp}}$-fields. They can be applied to H_{an}-fields.

In Chapter 8, we show that the reduct $\mathbb{T}_{\mathrm{re}}$ of $\mathbb{T}_{\mathrm{an}}$ is model complete in the language $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$ and that the embedding $\mathbb{T} \to \mathbf{No}$ in [**6**] is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementary. These results essentially follow from the results in [**4**]; the only result from this chapter which is used is Proposition 7.43. Our proof in Chapter 8 uses facts about restricted elementary functions which do not hold for general restricted analytic functions, so we believe that the results in on H_T-fields in this chapter will be necessary to prove any sort of model completeness result for $\mathbb{T}_{\mathrm{an}}$ or $\mathbb{T}_{\mathrm{an,exp}}$.

Remark 7.7. The Hahn differential field $\boldsymbol{k}[[\mathfrak{M}]]$ considered in Example 6.4 is *not* an H_T-field unless $\mathfrak{M}$ and $\partial_{\boldsymbol{k}}$ are both trivial. Indeed, suppose $\mathfrak{M}$ is nontrivial and take $\mathfrak{m} \in \mathfrak{M}$ with $\mathfrak{m} \succ 1$. Then $\mathfrak{m} > \mathcal{O}$ but $\partial\mathfrak{m} = 0$, violating (H1). If $\mathfrak{M}$ is trivial, then $\mathcal{O} = \boldsymbol{k}$ and $\smallO = \{0\}$, so if (H2) holds, then $\partial_{\boldsymbol{k}}$ is trivial.

7.1. H_T-asymptotic fields

Definition 7.8. K is an H_T**-asymptotic field** if for all $g \in K$ with $g \succ 1$, we have

(HA1) $g^\dagger > 0$,
(HA2) $g^\dagger \succ f'$ for all $f \in \smallO$, and
(HA3) $g^\dagger \succcurlyeq f'$ for all $f \in \mathcal{O}^\times$.

The definition above differs slightly from the definition of an H-asymptotic field given in [**4**], though we claim that every H_T-asymptotic field is H-asymptotic. Indeed, (HA2) and (HA3) along with [**4**, 9.1.3] imply that every H_T-asymptotic field K is *asymptotic*, that is, $f \prec g \iff f' \prec g'$ for all $f, g \in K^\times$ with $f, g \prec 1$. To see that each H_T-asymptotic field K is H*-asymptotic* in the sense of [**4**], let $f, g \in K^\times$ with $f \prec g \prec 1$. We need to show that $f^\dagger \succcurlyeq g^\dagger$. Applying condition (HA1) to g^{-1} and g/f, we have

$$g^\dagger \ = \ -(g^{-1})^\dagger \ < \ 0, \qquad g^\dagger - f^\dagger \ = \ (g/f)^\dagger \ > \ 0,$$

so $f^\dagger < g^\dagger < 0$. In particular, $f^\dagger \succcurlyeq g^\dagger$, as desired. Conversely, if $K \models T^{\mathcal{O},\partial}$ is asymptotic, then K satisfies (HA2) and (HA3) by [**4**, 9.1.3]. However, the H-asymptotic fields in [**4**] are not necessarily ordered, and not every H-asymptotic $K \models T^{\mathcal{O},\partial}$ satisfies (HA1).

For the remainder of this section, we assume that K is an H_T-asymptotic field. We say "H_T-asymptotic field extension" to mean "$T^{\mathcal{O},\partial}$-extension which is an H_T-asymptotic field." Note that any H_T-asymptotic field with nontrivial valuation must also have nontrivial derivation by (HA1). Indeed, (HA1) ensures that the constant field of any H_T-asymptotic field is contained in the valuation ring. We will collect a few facts from [**4**] about H-asymptotic fields for later use, and then we will examine the immediate H_T-asymptotic field extensions of K.

Fact 7.9 ([**4**], 9.1.4)**.** *Let $f, g \in K$ with $g \not\asymp 1$. If $f \prec g$, then $f' \prec g'$. If $f \preccurlyeq g$, then $f \sim g \iff f' \sim g'$.*

Let $f \in K^\times$ with $f \not\asymp 1$. As K is asymptotic, the values $v(f')$ and $v(f^\dagger)$ only depend on vf, so for $\gamma = vf$, we set

$$\gamma^\dagger \ := \ v(f^\dagger), \qquad \gamma' \ := \ v(f') \ = \ \gamma + \gamma^\dagger.$$

This gives us a map

$$\psi\colon \Gamma^{\neq} \to \Gamma, \qquad \psi(\gamma) \ := \ \gamma^\dagger.$$

Following Rosenlicht [**59**], we call the pair (Γ, ψ) the **asymptotic couple of** K. We have the following important subsets of Γ:

$$(\Gamma^<)' \ := \ \{\gamma' : \gamma \in \Gamma^<\}, \qquad (\Gamma^>)' \ := \ \{\gamma' : \gamma \in \Gamma^>\},$$
$$(\Gamma^{\neq})' \ := \ (\Gamma^<)' \cup (\Gamma^>)', \qquad \Psi \ := \ \psi(\Gamma^{\neq}) \ = \ \{\gamma^\dagger : \gamma \in \Gamma^{\neq}\}.$$

It is always the case that $(\Gamma^<)' < (\Gamma^>)'$ and that $\Psi < (\Gamma^>)'$. If there is $\beta \in \Gamma$ with $\Psi < \beta < (\Gamma^>)'$, then we call β a **gap in** K. There is at most one such β, and if Ψ has a largest element, then there is no such β. If K has trivial valuation, then the four important subsets above are empty and 0 is a gap in K. We say that K **is grounded** if Ψ has a largest element, and we say that K **is ungrounded** otherwise. Finally, we say that

K **has asymptotic integration** if $\Gamma = (\Gamma^{\neq})'$. If β is a gap in K or if $\beta = \max \Psi$, then $\Gamma \setminus (\Gamma^{\neq})' = \{\beta\}$. We have the following trichotomy for the structure of H_T-asymptotic fields:

Fact 7.10 ([**4**], 9.2.16)**.** *Exactly one of the following holds:*

(1) *K has asymptotic integration;*
(2) *K has a gap;*
(3) *K is grounded.*

Remark 7.11. Though it will not be used in this thesis, it is worth noting that if Δ is a nontrivial convex Λ-subspace of Γ, then the Δ-coarsening K_Δ is H_T-asymptotic by [**4**, 9.2.24]. If K has small derivation and $\psi(\Delta^{\neq}) \subseteq \Delta$, then the Δ-specialization of K is H_T-asymptotic by [**4**, 9.1.3]. The axiom (HA1) does not follow directly from [**4**, 9.2.24] or [**4**, 9.1.3], but it is easily verified in both cases, using that it holds in K.

Immediate extensions of H_T-asymptotic fields. By definition, any H_T-asymptotic field is a model of $T^{\mathcal{O},\partial}$, but the continuity assumption for $T^{\mathcal{O},\partial}$ is actually implied by the second condition. In the lemma below, we give a test for whether an immediate $T^{\mathcal{O},\partial}$-extension of K is H_T-asymptotic.

Lemma 7.12. *Let M be an immediate $T^{\mathcal{O},\partial}$-extension of K. If $f' \prec g^\dagger$ for all $f \in \smallo_M$ and all $g \in K$ with $g \succ 1$, then M is an H_T-asymptotic field.*

Proof. Let $h \in M$ with $h \succ 1$ and take $g \in K$ with $h \sim g$. For $\varepsilon \in \smallo_M$ with $h = g(1+\varepsilon)$, we have

$$h^\dagger - g^\dagger \;=\; (h/g)^\dagger \;=\; (1+\varepsilon)^\dagger \;=\; \frac{\varepsilon'}{1+\varepsilon} \;\sim\; \varepsilon'.$$

By assumption, $\varepsilon' \prec g^\dagger$, so $h^\dagger \sim g^\dagger$. As K is H_T-asymptotic, we have $g^\dagger > 0$, so $h^\dagger > 0$ as well. Additionally, we have $h^\dagger \sim g^\dagger \succ f'$ for all $f \in \smallo_M$, by assumption. Now let $f \in \mathcal{O}_M^\times$. Take $u \in K$ and $\delta \in \smallo_M$ with $f = u + \delta$, so $f' = u' + \delta'$. We have $\delta' \prec g^\dagger$ by assumption and $u' \preccurlyeq g^\dagger$, since K is H_T-asymptotic and $u, g \in K$, so $f' \preccurlyeq g^\dagger \sim h^\dagger$. □

We can use the above lemma to relate H_T-asymptotic field extensions to strict extensions, as considered in Chapter 6. First, let us consider the relationship between $\Gamma(\partial)$ and Ψ.

Lemma 7.13. *If K is grounded or has asymptotic integration, then $\Gamma(\partial) = \Psi^\downarrow$. If β is a gap in K, then $\Gamma(\partial) = \Psi^\downarrow \cup \{\beta\}$.*

Proof. Let $\gamma \in \Psi$ and take $g \in K$ with $g \succ 1$ and $v(g^\dagger) = \gamma$. Then $g^\dagger \succ f'$ for all $f \in \smallo$ by (HA2), so $\gamma \in \Gamma(\partial)$. This shows that $\Psi^\downarrow \subseteq \Gamma(\partial)$. Now let $\beta \in \Gamma \setminus \Psi^\downarrow$. If β is not a gap in K, then $\beta \in (\Gamma^>)'$, so $\beta \notin \Gamma(\partial)$. If β is a gap in K, then $\beta = \sup \Psi$, so $\beta \in \Gamma(\partial)$ by Lemma 6.9. □

Corollary 7.14. *Let M be an immediate $T^{\mathcal{O},\partial}$-extension of K. Then M is a strict extension of K if and only if M is H_T-asymptotic.*

Proof. Suppose that M is a strict $T^{\mathcal{O},\partial}$-extension of K and let $f \in \smallo_M$. As $\Psi \subseteq \Gamma(\partial)$ by Lemma 7.13, we have $f' \prec \phi$ for all $\phi \in K^\times$ with $v\phi \in \Psi$. Thus, M is an H_T-asymptotic field by Lemma 7.12. Conversely, suppose that M is H_T-asymptotic and let $\phi \in K^\times$ with $v\phi \in \Gamma(\partial)$. Lemma 7.13 gives $v\phi < (\Gamma^>)'$. We claim that $\partial\smallo_M \subseteq \phi\smallo_M$, from which strictness follows by Fact 6.8. Let $f \in \smallo_M$ and take $g \in \smallo$ with $f \sim g$. Then $v(g') \in (\Gamma^>)'$, so $f' \sim g' \prec \phi$ by Fact 7.9. □

Since K is H_T-asymptotic, we have $S(\partial) = \{0\}$ by [**5**, 1.14], so K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension by Theorem 6.24. By Corollary 7.14, this extension is H_T-asymptotic. Thus, we have the following:

Corollary 7.15. *K has a spherically complete immediate H_T-asymptotic field extension.*

In this chapter, we are primarily interested in simple extensions. Since $S(\partial) = \{0\}$, we can use Proposition 6.23 to get a handle on certain simple immediate H_T-asymptotic field extensions of K:

Proposition 7.16. *Let $G\colon K \to K$ be an $\mathcal{L}(K)$-definable function, let (a_ρ) be a divergent pc-sequence in K, and suppose $a'_\rho - G(a_\rho) \rightsquigarrow 0$. Then K has an immediate H_T-asymptotic field extension $K\langle a\rangle$ with $a_\rho \rightsquigarrow a$ and $a' = G(a)$. If b is a pseudolimit of (a_ρ) in an H_T-asymptotic field extension M of K with $b' = G(b)$, then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b.*

Proof. Since there is a divergent pc-sequence in K, we have $\mathcal{O} \neq K$, so ∂ is nontrivial by (HA1) and Assumption 6.2 is satisfied. This allows us to freely use the results in Chapter 6. Let $F\colon K^2 \to K$ be the function $F(Y_0, Y_1) := Y_1 - G(Y_0)$, so F is in implicit form. Let ℓ be a pseudolimit of (a_ρ) in an elementary $T^{\mathcal{O},\partial}$-extension of K. We claim that $F \in Z_1(K, \ell)$. If not, then since $Z_0(K, \ell) = \emptyset$ by Lemma 6.16, we may apply Proposition 6.18, to get that

$$y' - G(y) \;=\; F(y, y') \;\sim\; F(\ell, \ell')$$

for all $y \in K$ sufficiently close to ℓ. In particular, $a'_\rho - G(a_\rho) \sim a'_\sigma - G(a_\sigma)$ for all sufficiently large indices σ, ρ, contradicting our assumption that $a'_\rho - G(a_\rho) \rightsquigarrow 0$. Thus, $F \in Z_1(K, \ell)$ as claimed, so Proposition 6.23 gives an immediate strict $T^{\mathcal{O},\partial}$-extension $K\langle a\rangle$ of K with $a' = G(a)$ and $v(a - y) = v(\ell - y)$ for each $y \in K$. In particular, $v(a - a_\rho) = v(\ell - a_\rho)$ for all indices ρ, so a is a pseudolimit of (a_ρ). The $T^{\mathcal{O},\partial}$-model $K\langle a\rangle$ is an H_T-asymptotic field by Corollary 7.14. For b and M as in the statement of the proposition, Corollary 5.12 gives a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. This is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Lemma 3.31. □

Corollary 7.17. *Let $s \in K$ with $vs \in (\Gamma^{>})'$ and $s \notin \partial\smallo$ and suppose $v(s - \partial\smallo)$ has no largest element. Then K has an immediate H_T-asymptotic field extension $K\langle a\rangle$ with $a \prec 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K with $s \in \partial\smallo_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$.*

Proof. Let (a_ρ) be a well-indexed sequence in $\smallo$ such that $v(s - a'_\rho)$ is strictly increasing as a function of ρ and cofinal in $v(s - \partial\smallo)$. The proof of [**4**, 10.2.4] gives that (a_ρ) is a divergent pc-sequence in K. We apply Proposition 7.16 where G is the constant function s to get an immediate H_T-asymptotic field extension $K\langle a\rangle$ of K with $a_\rho \rightsquigarrow a$ and $a' = s$. Let M be an H_T-asymptotic field extension of K and let $b \in \smallo_M$ with $b' = s$. Then for $\rho < \sigma$, we have

$$(b - a_\rho)' \;=\; s - a'_\rho \;\sim\; (a_\sigma - a_\rho)'.$$

Since $b - a_\rho, a_\sigma - a_\rho \prec 1$, Fact 7.9 gives us that $b - a_\rho \sim a_\sigma - a_\rho$, so $a_\rho \rightsquigarrow b$. Proposition 7.16 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath\colon K\langle a\rangle \to M$ sending a to b. For uniqueness, let $\jmath\colon K\langle a\rangle \to M$ be an arbitrary $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\jmath(a) - b \in C_M$ since $\jmath(a)' = s = b'$. Since $\jmath(a), b \prec 1$ and $C_M^\times \subseteq \mathcal{O}_M^\times$, we must have $\jmath(a) = b$. This shows that $\jmath = \imath$. □

Corollary 7.18. *Let $s \in K$ with $v(s - \partial K) < (\Gamma^{>})'$ and suppose $v(s - \partial K)$ has no largest element. Then K has an immediate H_T-asymptotic field extension $K\langle a\rangle$ with $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b.*

Proof. Let (a_ρ) be a well-indexed sequence in K such that $v(s-a'_\rho)$ is strictly increasing as a function of ρ and cofinal in $v(s-\partial K)$ and such that $s-a'_\rho \prec s$ for each ρ. The proof of [**4**, 10.2.6] gives that (a_ρ) is a divergent pc-sequence in K. We apply Proposition 7.16 where G is the constant function s to get an immediate H_T-asymptotic field extension $K\langle a\rangle$ of K with $a_\rho \rightsquigarrow a$ and $a'=s$. Let M be an H_T-asymptotic field extension of K and let $b\in M$ with $b'=s$. For $\rho<\sigma$, we have

$$(b-a_\rho)' \;=\; s-a'_\rho \;\sim\; (a_\sigma-a_\rho)',$$

so $v(b-a_\rho)'\in(\Gamma^<)'$ and $b-a_\rho \succ 1$. Fact 7.9 gives $b-a_\rho \sim a_\sigma-a_\rho$, so $a_\rho \rightsquigarrow b$ and Proposition 7.16 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath\colon K\langle a\rangle \to M$ sending a to b. ☐

7.2. λ-freeness

In this section, let K be an ungrounded H_T-asymptotic field with $\Gamma\neq 0$. A **logarithmic sequence in** K is a well-indexed sequence (ℓ_ρ) in K such that:

(1) $\ell'_{\rho+1} \asymp \ell^\dagger_\rho$ for all ρ;
(2) $\ell_\rho \succ \ell_\sigma \succ 1$ for all $\sigma>\rho$;
(3) $(v\ell_\rho)$ is cofinal in $\Gamma^<$.

Logarithmic sequences can be constructed by transfinite recursion. Note that if M is an H_T-asymptotic field extension of K with $\Gamma^<$ cofinal in $\Gamma^<_M$, then any logarithmic sequence in K is a logarithmic sequence in M.

A **λ-sequence in** K is a sequence (λ_ρ) where $\lambda_\rho = -\ell^{\dagger\dagger}_\rho$ for some logarithmic sequence (ℓ_ρ) in K. By [**4**, 11.5.3], any two λ-sequences are equivalent as pc-sequences. We say K is **λ-free** if no λ-sequence in K has a pseudolimit in K.

Lemma 7.19. *If K is an increasing union of λ-free H_T-asymptotic fields, then K is λ-free.*

Proof. By [**4**, 11.6.1], K is λ-free if and only if for all $s\in K$, there is $g\in K$ with $g\succ 1$ and $s-g^{\dagger\dagger}\succcurlyeq g^\dagger$. This equivalent condition is preserved by increasing unions. ☐

For the remainder of this section, let (ℓ_ρ) be a logarithmic sequence in K with corresponding λ-sequence (λ_ρ). Nothing here will depend on the specific choice of (ℓ_ρ). Below are two consequences of λ-freeness from [**4**]

Fact 7.20 ([**4**], 11.5.2 and 11.6.1 (v))**.** *If K is λ-free, then K has asymptotic integration. If K is λ-free and λ is a pseudolimit of (λ_ρ) in an H_T-asymptotic field extension of K, then $v(\lambda-K)=\Psi^\downarrow$.*

For us, the importance of λ-freeness comes from its relation to gaps:

Lemma 7.21. *Suppose K has asymptotic integration, let $s\in K$, and let $M=K\langle f\rangle$ be an H_T-asymptotic field extension of K with $f\neq 0$ and $f^\dagger=s$. Then vf is a gap in $K\langle f\rangle$ if and only $\lambda_\rho\rightsquigarrow -s$.*

Proof. One direction is by [**4**, 11.5.12]: if vf is a gap in M then $\lambda_\rho \rightsquigarrow -s$. For the other direction, suppose $\lambda_\rho\rightsquigarrow -s$. Then [**4**, 11.5.13] with $-s$ in place of λ gives

(i) $v\big(s-(K^\times)^\dagger\big)$ is a cofinal subset of $\Psi^\downarrow$, and
(ii) $vf\notin\Gamma$.

We claim that $\Psi_M \subseteq \Psi^{\downarrow}$. The Wilkie inequality and (ii) above give $\Gamma_M = \Gamma \oplus \Lambda vf$, so we need to show that $\psi(\gamma + \lambda vf) \in \Psi^{\downarrow}$ for each $\gamma \in \Gamma$ and each $\lambda \in \Lambda$. Let γ and λ be given and take $y \in K^{>}$ with $vy = \gamma$. We may assume $\lambda \neq 0$, and we set $z := y^{-1/\lambda}$, so $y^{\dagger} = -\lambda z^{\dagger}$. We have

$$\psi(\gamma + \lambda vf) \;=\; v(yf^{\lambda})^{\dagger} \;=\; v(y^{\dagger} + \lambda s) \;=\; v(\lambda s - \lambda z^{\dagger}) \;=\; v(s - z^{\dagger}),$$

so $\psi(\gamma + \lambda vf) \in \Psi^{\downarrow}$ by (i) above and $\Psi_M \subseteq \Psi^{\downarrow}$ as claimed. Now suppose toward contradiction that vf is not a gap in M. Then Ψ_M, being a cofinal subset of $\Psi^{\downarrow}$, has no maximum and so $vf \in (\Gamma_M^{\neq})'$. Take $\beta \in \Gamma_M^{\neq}$ with $\beta' = vf$ and take $y \in K$ with $\beta^{\dagger} < vy \in \Psi$. Our assumption that $\boldsymbol{\lambda}_{\rho} \rightsquigarrow -s$ along with [**4**, 11.5.6 (iii)] gives $s - y^{\dagger} \prec y$, so

$$\psi(vf - vy) \;=\; v(f/y)^{\dagger} \;=\; v(s - y^{\dagger}) > vy,$$

contradicting [**4**, 9.2.2] (with $\alpha = vf$ and $\gamma = vy$). □

In [**41**], Gehret defines a property—the yardstick property—which allows us to check whether $\boldsymbol{\lambda}$-freeness is preserved in various extensions. Let S be a nonempty convex subset of Γ without a largest element.

(1) We say that S has the **yardstick property** if there is $\beta \in S$ such that $\gamma - \chi(\gamma) \in S$ for all $\gamma \in S^{>\beta}$, where $\chi(0) = 0$ and $\chi(\alpha)$ is the unique element of $\Gamma^{<}$ with $\chi(\alpha)' = \psi(\alpha)$ for $\alpha \neq 0$.
(2) We say that S is **jammed** if for every nontrivial convex subgroup $\{0\} \neq \Delta \subseteq \Gamma$, there is $\beta \in S$ such that $\gamma - \beta \in \Delta$ for all $\gamma \in S^{>\beta}$.

Note that if S is jammed, then so is $\gamma + S$ for any $\gamma \in \Gamma$. Being jammed and having the yardstick property are incompatible, except in the following case:

Fact 7.22 ([**41**], 3.17)**.** *Let S be a nonempty convex subset of Γ without a largest element which has the yardstick property. Then S is jammed if and only if $S^{\downarrow} = \Gamma^{<}$.*

The lemma below is an analog of [**41**, 6.19] with virtually the same proof; only minor modifications and substitutions are required.

Lemma 7.23. *Let $K\langle a \rangle$ be a simple immediate H_T-asymptotic field extension of K. If K is $\boldsymbol{\lambda}$-free and $v(a - K) \subseteq \Gamma$ has the yardstick property, then $K\langle a \rangle$ is $\boldsymbol{\lambda}$-free.*

Proof. Suppose toward contradiction that $K\langle a \rangle$ is not $\boldsymbol{\lambda}$-free and take $\boldsymbol{\lambda} \in K\langle a \rangle$ with $\boldsymbol{\lambda}_{\rho} \rightsquigarrow \boldsymbol{\lambda}$. Since $\boldsymbol{\lambda} \notin K$, Lemma 5.7 gives $\gamma \in \Gamma$ with $v(\boldsymbol{\lambda} - K) = \gamma + v(a - K)$. Fact 7.20 gives $v(\boldsymbol{\lambda} - K) = \Psi^{\downarrow}$, so $v(\boldsymbol{\lambda} - K)$ is jammed by [**41**, 3.11]. Thus, $v(a - K)$ is jammed as well, so $v(a - K) = \Gamma^{<}$ by Fact 7.22. In particular, $v(a - K)$ has a supremum in Γ, so $v(\boldsymbol{\lambda} - K) = \Psi^{\downarrow}$ also has a supremum in Γ, contradicting that K has asymptotic integration by Fact 7.20. □

$\boldsymbol{\omega}$-freeness. An **$\boldsymbol{\omega}$-sequence in** K is a sequence $(\boldsymbol{\omega}_{\rho})$ where $\boldsymbol{\omega}_{\rho} = -(2\boldsymbol{\lambda}_{\rho}' + \boldsymbol{\lambda}_{\rho}^2)$ for some $\boldsymbol{\lambda}$-sequence $(\boldsymbol{\lambda}_{\rho})$ in K. We say K is **$\boldsymbol{\omega}$-free** if no $\boldsymbol{\omega}$-sequence in K has a pseudolimit in K. If $\boldsymbol{\lambda}_{\rho} \rightsquigarrow \boldsymbol{\lambda} \in K$, then the corresponding $\boldsymbol{\omega}$-sequence $(\boldsymbol{\omega}_{\rho})$ has pseudolimit $-(2\boldsymbol{\lambda}' + \boldsymbol{\lambda}^2) \in K$, so $\boldsymbol{\omega}$-freeness implies $\boldsymbol{\lambda}$-freeness. The property of $\boldsymbol{\omega}$-freeness plays a much larger role than $\boldsymbol{\lambda}$-freeness in [**4**], but in this chapter, $\boldsymbol{\lambda}$-freeness is the more central concept. Even so, $\boldsymbol{\omega}$-freeness makes an appearance in Corollary 7.35 and Proposition 7.43 below. Proposition 7.43 will be used in Chapter 8, and we discuss $\boldsymbol{\omega}$-freeness a bit more in the context of H-fields in Section 8.1.

7.3. H_T-fields and pre-H_T-fields

Definition 7.24. K is a **pre-H_T-field** if for all $g \in K$ with $g \succ 1$, we have

(PH1) $g^\dagger > 0$, and
(PH2) $g^\dagger \succ f'$ for all $f \in \mathcal{O}$.

Every pre-H_T-field is H_T-asymptotic, and if K is H_T-asymptotic and $g^\dagger \succ f'$ for all $f, g \in K$ with $g \succ f \asymp 1$, then K is a pre-H_T-field. As with H_T-asymptotic fields, every pre-H_T-field is a pre-H-field, as defined in [**4**]. In the case of pre-H_T-fields, the converse also holds: every model of $T^{\mathcal{O},\partial}$ which is a pre-H-field is a pre-H_T-field. To see this, use [**4**, 10.1.1] and note that (PH1) is equivalent to the condition that $g' > 0$ for all $g \in K$ with $g > \mathcal{O}$. If K is an $\mathcal{L}^{\mathcal{O},\partial}$-substructure of a pre-$H_T$-field, then K is a pre-H_T-field.

Recall from the beginning of this chapter that K is an **H_T-field** if

(H1) for all $f \in K$, if $f > \mathcal{O}$, then $f' > 0$, and
(H2) $\mathcal{O} = C + \smallo$.

Note that if K is an H_T-field, then C is a lift of $\operatorname{res}(K)$.

Lemma 7.25. *The following are equivalent:*

(1) *K is a pre-H_T-field and $\mathcal{O} = C + \smallo$;*
(2) *K is an H_T-asymptotic field and $\mathcal{O} = C + \smallo$;*
(3) *K is an H_T-field.*

Proof. It is immediate that (1) implies (2). Suppose (2) holds and let $f \in K$ with $f > \mathcal{O}$. Then $f \succ 1$ so $f^\dagger > 0$ by (HA1). As $f > 0$, this gives $f' > 0$, so (H1) is satisfied. Of course (H2) is satisfied by assumption, so (3) holds. To see that (3) implies (1), we need to show that every H_T-field is a pre-H_T-field. For (PH1), let $f \in K$ with $f \succ 1$. Then $|f| > \mathcal{O}$, so $|f|' > 0$. Since $f^\dagger = |f|^\dagger$, this gives $f^\dagger > 0$. Now for (PH2), let $f, g \in K$ with $g \succ 1$ and $f \preccurlyeq 1$. We need to show that $g^\dagger \succ f'$. This is shown in [**4**, 10.5.1], but we repeat the proof here. First, by replacing g with $-g$ if need be, we may assume that $g > 0$. As $\mathcal{O} = C + \smallo$, we may subtract a constant from f to arrange that $f \prec 1$. Let $c \in C^>$, so $0 < c + f, c - f \asymp 1$. This gives $g(c+f), g(c-f) > \mathcal{O}$, so $g'(c+f) + gf', g'(c-f) - gf' > 0$ by (H1), yielding

$$g'(c-f) \ > \ gf' \ > \ -g'(c+f).$$

Dividing by g gives

$$g^\dagger(c-f) \ > \ f' \ > \ -g^\dagger(c+f).$$

As $f \prec 1$ and $c \in C^>$ can be taken to be arbitrarily small, we see that $f' \prec g^\dagger$ as desired. ☐

Corollary 7.26. *Let K be an H_T-field and let M be an H_T-asymptotic field extension of K with* $\operatorname{res}(M) = \operatorname{res}(K)$. *Then M is an H_T-field with $C_M = C$.*

Proof. We have $C \subseteq C_M$ and by (HA1), we have $C_M \subseteq \mathcal{O}_M$. As C is a lift of $\operatorname{res}(K) = \operatorname{res}(M)$, it is maximal among the elementary $\mathcal{L}$-substructures of M contained in $\mathcal{O}_M$, so $C = C_M$ and $\mathcal{O}_M = C + \smallo_M$; see [**29**, 2.11 and 2.12]. We conclude that M is an H_T-field by Lemma 7.25. ☐

Lemma 7.27. *Let K be a pre-H_T-field and let M be an immediate H_T-asymptotic field extension of K. Then M is a pre-H_T-field. If K is an H_T-field, then M is as well.*

Proof. Let $f, g \in M$ with $g \succ f \asymp 1$. We need to show that $g^\dagger \succ f'$. Using that $\Gamma_M = \Gamma$, take $a \in K$ with $g \asymp a$, so $g^\dagger \asymp a^\dagger$. Using that $\operatorname{res}(M) = \operatorname{res}(K)$, take $b \in K$ with $f - b \prec 1$, so $(f-b)' \prec a^\dagger$, as M is H_T-asymptotic. As K is a pre-H_T-field, we also have $b' \prec a^\dagger$, so

$$f' \ = \ (f-b)' + b' \ \prec \ a^\dagger \ \asymp \ g^\dagger.$$

If K is an H_T-field, then Corollary 7.26 gives that M is an H_T-field as well. □

As with H_T-asymptotic fields, we say "(pre)-H_T-field extension" to mean "$T^{\mathcal{O},\partial}$-extension which is a (pre)-H_T-field." Using Lemma 7.27, we have the following consequence of Corollary 7.15:

Corollary 7.28. *Every pre-H_T-field has a spherically complete immediate pre-H_T-field extension. Every H_T-field has a spherically complete immediate H_T-field extension.*

The next lemma gives a useful test for whether a simple extension of K is a pre-H_T-field.

Lemma 7.29. *Let K be a pre-H_T-field and let $M = K\langle a \rangle$ be a $T^{\mathcal{O}}$-extension of K with $va \notin \Gamma$. Suppose M is equipped with a T-derivation extending the T-derivation on K such that for all $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$, we have*

(i) $(ga^\lambda)^\dagger > 0$,

(ii) $(ga^\lambda)^\dagger \succ f'$ *for all* $f \in K$ *with* $f \preccurlyeq 1$,

(iii) $(ga^\lambda)^\dagger \succ F(a)'$ *for all* $\mathcal{L}(K)$*-definable functions* $F\colon K \to K$ *with* $F(a) \prec 1$ *and* $F(a) \notin K$.

Then M is a pre-H_T-field. If K is an H_T-field, then so is M.

Proof. Let $h \in M$ with $h \succ 1$ and take $g \in K^\times$ and $\lambda \in \Lambda$ with $h \asymp ga^\lambda$. By the Wilkie inequality, we have $\operatorname{res}(M) = \operatorname{res}(K)$, so by multiplying g with an element in $\mathcal{O}^\times$, we may even assume that $h \sim ga^\lambda$. Take $\varepsilon \in o_M$ with $h = ga^\lambda(1+\varepsilon)$. Then

$$h^\dagger - (ga^\lambda)^\dagger \ = \ (1+\varepsilon)^\dagger \ = \ \frac{\varepsilon'}{1+\varepsilon} \ \sim \ \varepsilon'.$$

We have $\varepsilon' \prec (ga^\lambda)^\dagger$ by (ii) and (iii), so $h^\dagger \sim (ga^\lambda)^\dagger$. In particular, $h^\dagger > 0$ by (i), so (PH1) holds. For (PH2), let $f \in \mathcal{O}_M$. We need to show that $h^\dagger \succ f'$. Since $h^\dagger \sim (ga^\lambda)^\dagger$, it suffices to show that $(ga^\lambda)^\dagger \succ f'$. This follows from (ii) if $f \in K$, so we may assume $f \in M \setminus K$. As $\operatorname{res}(M) = \operatorname{res}(K)$, we may take $u \in \mathcal{O}$ with $f - u \prec 1$. Take an $\mathcal{L}(K)$-definable function $F\colon K \to K$ with $F(a) = f - u$. Then $f' = u' + F(a)' \prec (ga^\lambda)^\dagger$ by (ii) and (iii). Finally, if K is an H_T-field, then M is as well by Corollary 7.26, since $\operatorname{res}(M) = \operatorname{res}(K)$. □

Remark 7.30. Let K be an H_T-field. If $E \models T$ is an $\mathcal{L}^{\mathcal{O},\partial}$-substructure of K, then E may not be an H_T-field, as the (existential) condition (H2) may not be met. Of course, E will be a pre-H_T-field. Let Δ be a nontrivial convex Λ-subspace of Γ. Then the Δ-coarsening K_Δ is an H_T-asymptotic field by Remark 7.11, but it is not an H_T-field, as there are elements in the valuation ring $\dot{\mathcal{O}}$ of K_Δ which are greater than the constant field $C_{K_\Delta} = C$. It may or may not be the case that K_Δ is a pre-H_T-field; see [**4**, 10.1.5] for necessary and sufficient conditions for this to hold. If K has small derivation and $\psi(\Delta^{\neq}) \subseteq \Delta$, then $\operatorname{res}(K_\Delta)$ is an H_T-field by [**4**, 10.1.18].

Remark 7.31. Here is one observation worth making in connection with our discussion of dense pairs in Section 4.5: the $\mathcal{L}^P$ reduct (K, P) of an H_T-field K, where $P(K) = C$, is a *tame pair*, as defined in [**29**].

7.4. Extensions by integrals and the H_T-field hull

In this section, let K be a pre-H_T-field. At the end of this section, we show that K has an H_T-field extension, called the *H_T-field hull*, which embeds uniquely over K into any H_T-field extension of K. Before this, we consider various extensions by integrals. Some of these extensions will be used for the H_T-field hull, but others will not be used until Section 7.6.

Adjoining integrals.

Lemma 7.32. *Let $s \in K$ and suppose vs is a gap in K. Then K has a pre-H_T-field extension $K\langle a\rangle$ with $a \prec 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K with $s \in \partial\smallo_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$. The pre-H_T-field $K\langle a\rangle$ is grounded with*

$$\operatorname{res} K\langle a\rangle \ = \ \operatorname{res} K, \qquad \Gamma_{K\langle a\rangle} \ = \ \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a\rangle} \ = \ \Psi \cup \{va^\dagger\}, \qquad va^\dagger \ > \ \Psi.$$

Proof. By replacing s with $-s$ if need be, we arrange that $s < 0$. Let $K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K where $a > 0$ and $0 < va < \Gamma^>$. The Wilkie inequality gives $\Gamma_{K\langle a\rangle} = \Gamma \oplus \Lambda va$ and $\operatorname{res} K\langle a\rangle = \operatorname{res} K$. Using Lemma 3.31, we equip $K\langle a\rangle$ with the unique T-derivation that extends the derivation on K and satisfies $a' = s$. We need to show that $K\langle a\rangle$ is a pre-H_T-field extension of K. Let $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$. By Lemma 7.29, it suffices to verify the following:

(i) $(ga^\lambda)^\dagger > 0$;
(ii) $(ga^\lambda)^\dagger \succ f'$ for all $f \in K$ with $f \preccurlyeq 1$;
(iii) $(ga^\lambda)^\dagger \succ F(a)'$ for all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.

Since $(\Gamma^<)' < vs$ and $0 < va < \Gamma^>$, we have

$$(\Gamma^<)' \ < \ vs - va \ < \ vs.$$

If $g \asymp 1$, then $g^\dagger \asymp g' \preccurlyeq s \prec a^\dagger$, so

$$(ga^\lambda)^\dagger \ = \ g^\dagger + \lambda a^\dagger \ \sim \ \lambda a^\dagger.$$

Since $a \prec 1$ and $ga^\lambda \asymp a^\lambda \succ 1$ by assumption, we must have $\lambda < 0$. Since $a^\dagger = s/a < 0$, we have $\lambda a^\dagger > 0$, so $(ga^\lambda)^\dagger > 0$ as well. If $g \not\asymp 1$, then we must have $g \succ 1$, so $g^\dagger > 0$. Since $vg^\dagger \in (\Gamma^<)'$, we have

$$(ga^\lambda)^\dagger \ = \ g^\dagger + \lambda a^\dagger \ \sim \ g^\dagger \ > \ 0.$$

This takes care of (i) and also gives us that $(ga^\lambda)^\dagger \succcurlyeq a^\dagger$. We can use this to quickly take care of (ii): if $f \in \mathcal{O}$, then $f' \preccurlyeq s \prec a^\dagger \preccurlyeq (ga^\lambda)^\dagger$.

Now we turn to (iii). Let $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$. We need to show that $F(a)' \prec a^\dagger = s/a$. We consider two cases. First, suppose $\mathcal{O} = K$. Take an $\mathcal{L}(\emptyset)$-definable function $G\colon K^{1+n} \to K$ and an $\mathcal{L}(\emptyset)$-independent tuple $b = (b_1, \ldots, b_n) \in K^n$ with $F(a) = G(a, b)$. Then

$$F(a)' \ = \ G(a,b)' \ = \ \mathbf{J}_G(a,b)(s, b_1', \ldots, b_n'),$$

so by applying Corollary 5.24 with $(s, b_1', \ldots, b_n') \in K^{1+n}$ in place of d, we get $F(a)' \prec a^{-1}$. Since $s \asymp 1$, this gives $F(a)' \prec s/a$, as desired. Now suppose $\mathcal{O} \neq K$. We need to show that $F^{[\partial]}(a) + F'(a)s \prec s/a$. Proposition 5.23 gives $F'(a) \preccurlyeq a^{-1}F(a) \prec a^{-1}$, so $F'(a)s \prec s/a$ and it remains to show that $F^{[\partial]}(a) \prec s/a$. Since $K\langle a\rangle$ is an elementary $T^{\mathcal{O}}$-extension of K, it suffices to show that for each $\mathcal{L}^{\mathcal{O}}(K)$-definable set $A \subseteq K$ with $a \in A^{K\langle a\rangle}$, there is $y \in A$ with $F^{[\partial]}(y) \prec s/y$. Let A be such a set and, by shrinking A if need be, assume that F is $\mathcal{C}^1$ on A and that $y, F(y) \prec 1$ for all $y \in A$. Since $F'(a) \prec a^{-1}$, we can use $\mathcal{L}^{\mathcal{O}}$-elementarity to take $y \in A$ with $F'(y) \prec y^{-1}$. Multiplying by y' gives $F'(y)y' \prec y^\dagger$ for this y. Since $F(y) \prec 1$ and K is a pre-H_T-field, we have $F(y)' \prec y^\dagger$. Thus,

$$F^{[\partial]}(y) \;=\; F(y)' - F'(y)y' \;\prec\; y^\dagger.$$

Since $y \prec 1$ and vs is a gap in K, we have $y' \prec s$, so $F^{[\partial]}(y) \prec y^\dagger \prec s/y$, as desired.

Finally, let M be an H_T-asymptotic field extension of K and let $b \in \mathcal{o}_M$ with $b' = s$. Then $b^\dagger = s/b$ must be negative by (HA1), so b is positive since s is negative. Moreover, vb must realize the cut $\Gamma^{\leqslant}$ since $vs \in (\Gamma_M^>)'$ and $vs < (\Gamma^>)'$. Lemma 5.20 gives a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $\imath\colon K\langle a\rangle \to M$ sending a to b and Lemma 3.31 tells us that $\imath$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Let $\jmath\colon K\langle a\rangle \to M$ be an arbitrary $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\jmath(a) - b \in C_M$ since $\jmath(a)' = s = b'$. Since $\jmath(a), b \prec 1$, we see that $\jmath(a) = b$. This shows that $\jmath = \imath$, so $\imath$ is unique. □

Lemma 7.33. *Let K be an H_T-field, let $s \in K$, and suppose vs is a gap in K. Then K has an H_T-field extension $K\langle a\rangle$ with $a \succ 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b \succ 1$ and $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. The H_T-field $K\langle a\rangle$ is grounded with*

$$\operatorname{res} K\langle a\rangle \;=\; \operatorname{res} K, \qquad \Gamma_{K\langle a\rangle} \;=\; \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a\rangle} \;=\; \Psi \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi.$$

Proof. We may assume that $s > 0$. Let $K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K where $a > 0$ and $\Gamma^< < va < 0$, so $\Gamma_{K\langle a\rangle} = \Gamma \oplus \Lambda va$ and $\operatorname{res} K\langle a\rangle = \operatorname{res} K$ by the Wilkie inequality. Using Lemma 3.31, we equip $K\langle a\rangle$ with the unique T-derivation that extends the derivation on K and satisfies $a' = s$. To see that $K\langle a\rangle$ is an H_T-field extension of K, let $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$. By Lemma 7.29, it suffices to verify the following:

(i) $(ga^\lambda)^\dagger > 0$;
(ii) $(ga^\lambda)^\dagger \succ f'$ for all $f \in K$ with $f \preccurlyeq 1$;
(iii) $(ga^\lambda)^\dagger \succ F(a)'$ for all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.

Since $vs < (\Gamma^>)'$ and $\Gamma^< < va < 0$, we have

$$vs \;<\; vs - va \;<\; (\Gamma^>)'.$$

Since K is an H_T-field, we have for each $f \in \mathcal{O}$ some $c \in C$ with $f - c \in \mathcal{o}$, which gives

$$v(f') \;=\; v(f-c)' \;\in\; (\Gamma^>)' \;>\; va^\dagger.$$

Therefore if $g \asymp 1$, then $v(g^\dagger) = v(g') > v(a^\dagger)$ and

$$(ga^\lambda)^\dagger \;=\; g^\dagger + \lambda a^\dagger \;\sim\; \lambda a^\dagger.$$

Since λ and $a^\dagger = s/a$ are both positive, we have $(ga^\lambda)^\dagger \sim \lambda a^\dagger > 0$. If $g \not\asymp 1$, then $g \succ 1$ and $g^\dagger \succ a^\dagger$, which gives

$$(ga^\lambda)^\dagger \;=\; g^\dagger + \lambda a^\dagger \;\sim\; g^\dagger \;>\; 0.$$

This takes care of (i) and tells us that $(ga^\lambda)^\dagger \succcurlyeq a^\dagger$. Then (ii) follows, since $v(f') > va^\dagger \geqslant v(ga^\lambda)^\dagger$ for $f \in \mathcal{O}$.

Now we turn to (iii). Let $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$. As in the proof of Lemma 7.32, we need to show that $F(a)' = F^{[\partial]}(a) + F'(a)s \prec s/a$. Proposition 5.23 gives $F'(a) \preccurlyeq a^{-1}F(a) \prec a^{-1}$, so $F'(a)s \prec s/a$ and it remains to show that $F^{[\partial]}(a) \prec s/a$. We claim that $|F^{[\partial]}(a)| < s/a^2 \prec s/a$. Since $F^{[\partial]}$ is $\mathcal{L}(K)$-definable, it suffices to show that for each interval $I \subseteq K$ with $a \in I^{K\langle a\rangle}$, there is $y \in I$ with

$|F^{[\partial]}(y)| < s/y^2$. Let I be such an interval and, by shrinking I if need be, assume that F is $\mathcal{C}^1$ on I and that $|F(y)| < 1$ for all $y \in I$. Since K is an H_T-field and a realizes the cut $\mathcal{O}^{\downarrow}$, the interval I contains a constant $c \in C^{>}$. Since $|F(c)| < 1$ and s is a gap in K, we have

$$F(c)' \ = \ F^{[\partial]}(c) + F'(c)c' \ = \ F^{[\partial]}(c) \ \prec \ s.$$

Since $c^2 \asymp 1$, we have $c^2 F^{[\partial]}(c) \prec s$, which yields $|F^{[\partial]}(c)| < s/c^2$, as desired.

Finally, let M be an H_T-asymptotic field extension of K and let $b \in M$ with $b \succ 1$ and $b' = s$. Then $b^\dagger = s/b$ must be positive, so b is positive since s is positive. Since $vs \in (\Gamma_M^{<})'$ and $vs > (\Gamma^{<})'$, we see that vb must realize the cut $\Gamma^{<}$. Lemma 5.20 gives a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $\imath \colon K\langle a\rangle \to M$ sending a to b, and this is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Lemma 3.31. □

Lemma 7.34. *Let K be an H_T-field, let $s \in K$ and suppose $vs = \max \Psi$. Then K has an H_T-field extension $K\langle a\rangle$ with $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. The H_T-field $K\langle a\rangle$ is grounded with*

$$\operatorname{res} K\langle a\rangle \ = \ \operatorname{res} K, \qquad \Gamma_{K\langle a\rangle} \ = \ \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a\rangle} \ = \ \Psi \cup \{va^\dagger\}, \qquad va^\dagger \ > \ \Psi.$$

Proof. Identical to the proof of Lemma 7.33. For the embedding part of the lemma, one needs to note that for $b \in M$ with $b' = s$, it must be the case that $b \succ 1$. □

The assumption that K is an H_T-field in Lemma 7.33 is necessary, but this assumption can be removed in Lemma 7.34; see Remark 7.40 below. Lemma 7.34 can be used to associate to each grounded H_T-field a canonical ungrounded ω-free extension. First, let us say that K is **closed under logarithms** if for each $a \in K^\times$, there is $b \in K$ with $b' = a^\dagger$.

Corollary 7.35. *Let K be a grounded H_T-field. Then K has an ungrounded ω-free (hence, λ-free) H_T-field extension K_ω with $\operatorname{res}(K_\omega) = \operatorname{res}(K)$ which embeds over K into any H_T-asymptotic field extension of K which is closed under logarithms.*

Proof. Let $s \in K$ with $vs^\dagger = \max \Psi$. Using Lemma 7.34, we take an H_T-field extension $K\langle a\rangle$ where $a' = s^\dagger$. We have

$$\operatorname{res} K\langle a\rangle \ = \ \operatorname{res} K, \qquad \Gamma_{K\langle a\rangle} \ = \ \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a\rangle} \ = \ \Psi \cup \{va^\dagger\}, \qquad va^\dagger \ > \ \Psi.$$

Repeating this process, we construct for each n an H_T-field extension K_n of K with

$$K_0 \ = \ K, \qquad K_{n+1} \ = \ K_n\langle a_n\rangle, \qquad a_0 \ = \ a, \qquad a_{n+1}' \ = \ a_n^\dagger.$$

Set $K_\omega := \bigcup_n K_n$. Then $\operatorname{res}(K_\omega) \ = \ \operatorname{res}(K)$ and

$$\Gamma_{K_\omega} \ = \ \Gamma \oplus \bigoplus_n \Lambda va_n, \qquad \Psi_{K_\omega} \ = \ \Psi \cup \{va_0^\dagger, va_1^\dagger, \ldots\}, \qquad \Psi \ < \ va_0^\dagger \ < \ va_1^\dagger \ < \ \cdots .$$

Moreover, K_ω is ω-free by [**4**, 11.7.15], since K_ω is ungrounded and each K_n is grounded. Let M be an H_T-asymptotic field extension of K which is closed under logarithms. Then there are elements $b_0, b_1, \ldots \in M$ with $b_0' = s^\dagger$ and $b_{n+1}' = b_n^\dagger$ for each n. Repeated use of the embedding property in Lemma 7.34 allows us construct an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_\omega \to M$ which sends a_n to b_n for each n. □

We can use Corollaries 7.17 and 7.18 to say something about immediate extensions of K by integrals.

Corollary 7.36. *Let $s \in K$ with $vs \in (\Gamma^{>})'$ and $s \notin \partial\smallo$. Then K has an immediate pre-H_T-field extension $K\langle a\rangle$ with $a \prec 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K with $s \in \partial\smallo_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$. If K is ungrounded and λ-free, then so is $K\langle a\rangle$.*

PROOF. Let $S := v(s - \partial\smallo)$ and let $y \in \smallo$. Then $v(s - y') \in (\Gamma^{>})'$, so we may take $b \in \smallo$ with $s - y' \asymp b'$. Take $u \in \mathcal{O}^{\times}$ with $s - y' = ub'$. Then

$$s - (y + ub)' \ = \ s - y' - ub' - u'b \ = \ -u'b \ \prec \ b' \ \asymp \ s - y',$$

since $u' \prec b^{\dagger}$ by (PH2). Thus, S has no largest element and Corollary 7.17 gives an immediate H_T-asymptotic field extension $K\langle a\rangle$ of K with $a \prec 1$, $a' = s$, and the desired embedding property. By Lemma 7.27, $K\langle a\rangle$ is itself a pre-H_T-field. By [**41**, 8.5], the set $v(a - K)$ has the yardstick property, so if K is ungrounded and λ-free, then $K\langle a\rangle$ is as well by Lemma 7.23. □

Corollary 7.37. *Let $s \in K$ with $v(s - \partial K) \subseteq (\Gamma^{<})'$. Then K has an immediate pre-H_T-field extension $K\langle a\rangle$ with $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. If K is ungrounded and λ-free, then so is $K\langle a\rangle$.*

PROOF. Let $S := v(s - \partial K)$ and let $y \in K$. Then $v(s - y') \in (\Gamma^{<})'$, so we may take $b \succ 1$ with $s - y' \asymp b'$. As in the proof of Corollary 7.36, we have $s - (y + ub)' \prec s - y'$ for $u := (s - y')/b' \in \mathcal{O}^{\times}$, so S has no largest element. Corollary 7.18 gives an immediate H_T-asymptotic field extension $K\langle a\rangle$ of K with $a' = s$ and the desired embedding property. By Lemma 7.27, $K\langle a\rangle$ is itself a pre-H_T-field. By [**41**, 9.6], the set $v(a - K)$ has the yardstick property, so if K is ungrounded and λ-free, then $K\langle a\rangle$ is as well by Lemma 7.23. □

The H_T-field hull. We now show that K has a minimal H_T-field extension. We say that $\beta \in \Gamma$ is a **fake gap in** K if β is a gap in K and $\beta = v(b')$ for some $b \in K$. Then necessarily $b \asymp 1$, for otherwise $\beta \in (\Gamma^{\neq})'$. Likewise, $b \not\sim c$ for any $c \in C$, for otherwise $b' = (b - c)'$ and $\beta \in (\Gamma^{>})'$. Thus, no H_T-field has a fake gap. Of course, if K is grounded or has asymptotic integration, then K does not have a fake gap. Suppose K does not have a fake gap and let M be an immediate pre-H_T-field extension of K. We claim that M does not have a fake gap. Let $b \in M$ with $b \asymp 1$ and take $a \in K$ with $b - a \prec 1$. Then $v(b - a)' \in (\Gamma^{>})'$. As K has no fake gap, we also have $v(a') \in (\Gamma^{>})'$, so

$$v(b') \ = \ v\big((b - a)' + a'\big) \ \geqslant \ \min\big\{v(b - a)', v(a')\big\} \in (\Gamma^{>})'.$$

Theorem 7.38. *K has an H_T-field extension $H_T(K)$ such that for any H_T-field extension M of K, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $H_T(K) \to M$. For $L := H_T(K)$, we have*

$$L \ = \ K\langle C_L\rangle, \qquad \operatorname{res}(L) = \operatorname{res}(K).$$

PROOF. We first construct a pre-H_T-field extension K_0 of K which does not have a fake gap as follows: if K does not have a fake gap, then we let $K_0 := K$. If K has a fake gap $\beta = v(b')$, then we apply Lemma 7.32 with $s = b'$ to get a pre-H_T-field extension $K\langle a\rangle$ of K with $a \prec 1$ and $a' = b'$. Then $K\langle a\rangle$ does not have a fake gap as it is grounded, and we set $K_0 := K\langle a\rangle$. Given an H_T-field extension M of K, we claim that there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_0 \to M$. This is clear if $K_0 = K$. If not, then let b be as above and take $c \in C_M$ with $b \sim c$. Then $b' = (b - c)' \in \partial\smallo_M$, so Lemma 7.32 gives a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_0 \to M$. Note that $K_0 = K\langle b - a\rangle$ and $b - a \in C_{K_0}$, so $K_0 = K\langle C_{K_0}\rangle$. Lemma 7.32 also gives $\operatorname{res}(K_0) = \operatorname{res}(K)$.

Suppose K_0 is not an H_T-field, so there is $b \in \mathcal{O}_{K_0}$ with $b \notin C_{K_0} + \smallo_{K_0}$. Then $b' \notin \partial\smallo_{K_0}$, for otherwise we would have $b - \varepsilon \in C_{K_0}$ for some $\varepsilon \in \smallo_{K_0}$. Since $v(b')$ is not a fake gap, we have $v(b') \in (\Gamma^{>}_{K_0})'$. Corollary 7.36

gives an immediate pre-H_T-field extension $K^* := K_0\langle a\rangle$ of K_0 with $a \prec 1$ and $a' = b'$. Given an H_T-field extension M of K_0, take $c \in C_M$ with $b \sim c$. Then $b' = (b-c)' \in \partial\smallo_M$, so Corollary 7.36 gives a unique $\mathcal{L}^{\mathcal{O},\partial}(K_0)$-embedding $K^* \to M$. Note that $K^* = K_0\langle b-a\rangle$ and $b - a \in C_{K^*}$, so $K^* = K_0\langle C_{K^*}\rangle = K\langle C_{K^*}\rangle$. As K^* is an immediate extension of K_0, there is no fake gap in K^*. By iterating this process, we build an immediate H_T-field extension L of K_0 such that for any H_T-field extension M of K_0, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K_0)$-embedding $L \to M$. Using also the embedding property for K_0 over K, we see that for any H_T-field extension M of K, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L \to M$. Moreover, $\operatorname{res}(L) = \operatorname{res}(K_0) = \operatorname{res}(K)$ and $L = K\langle C_L\rangle$ by construction. We let $H_T(K)$ be this extension L. □

The universal property in Theorem 7.38 determines $H_T(K)$ uniquely up to unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. We call $H_T(K)$ the **H_T-field hull of** K. If K does not have a fake gap, then $H_T(K)$ is an immediate extension of K; in particular, $\Gamma_{H_T(K)} = \Gamma$. If β is a fake gap in K, then $\Gamma_{H_T(K)} = \Gamma \oplus \Lambda va$ for $a \in H_T(K)$ with $\Gamma^< < va < 0$ and $v(a') = \beta$. The following consequence of Theorem 7.38 is not used anywhere, but it may be worth noting.

Corollary 7.39. *The following are equivalent:*

(1) *every spherically complete immediate H_T-asymptotic field extension of K is an H_T-field;*
(2) *K has a spherically complete immediate H_T-field extension;*
(3) *K does not have a fake gap.*

Proof. By Corollary 7.15, we know that K has a spherically complete immediate H_T-asymptotic field extension, so (1) implies (2). Suppose (2) holds and let M be a spherically complete immediate H_T-field extension of K. By the universal property of the H_T-field hull, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $H_T(K) \to M$. Then $H_T(K)$ is an immediate extension of K, so $\Gamma_{H_T(K)} = \Gamma$ and K does not have a fake gap by the remarks preceding this corollary. Finally, suppose (3) holds and let M be a spherically complete immediate H_T-asymptotic field extension of K. Then M is a pre-H_T-field by Lemma 7.27, and the remarks before Theorem 7.38 tell us that M does not have a fake gap. Thus, $H_T(M)$ is an immediate extension of M, so $M = H_T(M)$, as M has no proper immediate extensions. □

Remark 7.40. We can use the H_T-field hull to remove the assumption that K is an H_T-field in Lemma 7.34. Here is how: suppose K is a pre-H_T-field and let $s \in K$ with $vs = \max\Psi$. Let $L = H_T(K)$, so L is an immediate extension of K and $vs = \max\Psi_L$, since K is grounded. Apply Lemma 7.34 to L to get an H_T-field extension $L\langle a\rangle$ with $a' = s$. Then $K\langle a\rangle$, being an $\mathcal{L}^{\mathcal{O},\partial}$-substructure of an H_T-field, is a pre-H_T-field. Of course, $K\langle a\rangle$ is grounded with

$$\operatorname{res} K\langle a\rangle \;=\; \operatorname{res} K, \qquad \Gamma_{K\langle a\rangle} \;=\; \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a\rangle} \;=\; \Psi \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi.$$

Moreover, if b is an element in an H_T-asymptotic field extension M of K with $b' = s$, then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. This stronger version of Lemma 7.34 can be used to strengthen Corollary 7.35 accordingly.

If K is a pre-H_T-field and vs is a gap in K with $s \in K$, we may wonder whether K has a pre-H_T-field extension $K\langle a\rangle$ with $a \succ 1$ and $a' = s$ as in Lemma 7.33. The above argument shows that this can be done so long as vs is not a fake gap in K. If vs is a fake gap in K, then s can not have an infinite integral, so there is no such extension.

7.5. Exponential integral and constant field extensions

In this section, let K be a pre-H_T-field. We begin by looking at two types of extensions of K by *exponential integrals*, that is, elements a with $a^\dagger \in K$. After this, we consider extensions of H_T-fields by constants.

Adjoining exponential integrals.

Lemma 7.41. *Let $s \in K$ with $vs \in (\Gamma^>)'$ and suppose that $s \neq y^\dagger$ for all $y \in K^\times$. Then K has an immediate pre-H_T-field extension $K\langle a\rangle$ with $a \sim 1$ and $a^\dagger = s$ such that for any H_T-asymptotic field extension M of K with $s \in (1+\smallo_M)^\dagger$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$. If K is ungrounded and $\boldsymbol{\lambda}$-free, then so is $K\langle a\rangle$.*

Proof. Let $S := v\big(s-(1+\smallo)^\dagger\big) \subseteq (\Gamma^>)'$. By the proof of [**4**, 10.4.3], S has no largest element. Let (a_ρ) be a well-indexed sequence in $1+\smallo$ such that $v(s-a_\rho^\dagger)$ is strictly increasing in S as a function of ρ. Then (a_ρ) is a divergent pc-sequence in K, again by the proof of [**4**, 10.4.3]. We apply Proposition 7.16 with $G(Y) = sY$ to get an immediate pre-H_T-field extension $K\langle a\rangle$ of K with $a_\rho \rightsquigarrow a$ and $a^\dagger = s$. Note that $a \sim 1$, since each $a_\rho \sim 1$. Let M be an H_T-asymptotic field extension of K and let $b \in M$ with $b \sim 1$ and $b^\dagger = s$. Then $a_\rho^\dagger \rightsquigarrow b^\dagger$, and so $a_\rho \rightsquigarrow b$ by the proof of [**4**, 10.4.3]. Proposition 7.16 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath\colon K\langle a\rangle \to M$ that sends a to b. For uniqueness, let $\jmath\colon K\langle a\rangle \to M$ be an arbitrary $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\jmath(a)/b \in C_M^\times$ since $\jmath(a)^\dagger = s = b^\dagger$. Since $\jmath(a) \sim 1 \sim b$, we see that $\jmath(a) = b$, so $\jmath = \imath$. By [**41**, 7.6], the set $v(a-K)$ has the yardstick property, so if K is ungrounded and $\boldsymbol{\lambda}$-free, then $K\langle a\rangle$ is as well by Lemma 7.23. □

Lemma 7.42. *Let $s \in K$ with $v\big(s-(K^\times)^\dagger\big) \subseteq \Psi^\downarrow$. Then K has a pre-H_T-field extension $K\langle a\rangle$ with $a > 0$ and $a^\dagger = s$ such that for any pre-H_T-field extension M of K and $b \in M^>$ with $b^\dagger = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. Moreover, the extension $K\langle a\rangle$ has the following properties:*

1. *$va \notin \Gamma$ and $\Gamma_{K\langle a\rangle} = \Gamma \oplus \Lambda va$;*
2. *$\operatorname{res} K\langle a\rangle = \operatorname{res} K$;*
3. *Ψ is cofinal in $\Psi_{K\langle a\rangle}$;*
4. *a gap in K remains a gap in $K\langle a\rangle$;*
5. *if K is ungrounded and $\boldsymbol{\lambda}$-free, then so is $K\langle a\rangle$.*

Proof. Let b be an element in a pre-H_T-field extension M of K with $b > 0$ and $b^\dagger = s$. Then $vb \notin \Gamma$; otherwise there is $f \in K$ and $u \in \mathcal{O}_M^\times$ with $b/f = u$, so $s - f^\dagger = u^\dagger \asymp u'$ and $v(s-f^\dagger) > \Psi$, a contradiction. Let $y \in K^\times$ with $y \prec b$. Then $y/b \prec 1$ so $y^\dagger < b^\dagger = s$. Likewise if $y \in K^\times$ with $y \succ b$, then $y^\dagger > s$. Thus, vb realizes the cut

$$S \ := \ \{vy : y^\dagger > s\} \ \subseteq \Gamma.$$

Let $K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K where $a > 0$ and va realizes the cut S. The Wilkie inequality gives $\Gamma_{K\langle a\rangle} = \Gamma \oplus \Lambda va$ and $\operatorname{res} K\langle a\rangle = \operatorname{res} K$. Using Lemma 3.31, we equip $K\langle a\rangle$ with the unique T-derivation which extends the derivation on K and satisfies $a^\dagger = s$. If we can show that $K\langle a\rangle$ is a pre-H_T-field, then the embedding property of $K\langle a\rangle$ follows from Lemmas 3.31 and 5.20 and the discussion above.

To see that $K\langle a\rangle$ is a pre-H_T-field, let $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$. By Lemma 7.29, it suffices to verify the following:

(i) $(ga^\lambda)^\dagger > 0$;
(ii) $(ga^\lambda)^\dagger \succ f'$ for all $f \in K$ with $f \preccurlyeq 1$;
(iii) $(ga^\lambda)^\dagger \succ F(a)'$ for all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.

First we deal with (i). If $\lambda = 0$, then $g \succ 1$ and $(ga^\lambda)^\dagger = g^\dagger > 0$. If $\lambda > 0$, then since $ga^\lambda \succ 1$ we have $a \succ g^{-\lambda^{-1}}$, so $s > (g^{-\lambda^{-1}})^\dagger = -\lambda^{-1}g^\dagger$. This gives $\lambda^{-1}g^\dagger + s > 0$, so

$$(ga^\lambda)^\dagger \;=\; g^\dagger + \lambda s \;=\; \lambda(\lambda^{-1}g^\dagger + s) \;>\; 0.$$

On the other hand, if $\lambda < 0$, then $a \prec g^{-\lambda^{-1}}$, so $\lambda^{-1}g^\dagger + s < 0$ and again,

$$(ga^\lambda)^\dagger \;=\; \lambda(\lambda^{-1}g^\dagger + s) \;>\; 0.$$

Note that since

$$(ga^\lambda)^\dagger \;=\; \lambda(\lambda^{-1}g^\dagger + s) \;=\; \lambda\big(s - (g^{-\lambda^{-1}})^\dagger\big) \;\asymp\; s - (g^{-\lambda^{-1}})^\dagger,$$

we have $v(ga^\lambda)^\dagger \in \Psi^\downarrow$. Thus, for (ii) it suffices to note that $h^\dagger \succ f'$ for all $f, h \in K$, since K is a pre-H_T-field. Likewise, for (iii), it suffices to show that $h^\dagger \succ F(a)'$ for all $h \in K$ with $h \succ 1$ and all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$. Suppose toward contradiction that there are F, h for which this does not hold, so $F(a) \prec 1$ but

$$F(a)' \;=\; F^{[\partial]}(a) + F'(a)as \;\succcurlyeq\; h^\dagger.$$

By replacing F with $-F$ if necessary, we may assume that $F(a)' > 0$. Since $\operatorname{res} K\langle a\rangle = \operatorname{res} K$, we have $F^{[\partial]}(a) + F'(a)as > uh^\dagger > 0$ for some $u \in K$ with $u \asymp 1$. We may take an interval $I \subseteq K^>$ with $a \in I^{K\langle a\rangle}$ such that

$$|F(y)| \;<\; 1, \qquad F^{[\partial]}(y) + F'(y)ys \;>\; uh^\dagger$$

for all $y \in I$. For $y \in I$, we have $F(y) \preccurlyeq 1$, so $F(y)' = F^{[\partial]}(y) + F'(y)y' \prec h^\dagger$ since K is a pre-H_T-field. This gives

$$\big(F^{[\partial]}(y) + F'(y)ys\big) - \big(F^{[\partial]}(y) + F'(y)y'\big) \;=\; F'(y)y(s - y^\dagger) \;>\; \frac{1}{2}uh^\dagger \;>\; 0.$$

In particular, the function $F'(y)y(s - y^\dagger)$ has constant sign on I. By shrinking I, we may assume that $F'(y)$ and y have constant sign on I, so $s - y^\dagger$ has constant sign on I as well. This is a contradiction: if $y \in I$ is greater than a, then $y \succ a$ and $y^\dagger > s$ and if $y \in I$ is less than a then $y \prec a$ and $y^\dagger < s$.

Now that we know that $K\langle a\rangle$ is a pre-H_T-field extension of K with the required embedding property, all that remains is to check that $K\langle a\rangle$ satisfies properties (1)–(5). We have already verified properties (1) and (2). For (3), let $h \in K\langle a\rangle$ with $h \succ 1$. Then $h \asymp ga^\lambda$ for some $g \in K$ and some $\lambda \in \Lambda$ by (1), so $v(h^\dagger) = v(ga^\lambda)^\dagger$, since $K\langle a\rangle$ is H_T-asymptotic. We have already shown that $v(ga^\lambda)^\dagger \in \Psi^\downarrow$, so $v(h^\dagger) \in \Psi^\downarrow$ as well. As for (4), let β be a gap in K. Then $\beta > \Psi$, so $\beta > \Psi_{K\langle a\rangle}$ since Ψ is cofinal in $\Psi_{K\langle a\rangle}$. Suppose toward contradiction that β is not a gap in $K\langle a\rangle$, so $\beta = \alpha'$ for some $\alpha \in \Gamma^>_{K\langle a\rangle}$. The universal property in Lemma 7.32 gives $\max \Psi_{K\langle a\rangle} = \alpha^\dagger > \Psi$, contradicting that Ψ is cofinal in $\Psi_{K\langle a\rangle}$.

Finally, suppose K is ungrounded and let (ℓ_ρ) be a logarithmic sequence in K with corresponding $\boldsymbol{\lambda}$-sequence $(\boldsymbol{\lambda}_\rho)$. Then $K\langle a\rangle$ is ungrounded since Ψ is cofinal in $\Psi_{K\langle a\rangle}$. It follows that $\Gamma^<$ is cofinal in $\Gamma^<_{K\langle a\rangle}$. To see this, let $f \in K\langle a\rangle$ with $f \succ 1$ and suppose toward contradiction that $f \prec g$ for all $g \in K$ with $g \succ 1$. Then $f^\dagger \preccurlyeq g^\dagger$ for such g, so $v(f^\dagger) > \Psi$ since K is ungrounded, contradicting (3). Therefore, (ℓ_ρ) remains a logarithmic sequence in $K\langle a\rangle$ and $(\boldsymbol{\lambda}_\rho)$ remains a $\boldsymbol{\lambda}$-sequence in $K\langle a\rangle$. Suppose toward contradiction that K is $\boldsymbol{\lambda}$-free and that $\boldsymbol{\lambda}_\rho \rightsquigarrow \boldsymbol{\lambda} \in K\langle a\rangle$. Then $\boldsymbol{\lambda} \notin K$, so $a \in K\langle\boldsymbol{\lambda}\rangle$. This is a contradiction, as $K\langle\boldsymbol{\lambda}\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K by Lemma 5.10 and $va \notin \Gamma$. This proves (5). $\square$

We can use Lemma 7.42 along with Corollary 7.28 and Lemma 7.21 to prove the following extension result for pre-H_T-fields. This result will be used in Chapter 8.

Proposition 7.43. *K has an ungrounded ω-free H_T-field extension.*

Proof. By passing to the H_T-field hull of K, we may assume that K is an H_T-field. First, we will show that every H_T-field with asymptotic integration has an H_T-field extension with a gap. Then, we will show that every H_T-field with a gap has a grounded H_T-field extension. Finally, we will show that every grounded H_T-field has an ungrounded ω-free H_T-field extension.

For the first part, suppose K has asymptotic integration. As having asymptotic integration is a property of the asymptotic couple of K, every immediate extension of K also has asymptotic integration. By applying Corollary 7.28 we can pass to a spherically complete extension of K, so we may assume that K is spherically complete. Let (λ_ρ) be a λ-sequence in K, so (λ_ρ) has a pseudolimit $\lambda \in K$ by spherical completeness. The set $v\big(\lambda + (K^\times)^\dagger\big)$ is a cofinal subset of $\Psi^\downarrow$ by [**4**, 11.5.13], so Lemma 7.42 gives an H_T-field extension $K\langle a\rangle$ of K with $a^\dagger = -\lambda$. This extension has a gap, namely va, by Lemma 7.21.

Now, assume that K has a gap $\beta \in \Gamma$. Take $s \in K$ with $vs = \beta$ and use Lemma 7.32 to get a grounded H_T-field extension $K\langle a\rangle$ of K with $a' = s$. Finally, if K is grounded, apply Corollary 7.35 to get an ungrounded ω-free H_T-field extension K_ω of K. □

Constant field extensions.

Proposition 7.44. *Let K be an H_T-field and let E be a T-extension of C. Then there is an H_T-field extension L of K where C_L is $\mathcal{L}(C)$-isomorphic to E such that for any H_T-field extension M of K and any $\mathcal{L}(C)$-embedding $\imath\colon C_L \to C_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L \to M$ extending $\imath$.*

Proof. It suffices to consider the case $E = C\langle f\rangle$ where $f \notin C$. Let $L = K\langle a\rangle$ be a simple T-extension of K where a realizes the cut

$$(C^{<f} + o)^\downarrow \;=\; \{y \in K : y < \mathcal{O}\} \cup \{c + \varepsilon : c \in C^{<f} \text{ and } \varepsilon \in o\}.$$

We expand L to an $\mathcal{L}^{\mathcal{O}}$-structure by letting

$$\mathcal{O}_L \;:=\; \big\{y \in L : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

This expansion of L is a $T^{\mathcal{O}}$-extension of K by Fact 5.3. Note that $a \in \mathcal{O}_L$ and that $\bar a \notin \operatorname{res}(K)$, so the Wilkie inequality gives $\Gamma_L = \Gamma$. Using Lemma 3.31, we extend ∂ uniquely to a T-derivation on L with $a' = 0$.

We claim that L is an H_T-field extension of K. To see that L satisfies (H1), let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) > \mathcal{O}_L$. We need to show

$$F(a)' \;=\; F^{[\partial]}(a) + F'(a)a' \;=\; F^{[\partial]}(a) \;>\; 0,$$

As $F^{[\partial]}$ is $\mathcal{L}(K)$-definable, it suffices to show that for any subinterval $I \subseteq K$ with $a \in I^L$, there is $y \in I$ with $F^{[\partial]}(y) > 0$. Let I be such a subinterval. Using that $\Gamma_L = \Gamma$ and that $\Gamma^>$ has no least element, take $d \in K$ with $F(a) > d > \mathcal{O}_L$. By shrinking I, we arrange that $F(y) > d$ for all $y \in I$. Since $\bar a \in \bar I^{\operatorname{res}(L)}$, we see that $\bar I$ must be infinite. Thus, $I \cap C$ is infinite, so take $c \in I \cap C$. As $c \in C$, we have $F(c)' = F^{[\partial]}(c)$. As $c \in I$, we have $F(c) > d > \mathcal{O}$, so $F(c)' = F^{[\partial]}(c) > 0$, as desired. By (H1), we have $C_L \subseteq \mathcal{O}_L$. Clearly, C_L contains $C\langle a\rangle$. Since $C\langle a\rangle$ is a lift of $\operatorname{res} L$, it is maximal among the elementary $\mathcal{L}$-substructures of L contained in $\mathcal{O}_L$, so $C\langle a\rangle = C_L$ and $\mathcal{O}_L = C_L + o_L$; see [**29**, 2.11 and 2.12]. This establishes (H2) and completes the proof that L is an H_T-field. It also tells us that C_L is $\mathcal{L}(C)$-isomorphic to E.

Given an H_T-field extension M of K and an $\mathcal{L}(C)$-embedding $\imath\colon C_L \to M$, there is at most one possible $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\jmath\colon L \to M$ which extends $\imath$, namely the one which sends a to $\imath(a)$. Let us show that this is actually an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. By assumption, a and $\imath(a)$ realize the same cut over C. Since $a - c \notin \smallo_L$ and $\imath(a) - c \notin \smallo_M$ for all $c \in C$, this assumption gives that a and $\imath(a)$ realize the same cut over $\mathcal{O}$. As $a \in \mathcal{O}_L$ and $\imath(a) \in \mathcal{O}_M$, we see that a and $\imath(a)$ realize the same cut over K, so $\jmath$ is an $\mathcal{L}(K)$-embedding. Lemma 3.31 ensures that $\jmath$ is an $\mathcal{L}^{\partial}(K)$-embedding. To see that $\jmath$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, let $f \in K\langle a\rangle$. If $f \in \mathcal{O}_L$, then $|f| < c$ for some $c \in C_L$, so $|\jmath(f)| < \imath(c) \in C_M$, which gives $\jmath(f) \in \mathcal{O}_M$. Conversely, if $f \notin \mathcal{O}_L$, then $|f| > d$ for some $d \in K$ with $d > \mathcal{O}$, so $|\jmath(f)| > d$, which gives $\jmath(f) \notin \mathcal{O}_M$. □

7.6. Liouville closed H_T-fields

In this section, K is an H_T-field.

Definition 7.45. We say that K is **Liouville closed** if for each $y \in K$ there is $f \in K$ and $g \in K^{\times}$ with $f' = g^{\dagger} = y$. A **T-Liouville extension** of K is an H_T-field extension L of K where

(1) $C_L = C$, and
(2) each $a \in L$ is contained in an H_T-subfield $K\langle t_1, \ldots, t_n\rangle \subseteq L$ where for $i = 1, \ldots, n$, either $t_i' \in K\langle t_1, \ldots, t_{i-1}\rangle$ or $t_i \neq 0$ and $t_i^{\dagger} \in K\langle t_1, \ldots, t_{i-1}\rangle$.

A **T-Liouville closure** of K is a T-Liouville extension of K which is Liouville closed.

Below we list some easily verified facts about T-Liouville extensions of K.

Fact 7.46.

(1) *If L is a T-Liouville extension of K and M is a T-Liouville extension of L, then M is a T-Liouville extension of K.*
(2) *If M is a T-Liouville extension of K and L is an H_T-field extension of K contained in M, then M is a T-Liouville extension of L.*
(3) *If $(L_i)_{i\in I}$ is an increasing chain of T-Liouville extensions of K, then the union $\bigcup_{i\in I} L_i$ is a T-Liouville extension of K.*
(4) *Every T-Liouville extension of K has the same cardinality as K.*

Let $K\langle a\rangle$ be one of the extensions constructed in Lemmas 7.32, 7.33, 7.34, 7.41 and 7.42 and in Corollaries 7.36 and 7.37. Then $K\langle a\rangle$ is an H_T-field extension of K with constant field C by Corollary 7.26. Thus, $K\langle a\rangle$ is a T-Liouville extension of K. Suppose K is grounded and let K_{ω} be the H_T-field extension of K constructed in Corollary 7.35. Then K_{ω} is the union of an increasing chain of T-Liouville extensions of K, so K_{ω} is an T-Liouville extension of K by Fact 7.46.

T-Liouville towers.

Definition 7.47. A **T-Liouville tower on** K is a strictly increasing chain $(K_\mu)_{\mu\leqslant\nu}$ of H_T-fields such that:

(1) $K_0 = K$;
(2) if $\mu \leqslant \nu$ is an infinite limit ordinal, then $K_\mu = \bigcup_{\eta<\mu} K_\eta$;
(3) if $\mu < \nu$, then $K_{\mu+1} = K_\mu\langle a_\mu\rangle$ with $a_\mu \notin K_\mu$ and one of the following holds:
    (a) $a_\mu' = s_\mu \in K_\mu$ with $a_\mu \prec 1$ and vs_μ is a gap in K_μ;
    (b) $a_\mu' = s_\mu \in K_\mu$ with $a_\mu \succ 1$ and vs_μ is a gap in K_μ;
    (c) $a_\mu' = s_\mu \in K_\mu$ with $vs_\mu = \max \Psi_{K_\mu}$;
    (d) $a_\mu' = s_\mu \in K_\mu$ with $a_\mu \prec 1$, $vs_\mu \in (\Gamma_{K_\mu}^{>})'$, and $s_\mu \notin \partial\smallo_{K_\mu}$;
    (e) $a_\mu' = s_\mu \in K_\mu$ with $v(s_\mu - \partial K_\mu) \subseteq (\Gamma_{K_\mu}^{<})'$;

(f) $a_\mu^\dagger = s_\mu \in K_\mu$ with $a_\mu \sim 1$, $vs_\mu \in (\Gamma_{K_\mu}^{>})'$, and $s_\mu \neq y^\dagger$ for all $y \in K_\mu^\times$;

(g) $a_\mu^\dagger = s_\mu \in K_\mu$ with $a_\mu > 0$ and $v\big(s_\mu - (K_\mu^\times)^\dagger\big) \subseteq \Psi_{K_\mu}^{\downarrow}$.

The H_T-field K_ν is called the *top of the tower* $(K_\mu)_{\mu\leqslant\nu}$.

Note that (a), (b), (c), (f), and (g) correspond to Lemmas 7.32, 7.33, 7.34, 7.41, and 7.42 and that (d) and (e) correspond to Corollaries 7.36 and 7.37, respectively. Thus, if $(K_\mu)_{\mu\leqslant\nu}$ is a T-Liouville tower on K, then each K_μ is a T-Liouville extension of K. Since each T-Liouville extension of K has the same cardinality as K, maximal T-Liouville towers on K exist by Zorn's lemma.

Lemma 7.48. *Let L be the top of a maximal T-Liouville tower on K. Then L is Liouville closed and, therefore, L is a T-Liouville closure of K.*

Proof. By (a) and (b), L does not have a gap, and by (c), L is not grounded, so L has asymptotic integration by Fact 7.10. Let $s \in L$. If s has no integral in L, then $v(s - \partial L) \not\subseteq (\Gamma_L^{<})'$ by (e), so there is $y \in L$ with $v(s - y') > (\Gamma_L^{<})'$. Since L has asymptotic integration, we have $v(s - y') \in (\Gamma_L^{>})'$ so by (d), there is $f \in \mathcal{O}_L$ with $f' = s - y'$. Then $s = (f + y)'$, a contradiction. If s has no exponential integral in $K^\times$, then $v\big(s - (L^\times)^\dagger\big) \not\subseteq \Psi_L^{\downarrow}$ by (g), so there is $b \in L^\times$ with $v(s - b^\dagger) > \Psi_L^{\downarrow}$. Again, asymptotic integration gives $v(s - b^\dagger) \in (\Gamma_L^{>})'$ so by (f), there is $g \in L^\times$ with $g \sim 1$ and $g^\dagger = s - b^\dagger$. Then $s = (bg)^\dagger$, another contradiction. □

Lemma 7.48 gives the existence of T-Liouville closures. In the remainder of this section, we investigate uniqueness. First, let us use T-Liouville towers to prove some additional facts about Liouville closed H_T-fields.

Lemma 7.49. *K is Liouville closed if and only if K has no proper T-Liouville extensions.*

Proof. If K is not Liouville closed, let $(K_\mu)_{\mu\leqslant\nu}$ be a maximal T-Liouville tower on K. Then K_ν is a T-Liouville closure of K by Lemma 7.48, so in particular, K_ν is a proper T-Liouville extension of K. Now suppose K is Liouville closed, let L be a T-Liouville extension of K, and let $a \in L$. We will show that $a \in K$. By definition, a is contained in an H_T-subfield $K\langle t_1, \ldots, t_n\rangle \subseteq L$ where for $i = 1, \ldots, n$, either $t_i' \in K\langle t_1, \ldots, t_{i-1}\rangle$ or $t_i^\dagger \in K\langle t_1, \ldots, t_{i-1}\rangle$. We show by induction that $K\langle t_1, \ldots, t_i\rangle = K$ for each $i \leqslant n$. Fix $i \leqslant n$ and suppose that $K\langle t_1, \ldots, t_{i-1}\rangle = K$. If $t_i' \in K$, then since K is Liouville closed, there is $f \in K$ with $f' = t_i'$. Then $t_i = f + c$ for some $c \in C_L = C$, so $t_i \in K$ as well. Likewise, if $t_i^\dagger \in K$, then there is $g \in K^\times$ with $g^\dagger = t_i^\dagger$, so $t_i = cg$ for some $c \in C_L^\times = C^\times$, again giving $t_i \in K$. □

Lemma 7.50. *Let L be a Liouville closed H_T-field extension of K and let $(K_\mu)_{\mu\leqslant\nu}$ be a T-Liouville tower on K. Suppose that $(K_\mu)_{\mu\leqslant\nu}$ is a tower in L, that is, each K_μ is an H_T-subfield of L. Suppose also that $(K_\mu)_{\mu\leqslant\nu}$ cannot be extended to a T-Liouville tower $(K_\mu)_{\mu\leqslant\nu+1}$ in L. Then $(K_\mu)_{\mu\leqslant\nu}$ is a maximal T-Liouville tower on K.*

Proof. By Lemma 7.49, it suffices to show that K_ν is Liouville closed. If vs is a gap in K_ν for some $s \in K_\nu$, then L contains an element a with $a' = s$. By subtracting a constant from a, we may assume that $a \not\asymp 1$. By Lemma 7.32 (if $a \prec 1$) or Lemma 7.33 (if $a \succ 1$), we see that $K_\nu\langle a\rangle \subseteq L$ is a T-Liouville extension of K_ν, contradicting the maximality of $(K_\mu)_{\mu\leqslant\nu}$ in L. Thus, K_ν has no gap and likewise, Lemma 7.34 shows that K_ν is ungrounded, so K_ν has asymptotic integration by Fact 7.10.

Fix $s \in K_\nu$. If $v(s - \partial K_\nu) \subseteq (\Gamma_{K_\nu}^{<})'$, then $K\langle f\rangle$ is a T-Liouville extension of K_ν contained in L for any $f \in L$ with $f' = s$ by Corollary 7.37, contradicting the maximality of $(K_\mu)_{\mu\leqslant\nu}$ in L. Therefore, we may take $y \in K_\nu$ with $v(s - y') > (\Gamma^<_{K_\nu})'$. As K_ν has asymptotic integration, we have $v(s - y') \in (\Gamma^>_{K_\nu})'$. If $s - y' \notin \partial\mathcal{O}_{K_\nu}$, then $K\langle g\rangle$ is a T-Liouville extension of K_ν contained in L for any $g \in \mathcal{O}_L$ with $g' = s - y'$ by Corollary 7.36, again contradicting the maximality of $(K_\mu)_{\mu\leqslant\nu}$ in L. Thus, $s - y' \in \partial\mathcal{O}_{K_\nu}$, so $s \in \partial K_\nu$. A similar argument, using Lemmas 7.41 and 7.42 shows that s has an exponential integral in $K_\nu^\times$. □

Corollary 7.51. *Let L be a Liouville closed H_T-field extension of K. Then L is a T-Liouville closure of K if and only if L has no proper Liouville closed H_T-subfields which contain K.*

Proof. Suppose that L is a T-Liouville closure of K and let M be a Liouville closed H_T-subfield of L containing K. Then L is an T-Liouville extension of M by Fact 7.46, so $M = L$ by Lemma 7.49. Now suppose that L is not a T-Liouville closure of K and let $(K_\mu)_{\mu\leqslant\nu}$ be a maximal T-Liouville tower on K in L. Then K_ν is a T-Liouville closure of K by Lemmas 7.48 and 7.50. In particular, K_ν is a proper Liouville closed H_T-subfield of L containing K. □

λ-freeness and the uniqueness of T-Liouville closures. Whether K has a unique T-Liouville closure up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism is closely tied to the existence of gaps, which is in turn related to λ-freeness.

Lemma 7.52. *Let $(K_\mu)_{\mu\leqslant\nu}$ be a T-Liouville tower on K and suppose K_μ does not have a gap for all $\mu < \nu$. Then K_ν embeds over K into any Liouville closed H_T-field extension of K.*

Proof. Let M be a Liouville closed H_T-field extension of K. We will construct an increasing chain of $\mathcal{L}^{\mathcal{O},\partial}(K)$-embeddings $(\imath_\mu\colon K_\mu \to M)_{\mu\leqslant\nu}$. We let $\imath_0\colon K_0 \to M$ be the identity on K, and we take increasing unions at limits. For successors, fix $\mu < \nu$ and let $\imath_\mu\colon K_\mu \to M$ be an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Since K_μ has no gap, $K_{\mu+1}$ is an extension of type (c), (d), (e), (f), or (g). The embedding properties in Lemmas 7.34, 7.41 and 7.42 and Corollaries 7.36 and 7.37 give an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath_{\mu+1}\colon K_{\mu+1} \to M$ extending $\imath_\mu$. □

Let $\Theta \subseteq \{(a),(b),\ldots,(g)\}$. Then a **Θ-tower on** K is a T-Liouville tower $(K_\mu)_{\mu\leqslant\nu}$ on K where for each $\mu < \nu$, the extension $K_{\mu+1}$ is one of the extensions from Θ. For example, if $\Theta = \{(a),(b),\ldots,(e)\}$, then the only extensions in a Θ-tower are extensions by integrals.

Proposition 7.53. *Suppose K is ungrounded and λ-free. Then K has a T-Liouville closure L which embeds over K into any Liouville closed H_T-field extension of K. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to L.*

Proof. Let $\Theta = \{(d),(e),(f),(g)\}$ and let $(K_\mu)_{\mu\leqslant\nu}$ be a maximal Θ-tower on K. Then for each $\mu \leqslant \nu$, K_μ is ungrounded and λ-free by Lemmas 7.19, 7.41, and 7.42 and Corollaries 7.36 and 7.37 and . In particular, each K_μ has asymptotic integration. This shows that K_ν has no extensions of type (a), (b), or (c), so $(K_\mu)_{\mu\leqslant\nu}$ is even a maximal T-Liouville tower on K. Set $L := K_\nu$, so L is a T-Liouville closure of K. Let M be a Liouville closed H_T-field extension of K. By Lemma 7.52, there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath\colon L \to M$. If, moreover, M is a T-Liouville closure of K, then $\imath(L)$ is a Liouville closed H_T-subfield of M containing K, so $\imath(L) = M$ by Corollary 7.51. Thus, L is unique up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. □

Proposition 7.54. *Suppose K is grounded. Then K has a T-Liouville closure L which embeds over K into any Liouville closed H_T-field extension of K. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to L.*

Proof. Let K_ω be as in Corollary 7.35. Then K_ω is an ungrounded and λ-free, so K_ω has a T-Liouville closure L which embeds over K_ω into any Liouville closed H_T-field extension of K_ω by Proposition 7.53. Then L is a T-Liouville closure of K as well, since K_ω is a T-Liouville extension of K. Let M be a Liouville closed

H_T-field extension of K. Clearly, any Liouville closed H_T-field is closed under logarithms, so Corollary 7.35 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_\omega \to M$ which further extends to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L \to M$. As in Proposition 7.53, uniqueness follows from this embedding property and Corollary 7.51. $\square$

Gaps and the nonuniqueness of T-Liouville closures. If K has a gap vs, then we have a choice to make. Either we can adjoin an integral a of s with $a \prec 1$, as is done in Lemma 7.32, or we can adjoin an integral b of s with $b \succ 1$, as in Lemma 7.33. This "fork in the road" prevents K from having a unique T-Liouville closure, but as we will see below, this is really the only obstruction to uniqueness.

Proposition 7.55. *Let $\beta \in \Gamma$ be a gap in K. Then K has T-Liouville closures L_1 and L_2 with $\beta \in (\Gamma^{>}_{L_1})'$ and $\beta \in (\Gamma^{<}_{L_2})'$. Let M be a Liouville closed H_T-field extension of K. If $\beta \in (\Gamma^{>}_{M})'$, then there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to M$. Likewise, if $\beta \in (\Gamma^{<}_{M})'$, then there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_2 \to M$. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2.*

PROOF. Let $s \in K$ with $vs = \beta$. Let $K_1 := K\langle a\rangle$ be the H_T-field extension of K given in Lemma 7.32, so $a \prec 1$ and $a' = s$, and let $K_2 := K\langle b\rangle$ be the H_T-field extension of K given in Lemma 7.33, so $b \succ 1$ and $b' = s$. Then K_1 is grounded, so it has a T-Liouville closure L_1 which embeds over K_1 into any Liouville closed H_T-field extension of K_1 by Proposition 7.54. Likewise, K_2 has a T-Liouville closure L_2 which embeds over K_2 into any Liouville closed H_T-field extension of K_2. Now let M be a Liouville-closed H_T-field extension of K. If $\beta \in (\Gamma^{>}_{M})'$, then the embedding property in Lemma 7.32 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_1 \to M$, which in turn extends to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to M$. If $\beta \in (\Gamma^{<}_{M})'$, then using the embedding property in Lemma 7.33 instead, we get an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_2 \to M$. If M is a T-Liouville closure of K, then M is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2 by Corollary 7.51 since M contains the $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic image of either L_1 or L_2 as a Liouville closed H_T-subfield. $\square$

We can use Lemma 7.21 to show that H_T-fields with asymptotic integration which are not λ-free also have two distinct T-Liouville closures.

Proposition 7.56. *Suppose that K has asymptotic integration and is not λ-free. Then K has T-Liouville closures L_1 and L_2 which are not $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic. If M is a Liouville closed H_T-field extension of K, then there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of either L_1 or L_2 into M. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2.*

PROOF. Let (λ_ρ) be a λ-sequence in K with pseudolimit $\lambda \in K$, so $v\big(\lambda + (K^\times)^\dagger\big)$ is a cofinal subset of $\Psi^\downarrow$ by [**4**, 11.5.13]. Lemma 7.42 gives an H_T-field extension $K\langle a\rangle$ of K with $a > 0$ and $a^\dagger = -\lambda$. By Lemma 7.21, va is a gap in $K\langle a\rangle$. By Proposition 7.55, $K\langle a\rangle$ has T-Liouville closures L_1 and L_2 with $va \in (\Gamma^{>}_{L_1})'$ and $va \in (\Gamma^{<}_{L_2})'$, one of which embeds over $K\langle a\rangle$ into any Liouville closed H_T-field extension of $K\langle a\rangle$. We claim that there is no $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to L_2$; in particular, L_1 and L_2 are nonisomorphic over K. To see this, take $b_1 \in L_1$ and $b_2 \in L_2$ with $b_1 \prec 1$, $b_2 \succ 1$, and $b_1' = b_2' = a$. Then $(b_1')^\dagger = (b_2')^\dagger = a^\dagger = -\lambda$. Suppose toward contradiction that $\imath\colon L_1 \to L_2$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\imath(b_1')^\dagger = (b_2')^\dagger$ so $\imath(b_1) = c_1 b_2 + c_2$ for some $c_1, c_2 \in C_{L_2}$ with $c_1 \neq 0$. Since $b_2 \succ 1$, this gives $\imath(b_1) \succ 1$, contradicting that $b_1 \prec 1$.

Let M be a Liouville closed H_T-field extension of K. Lemma 7.42 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ and Proposition 7.55 allows us to extend this embedding to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of either L_1 or L_2 into M. As in Proposition 7.55, we may use Corollary 7.51 to see that any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2. $\square$

Putting together the above propositions, we can now precisely state our main theorem on the existence and uniqueness of T-Liouville closures.

Theorem 7.57. *If K is grounded or if K is ungrounded and $\uplambda$-free, then K has exactly one T-Liouville closure up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. If K is ungrounded and not $\uplambda$-free, then K has exactly two T-Liouville closures up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. For any Liouville closed H_T-field extension M of K, there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of some T-Liouville closure of K into M.*

An application to $\mathbb{R}_{\text{an}}$-Hardy fields. In this subsection, let $\mathcal{R}$, $\mathcal{L}_{\mathcal{R}}$, and $T_{\mathcal{R}}$ be as in Example 7.4 and let $\mathcal{H}$ be an $\mathcal{R}$-Hardy field containing $\mathbb{R}$ (so $\mathcal{H}$ is an $H_{T_{\mathcal{R}}}$-field). Let $[f]$ be a germ of a real-valued unary function at $+\infty$. Then $[f]$ is said to be *comparable to* $\mathcal{H}$ if for each $[g] \in \mathcal{H}$, either $g(x) < f(x)$ eventually, or $g(x) > f(x)$ eventually, or $g(x) = f(x)$ eventually (where *eventually* means *for all sufficiently large* x). If $[f]$ is comparable to $\mathcal{H}$, then we let

$$\mathcal{H}\langle[f]\rangle \;:=\; \big\{t([g]) : t \text{ is a unary } \mathcal{L}_{\mathcal{R}}(\mathcal{H})\text{-term}\big\}.$$

If, in addition to being comparable with $\mathcal{H}$, the function f is eventually $\mathcal{C}^1$ and $[f'] \in \mathcal{H}\langle[f]\rangle$, then $\mathcal{H}\langle[f]\rangle$ is an $\mathcal{R}$-Hardy field; see [**30**, 5.12] and the remarks at the end of [**30**]. If $f' = g$ or $f = \exp(g)$ for some $[g] \in \mathcal{H}$, then $[f]$ is comparable to $\mathcal{H}$ by Boshernitzan [**9**, 5.3]; see also [**60**]. In both cases, f is eventually $\mathcal{C}^1$ and $[f'] \in \mathcal{H}\langle[f]\rangle$, so it follows that

- $\mathcal{H}\langle[\exp g]\rangle$ is an $\mathcal{R}$-Hardy field for $[g] \in \mathcal{H}$, and
- $\mathcal{H}\langle[f]\rangle$ is an $\mathcal{R}$-Hardy field if $[f'] \in \mathcal{H}$.

Since any increasing union of $\mathcal{R}$-Hardy fields is an $\mathcal{R}$-Hardy field and since Hardy fields are bounded in size, Zorn's lemma and the remarks above give us a *Liouville closed* $\mathcal{R}$-Hardy field extension of $\mathcal{H}$ where every germ has an integral and an exponential (thus, every germ also has a nonzero exponential integral). We denote by $\mathrm{Li}_{\mathcal{R}}(\mathcal{H})$ the intersection of all Liouville closed $\mathcal{R}$-Hardy field extensions of $\mathcal{H}$. Then $\mathrm{Li}_{\mathcal{R}}(\mathcal{H})$ is a $T_{\mathcal{R}}$-Liouville closure of $\mathcal{H}$ in the sense defined above; this follows from Corollary 7.51 when $T_{\mathcal{R}}$ is power bounded, but it is also true in general (with the appropriate generalization of a T-Liouville closure).

Here is an application when $\mathcal{R} = \mathbb{R}_{\text{an}}$. The appropriate language here is $\mathcal{L}_{\mathcal{R}} = \mathcal{L}^*_{\text{an}}$ (the extension of $\mathcal{L}_{\text{an}}$ by function symbols for multiplicative inversion and n^{th} roots). For an $\mathbb{R}_{\text{an}}$-Hardy field $\mathcal{H}$, let us use $\mathrm{Li}_{\text{an}}(\mathcal{H})$ instead of $\mathrm{Li}_{\mathbb{R}_{\text{an}}}(\mathcal{H})$. The following theorem is an analog of a theorem on Hardy fields from [**2**].

Theorem 7.58. *Let $\mathcal{H}$ be an $\mathbb{R}_{\text{an}}$-Hardy field and let $\imath\colon \mathcal{H} \to \mathbb{T}_{\text{an}}$ be an $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-embedding. Then $\imath$ extends to an $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-embedding $\mathrm{Li}_{\text{an}}(\mathcal{H}) \to \mathbb{T}_{\text{an}}$.*

Proof. Note that any $\mathbb{R}_{\text{an}}$-Hardy field contains $\mathbb{R}_{\text{an}}$ as an $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-substructure. If $\mathcal{H} = \mathbb{R}_{\text{an}}$, then we extend $\imath$ to an $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-embedding of

$$\mathcal{H}(\mathbb{R}_{\text{an}}) \;:=\; \big\{[t] : t \text{ is a unary } \mathcal{L}^*_{\text{an}}(\emptyset)\text{-term}\big\}$$

by sending $[t] \in \mathcal{H}(\mathbb{R}_{\text{an}})$ to $t(x) \in \mathbb{T}_{\text{an}}$ where $x \in \mathbb{T}_{\text{an}}$ is the distinguished positive infinite element with derivative $x' = 1$. One can easily verify that this is an $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-embedding. Thus, by replacing $\mathcal{H}$ by $\mathcal{H}(\mathbb{R}_{\text{an}})$ if need be, we may assume that $\mathcal{H}$ is a proper extension of $\mathbb{R}_{\text{an}}$. Let $K := \imath(\mathcal{H}) \in \mathbb{T}_{\text{an}}$ and let $(K_\mu)_{\mu\leqslant\nu}$ be a maximal T_{an}-Liouville tower on $\imath(\mathcal{H})$ in $\mathbb{T}_{\text{an}}$. Lemma 7.50 tells us that K_ν is a T_{an}-Liouville closure of K. Moreover, none of the H_T-fields K_μ have a gap by [**2**, 6.6], so Lemma 7.52 gives an $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-embedding $\jmath\colon K_\nu \to \mathrm{Li}_{\text{an}}(\mathcal{H})$ which extends $\imath^{-1}$. Since $\mathrm{Li}_{\text{an}}(\mathcal{H})$ is a T_{an}-Liouville closure of $\mathcal{H}$, we have $\jmath(K_\nu) = \mathrm{Li}_{\text{an}}(\mathcal{H})$ by Corollary 7.51. Thus, we may take $\jmath^{-1}\colon \mathrm{Li}_{\text{an}}(\mathcal{H}) \to \mathbb{T}_{\text{an}}$ to be our $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-embedding. ☐

7.7. The order 1 intermediate value property

In this section, let K be a pre-H_T-field. Our goal is to prove the following extension result:

Theorem 7.59. *K has an H_T-field extension M with the following property: for every $\mathcal{L}(M)$-definable continuous function $F\colon M \to M$ and every $b_1, b_2 \in M$ with*

$$b_1' \ <\ F(b_1), \qquad b_2' \ >\ F(b_2),$$

there is $a \in M$ between b_1 and b_2 with $a' = F(a)$.

Before proving this theorem, we need a lemma about extensions of pre-H_T-fields:

Lemma 7.60. *Suppose K is ungrounded, let L be a $T^{\mathcal{O}}$-extension of K with*

$$\mathcal{O}_L \ :=\ \big\{y \in L : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\},$$

and let ∂_L be a T-derivation on L extending ∂ such that $g' > 0$ for all $g \in L$ with $g > \mathcal{O}_L$. Then L is a pre-H_T-field extension of K.

Proof. Let $g \in L$ with $g \succ 1$. Then $|g|' > 0$ by assumption, so $g^\dagger = |g|^\dagger > 0$, proving (PH1). For (PH2), take $f \in \mathcal{O}_L$. We need to show that $f' \prec g^\dagger$. We will do this by showing that $v(f') > \Psi$ and that $vg^\dagger \in \Psi^\downarrow$. To see that $vf' > \Psi$, let $d \in K$ with $d > \mathcal{O}$. Then $d + f, d - f > \mathcal{O}_L$, so $d' + f', d' - f' > 0$. This gives $-d' < f' < d'$, so $v(f') \geqslant v(d')$. Since this holds for all $d \in K$ with $d > \mathcal{O}$, we have $v(f') > (\Gamma^<)'$. As K is ungrounded, we have $\Psi \subseteq (\Gamma^<)'$, so $v(f') > \Psi$. To see that $vg^\dagger \in \Psi^\downarrow$, take $d \in K$ with $|g| > d > \mathcal{O}$. Then $|g|d^{-1/2} > d^{1/2} > \mathcal{O}$ so $\big(|g|d^{-1/2}\big)^\dagger = g^\dagger - d^\dagger/2 > 0$, which gives $g^\dagger > d^\dagger/2$. As $d^\dagger > 0$, we have $vg^\dagger \leqslant vd^\dagger \in \Psi$, so $vg^\dagger \in \Psi^\downarrow$. □

Theorem 7.59 is a consequence of the following proposition; an analog of [**3**, 4.3]. See also [**27**], where van den Dries proves this for $\mathcal{R}$-Hardy fields.

Proposition 7.61. *Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable continuous function and let $b_1, b_2 \in K$ with*

$$b_1' \ <\ F(b_1), \qquad b_2' \ >\ F(b_2).$$

Then there is an H_T-field extension M of K and $a \in M$ between b_1 and b_2 with $a' = F(a)$.

Proof. If K is grounded, then we may use Corollary 7.35 to replace K with an ungrounded pre-H_T-field extension, so we assume that K is ungrounded. Let us handle the case that $b_1 < b_2$. Let I be the interval (b_1, b_2) and set

$$A \ :=\ \big\{y \in I : y' < F(y)\big\}.$$

Since $b_1' < F(b_1)$ and since F and ∂ are continuous, we have $y' < F(y)$ for all $y \in I$ sufficiently close to b_1. Thus, A is nonempty. Likewise, $y' > F(y)$ for all $y \in I$ sufficiently close to b_2, so A is not cofinal in I. If A has a supremum $b \in I$, then $b' = F(b)$ by continuity, and we may take $M = H_T(K)$. Thus, we may assume that A has no supremum in I. Let $L := K\langle a\rangle$ be a simple T-extension of K where a realizes the cut $A^\downarrow$. Using Lemma 3.31, we extend the T-derivation on K uniquely to a T-derivation on L by setting $a' := F(a)$. We also equip L with the T-convex valuation ring

$$\mathcal{O}_L \ :=\ \big\{y \in L : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

We claim that L is a pre-H_T-field extension of K. If we can show this, then we may take $M = H_T(L)$. By Lemma 7.60, it is enough to show that $g' > 0$ for all $g \in L$ with $g > \mathcal{O}_L$. Let $G\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $G(a) > \mathcal{O}_L$. We may assume $G(a) \notin K$. Suppose toward contradiction that

$$G(a)' \;=\; G^{[\partial]}(a) + G'(a)F(a) \;\leqslant\; 0.$$

Take $d \in K$ with $G(a) > d$ and take a subinterval $J \subseteq I$ with $a \in J^L$ such that G is $\mathcal{C}^1$ on J and such that for all $y \in J$, we have

$$G(y) \;>\; d, \qquad G^{[\partial]}(y) + G'(y)F(y) \;\leqslant\; 0.$$

Let $y \in J$. Since $G(y) > d > \mathcal{O}$ and K is a pre-H_T-field, we have $G(y)' = G^{[\partial]}(y) + G'(y)y' > 0$, so

$$\big(G^{[\partial]}(y) + G'(y)y'\big) - \big(G^{[\partial]}(y) + G'(y)F(y)\big) \;=\; G'(y)\big(y' - F(y)\big) \;>\; 0.$$

By shrinking J, we may assume that $G'(y)$ has constant sign on J, so $y' - F(y)$ has constant sign on J as well. This is a contradiction, as J contains elements of A as well as elements of $I \setminus A$. The case that $b_1 > b_2$ is virtually identical; we instead let I be the interval (b_2, b_1) and let

$$A \;:=\; \big\{y \in I : y' > F(y)\big\}. \qquad \square$$

CHAPTER 8

Model completeness for $\mathbb{T}_{\text{re}}$

In [**4**], Aschenbrenner, van den Dries, and van der Hoeven showed that the $\mathcal{L}^{\mathcal{O},\partial}_{\text{ring}}$-theory of the ordered valued differential field $\mathbb{T}$ of logarithmic-exponential transseries is model complete. Recall from Example 7.5 that $\mathbb{T}$ admits an expansion to an $H_{\text{an,exp}}$-field, denoted $\mathbb{T}_{\text{an,exp}}$. While a model completeness result for $\mathbb{T}_{\text{an,exp}}$ (or even its $\mathcal{L}^{\mathcal{O},\partial}_{\text{an}}$-reduct $\mathbb{T}_{\text{an}}$) is still out of reach, we show in this chapter how the proof in [**4**] can be amended slightly to show that the $\mathcal{L}^{\mathcal{O},\partial}_{\text{re}}$-reduct $\mathbb{T}_{\text{re}}$ is model complete.

In addition to model completeness, it is shown in [**4**] that the $\mathcal{L}^{\mathcal{O},\partial}_{\text{ring}}$-theory of $\mathbb{T}$ is one completion of the model complete theory of ω-*free newtonian Liouville closed H-fields*. This theory is the model companion of the theory of H-fields. Recall from Example 7.3 that the class of H_{RCF}-fields is the same as the class of real closed H-fields. Any Liouville closed H-field is real closed by definition (not our definition in Chapter 7, but the definition given in [**4**]; see Section 8.1 below). Thus, the theory of ω-free newtonian Liouville closed H-fields is also the model companion of the theory of H_{RCF}-fields. Note that $\mathbb{T}_{\text{re}}$ is an H_{re}-field. We can leverage our proof of model completeness for $\mathbb{T}_{\text{re}}$ to show that the theory of ω-free newtonian Liouville closed H_{re}-fields is the model companion of the theory of H_{re}-fields.

While these results are obviously related to the other topics in this thesis, the proofs of these results rely primarily on material from [**4**]. The only result from previous chapters which will play a role is Proposition 7.43, and even this can likely be circumvented. Accordingly, we drop our long-standing assumption that K and L are models of T and adopt a new assumption for this chapter:

Assumption 8.1. *For the remainder of this chapter K and L are H-fields.*

Recall from Chapter 1 that we view $K = (K, \mathcal{O}, \partial)$ as an $\mathcal{L}^{\mathcal{O},\partial}_{\text{ring}}$-structure. We still use all of our H_T-field notation for the H-field K: we use C for the constant field of K, we use $\smallo$ for the maximal ideal of $\mathcal{O}$, we use K^{ϕ} to denote the compositional conjugate of K by $\phi \in K^{\times}$, and so on. Though these notions and notations have not been formally defined for H-fields in this thesis, the definitions are the obvious analogs of the definitions made for H_T-fields. Everything is, of course, formally defined in [**4**].

As mentioned above, our proof that $\mathbb{T}_{\text{re}}$ is model complete involves minor variations on the proof in [**4**] that $\mathbb{T}$ is model complete, so we use Section 8.1 as an opportunity to give an overview of model completeness for $\mathbb{T}$. In Section 8.2, we introduce restricted elementary functions on H-fields and exploit the differential-algebraic nature of these functions to strengthen various embedding lemmas. We end this section by proving that the $\mathcal{L}^{\mathcal{O},\partial}_{\text{re}}$-theory of ω-free newtonian Liouville closed restricted elementary H-fields is model complete. In Section 8.3, we use the results in Section 8.2 to prove our main theorems: the $\mathcal{L}^{\mathcal{O},\partial}_{\text{re}}$-theory of $\mathbb{T}_{\text{re}}$ is model complete and the theory of ω-free newtonian Liouville closed H_{re}-fields is the model companion of the theory of H_{re}-fields.

8.1. An overview of model completeness for $\mathbb{T}$

In this section, we walk through the proof in [**4**] that the theory of ω-free newtonian Liouville closed H-fields is the model companion of the theory of H-fields. We also use this proof to characterize the completions of this theory. First, let us say something about the axioms.

The definition of Liouville closedness in the H-field setting is essentially the same as in the H_T-field setting—every element has an integral and an exponential integral. However, we also require that Liouville closed H-fields be real closed (this holds automatically for H_T-fields). The definition of ω-freeness is exactly the same as the definition given in Section 7.2. In that section, ω-freeness was only defined for ungrounded H_T-asymptotic fields, so let us make the convention that by an *ω-free H-field*, we mean an ungrounded H-field which is ω-free. Newtonianity is rather subtle, but when coupled with ω-freeness, it plays a similar role to differential henselianity.

For the remainder of this chapter, let NLC denote the $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}$-theory of ω-free newtonian Liouville closed H-fields. In [**4**], the authors first showed that NLC is the model *completion* of the theory of ω-free H-fields. Proving this uses our Model Completion Criterion. One needs to show that

(1) every ω-free H-field extends to a model of NLC, and
(2) for any $K, L \models$ NLC where L is $|K|^+$-saturated, any ω-free H-subfield $E \subseteq K$, and any $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}$-embedding $\imath\colon E \to L$, there is an $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}(E)$-embedding $K \to L$ which extends $\imath$.

Item (1) is handled by the following fact:

Fact 8.2 ([**4**], 14.5.10)**.** *If K is an ω-free H-field, then K has an ω-free newtonian Liouville closed H-field extension L where the constant field C_L is a real closure of C.*

Let K, L, E, and $\imath$ be as in (2). Since L is real closed, its constant field C_L is real closed as well. As L is assumed to be $|K|^+$-saturated, the constant field C_L is $|C|^+$-saturated as an $\mathcal{L}_{\text{ring}}$-structure. Since RCF is the model completion of the theory of ordered fields, we may extend the map $\imath|_{C_E}\colon C_E \to C_L$ to an $\mathcal{L}_{\text{ring}}$-embedding $\jmath\colon C \to C_L$. We need another fact:

Fact 8.3 ([**4**], 10.5.15 and 10.5.16)**.** *Let K, L be H-fields, let E be an H-subfield of K, let $\imath\colon E \to L$ be an $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}$-embedding, and let $\jmath\colon C \to C_L$ be an $\mathcal{L}_{\text{ring}}$-embedding with $\jmath|_{C_E} = \imath|_{C_E}$. Then there is a unique $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}$-embedding of the H-subfield $E(C)$ of K into L which extends both $\imath$ and $\jmath$.*

As $E(C)$ is a d-algebraic extension of K, it is also ω-free by [**4**, 13.6.1]. Thus, by replacing E with $E(C)$ and $\imath$ with the embedding $E(C) \to L$ guaranteed by Fact 8.3, we may assume that $C_E = C$. To finish off the proof of (2), we need one more fact:

Fact 8.4 ([**4**],16.2.3)**.** *Let $K, L \models$ NLC and suppose L is $|K|^+$-saturated. Let E be an ω-free H-subfield of K with $C_E = C$ and let $\imath\colon E \to L$ be an $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}$-embedding. Then $\imath$ extends to an $\mathcal{L}_{\text{ring}}^{\mathcal{O},\partial}$-embedding $K \to L$.*

This concludes the proof that NLC is the model completion of the theory of ω-free H-fields. To see that NLC is the model companion of the theory of H-fields, one needs to use [**4**, 11.7.18], which states that every H-field has an ω-free H-field extension.

In [**4**], the authors showed that NLC has two completions: NLC_{sm}, whose models are the models of NLC with small derivation, and NLC_{lg}, whose models do not have small derivation (an H-field which does not have small derivation is said to have *large derivation*). To prove this, the authors made use of a quantifier elimination result, but we can actually characterize the completions directly using Fact 8.4. To do this, we need a proposition:

Proposition 8.5. *Let $K, L \models \mathrm{NLC}$, suppose L is $|K|^+$-saturated, and let $\imath\colon C \to C_L$ be an $\mathcal{L}_{\mathrm{ring}}$-embedding. If both K and L have small derivation, then $\imath$ extends to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding $K \to L$. The same is true if both K and L have large derivation.*

Proof. We first consider the case that K and L have small derivation. Since K is Liouville closed, there is $x \in K$ with $x' = 1$. Since K has small derivation we must have $x \succ 1$. Since L is also Liouville closed with small derivation, there is $f \in L$ with $f \succ 1$ and $f' = 1$. We view C as an H-subfield of K with trivial derivation and gap 0, so $\imath$ is an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding. By [**4**, 10.2.2 and 10.5.11], the H-field $C(x)$ is grounded and $\imath$ extends to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding $C(x) \to L$ sending x to f. By [**4**, 11.7.17], this further extends to an embedding an ω-free H-field E into L. Since E contains C, we have $C_E = C$, so it remains to use Fact 8.4.

Now suppose K and L have large derivation. Again, since K is Liouville closed, there is $y \in K$ with $y' = 1$. Then $y \preccurlyeq 1$ since K has large derivation, so by subtracting a constant from y, we may assume that $y \prec 1$. Since L is also Liouville closed with large derivation, there is $g \in L$ with $g \prec 1$ and $g' = 1$. This time, we use [**4**, 10.2.1 and 10.5.10] to extend $\imath$ to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding $C(y) \to L$ by sending y to g. Again, $C(y)$ is grounded, so [**4**, 11.7.17] and Fact 8.4 allow us to further extend to an embedding $K \to L$. □

To see that $\mathrm{NLC}_{\mathrm{sm}}$ is complete, let $K, L \models \mathrm{NLC}_{\mathrm{sm}}$. We need to show that K and L are $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-elementarily equivalent. By replacing L with an elementary extension, we may assume that L is $|K|^+$-saturated. Then C_L is $|C|^+$-saturated, so there is an $\mathcal{L}_{\mathrm{ring}}$-embedding $\imath\colon C \to C_L$ since C and C_L are both models of the complete theory RCF; see [**4**, B.9.5]. This in turn extends to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding $\jmath\colon K \to L$ by Proposition 8.5. As NLC is model complete, $\jmath$ is an elementary embedding, so K and L are $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-elementarily equivalent. The same proof shows that $\mathrm{NLC}_{\mathrm{lg}}$ is complete. By [**4**, 15.0.2], the ordered valued differential field $\mathbb{T}$ is a model of $\mathrm{NLC}_{\mathrm{sm}}$. If $K \models \mathrm{NLC}_{\mathrm{lg}}$, then $K^{\phi} \models \mathrm{NLC}_{\mathrm{sm}}$ for any $\phi \in K^{>}$ with $v\phi \in \Psi$. Thus we have the following:

Corollary 8.6. *Every model of $\mathrm{NLC}_{\mathrm{sm}}$ is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-elementarily equivalent to $\mathbb{T}$. Every model of $\mathrm{NLC}_{\mathrm{lg}}$ has a compositional conjugate which is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-elementarily equivalent to $\mathbb{T}$.*

8.2. Restricted elementary H-fields

Definition 8.7. Let $\sin, \cos, \exp\colon K \to K$. We say that sin, cos, and exp are **restricted elementary functions on** K if all three functions are identically zero on $K \setminus [-1,1]$ and the following axioms are satisfied:

(RE1) for all $a \in [-1,1]$, we have $(\sin a)' = (\cos a)a'$, $(\cos a)' = -(\sin a)a'$, $(\exp a)' = (\exp a)a'$;
(RE2) $\sin(\smallo) \subseteq \smallo$ and $\cos(\smallo), \exp(\smallo) \subseteq 1 + \smallo$;
(RE3) for all $a, b \in [-1,1]$ with $a + b \in [-1,1]$, we have

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b, \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b, \\ \exp(a+b) &= \exp a \exp b. \end{aligned}$$

Let sin, cos, and exp be restricted elementary functions on K. Then sin, cos, and exp remain restricted elementary functions in any compositional conjugate of K. For each $c \in [-1,1]_C$ we have $(\sin c)' = (\cos c)c' = 0$ so $\sin c \in C$. Likewise, $\cos c, \exp c \in C$ for all $c \in [-1,1]_C$. With (RE2), this gives us $\sin(0) = 0$ and

$\cos(0) = \exp(0) = 1$. The next lemma shows that the restrictions of sin, cos, and exp to $\mathcal{o}$ are definable in the underlying H-field of K.

Lemma 8.8. *Let* sin, cos, *and* exp *be restricted elementary functions on K, let $a \in \mathcal{o}$, and let A and B be the homogeneous linear differential polynomials*

$$A(Y) \;=\; -(a')^3Y + a''Y' - a'Y'', \qquad B(Y) \;=\; a'Y - Y'.$$

Then $\sin a$ *is the unique zero of A in $a(1+\mathcal{o})$,* $\cos a$ *is the unique zero of A in $1+a\mathcal{o}$, and* $\exp a$ *is the unique zero of B in $1+\mathcal{o}$.*

Proof. We begin with sin and cos. Using (RE1), we have

$$\begin{aligned} A(\sin a) \;&=\; -(a')^3\sin a + a''(\sin a)' - a'(\sin a)'' \;=\; -(a')^3\sin a + a'a''\cos a - a'(a'\cos a)' \\ &=\; -(a')^3\sin a + a'a''\cos a + (a')^3\sin a - a'a''\cos a \;=\; 0. \end{aligned}$$

Likewise, $A(\cos a) = 0$. Now we show that $\sin a \in a(1+\mathcal{o})$ and that $\cos a \in 1 + a\mathcal{o}$. This is clear if $a = 0$, so we assume that $a \neq 0$. Since $\cos a \sim 1$ by (RE2), we have

$$(\sin a)' \;=\; (\cos a)a' \;\sim\; a'.$$

Then $\sin a \sim a$ since K is asymptotic and $a, \sin a \prec 1$; see [**4**, 9.1.4]. Since $\sin a \sim a \prec 1$, we have

$$(\cos a - 1)' \;=\; -(\sin a)a' \;\prec\; a'.$$

Again, this gives $\cos a - 1 \prec a$ since K is asymptotic. Finally, we show uniqueness. Again, we assume that $a \neq 0$. Since A is an order two homogeneous linear differential polynomial, the set of zeros of A in K is a C-linear subspace of K of dimension at most 2; see [**4**, 4.1.14]. Moreover, $\sin a$ and $\cos a$ are C-linearly independent since $c\sin a \in \mathcal{o}$ for all $c \in C$, so the set $\{\sin a, \cos a\}$ forms a basis for this subspace. Let

$$b \;=\; c_1\sin a + c_2\cos a$$

be an arbitrary zero of A in K, where $c_1, c_2 \in C$. If $b \sim a$, then c_2 must be 0, since otherwise $b \sim c_2 \asymp 1$. This gives $b = c_1\sin a \sim c_1 a$, so c_1 must be 1 and $b = \sin a$. If $b - 1 \prec a$, then c_2 must be 1, since otherwise $b - 1 \sim c_2 - 1 \asymp 1$. This gives

$$b - 1 \;=\; c_1\sin a + \cos a - 1 \;\in\; c_1 a + a\mathcal{o},$$

so c_1 must be 0 and $b = \cos a$.

The situation for exp is similar, but much simpler. We have $B(\exp a) = a'\exp a - (\exp a)' = 0$ by (RE1) and $\exp a \in 1 + \mathcal{o}$ by (RE2). Let b be an arbitrary zero of B in K, so $b = c\exp a$ for some $c \in C$. If $b \sim 1 \sim \exp a$, then c must be 1, so $b = \exp a$. □

Remark 8.9. Suppose $\mathcal{O} \neq K$. By extending the method used in the proof of Lemma 8.8, we can show that any restricted elementary functions on K are differentiable on the interval $(-1, 1)$. Let restricted elementary functions sin, cos, and exp be given and let $h \in \mathcal{o}$ with $h \neq 0$. By (RE1) and (RE2), we have

$$(\sin h - h)' \;=\; (\cos h - 1)h' \;\prec\; h',$$

so $\sin h - h \prec h$, since K is asymptotic. Likewise, (RE1) and (RE2) give

$$(\cos h - 1)' \;=\; -(\sin h)h' \;\prec\; h', \qquad (\exp h - 1 - h)' \;=\; (\exp h - 1)h' \;\prec\; h',$$

so $\cos h - 1, \exp h - 1 - h \prec h$. Thus,

$$\lim_{h\to 0}\frac{\sin h}{h} \;=\; 1, \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} \;=\; 0, \qquad \lim_{h\to 0}\frac{\exp h - 1}{h} \;=\; 1.$$

Let $a \in (-1,1)$ be given. Using the above identities and (RE3), we see that

$$\begin{aligned}
\lim_{h\to 0}\frac{\sin(a+h)-\sin a}{h} &= \lim_{h\to 0}\frac{\sin a\cos h - \cos a\sin h - \sin a}{h} \;=\; \cos a,\\
\lim_{h\to 0}\frac{\cos(a+h)-\cos a}{h} &= \lim_{h\to 0}\frac{\cos a\cos h - \sin a\sin h - \cos a}{h} \;=\; -\sin a,\\
\lim_{h\to 0}\frac{\exp(a+h)-\exp a}{h} &= \lim_{h\to 0}\frac{\exp a\exp h - \exp a}{h} \;=\; \exp a.
\end{aligned}$$

Thus, all three functions are differentiable at a. With obvious modifications, one can show that all three functions are left differentiable at 1 and right differentiable at -1.

Definition 8.10. Let sin, cos, and exp be restricted elementary functions on K. We say that K is a **restricted elementary H-field** if the expansion of the constant field C by $\sin|_C$, $\cos|_C$, and $\exp|_C$ is a model of T_{re}. The class of restricted elementary H-fields is axiomatized in the language $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$.

Corollary 8.11. *Let K and L be restricted elementary H-fields, let $\imath\colon K \to L$ be an $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-embedding, and suppose $\imath|_C\colon C \to C_L$ is an $\mathcal{L}_{\mathrm{re}}$-embedding. Then $\imath$ is an $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-embedding.*

Proof. We need to show that $\imath(\sin f) = \sin \imath(f)$, $\imath(\cos f) = \cos \imath(f)$, and $\imath(\exp f) = \exp \imath(f)$ for all $f \in [-1,1]$. Let $f \in [-1,1]$ and take $c \in [-1,1]_C$ and $a \in \smallo$ with $f = c + a$. We have

$$\imath(\sin f) \;=\; \imath(\sin c\cos a + \cos c\sin a) \;=\; \imath(\sin c)\imath(\cos a) + \imath(\cos c)\imath(\sin a)$$

by (RE3). Likewise,

$$\imath(\cos f) \;=\; \imath(\cos c)\imath(\cos a) - \imath(\sin c)\imath(\sin a), \qquad \imath(\exp f) \;=\; \imath(\exp c)\imath(\exp a).$$

By our assumption on $\imath$, it is enough to show that $\imath(\sin a) = \sin \imath(a)$, $\imath(\cos a) = \cos \imath(a)$, and $\imath(\exp a) = \exp \imath(a)$. Let A and B be the homogeneous linear differential polynomials over K from Lemma 8.8 and let $\imath A$ and $\imath B$ be the images of A and B under $\imath$, that is,

$$\imath A(Y) \;=\; -\imath(a')^3 Y + \imath(a'')Y' - \imath(a')Y'', \qquad \imath B(Y) \;=\; \imath(a')Y - Y'.$$

By Lemma 8.8 we know that $\sin a$ is a zero of A and that $\sin a \in a(1+\smallo)$, so $\imath(\sin a)$ is a zero of $\imath A$ and $\imath(\sin a) \in \imath(a)(1+\smallo_L)$. Then $\imath(\sin a) = \sin \imath(a)$, since $\sin \imath(a)$ is the unique zero of $\imath A$ in $\imath(a)(1+\smallo_L)$. Likewise, $\imath(\cos a) = \cos \imath(a)$ and $\imath(\exp a) = \exp \imath(a)$. □

Let us say something about the relationship between restricted elementary H-fields and H_{re}-fields.

Lemma 8.12. *Every H_{re}-field is a restricted elementary H-field. In particular, $\mathbb{T}_{\mathrm{re}}$ is a restricted elementary H-field.*

Proof. Let K be an H_{re}-field. Then ∂ is a T_{re}-derivation on K, so (RE1) is satisfied. (RE2) holds since continuity of sin, cos, and exp at 0 is an $\mathcal{L}_{\mathrm{re}}$-elementary property, hence true in all models of T_{re}. Likewise, (RE3) is an $\mathcal{L}_{\mathrm{re}}$-elementary property. □

Let $T_{\mathrm{re}}^{\mathrm{nl}}$ be the $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-theory of ω-free newtonian Liouville closed restricted elementary H-fields. We can use Lemma 8.12 to show that certain models of NLC admit expansions to models of $T_{\mathrm{re}}^{\mathrm{nl}}$:

Proposition 8.13. *Let $K \models \mathrm{NLC}$ and let C_{re} be an expansion of C to a model of T_{re}. Then there is a unique $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-expansion K_{re} of K to a restricted elementary H-field which contains C_{re} as an $\mathcal{L}_{\mathrm{re}}$-substructure. This expansion models $T_{\mathrm{re}}^{\mathrm{nl}}$.*

Proof. First, consider the case that K has small derivation. Then K is $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-elementarily equivalent to $\mathbb{T}$ by Corollary 8.6. Consider the $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-sentence φ which states that for all $a \in \smallo$, there is a unique $y \in a(1+\smallo)$ with $a'y'' = -(a')^3y + a''y'$. By Lemmas 8.8 and 8.12, the sentence φ holds in $\mathbb{T}_{\mathrm{re}}$, so it also holds in K. Let $a \in \smallo^{\neq}$. We let $\sin a$ be this unique $y \in a(1+\smallo)$ with $a'y'' = -(a')^3y + a''y'$ guaranteed by φ. A similar trick allows us to define $\cos a$ as the unique $y \in 1 + a\smallo$ with $a'y'' = -(a')^3y + a''y'$ and to define $\exp a$ as the unique $y \in 1+\smallo$ with $y' = a'y$.

We now expand K to an $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-structure—denoted K_{re}—as follows: let $f \in [-1,1]$, take $c \in [-1,1]_C$ and $a \in \smallo$ with $f = a + c$, and define $\sin f$, $\cos f$, and $\exp f$ by

$$\sin f := \sin c \cos a + \cos c \sin a, \qquad \cos f := \cos c \cos a - \sin c \sin a, \qquad \exp f := \exp c \exp a, \tag{8.1}$$

where $\sin c$, $\cos c$, and $\exp c$ are as defined in the expansion C_{re} and where $\sin a$, $\cos a$, and $\exp a$ are as defined above. Of course, these functions extend the restricted elementary functions on C_{re}. We need to check that sin, cos, and exp are restricted elementary functions on K. The axiom (RE2) holds by definition, so we really only need to check (RE1) and (RE3). Since the restrictions of sin, cos and exp to $\smallo$ are $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-definable, we can deduce that the identities in (RE1) and (RE3) hold for all $a, b \in \smallo$ by observing that they hold for all infinitesimal elements of $\mathbb{T}$. Showing that these identities hold for arbitrary elements in $[-1,1]$ follows easily (though somewhat tediously), using (8.1). For example, let $f \in [-1,1]$ and take $a \in \smallo$ and $c \in [-1,1]_C$ with $f = c + a$. Then

$$\begin{aligned}(\sin f)' &= (\sin c \cos a + \cos c \sin a)' = (\sin c)(\cos a)' + (\cos c)(\sin a)' \\ &= -(\sin c \sin a)a' + (\cos c \cos a)a' = (\cos f)a' = (\cos f)f'.\end{aligned}$$

By assumption, $K \models \mathrm{NLC}$, so $K_{\mathrm{re}} \models T_{\mathrm{re}}^{\mathrm{nl}}$. For uniqueness, let K_{re}^* be another expansion of K to a restricted elementary H-field which contains C_{re} as an $\mathcal{L}_{\mathrm{re}}$-substructure. Let $\imath\colon K_{\mathrm{re}} \to K_{\mathrm{re}}^*$ be the identity map, so $\imath$ is an $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-isomorphism and $\imath|_C$ is an $\mathcal{L}_{\mathrm{re}}$-isomorphism. Then $\imath$ is an $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-isomorphism by Corollary 8.11, so $K_{\mathrm{re}} = K_{\mathrm{re}}^*$.

If K has large derivation, then take $\phi \in K^>$ such that K^ϕ has small derivation and expand K^ϕ in place of K. The restricted elementary functions defined on K^ϕ work just as well for K. □

We can combine facts from Section 8.1 with Corollary 8.11 to prove the following embedding result:

Proposition 8.14. *Let $K, L \models T_{\mathrm{re}}^{\mathrm{nl}}$ and suppose L is $|K|^+$-saturated. Let E be an ω-free restricted elementary H-subfield of K and let $\imath\colon E \to L$ be an $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-embedding. Then $\imath$ extends to an $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-embedding $K \to L$.*

Proof. By assumption, C, C_E, and C_L are all models of the model complete theory T_{re}. As L is assumed to be $|K|^+$-saturated, the constant field C_L is $|C|^+$-saturated, so we may extend $\imath|_{C_E}\colon C_E \to C_L$ to an $\mathcal{L}_{\mathrm{re}}$-embedding $\jmath\colon C \to C_L$. By Fact 8.3, there is a unique $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-embedding of the H-subfield $E(C)$ of K into L which extends both $\imath$ and $\jmath$. As $E(C)$ is a d-algebraic extension of K, it is also ω-free by [**4**, 13.6.1], so we may use Fact 8.4 to further extend to an $\mathcal{L}_{\mathrm{ring}}^{\mathcal{O},\partial}$-embedding $\imath^*\colon K \to L$. Since $\imath^*|_C = \jmath$ is an $\mathcal{L}_{\mathrm{re}}$-embedding, Corollary 8.11 gives that $\imath^*$ is an $\mathcal{L}_{\mathrm{re}}^{\mathcal{O},\partial}$-embedding, as desired. □

Let us collect some consequences of Propositions 8.13 and 8.14.

Corollary 8.15. *$T^{\mathrm{nl}}_{\mathrm{re}}$ is the model completion of the theory of ω-free restricted elementary H-fields.*

PROOF. By Proposition 8.14 and the Model Completion Criterion, we need only show that every ω-free restricted elementary H-field extends to a model of $T^{\mathrm{nl}}_{\mathrm{re}}$. Let K be an ω-free restricted elementary H-field. Then C is real closed, so K extends to an ω-free newtonian Liouville closed H-field L with $C_L = C$ by Fact 8.2. By Proposition 8.13, there is a unique expansion of L to a model of $T^{\mathrm{nl}}_{\mathrm{re}}$ which contains C as an $\mathcal{L}_{\mathrm{re}}$-substructure. Using Corollary 8.11 where $\imath\colon K \to L$ is the inclusion map, we see that this expansion is a restricted elementary H-field extension of K. □

Corollary 8.16. *$T^{\mathrm{nl}}_{\mathrm{re}}$ has two completions: $T^{\mathrm{nl}}_{\mathrm{re,sm}}$, whose models are the models of $T^{\mathrm{nl}}_{\mathrm{re}}$ with small derivation, and $T^{\mathrm{nl}}_{\mathrm{re,lg}}$, whose models have large derivation.*

PROOF. Let $K, L \models T^{\mathrm{nl}}_{\mathrm{re,sm}}$. We need to show that K and L are $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementarily equivalent. By replacing L with an elementary extension, we may assume that L is $|K|^+$-saturated. Then C_L is $|C|^+$-saturated, so there is an $\mathcal{L}_{\mathrm{re}}$-embedding $\imath\colon C \to C_L$ since C and C_L are both models of T_{re}; see [**4**, B.9.5]. By Proposition 8.5, $\imath$ extends to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding $\jmath\colon K \to L$. Corollary 8.11 gives that $\jmath$ is even an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-embedding. As $T^{\mathrm{nl}}_{\mathrm{re}}$ is model complete by Corollary 8.15, $\jmath$ is an elementary embedding, so K and L are $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementarily equivalent. The same proof shows that $T^{\mathrm{nl}}_{\mathrm{re,lg}}$ is complete. □

8.3. The model companion of the theory of H_{re}-fields

Since $\mathbb{T}_{\mathrm{re}} \models T^{\mathrm{nl}}_{\mathrm{re,sm}}$ by [**4**, 15.0.2] and Lemma 8.12, we have the following:

Theorem 8.17. *The $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-theory of $\mathbb{T}_{\mathrm{re}}$ is model complete.*

As mentioned at the beginning of Chapter 7, there is a canonical $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-embedding $\imath\colon \mathbb{T} \to \mathbf{No}$. The embedding $\imath$ was shown to be $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}$-elementary in [**6**], so $\mathbf{No} \models \mathrm{NLC}$. Since $\imath$ is strongly additive and $\mathbb{R}$-linear and since the restricted sine, cosine, and exponential functions for both $\mathbb{T}_{\mathrm{re}}$ and $\mathbf{No}_{\mathrm{re}}$ are defined via Taylor expansion, $\imath$ is an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-embedding. Using that $\mathbf{No}_{\mathrm{re}}$ is also a model of the model complete theory $T^{\mathrm{nl}}_{\mathrm{re,}}$, we have:

Corollary 8.18. *The embedding $\imath\colon \mathbb{T}_{\mathrm{re}} \to \mathbf{No}_{\mathrm{re}}$ constructed in* [**6**] *is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementary.*

Remark 8.19. Apart from the numerous results from [**4**] in Section 8.1, the proof of Theorem 8.17 relies on model completeness for T_{re} and the following observations:

(1) The restrictions of sin, cos, and exp to the maximal ideal of any restricted elementary H-field are $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{ring}}(\emptyset)$-definable.
(2) Restricted elementary functions are completely determined by their restrictions to the maximal ideal and the constant field of any restricted elementary H-field.

With the obvious changes, one can also prove that the theory of the ordered valued differential field $\mathbb{T}$ expanded by *just* the restricted exponential function or *just* the restricted sine and cosine functions is model complete. This uses that the theory of $\mathbb{R}$ expanded by the restricted exponential function is model complete, as is the theory of $\mathbb{R}$ expanded by restricted sine and cosine functions; see [**40, 68**].

Here is a useful consequence of Theorem 8.17:

Corollary 8.20. *Every model of $T^{\mathrm{nl}}_{\mathrm{re,sm}}$ is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementarily equivalent to $\mathbb{T}_{\mathrm{re}}$. Every model of $T^{\mathrm{nl}}_{\mathrm{re,lg}}$ has a compositional conjugate which is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementarily equivalent to $\mathbb{T}_{\mathrm{re}}$.*

We can use Corollary 8.20 to show prove a partial converse to Lemma 8.12.

Corollary 8.21. *Let $K \models T^{\mathrm{nl}}_{\mathrm{re}}$. Then K is an H_{re}-field.*

Proof. Use that $\mathbb{T}_{\mathrm{re}}$ is an H_{re}-field, that K has a compositional conjugate which is $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{re}}$-elementarily equivalent to $\mathbb{T}_{\mathrm{re}}$, and that if K^{ϕ} is an H_{re}-field for $\phi \in K^{>}$, then K is an H_{re}-field. □

Remark 8.22. It is not true in general that every restricted elementary H-field is an H_{re}-field. Here is a counterexample: consider the Hahn field $\mathbb{R}[[x^{\mathbb{Z}}]]$, equipped with the derivation and valuation ring

$$\partial\Big(\sum_{d\in\mathbb{Z}} c_d x^d\Big) \ := \ \sum_{d\in\mathbb{Z}} c_d d x^{d-1}, \qquad \mathcal{O} \ := \ \Big\{\sum_{d\in\mathbb{Z}} c_d x^d : c_d = 0 \text{ whenever } d > 0\Big\}.$$

Let $f \in \mathbb{R}[[x^{\mathbb{Z}}]]$ with $|f| \leqslant 1$, take $c \in \mathbb{R}$ and $\varepsilon \prec 1$ such that $f = c + \varepsilon$, and let

$$\sin f \ := \ \sum_n \frac{\sin^{(n)}(r)}{n!}\varepsilon^n, \qquad \cos f \ := \ \sum_n \frac{\cos^{(n)}(r)}{n!}\varepsilon^n, \qquad \exp f \ := \ \sum_n \frac{\exp^{(n)}(r)}{n!}\varepsilon^n.$$

It is not too difficult to check that $\mathbb{R}[[x^{\mathbb{Z}}]]$ with the derivation, valuation ring, sin, cos, and exp functions as defined above is a restricted elementary H-field. However, it is not real closed (for example, x has no square root) and any H_{re}-field is real closed.

We can now show that the theory of H_{re}-fields has a model companion.

Theorem 8.23. *The theory of ω-free newtonian Liouville closed H_{re}-fields is the model companion of the theory of H_{re}-fields.*

Proof. Lemma 8.12 and Corollaries 8.15 and 8.21 tell us that the theory of ω-free newtonian Liouville closed H_{re}-fields is the model completion of the theory of ω-free H_{re}-fields. It remains to use Proposition 7.43, which tells us that every H_{re}-field has an ω-free H_{re}-field extension. □

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