RECEDING-HORIZON SWITCHED LINEAR SYSTEM DESIGN: A SEMIDEFINITE PROGRAMMING APPROACH WITH DISTRIBUTED COMPUTATION

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mechanical Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

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Abstract

This dissertation presents a framework for analysis and controller synthesis problems for switched linear systems. These are multi-modal systems whose parameters vary within a finite set according to the state of a discrete time automaton; the switching signal may be unconstrained or may be drawn from a language of admissible switching signals. This model of system dynamics and discrete logic has many applications in a number of engineering contexts. A receding-horizon type approach is taken by designing controllers with access to a finite-length preview of future modes and finite memory of past modes; the length of both preview and memory are taken as design choices. The results developed here take the form of nested sequences of SDP feasibility problems. These conditions are exact in that the feasibility of any element of the sequence is sufficient to construct a suitable controller, while the existence of a suitable controller necessitates the feasibility of some element of the sequence.

Considered first is the problem of controller synthesis for the stabilization of switched systems. These developments serve both as a control problem of interest and a demonstration of the methods used to solve subsequent switched control problems. Exact conditions for the existence of a controller are developed, along with converse results which rule out levels of closed-loop stability based on the infeasibility of individual SDP problems. This permits the achievable closed-loop performance level to be approximated to arbitrary accuracy.

Examined next are two different performance problems: one of disturbance attenuation and one of windowed variance. For each problem, controller synthesis conditions are presented exactly in

the form of SDP feasibility problems which may be optimized to determine levels of performance. In both cases, the performance level may be taken as uniform or allowed to vary based on the switching path encountered.

The controller synthesis conditions presented here can grow both large and computationally intensive, but they share a common structural sparsity which may be exploited. The last part of this dissertation examines this structure and presents a distributed approach to solving such problems. This maintains the tractability of these results even at large scales, expanding the scope of systems to which these methods can be applied.

Acknowledgements

My first and sincerest thanks to my adviser, Prof. Geir Dullerud, for his support, advice, and encouragement throughout my graduate studies at Illinois. Simply put, this work would not have been possible without him. I am also grateful to the members of my committee for giving their time, expertise, and feedback to this work: Prof. Daniel Liberzon, Prof. Srinivasa Salapaka, Prof. Petros Voulgaris, and Prof. Raphaël Jungers.

My time at Illinois has been filled with wonderful collaboration and conversation on all manner of subjects. A special thanks to Prof. Ji-Woong Lee, Prof. Raphaël Jungers, and Matthew Philippe for their wonderful conversations and collaboration. I would also like to thank the many members of the Illinois community for productive and stimulating technical conversations and feedback, including Anshuman, Seung Ho, Peter, Yu, Joao, Judy, and many others.

I am also grateful to the many members of the MechSE community who have supported my development as a teacher. The advice and mentoring of Prof. Matt West, Prof. Mariana Silva, and the other faculty members of the TAM 2XX project have benefited me immeasurably.

My family have long supported my education and encouraged me to pursue my interests; I would not have made it this far without the beginning they gave me. Last, but certainly not least, my wife Jenny has been beside me throughout this work, providing understanding, support, and the occasional encouragement to step back and take a break. I am truly fortunate to have her as a partner for both this dissertation and beyond.

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Chapter 1

Introduction

This dissertation presents a framework for the design and synthesis of controllers for switched linear systems. Switched systems are multi-modal systems which exhibit nondeterministic switching between operating modes and have been studied by several authors (see, e.g., [45, 54, 55, 56, 79]). Such systems are characterized by a finite set of operating modes, each with a corresponding state-space model. The active mode at each time is determined by a switching signal which chooses from the set of modes. A common representation of the switching signal is through the evolution of a finite-state automaton whose state at each time determines the active mode. The behavior of such an automaton can be modeled equivalently as a walk through a strongly connected, directed graph. The constraints on the possible switching form a switching language; switching languages generated by automata are also called *regular languages* in automaton theory. A special case of these is the unrestricted switching case, where the system may take any mode at each time step; this type of system is also called a *discrete linear inclusion*. More general switching languages are also possible (see Chapter 5). As a result, the exact sequence of system parameters is not known at any time during the evolution of the system. This combination of continuous-state dynamics and discrete logic forms a hybrid system model with application in many different engineering contexts. Some application domains include networked control systems [15, 40, 48], distributed

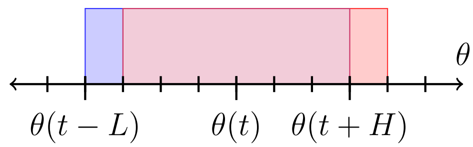

Figure 1.1: The window of switching information available to the controller at time t (in blue) and $t+1$ (in red). Information preserved by the controller over this transition consists exactly of the overlap in these two sequences (in purple).

networks of autonomous vehicles [43, 62], macroeconomic models [23, 83], and biological and chemical processes [5, 74].

This work takes a receding-horizon type approach to the problem of switched system control by designing controllers with access to finitely many future switching modes as well as memory of finitely many past modes. The resulting controllers may then select their gains according to a window of switching knowledge (see Figure 1.1), in addition to any measured system outputs. The length of both the preview and memory of the controller are taken as design choices; access to future switching modes may be limited by the system being controlled, while memory of past modes may be limited by factors such as controller memory or complexity. The work presented here resembles – and indeed is inspired directly by – results for path-dependent controllers without access to a preview of future modes. This includes results on stabilization of switched systems ([53]), disturbance attenuation ([52]), and windowed variance ([50]) as well as path-by-path performance ([51]). The work of this dissertation serves as an extension of these results to consider controllers with preview; whenever the preview length is set to zero, these past results are recovered exactly.

This type of controller information structure resembles previous results on receding-horizon type control for switched systems (such as [4, 11, 23, 35, 56, 68]), in which the controller gains are dependent on the switching signal. However, these results consider controllers which depend only on the current mode, even though controllers with memory of past switching modes are known to improve performance ([18, 54]).

The controller dependence on finitely-many future switching modes also suggests a comparison to model-predictive control (MPC) or receding-horizon control methods. Such methods typically require the online solution of a finite-horizon optimization problem to determine the control action at each time step. Specific applications of MPC to switched systems have been considered previously ([16, 59, 61]), each with different assumptions. In [16] the switching signal is treated as a design choice rather than an exogenous signal. In [59] the entire switching signal must be known *a priori*, whereas [61] supposes no switching knowledge beyond the current mode. These approaches also make no assumptions or restrictions on allowable switching sequences. In comparison with these past results, this thesis occupies a middle ground by permitting an exact preview of only finitely many future modes. It is worth noting that the length of the preview may be a design choice or may be fixed (or bounded) by the problem formulation. In contrast, the controller memory length is always a design choice (though possibly limited by design criteria such as onboard memory restrictions).

Another difference between the approach presented here and mainstream MPC is in the computation of controller gains. Many MPC approaches either require (repeated) solving of an online optimization problem at each time step; efforts are often made to simplify or approximate this optimization problem for speed (see, e.g., [8, 22, 33, 38, 46, 49, 77, 81]). In contrast, the results presented here allow controller gains to be computed offline. The online implementation of such controllers requires only the selection of an appropriate gain (as informed by the observed switching signal) at each time. The offline approach allows for guaranteed stabilization and performance bounds, which are typically only achieved heuristically by online MPC. However, the approach presented here is currently unable to accommodate non-convex input or output saturation constraints; this is a key attribute of the MPC approach.

In this work, controller synthesis conditions are presented as nested sequences of semidefinite programming problems (SDPs). The conditions are exact in that the feasibility of any element of this sequence is sufficient to demonstrate the existence of a satisfactory controller; indeed, such a controller can be constructed directly using the feasible solution. They are also asymptotically necessary, in that the existence of a suitable controller guarantees that some element of the sequence is feasible. Using these conditions to search for suitable controllers is then accomplished by constructing and solving each element in the sequence until a feasible solution is found. The size of these SDPs grows combinatorially in both the number of system modes and the length of the controller information window; this makes both the construction and solving of these problems computationally intense in time and memory requirements, even for modestly-sized problems. This leads to an interest in decomposing these problems such that distributed computing methods can be applied.

One approach to this computational problem comes from the relationship between sparse semidefinite matrices and chordal graphs characterized by [1, 39]. These results can be applied to SDPs whose parameters are sparse [58] or block-sparse [84] which improve efficiency by reducing a large semidefinite constraint to a set of smaller-dimension constraints. This is coupled with equality constraints that ensure consistency. This approach is of limited use for the switched conditions considered here, however, as it requires that the state-space models of the system themselves have structure; the results presented here make no such assumptions. A more attractive approach is to make use of the sparsity structure of the individual constraints; each constraint contains only a small number of decision variables, and each decision variable appears in only a small number of constraints. A distributed approach to solving sparsely-coupled optimization problems was presented in [65, 66] to solve linear programming (LP) problems; the coupling between constraints is used directly to subdivide the problem across multiple computation agents. This decomposition applies equally to SDPs, and was presented for the standard primal form in [64, 67].

In order to investigate the structure of switched controller problems and their decomposition, this dissertation first considers the problem of stabilization of *positive* switched systems. The restriction to positive systems provides controller synthesis conditions which are expressed as LPs rather than SDPs. With this simplification, the decomposition algorithm for LPs is adapted to the particular structure and needs of switched systems. This decomposition applies equally to the

SDPs developed for general switched control, with the qualification that these appear in the dual form instead. The intuition needed to decompose the problem is the same, however, and can be readily used to develop corresponding tools for this class of controller synthesis problems.

The organization of this dissertation is as follows: the remainder of this chapter presents the notation used throughout as well as some preliminary lemmas used to prove the main results. In Chapter 2, a complete examination of the stabilization of swiched linear systems is given, including exact, convex conditions for controller existence and synthesis and converse results about achievable closed-loop decay rates. These developments may also be seen as a prototype for the performance results that follow. Chapters 3 and 4 apply the tools developed in Chapter 2 to two separate performance problems; in Chapter 3 disturbance attenuation via ℓ_2-induced gain is examined, while in Chapter 4 a windowed variance performance measure is considered. The controller existence conditions of the preceding chapters are extended to more general, non-regular switching languages in Chapter 5. In Chapter 6 the stabilization of positive switched systems is discussed; in this special setting the controller existence conditions take the form of linear programming problems instead of semidefinite programming problems. The computational tractability of the results presented is examined in Chapter 7; here a scalable decomposition of the controller existence conditions is presented and an open-source implementation of the resulting algorithm are discussed. Finally, concluding remarks are given in Chapter 8.

1.1 Notation

In the developments that follow, it will be convenient to represent time-varying systems using block-diagonal operators on sequences in $\ell_2(\mathbb{R}^n)$ (following the example of [24]). The space $\ell_2(\mathbb{R}^n)$ consists of sequences $x = (x_0, x_1, x_2, \ldots)$ where each $x_k \in \mathbb{R}^n$ and

$$\sum_{k=0}^{\infty} \|x_k\|^2 < \infty.$$

For elements in $\ell_2(\mathbb{R}^n)$, the norm is given by

$$\|x\|_{\ell_2(\mathbb{R}^n)} := \left(\sum_{k=0}^{\infty} \|x_k\|^2 \right)^{1/2}$$

and the inner product is given by

$$\langle x, y \rangle_{\ell_2(\mathbb{R}^n)} := \sum_{k=0}^{\infty} x_k^* y_k.$$

When the dimension of the underlying vector space is clear, the space will simply be referred to as ℓ_2 and the subscript on the norm and inner product dropped. A useful operator on this space is the unilateral shift operator Z, defined such that

$$Z(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots).$$

Also useful will be the particular vector $\mathbf{1} := (1, 1, 1, \dots)$ in which every entry is equal to one (though this is not a member of $\ell_2(\mathbb{R}^n)$).

A bounded operator $Q : \ell_2(\mathbb{R}^n) \mapsto \ell_2(\mathbb{R}^m)$ is called *block-diagonal* if there exists a sequence of operators $Q_k : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that, if $y = Qx$ then $y_k = Q_k x_k$ for every index $k \geq 0$. Such operators have the representation

$$Q = \begin{bmatrix} Q_0 & 0 & 0 & \dots \\ 0 & Q_1 & 0 & \\ 0 & 0 & Q_2 & \\ \vdots & & & \ddots \end{bmatrix}.$$

Conversely, any uniformly bounded sequence of operators $Q_k : \mathbb{R}^n \mapsto \mathbb{R}^m$ may be used to construct the block-diagonal operator $Q = \operatorname{diag}(Q_0, Q_1, Q_2, \dots)$. In particular, the unilateral shift Z

can be represented in this way as

$$Z = \begin{bmatrix} 0 & I & 0 & \dots \\ 0 & 0 & I & \\ 0 & 0 & 0 & \ddots \\ \vdots & & & \ddots \end{bmatrix}.$$

For any operator Q, the adjoint is given by Q^* while Q^k denotes repeated multiplication. The inequality $Q \succ 0$ denotes a positive coercive operator; i.e., there exists a $c \geq 0$ such that $\langle x, Qx \rangle \geq c\|x\|$ for every $x \in \ell_2$. The nonstrict inequality $Q \succeq 0$ denotes that $\langle x, Qx \rangle \geq 0$ for every $x \in \ell_2$. The inequalities $Q \prec 0$ and $Q \preceq 0$ are defined in a corresponding manner. Any operator $X \succ 0$ induces a norm on ℓ_2 given by

$$\|y\|_X := y^* X y.$$

For any matrix $X \in \mathbb{R}^{n \times m}$, the image, kernel, null space, and rank of the matrix are denoted by $\mathrm{Im}\,(X)$, $\mathrm{Ker}\,(X)$, $\mathrm{Null}(X)$, and $\mathrm{rank}(X)$, respectively. The trace of a matrix is denoted by $\mathrm{Tr}\,(X)$. The notation $N(X)$ denotes any full-rank matrix satisfying $\mathrm{Im}\,(N(X)) = \mathrm{Ker}\,(X)$. The largest singular value of X is denoted $\sigma(X)$.

For a symmetric matrix $X \in \mathbb{R}^{n \times n}$, the inequality $X \succ 0$ ($X \succeq 0$) denotes that X is positive definite (positive semidefinite). Any matrix $X \succ 0$ induces a norm given by

$$\|y\|_X := y^* X y.$$

For any indexed sequence $x = (x_0, x_1, x_2, \dots)$, let $x_{(a:b)}$ denote the subsequence $(x_a, \dots, x_b)$ when $a < b$ or the element x_a when $a = b$. Likewise, for a time-indexed sequence $\theta(t)$, let $\theta_{(a:b)}$ denote the finite subsequence $(\theta(a), \dots, \theta(b))$ when $a < b$ or the element $\theta(a)$ when $a = b$. For any nonnegative integer N, the notation $[N]$ denotes the set of indices $\{1, \dots, N\}$.

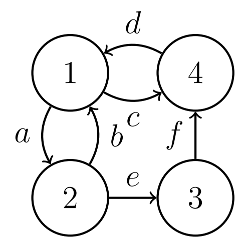

Figure 1.2: A directed graph with four modes which generates a constrained switching language. When interpreted as an automaton, the state of the automaton determines the switching mode for the system at each time.

Particularly in Chapter 6, the inequality $\geq$ (or $>$) is used to denote *component-wise* inequalities; i.e., for $x \in \mathbb{R}^n$, $x > 0$ if and only if $x_k > 0$ for each component $0 \leq k \leq n$. This notation extends to entry-wise inequalities on matrices and block-diagonal operators.

A swiched linear system is described by a finite, indexed collection of parameters

$$\mathcal{M} := \{(A_1, B_1, C_1, D_1), \ldots, (A_N, B_N, C_N, D_N)\} . \tag{1.1}$$

At each time $t \geq 0$, the system parameters are chosen by a switching signal $\theta : \mathbb{Z}^+ \mapsto [N]$. The resulting state-space dynamics of the system are given by

$$\begin{aligned} x_{t+1} &= A_{\theta(t)} x_t + B_{\theta(t)} w_t \\ z_t &= C_{\theta(t)} x_t + D_{\theta(t)} w_t. \end{aligned} \tag{1.2}$$

The switching signal is drawn from a set of admissible switching sequences Θ; throughout this dissertation, the admissible switching sequences will be drawn from a *regular language* (with the exception of the developments of Chapter 5). In this case, the switching constraint may be realized by a strongly connected, directed graph such as that of Figure 1.2. Such a graph can also be represented by an adjacency matrix $Q \in \{0, 1\}^{N \times N}$ which has $Q_{i,j} = 1$ if and only if the system may switch from mode i to mode j.

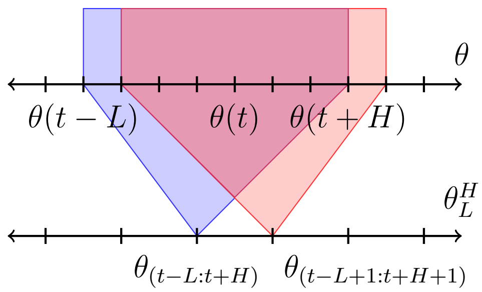

Figure 1.3: The upper time line shows the underlying switching sequence θ with finite-length observations at time t (in blue) and $t+1$ (in red). These finite-length paths can be interpreted as individual modes of an induced switching signal shown on the lower time line.

When a path-dependent controller, connected in feedback, forms a closed-loop system like that of (1.2), the closed-loop system parameters depend not only on the current mode (which selects the plant parameters) but on the entire switching path known to the controller. The dependence on a finite-length path can be formalized by an *induced switching sequence* whose values are drawn from the admissible finite switching paths. The relationship between the plant switching signal and closed-loop induced switching signal can be seen in Figure 1.3. The admissible transitions for the induced switching signal are precisely those which preserve switching information over a single time step; this can be described by an *induced switching graph* as shown in Figure 1.4.

When performing analysis on induced switched systems, the results presented will on occasion depend on induced switching modes or finite-length paths of induced switching modes when the individual parameters depend only on the current mode of the underlying switching signal. To extract this information, the function $\phi : [N]^{L+H+1} \mapsto [N]$ will be used, defined as

$$\phi\left(\theta_{(t-L:t+H)}\right) = \theta(t), \quad \phi\left((i_{-L}, \ldots, i_0, \ldots, i_H)\right) = i_0. \tag{1.3}$$

In essence, the function ϕ selects the current mode from a sequence which also includes past and/or future modes. This would correspond precisely to selecting the mode indicated in angle brackets for each induced mode in Figure 1.4.

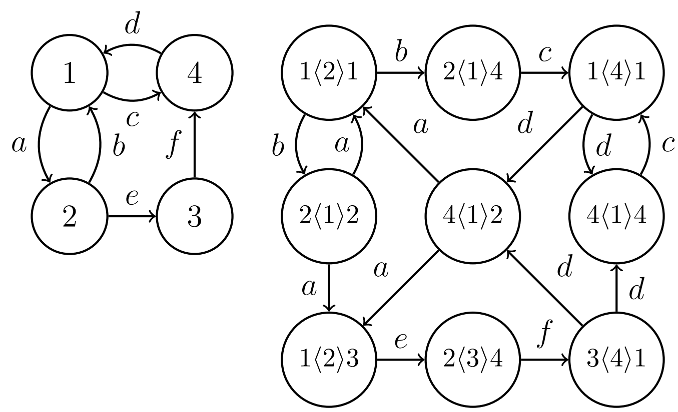

Figure 1.4: The four-mode switching graph of Figure 1.2 next to an induced switching graph for paths with memory of length one and horizon of length one. The modes of the induced graph are labeled by finite-length paths, with the current mode of the underlying sequence marked in angle brackets. The edge labels in the induced graph match those taken by the underlying switching signal.

1.2 Preliminaries from Linear Algebra

The development of controller synthesis conditions that follows relies on several standard results from linear algebra and system theory. The proofs are not directly related to the topic of this dissertation and are omitted; references are provided which contain both a proof and additional details. The first required result is the well-known Schur complement formula;

Lemma 1.1. *Consider a partitioned matrix given by*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Then the following are equivalent:

(a) $X \succ 0$*;*

(b) $X_{22} \succ 0$ *and* $X_{11} - X_{21}X_{22}^{-1}X_{12} \succ 0$

(c) $X_{11} \succ 0$ *and* $X_{22} - X_{12}X_{11}^{-1}X_{21} \succ 0$*.*

Proof. See, e.g., [25], Theorem 1.10 for a proof. □

The next result is an elimination lemma which is used to create equivalent linear matrix inequalities from nonlinear ones by removing one of the variables.

Lemma 1.2. *Given matrices R, S, and symmetric matrix Q, there exists a matrix J of compatible dimension such that*

$$Q + R^* J S + S^* J^* R \prec 0 \tag{1.4}$$

if and only if the following inequalities are satisfied:

$$N(R)^* Q N(R) \succ 0; \tag{1.5a}$$

$$N(S)^* Q N(S) \succ 0. \tag{1.5b}$$

Proof. Proofs of this result appear in both [37] and [63]. □

The last result of this section is also used to create equivalent linear matrix inequalities from nonlinear ones; it provides conditions under which a larger matrix can be reconstructed from its upper left block and that of its inverse.

Lemma 1.3. *Suppose that $R, S \in \mathbb{R}^{n\times n}$ are positive definite matrices. Then for any integer $\tilde{n} \geq 0$, there is a positive definite matrix $X \in \mathbb{R}^{(n+\tilde{n})\times(n+\tilde{n})}$ such that*

$$X = \begin{bmatrix} S & S_2 \\ S_2^* & S_3 \end{bmatrix}; \quad X^{-1} = \begin{bmatrix} R & R_2 \\ R_2^* & R_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \succeq 0 \quad \textit{and} \quad \textit{rank} \begin{bmatrix} R & I \\ I & S \end{bmatrix} \leq n + \tilde{n}.$$

Proof. A proof of this result can be found in [63]. □

Remark 1.4. In the previous result, the rank condition is non-convex and difficult in general to test. Whenever $\tilde{n} \geq n$, this condition is trivally satisfied. This informs the choices of controller rank throughout this work; when this condition is trivially satisfied, it will be omitted from the results presented.

Chapter 2

Stabilization of Switched Linear Systems

This chapter provides a complete treatment of the existence and synthesis of uniformly stabilizing controllers for switched linear systems. This simpler setting also provides for a clear demonstration of the intuition behind our controller synthesis methods. The chapter begins by developing exact conditions for the existence of stabilizing controllers in Section 2.1. These conditions are only asymptotically necessary, so converse results that provide information from infeasible LMIs are given in Section 2.2. A numerical example of these results using a phyically-motivated system are given in Section 2.3.

The stabilization results presented in Section 2.1 appeared originally in [26]; they were subsequently presented in [29] as well as in [30]. The converse results on the minimum achievable decay rate in Section 2.2 were presented in [27] for the unrestricted case, and [32] for the general case of restricted switching.

2.1 Uniformly Stabilizing Controllers

This section develops exact, convex conditions for the existence of a path-dependent stabilizing controller for switched systems. The approach used here is followed in Sections 3.1 and 4.1 for controllers with performance guarantees, as well as the positive system results of Chapter 6.

2.1.1 Preliminaries from LTV Stability Theory

Before proceeding to switched systems, some results for linear time-varying (LTV) systems are required. To this end, consider first the LTV system described by

$$x_{t+1} = A_t x_t \tag{2.1}$$

where $A_t \in \mathbb{R}^{n\times n}$ for all $t \geq 0$. It is assumed throughout this section that the A_t are uniformly bounded; in the case of switched systems each A_t is drawn from a finite set, so this assumption is not restrictive. Therefore, let $A := \text{diag}(A_0, A_1, \dots)$ be the block-diagonal operator constructed from this sequence of matrices. Then this LTV system has the equivalent representation

$$x = ZAx. \tag{2.2}$$

Definition 2.1. The LTV system in (2.2) is called *uniformly exponentially stable* if there exist constants $c \geq 1$ and $\lambda \in (0, 1)$ such that, for all $k \geq 0$,

$$\|(ZA)^k\| \leq c\lambda^k. \tag{2.3}$$

Remark 2.2. The previous definition is equivalent to the requirement that $\|A_{t+k} \cdot \ldots A_t\| \leq c\lambda^k$ for every $t \geq 0$ and every $k \geq 0$.

The uniform exponential stability of (2.2) can be characterized by the existence of a solution to a block-diagonal Lyapunov inequality of the same form as the LTI case.

Lemma 2.3. *The system of* (2.2) *is uniformly exponentially stable if and only if there exists a* *block-diagonal operator* $Y \succ 0$ *such that*

$$(ZA)Y(ZA)^* - Y \prec 0. \tag{2.4}$$

The previous condition holds if and only if there exists a block-diagonal operator $X \succ 0$ *such that*

$$(ZA)^* X(ZA) - X \prec 0. \tag{2.5}$$

Proof. These inequalities are operator versions of well-known results for LTI systems; a complete proof that the second condition is equivalent to uniform stability can be found in [24]. The equivalence of the first and second condition is straightforward, using two applications of the Schur complement formulas of Lemma 1.1 and the relationship $X^{-1} = Y$. □

Remark 2.4. Examining the block structure of, e.g., (2.4), provides the sequence of matrix inequalities

$$A_t Y_t A_t - Y_{t+1} \prec 0$$

for each $t \geq 0$; this is a familiar Lyapunov condition for time-varying systems.

Once a feasible solution to (2.4) is found, suitable constants c and λ required by Definition 2.1 may be recovered which are independent of the individual blocks of Y.

Lemma 2.5. *Suppose that* $Y \succ 0$ *is a solution to* (2.4)*, and* α*,* β*, and* γ *are positive constants satisfying*

$$\alpha I \preceq Y \preceq \beta I \quad \textit{and} \quad (ZA)Y(ZA)^* - Y \preceq -\gamma I.$$

Then (2.2) *is uniformly exponentially stable with* c *and* λ *given by*

$$c = \sqrt{\frac{\beta}{\alpha}} \quad \textit{and} \quad \lambda = \sqrt{1 - \frac{\gamma}{\beta}}.$$

Proof. The inequalities above demonstrate that, for any x,

$$\|(ZA)^x\|_Y^2 - \|x\|_Y^2 \leq -\gamma\|x\|^2$$

and that

$$\alpha\|x\|^2 \leq \|x\|_Y^2 \leq \beta\|x\|^2.$$

Rearranging the first inequality and applying the upper bound on $\|\cdot\|_Y$ show that $\beta \leq \gamma$ and

$$\|(ZA)x\|_Y^2 \leq \left(1 - \frac{\gamma}{\beta}\right)\|x\|_Y^2.$$

Since the norm $\|\cdot\|_Y$ is submultiplicative, repeated multiplication gives

$$\|(ZA)^k x\|_Y^2 \leq \left(1 - \frac{\gamma}{\beta}\right)^k \|x\|_Y^2.$$

Then the bounds on $\|\cdot\|_Y$ and rearranging the previous inequality provide

$$\frac{\|(ZA)^k x\|}{\|x\|} = \sqrt{\frac{\beta}{\alpha}}\left(\sqrt{1 - \frac{\gamma}{\beta}}\right)^k$$

which is the desired result. □

The final lemma of this section demonstrates, for stable systems, the existence of a finite-past dependent solution to (2.4). This finite-past dependent solution will play a key role in the development of a stability condition for switched systems.

Lemma 2.6. *Suppose that the system of* (2.2) *is uniformly exponentially stable. Then there exists a* $Y \succ 0$ *such that* (2.4) *is satisfied whose blocks depend on a finite number of past parameters.*

Proof. Suppose the system of (2.2) is uniformly exponentially stable and that c, λ are suitable constants as required by Definition 2.1. Consider the sequence of operators

$$Y^{(M)} := \sum_{k=0}^{M-1} (ZA)^k \left[(ZA)^*\right]^k . \tag{2.6}$$

Clearly $Y^{(M)} \succ 0$ for each M; note that, because the system is stable, this sequence converges to a limit $Y \succ 0$ which satisfies (2.4). Now pick M such that $c\lambda^M < 1$. Direct substitution gives

$$(ZA)Y^{(M)}(ZA)^* - Y^{(M)} = (ZA)^M \left[(ZA)^*\right]^M - I \leq -\left(1 - c^2\lambda^{2M}\right) I$$

so $Y^{(M)}$ satisfies (2.4). Inspection of the blocks of $Y^{(M)}$ shows that

$$Y_k^{(M)} = I + \sum_{s=\max\{0,k-M\}}^{k-1} \left(A_s \cdot \ldots A_{k-1}\right) \left(A_s \cdot \ldots A_{k-1}\right)^*$$

and so each block of $Y^{(M)}$ depends on at most M past parameters. □

Using the finite-past dependent solution to (2.4) required above and a Schur complement argument, it is straightforward to see that a similar result holds for the dual inequality (2.5).

Corollary 2.7. *Suppose that the system of* (2.2) *is uniformly exponentially stable. Then there exists an* $X \succ 0$ *such that* (2.5) *is satisfied whose blocks depend on a finite number of past parameters.*

2.1.2 Stability and Stabilization of Switched Systems

The LTV stability results above may now be used to produce stability conditions for switched systems. Consider now the switched system

$$x_{t+1} = A_{\theta(t)} x_t \tag{2.7}$$

where the system parameters are chosen from $\mathcal{A} = \{A_1, \dots, A_N\}$ and $\theta \in \Theta$ is any admissible switching sequence taking values in $[N]$. For any particular sequence, this system has the operator representation

$$x = ZA_\theta x \tag{2.8}$$

with $A_\theta := \operatorname{diag}(A_{\theta(0)}, A_{\theta(1)}, \dots)$. The definition of uniform exponential stability is extended to this setting by requiring that the stability bounds hold for every admissible switching sequence.

Definition 2.8. A switched linear system is called uniformly exponentially stable if there exist constants $c \geq 1$ and $\lambda \in (0, 1)$ such that, for every admissible switching sequence θ, the corresponding LTV system is uniformly exponentially stable in the sense of Definition 2.1.

For the remainder of this section the stability conditions will be formulated in terms of the dual Lyapunov inequality (2.5), making use of Corollary 2.7. The main stability result for switched systems can now be stated.

Theorem 2.9. *For $H \geq 0$ and $L \geq 0$, the system of* (2.7) *is uniformly exponentially stable if and only if there exist an integer $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{M+L+H}$ such that, for all admissible $i_{(-L:H)}$ and ϕ as in* (1.3)*,*

$$A^*_{\phi\left(i_{(-L:H)}\right)} X_{i_{(-L-M+1:H)}} A_{\phi\left(i_{(-L:H)}\right)} - X_{i_{(-L-M:H-1)}} \prec 0. \tag{2.9}$$

Remark 2.10. If $L = H = M = 0$ in the previous conditions, then the set of X_j consists of a single positive definite matrix X and (2.9) reads $A^*_{i_0} X A_{i_0} - X \prec 0$ for each $i_0 \in [N]$. In this case, X is a common quadratic Lyapunov function for the system; i.e., $V(x) = x^* X x$.

Proof. First consider sufficiency; suppose that an M and a family of X_j satisfying (2.9) are found. There are finitely many inequalities, so there exist positive constants α and β which satisfy

$$\alpha I \preceq X_j \preceq \beta I \quad \text{and} \quad A^*_{\phi\left(i_{(-L:H)}\right)} X_{i_{(-L-M+1:H)}} A_{\phi\left(i_{(-L:H)}\right)} - X_{i_{(-L-M:H-1)}} \preceq -\alpha I.$$

In particular, α and β do not depend on any particular admissible sequence. Choose any such sequence θ. Left-extend this sequence by selecting modes $\phi_{-L-M}, \ldots, \phi_{-1}$ such that the sequence $(\phi_{-L-M}, \ldots, \phi_{-1}, \theta(0))$ is an admissible sequence (this may always be done for a strongly connected switching constraint). Then define $\theta(-L-M) := \phi_{-L-M}, \ldots, \theta(-1) = \phi_{-1}$ such that $\theta(t)$ is now well-defined for $t \geq -L-M$. Now construct a block-diagonal operator X by selecting $X_t = X_{\theta_{(t-L-M:t+H-1)}}$ for each $t \geq 0$. Also let $A_\theta = \text{diag}(A_{\phi(\theta_{(t-L:t+H)})})$. A block-by-block argument using the inequalities of (2.9) shows that

$$\alpha I \preceq X \preceq \beta I \quad \text{and} \quad (ZA_\theta)^* X (ZA_\theta) - X \preceq -\alpha I.$$

Then by Lemma 2.3 this system is uniformly exponentially stable, and by Lemma 2.5 the necessary constants c and λ can be chosen from α and β, which were independent of the particular switching sequence.

Now consider necessity; suppose that the switched system (2.7) is uniformly exponentially stable. Lemma 2.6 provides for a block-diagonal X satisfying (2.5) whose blocks are dependent precisely on $A_{t-1}, \ldots, A_{t-M}$. These parameters are in turn dependent on the switching sequence; i.e., $A_t = A_{\theta(t)} = A_{\phi\left(\theta_{(t-L:t+H)}\right)}$. Therefore, relabel $X_t = X_{\theta_{(t-L-M:t+H-1)}}$ for each $t \geq L+M$. This construction must hold for every admissible switching sequence. Choose one which is *recurrent*; i.e., a sequence in which every finite path of length $M+1$ appears infinitely often. The existence of such a sequence is guaranteed for a strongly connected switching constraint. Then examining the block-by-block inequalities for $t \geq L+M$ recovers every inequality described in (2.9). $\square$

Remark 2.11. The previous proof makes use of two properties of Lyapunov inequalities for LTV systems. First, any feasible solution is sufficient to demonstrate that the system is stable. Second, the stability of the system necessarily requires a finite-past dependent solution to the Lyapunov inequality. The intuition of these two halves of the proof will be used in developing other classes of controller (see Chapters 3, 4, and 6). In each case, effort must first be made to establish these two properties for LTV systems before a switched condition can be developed.

Theorem 2.9 characterizes the stability of the switched system in terms of a sequence (in M) of semidefinite programming problems. Each element in the sequence contains finitely many inequalities, and a solution to any element of the sequence is sufficient to demonstrate stability of the system. This result may now be applied to a closed-loop switched system. These developments follow the standard methods of [37]. Consider the switched system with controlled input given by

$$\begin{aligned} x_{t+1} &= A_{\theta(t)}x_t + B_{\theta(t)}u_t \\ y_t &= C_{\theta(t)}x_t. \end{aligned} \tag{2.10}$$

This system will be connected in feedback with a controller of the form

$$\begin{aligned} \hat{x}_{t+1} &= \hat{A}_t\hat{x}_t + \hat{B}_t y_t \\ u_t &= \hat{C}_t\hat{x}_t + \hat{D}_t y_t \end{aligned} \tag{2.11}$$

with a controller state $\hat{x} \in \mathbb{R}^n$ and compatible dimensions for the remaining parameters. In order to write the closed-loop system parameters, define

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B}_i = \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix}; \quad \tilde{C}_i = \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix}$$

for each $i \in [N]$. Also collect the controller gains by defining

$$K_t = \begin{bmatrix} \hat{A}_t & \hat{B}_t \\ \hat{C}_t & \hat{D}_t \end{bmatrix}.$$

Then it is a straightforward exercise to see that the closed-loop system has state $x_C(t) = \begin{bmatrix} x_t^* & \hat{x}_t^* \end{bmatrix}^*$

and parameters

$$A_C(t) = \tilde{A}_{\theta(t)} + \tilde{B}_{\theta(t)} K_t \tilde{C}_{\theta(t)}. \tag{2.12}$$

The controller is given memory $L \geq 0$ and preview $H \geq 0$, so the controller gain matrix satisfies $K_t = K_{\theta_{(t-L:t+H)}}$; by extension, $A_C(t) = A_C(\theta_{(t-L:t+H)})$. Then the closed-loop system is exactly a switched system whose modes are the induced switching modes. This allows Theorem 2.9 to be applied directly.

Theorem 2.12. *For $H \geq 0$ and $L \geq 0$, the system of* (2.12) *is uniformly exponentially stable if and only if there exist and $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that, for all admissible $i_{(-L-M:H)}$,*

$$A_C\left(i_{(-L:H)}\right)^* X_{i_{(-L-M:H-1)}} A_C\left(i_{(-L:H)}\right) - X_{i_{(-L-M+1:H)}} \prec 0. \tag{2.13}$$

Proof. Apply the result of Theorem 2.9 to (2.12), noting that a path of M modes in the induced switched system depends precisely on $L+M+H$ modes of the underlying switching sequence. □

The inequalities of (2.13) are not simultaneously linear in both X and K and may be treated (for each admissible sequence) using the methods of [37]. Rewrite (2.13) as

$$H_i + F_{i_0}^* K_{i_{(-L:H)}} G_{i_0} + G_{i_0}^* K_{i_{(-L:H)}}^* F_{i_0} \prec 0 \tag{2.14}$$

in which

$$F_{i_0} = \begin{bmatrix} \tilde{B}_{i_0}^* & 0 \end{bmatrix}; \quad G_{i_0} = \begin{bmatrix} 0 & \tilde{C}_{i_0} \end{bmatrix};$$
$$H_i = \begin{bmatrix} -X_{i_{(-L-M+1:H)}}^{-1} & \tilde{A}_{i_0} \\ \tilde{A}_{i_0}^* & -X_{i_{(-L-M:H-1)}} \end{bmatrix}.$$

Remark 2.13. The inequalities above are formed to be compatible with (1.4), and progressing to (1.5a) and (1.5b) is always possible. However, the reverse direction is only guaranteed when there is precisely one controller matrix for each such inequality; i.e., when $M = 0$. Therefore, after the use of Lemma 1.2, the distinction between controller memory L and Lyapunov memory M is lost.

Lemma 2.14. *There exist controller gains satisfying* (2.14) *if and only if, for every admissible sequence* $i_{(-\bar{L}:H)}$,

$$N\left(F_{i_0}\right)^* H_i N\left(F_{i_0}\right) \prec 0 \tag{2.15a}$$

$$N\left(G_{i_0}\right)^* H_i N\left(G_{i_0}\right) \prec 0. \tag{2.15b}$$

When a controller satisfying (2.14) *is known, one may choose* $\bar{L} = L + M$*; when* (2.15) *are satisfied, a controller of memory* $L = \bar{L}$ *is guaranteed to exist.*

Proof. Apply Lemma 1.2 to (2.14), subject to the limitation noted in Remark 2.13. □

Consider a compatible partition of the X_j and their inverse such that

$$X_i = \begin{bmatrix} S_i & N \\ N^* & \cdot \end{bmatrix}; \quad X_i^{-1} = \begin{bmatrix} R_i & L \\ L^* & \cdot \end{bmatrix}. \tag{2.16}$$

Take an explicit representation of $N(F_{i_0})$ as

$$N(F_{i_0}) = \begin{bmatrix} N(B_{i_0}^*) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Substitution into (2.15a) along with the partition of X_i^{-1} yields

$$\begin{bmatrix} -N(B_{i_0}^*)^* R_{i_{(-L-M+1:H)}} N(B_{i_0}^*) & N(B_{i_0}^*)^* A_{i_0} & 0 \\ A_{i_0}^* N(B_{i_0}^*) & & \\ 0 & & -X_{i_{(-L-M:H-1)}} \end{bmatrix} \prec 0.$$

Applying the Schur complement formula to the lower-right block and again applying the partition produces (2.17a). An explicit representation of $N(G_{i_0})$ and similar manipulations produce (2.17b). These inequalities and Lemma 1.3 lead to the following result.

Theorem 2.15. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (2.12) *is uniformly exponentially stable if and only if there exist and integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that, for all admissible $i_{(-\bar{L}:H)}$,*

$$N(B_{i_0}^*)^* \left(A_{i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* - R_{i_{(-\bar{L}+1:H)}} \right) N(B_{i_0}^*) \prec 0 \tag{2.17a}$$

$$N(C_{i_0})^* \left(A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} A_{i_0} - S_{i_{(-\bar{L}:H-1)}} \right) N(C_{i_0}) \prec 0 \tag{2.17b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0. \tag{2.17c}$$

Furthermore, given solutions to the inequalities of (2.17)*, a controller may be chosen with memory at most $\bar{L}$.*

Proof. Given a stabilizing controller such that (2.13) is feasible for some M, the developments described above yield (2.17) with $\bar{L} = L + M$. Conversely, given solutions to (2.17), Lemma 1.3 allows for the construction of the X_j from the corresponding R_j and S_j according to (2.16). Then substituting these X_j into (2.14) produces inequalities in the controller gains K (with memory $\bar{L}$); solving this feasibility problem (which is guaranteed to be feasible) produces the desired controller gains. □

2.2 Minimum Achievable Decay Rates

Theorem 2.15 provides exact conditions for the existence of a uniformly stabilizing controller in the form of a sequence of nested SDP feasibility problems. The feasibility of any element of this sequence is sufficient to prove stabilizability and allows the direct construction of a suitable stabilizing controller. However, no additional guarantees are made about the closed-loop decay rate provided by this controller. Furthermore, the conditions are only asymptotically necessary; no finite number of infeasible results can conclusively demonstrate that no suitable controller exists (that is, the question of stability or stabilization is only *semidecidable*). This leads to two important questions about the stabilization results presented above:

1. What is the best possible closed-loop decay rate achievable for a path-dependent controller (of fixed memory and/or horizon)?

2. What information about achievable decay rates is obtained from a single infeasible result?

These two questions are of particular interest when applying these results to engineering design. Since transient performance is dependent on the decay rate of a system, an exponential decay rate only slightly less than one may lead to unacceptably long settling times for the system. The resulting design criteria may call for a stronger requirement than simple asymptotic stability. At the same time, the existence of a stabilizing controller guarantees that some element of this SDP sequence is feasible, but there is no *a priori* way to determine how many elements must be checked to locate it. Thus, many SDPs may need to be constructed and solved before a suitable controller is located.

The stability of switched systems is closely related to the *joint spectral radius* (JSR) introduced by [78] (see also [20, 41]). The JSR is a generalization of the spectral radius to switching systems and captures the worst-case growth rate of the system for any modal trajectory. In particular, it is well known that the stability of the discrete linear inclusion (that is, a switched system with no switching constraint) is equivalent to the parameter set having JSR strictly less than one. The joint spectral radius has been studied extensively for this reason (see [45] for many examples). However, exact computation of the JSR is known to be NP-hard, and the question $\rho(\mathcal{A}) \leq 1$ is undecidable even for two matrices [13]. Recent methods for approximating the JSR efficiently have appeared in [3, 2, 12], with the results of [3] using an SDP estimation scheme of nearly the same form as the results of Theorem 2.9. This allows the estimation scheme to be extended to consider closed-loop stability and the existence of a controller achieving a particular decay rate.

For more general switched systems with a switching constraint, the *constrained joint spectral radius* (CJSR) introduced in [19] (and in [47] as the Markovian joint spectral radius) generalizes the notion of worst-case growth rate to constrained switching systems. In particular, the addition of a switching constraint may result in a stable (or stabilizable) switched system where the unconstrained system was not, due to the elimination of ”badly behaved” sequences excluded by the switching constraint. Several important properties of the CJSR which are analogous to those of the JSR have been demonstrated, allowing the methods used for JSR estimation to be extended to this case. This has led to an extension of the estimation scheme of [3] to the CJSR estimation in [70]. These results are extended to closed loop systems in a similar way.

In both the constrained and unconstrained case, these results allow for the construction of a sequence of estimation intervals E_M which have the properties:

(a) The minimum decay rate is contained in each E_M,

(b) The intervals are nested; e.g., $E_{M+1} \subseteq E_M$,

(c) The intervals converge on the true decay rate; i.e., $\lim_{M\to\infty} |E_M| = 0$.

These intervals may be created directly from the feasibility of the SDP conditions derived below.

2.2.1 Minimum Achievable Decay Rates for Discrete Linear Inclusions

Consider the system of (2.7), first with unrestricted switching between the parameters in $\mathcal{A} = \{A_1, \ldots, A_N\}$. The stability of this system can be characterized via the joint spectral radius.

Definition 2.16. The joint spectral radius of the set $\mathcal{A}$ is given by

$$\rho(\mathcal{A}) := \lim_{k\to\infty} \rho_k(\mathcal{A})$$

where, for each $k \geq 0$,

$$\rho_k(\mathcal{A}) := \max_{i_{(1:k)}} \|A_{i_1} \cdot \ldots A_{i_k}\|^{1/k}.$$

The stability of (2.7) is known to be equivalent to $\rho(\mathcal{A}) < 1$; the following result provides a characterization of $\rho(\mathcal{A})$ in terms of the existence of a particular norm.

Lemma 2.17 (Kozyakin)**.** *The discrete linear inclusion* (2.7) *is uniformly exponentially stable if and only if there exists a norm* $\|\cdot\|_*$ *such that* $\|A_i\|_* < 1$ *for every* $A_i \in \mathcal{A}$.

Proof. The proof of this result is given in the appendix of [47]. □

By rescaling the parameters in $\mathcal{A}$, this result can be used to prove the existence of a norm which approximates $\rho(\mathcal{A})$ to any desired accuracy.

Corollary 2.18. *For every* $\epsilon > 0$, *there exists a norm* $\|\cdot\|_{*,\epsilon}$ *such that* $\|A_i\|_{*,\epsilon} < \rho(\mathcal{A}) + \epsilon$ *for every* $A_i \in \mathcal{A}$.

Proof. This is shown in [13]. □

While this norm is guaranteed to exist, computing it directly or searching the space of all norms is difficult (if not impossible) in general. An efficient approximation may be found by considering only quadratic norms. Consider the problem of finding a single $X \succ 0$ such that, for each $i \in [N]$,

$$A_i^* X A_i - \gamma^2 X \prec 0. \tag{2.18}$$

If a solution to these inequalities exists, then it is immediate that $\|A_i\|_X < \gamma$ for every $A_i \in \mathcal{A}$ and consequently that $\rho(\mathcal{A}) < \gamma$. This intuition leads to the *constant quadratic Lyapunov estimator* given below.

Definition 2.19. The constant quadratic Lyapunov estimator $\hat{\rho}_0(\mathcal{A})$ is defined by

$$\hat{\rho}_0(\mathcal{A}) := \inf \{\gamma : \text{A feasible solution to (2.18) exists}\} . \tag{2.19}$$

Clearly $\rho(\mathcal{A}) \leq \hat{\rho}_0(\mathcal{A})$, providing an upper bound. A lower bound is obtained using the following result, known as the *John ellipsoid*.

Lemma 2.20. *Let $K \subset \mathbb{R}^n$ be a compact, convex set with nonempty interior. Then there exists an ellipsoid E with center c such that the relationship $E \subseteq K \subseteq n(E - c) + c$ holds. If K is symmetric about the origin, then the constant n may be changed to $\sqrt{n}$.*

Proof. This result was initially presented in [44]. □

Applying this result to the level sets of the norm $\|\cdot\|_{*,\epsilon}$ implies the existence of a quadratic norm (whose level set is an ellipse) that satisfies the inequalities

$$\|\cdot\|_E \leq \|\cdot\|_{*,\epsilon} \leq \|\sqrt{n}\cdot\|_E. \tag{2.20}$$

Theorem 2.21. *The estimator $\hat{\rho}_0(\mathcal{A})$ satisfies the inequality*

$$\frac{1}{\sqrt{n}}\hat{\rho}_0(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}0(\mathcal{A}). \tag{2.21}$$

Proof. The proof (presented in [12]) uses Lemma 2.20 directly. □

A tighter bound on $\rho(\mathcal{A})$ is established by considering path-dependent Lyapunov functions. Modify (2.18) to search for a collection of $X_j \succ 0$ for $j \in [N]^M$ such that, for every sequence $i_{(0:M)}$,

$$A^*_{i_M} X_{i_{(1:M)}} A_{i_M} - \gamma^2 X_{i_{(0:M-1)}} \prec 0. \tag{2.22}$$

Likewise, introduce the path-dependent estimator which minimizes γ for this set of LMIs.

Definition 2.22. The M-path-dependent quadratic Lyapunov estimator $\hat{\rho}_M(\mathcal{A})$ is defined by

$$\hat{\rho}_M(\mathcal{A}) := \inf\{\gamma : \text{A feasible solution to (2.22) exists}\}. \tag{2.23}$$

Theorem 2.23. *The estimator $\hat{\rho}_M(\mathcal{A})$ satisfies the inequality*

$$\frac{1}{\sqrt[2(M+1)]{n}}\hat{\rho}_M(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}_M(\mathcal{A}). \tag{2.24}$$

Proof. This is Theorem 6.1 in [3]. □

Remark 2.24. A feasible solution to (2.22) allows the construction of a min-of-quadratics common Lyapunov function. If the dual form of these inequalities is used instead, then a max-of-quadratics common Lyapunov function may be obtained that is also a norm. This is one way to construct the norm $\|\cdot\|_{*,\epsilon}$ of Corollary 2.18.

The bounds in (2.24) show that the JSR lies in the interval

$$I_M := \left[\frac{\hat{\rho}_M(\mathcal{A})}{\sqrt[2(M+1)]{n}}, \hat{\rho}_M(\mathcal{A})\right]$$

which has length $(1 - 1/\sqrt[2(M+1)]{n})\hat{\rho}_M(\mathcal{A})$ and is monotonically decreasing in M. Therefore each estimator provides a better approximation of $\rho(\mathcal{A})$. Since it is not true in general that $I_{M+1} \subset I_M$ (e.g., if $\hat{\rho}_{M+1}(\mathcal{A})$ is significantly smaller than $\hat{\rho}_M(\mathcal{A})$), the best possible estimate comes from computing the intersection of these intervals for all M.

This estimation scheme may be applied to the closed-loop system of (2.12) by modifying Theorem 2.12 to explicitly specify a decay rate.

Theorem 2.25. *For $H \geq 0$, $L \geq 0$, the system of* (2.12) *has decay rate less than γ if and only if there exist and $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that, for all $i_{(-L-M:H)}$,*

$$A_C(i_{(-L:H)})^* X_{i_{(-L-M+1:H)}} A_C(i_{(-L:H)}) - \gamma^2 X_{i_{(-L-M:H-1)}} \prec 0. \tag{2.25}$$

Proof. Consider the scaled closed-loop parameters $\gamma^{-1} A_C(i_{(-L:H)})$ and apply Theorem 2.12; these conditions show the uniform exponential stability of the scaled inclusion. The bound on decay rate follows directly. □

For a particular controller, the previous result allows the estimation of the corresponding closed-loop decay rate by constructing the closed-loop parameters and applying Theorem 2.23; of interest is finding the smallest decay rate possible for any such controller.

Definition 2.26. The quantity $\rho(\mathcal{M}, L, H)$ is defined as the minimum closed-loop decay rate achieved for (2.12) by any controller with memory L and horizon H, where $\mathcal{M}$ are the plant parameters.

An estimation scheme for this minimum achievable decay rate may be constructed in terms of the Lyapunov memory used in (2.25).

Definition 2.27. The estimator $\hat{\rho}_M(\mathcal{M}, L, H)$ is defined as the minimum of the estimator $\hat{\rho}_M(\mathcal{A}_C)$ over all controllers with memory L and horizon H.

Lemma 2.28. *The estimator $\hat{\rho}_M(\mathcal{M}, L, H)$ satisfies the inequality*

$$\frac{1}{\sqrt[2(M+1)]{n}} \hat{\rho}_M(\mathcal{M}, L, H) \leq \rho(\mathcal{M}, L, H) \leq \hat{\rho}_M(\mathcal{M}, L, H). \tag{2.26}$$

Proof. For any particular controller of memory L and horizon H, construct the closed-loop parameter set $\mathcal{A}_C$ and corresponding estimators $\hat{\rho}_M(\mathcal{A}_C)$ which satisfy

$$\frac{1}{\sqrt[2(M+1)]{n}} \hat{\rho}_M(\mathcal{A}_C) \leq \rho(\mathcal{A}_C) \leq \hat{\rho}_M(\mathcal{A}_C).$$

Since this holds for all such controllers, the right-hand inequality is preserved by taking an infimum over controller gains, producing the upper bound of (2.26). For the lower bound, an infimum over all possible controllers must satisfy

$$\hat{\rho}_M(\mathcal{M}, L, H) \leq \hat{\rho}_M(\mathcal{A}_C)$$

for any particular controller, demonstrating the lower bound of (2.26). □

Since it is implausible to test each individual controller, the conditions of Theorem 2.25 are modified in the same way as those of Theorem 2.12, producing a scaled version of Theorem 2.15.

Theorem 2.29. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (2.12) *has decay rate less than γ if and only if there exist and integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that, for all $i_{(-\bar{L}:H)}$,*

$$N(B_{i_0}^*)^* \left(A_{i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* - \gamma^2 R_{i_{(-\bar{L}+1:H)}} \right) N(B_{i_0}^*) \prec 0 \tag{2.27a}$$

$$N(C_{i_0})^* \left(A_{i_0} S_{i_{(-\bar{L}+1:H)}} A_{i_0} - \gamma^2 S_{i_{(-\bar{L}:H-1)}} \right) N(C_{i_0}) \prec 0 \tag{2.27b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0. \tag{2.27c}$$

Furthermore, given solutions to the inequalities of (2.27)*, a controller may be chosen with memory at most $\bar{L}$.*

The conditions of (2.27) may be used either to explicitly construct a controller with decay rate γ or to rule out an achievable decay rate for any such controller.

Theorem 2.30. *Let $H \geq 0$ and $\gamma > 0$ and consider the inequalities of* (2.27)*. The following are true:*

(a) When these inequalities are feasible, then

$$\rho(\mathcal{M}, \bar{L}, H) \leq \hat{\rho}_0(\mathcal{M}, \bar{L}, H)$$

and a controller with horizon H and memory at most $\bar{L}$ can be chosen such that the closed-loop decay rate is less than γ.

(b) When the inequalities are infeasible, then for each pair L, M such that $L + M = \bar{L}$,

$$\gamma \leq \hat{\rho}_M(\mathcal{M}, L, H).$$

It follows immediately that, for each such pair,

$$\frac{\gamma}{\sqrt[2(M+1)]{n}} \leq \rho(\mathcal{M}, L, H).$$

Proof. For (a), feasibility of (2.27) allows direct construction of a controller with memory $\bar{L}$ such that the closed-loop has decay rate less than γ.

For (b), note that if $\hat{\rho}_M(\mathcal{M}, L, H) \leq \gamma$ then a controller of memory L exists such that the closed-loop satisfies (2.25). This directly implies the feasibility of (2.27) for $\bar{L} = L + M$. Consequently, the infeasibility of (b) for $\bar{L}$ implies that $\hat{\rho}_M(\mathcal{M}, L, H) > \gamma$ for each pair L, M such that $L + M = \bar{L}$. □

Remark 2.31. That the infeasibility of a single LMI in Theorem 2.30 provides bounds for a family of estimators is another consequence of using the elimination lemma as noted in Remark 2.31.

Using the infeasibility result of the previous theorem allows a stopping condition to be set when searching for a suitable stabilizing controller. If the lower bounds on achievable performance by controller either disallow the required decay rate, or necessitate a controller memory larger than design criteria permit, the search for a controller can be decide negatively.

2.2.2 Minimum Achievable Stability Radius for Constrained Switching

Using similar methods, the results of the previous section may be generalized to consider constrained switching systems. Here the worst-case growth rate of the system is measured by the *constrained joint spectral radius*.

Definition 2.32. The constrained joint spectral radius of the set $\mathcal{A}$ on switching language Θ is given by

$$\rho(\mathcal{A}, \Theta) := \lim_{k\to\infty} \rho_k(\mathcal{A}, \Theta)$$

where, for each $k \geq 0$,

$$\rho_k(\mathcal{A}, \Theta) := \max_{i_{(1:k)}\in\Theta} \|A_{i_1} \cdot \ldots A_{i_k}\|^{1/k}.$$

Similar to the JSR, the CJSR characterizes the stability of the constrained system. Indeed, uniform exponential stability is equivalent to $\rho(\mathcal{A}, \Theta) < 1$ (this is a central result of [19]).

While the key tool used for exploring unconstrained switching was a norm, the key tool for constrained switching systems is the multinorm (taken from [70], Definition 2).

Definition 2.33. A multinorm $\mathcal{H}$ for the system of (2.7) is a set of N norms, $\mathcal{H} = \{|\cdot|_1, \ldots, |\cdot|_N\}$. The *value* $\gamma^*(\mathcal{H})$ of a multinorm is defined as the minimum over all $\gamma \geq 0$ such that, for any admissible sequence (i, j),

$$|A_j x|_j \leq \gamma |x|_i, \quad \forall x \in \mathbb{R}^n. \tag{2.28}$$

Much like the result of Lemma 2.17 (and Corollary 2.18), multinorms characterize the CJSR for constrained systems.

Lemma 2.34. *For any multinorm $\mathcal{H}$, $\rho(\mathcal{A}, \Theta) \leq \gamma^*(\mathcal{H})$.*

Lemma 2.35. *For any $\epsilon > 0$, there exists a multinorm $\mathcal{H}_\epsilon$ such that $\gamma^*(\mathcal{H}_\epsilon \leq \rho(\mathcal{A}, \Theta)$.*

Proof. These are Lemmas 3.3 and 3.4 of [70]. □

As was the case with estimating the JSR via a single norm, a search over all multinorms would be difficult or impossible in general. Using the same restriction to quadratic multinorms leads to a search for mode indexed matrices $X_j \succ 0$ such that, for each admissible sequence (i, j),

$$A_j^* X_j A_j - \gamma^2 X_i \prec 0. \tag{2.29}$$

When these inequalities are feasible, the X_j provide a quadratic multinorm whose value is no larger than γ.

Definition 2.36. The modal quadratic Lyapunov estimator $\hat{\rho}_1(\mathcal{M}, \Theta)$ is defined by

$$\hat{\rho}_1(\mathcal{M}, \Theta) := \inf \{\gamma : \text{A feasible solution to (2.29) exists}\}. \tag{2.30}$$

An application of Lemma 2.20 to each member of the multinorm results in accuracy bounds for this estimator.

Theorem 2.37. *The estimator $\hat{\rho}_1(\mathcal{M}, \Theta)$ satisfies the inequality*

$$\frac{1}{\sqrt{n}} \hat{\rho}_1(\mathcal{M}, \Theta) \leq \rho(\mathcal{M}, \Theta) \leq \hat{\rho}_1(\mathcal{M}, \Theta). \tag{2.31}$$

Proof. The proof is of the same spirit as Theorem 2.21; for full details, see [70]. □

This estimator may also be extended to consider path-dependent quadratic forms; this allows tighter bounds to be obtained. Modify (2.29) to search for a collection of $X_j \succ 0$ for $j \in [N]^M$ such that, for every admissible sequence $i_{(0:M)}$,

$$A_{i_M}^* X_{i_{(0:M)}} A_{i_M} - \gamma^2 X_{i_{(0:M-1)}} \prec 0 \tag{2.32}$$

Also needed is the path-dependent estimator which minimizes γ for these LMIs.

Definition 2.38. The M-path-dependent quadratic Lyapunov estimator $\hat{\rho}_M(\mathcal{A}, \Theta)$ is defined by

$$\hat{\rho}_M(\mathcal{A}, \Theta) := \inf \{\gamma : \text{A feasible solution to (2.32) exists}\}. \tag{2.33}$$

Theorem 2.39. *The estimator $\hat{\rho}_M(\mathcal{A}, \Theta)$ satisfies*

$$\frac{1}{\sqrt[2M]{n}}\hat{\rho}_M(\mathcal{A}, \Theta) \leq \rho(\mathcal{A}, \Theta) \leq \hat{\rho}_M(\mathcal{A}, \Theta). \tag{2.34}$$

Proof. These bounds come from [70], Theorems 4.8 and 4.10. □

Remark 2.40. The bounds provided in (2.34) (for constrained switching) and (2.24) are very similar. The only difference lies in the power of the root taken as part of the lower bound. This is a consequence of requiring a single norm in the unconstrained case, but a multinorm in the constrained case.

The application of CJSR estimators to closed-loop systems proceeds in the same manner as in the unconstrained case, with parallels in both the definitions and resulting bounds. First, consider the constrained version of Theorem 2.25.

Theorem 2.41. *For $H \geq 0$, $L \geq 0$, the system of* (2.12) *has decay rate less than γ if and only if there exist and $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that, for all admissible $i_{(-L-M:H)}$,*

$$A_C(i_{(-L:H)})^* X_{i_{(-L-M+1:H)}} A_C(i_{(-L:H)}) - \gamma^2 X_{i_{(-L-M:H-1)}} \prec 0. \tag{2.35}$$

Proof. The proof is essentially identical to that of Theorem 2.25, subject to the additional switching constraint. □

A notion of the minimum achievable closed-loop stability radius for this system, as well as the corresponding estimators, follows accordingly.

Definition 2.42. The quantity $\rho(\mathcal{M}, \Theta, L, H)$ is defined as the minimum closed-loop decay rate achieved for (2.12) by any controller with memory L and horizon H, where $\mathcal{M}$ are the plant parameters and Θ is the switching language.

Definition 2.43. The estimator $\hat{\rho}_M(\mathcal{M}, \Theta, L, H)$ is defined as the minimum of $\hat{\rho}_M(\mathcal{A}_C, \Theta)$ over all controllers with memory L and horizon H.

Lemma 2.44. *The estimator $\hat{\rho}_M(\mathcal{M}, \Theta, L, H)$ satisfies the inequality*

$$\frac{1}{\sqrt[2M]{n}}\hat{\rho}_M(\mathcal{M}, \Theta, L, H) \leq \rho(\mathcal{M}, \Theta, L, H) \leq \hat{\rho}_M(\mathcal{M}, \Theta, L, H). \tag{2.36}$$

Proof. The argument is identical to the proof of Lemma 2.28. □

The conditions of Theorem 2.41 are modified to obtain the corresponding controller existence conditions.

Theorem 2.45. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (2.12) *has decay rate less than γ if and only if there exist and integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that, for all admissible $i_{(-\bar{L}:H)}$,*

$$N(B_{i_0}^*)^* \left(A_{i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* - \gamma^2 R_{i_{(-\bar{L}+1:H)}} \right) N(B_{i_0}^*) \prec 0 \tag{2.37a}$$

$$N(C_{i_0})^* \left(A_{i_0} S_{i_{(-\bar{L}+1:H)}} A_{i_0} - \gamma^2 S_{i_{(-\bar{L}:H-1)}} \right) N(C_{i_0}) \prec 0 \tag{2.37b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0. \tag{2.37c}$$

Furthermore, given solutions to the inequalities of (2.37)*, a controller may be chosen with memory at most $\bar{L}$.*

Theorem 2.46. *Let $H \geq 0$ and $\gamma > 0$ and consider the inequalities of* (2.37)*. The following are true:*

(a) When these inequalities are feasible, then

$$\rho(\mathcal{M}, \Theta, \bar{L}, H) \leq \hat{\rho}_0(\mathcal{M}, \Theta, \bar{L}, H)$$

and a controller with horizon H and memory at most $\bar{L}$ can be chosen such that the closed-loop decay rate is less than γ.

(b) When the inequalities are infeasible, then for each pair L, M such that $L + M = \bar{L}$,

$$\gamma \leq \hat{\rho}_M(\mathcal{M}, \Theta, L, H).$$

It follows immediately that, for each such pair,

$$\frac{\gamma}{\sqrt[2M]{n}} \leq \rho(\mathcal{M}, \Theta, L, H).$$

With the conditions for both unconstrained and constrained switching presented, a numerical example will compare the achievable stability radius for both.

2.3 Stabilization of a Switching Pendulum

This section provides a demonstration of the results of this chapter through a simple, physically motivated example of an inverted pendulum.

Example 2.47. Consider the system of Figure 2.1, which depicts a double-pendulum system with a barbell connected to the upper linkage using the operating point where all linkages are vertical. Each link has length $1\ \mathrm{m}$ and mass $1\ \mathrm{kg}$. A small, $0.1\ \mathrm{kg}$ mass may jump between the two ends of the barbell, producing two operating modes (transitions between the two operating modes correspond to a discontinuous drop or rise in the potential energy of the sysetm via the motion of

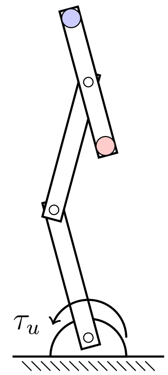

Figure 2.1: A barbell-double pendulum system with a moveable mass located at either the upper (red) or lower (blue) position.

the small mass). A controlled torque τ_u is applied to the bottom hinge as a controlled input. The continuous, nonlinear dynamics of the system may be described in terms of the angle each link makes with the vertical and the derivatives of these angles. The resulting nonlinear dynamics are then linearized around the vertical operating point. These continuous dynamics are then discretized using a time interval $t = 0.05$ s.

In the configuration shown in Figure 2.1, the barbell canbe stabilized by driving the topmost hinge (i.e., the upper end of the middle link) left if the umping mass is at the blue ned, or right if the jumping mass is at the red end; the correct control action depends on the location of the mass. However, the intermediate linkage delays the effect of an input choice on the position of this hinge. If the mass switches position after a choice has been made, then the control action will push the system away from equilibrium. Knowledge of the next position of the mass allows the controller to stabilize the system.

The values in Table 2.1 are found by applying the conditions of (2.27) repeatedly to find the largest lower bound such that the conditions are infeasible. These lower bounds are used along with Theorem 2.30 to provide a lower bound on the estimators $\hat{\rho}_M(\mathcal{M}, L, H)$ in Tables 2.2 and 2.3. Each entry in Table 2.1 yields a diagonal of information in one of the tables below.

The next example imposes a switching constraint on the pendulum system.

$\bar{L}$ / H	0	1	2
0	1.0008	1.0008	1.0000
1	1.0000	1.0000	0.8658
2	1.0000	0.9863	·

Table 2.1: A table of the largest values of γ for which the conditions of (2.27) are *infeasible* for the pendulum in Fig. 2.1
.

L / M	0	1	2
0	0.4086	0.6389	0.7418
1	0.4082	0.6389	·
2	0.4082	·	

L / M	0	1	2
0	0.4086	0.6389	0.7317
1	0.4082	0.6302	·
2	0.4027	·	

Table 2.2: A table of lower bounds on $\hat{\rho}_M(\mathcal{M}, L, 0)$ (left table) and $\hat{\rho}_M(\mathcal{M}, L, 1)$ (right table) obtained from the data in Table 2.1

L / M	0	1
0	0.4082	0.5532
1	0.3535	·

Table 2.3: A table of lower bounds on $\hat{\rho}_M(\mathcal{M}, L, 2)$ obtained from the data in Table 2.1

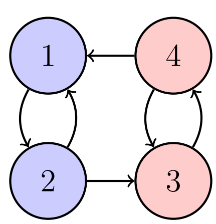

Figure 2.2: A four-mode switching logic for the pendulum system as described in Example 2.48.

$\bar{L}$ / H	0	1	2
0	1.0001	0.9744	0.9526
1	0.9813	0.9580	0.9218
2	0.9648	0.9464	·

Table 2.4: A table of the largest values of γ for which the conditions of (2.37) are *infeasible* for Example 2.48.

L / M	0	1	2
0	0.4087	0.6270	0.7157
1	0.4006	0.6165	·
2	0.3939	·	

L / M	0	1	2
0	0.3978	0.6121	0.7021
1	0.3911	0.6047	·
2	0.3864	·	

Table 2.5: Lower bounds on $\rho(\mathcal{M}, L, 0)$ (left table) and $\rho(\mathcal{M}, L, 1)$ (right table), obtained from the data in Table 2.4.

Example 2.48. Consider again the system of Figure 2.1, but introduce the switching constraint shown in Figure 2.2. Modes 1 and 2 (left, in blue) have parameters corresponding to the up position for the small mass, while modes 3 and 4 (right, in red) have parameters corresponding to the down position. The resulting switching behavior is such that the mass may only switch positions on every second time step, rather than every time step.

In the same way as the previous example, the largest infeasible values of γ for the constrained problem are presented in Table 2.4. One can compare the values in this table to those found for the unconstrained system from Table 2.1 to see the effect that the switching constraint has on stability. Whereas the unconstrained system required foreknowledge in order to achieve stabilization, the restriction to swiching modes every other time step permits a memory-only controller to stabilize the system in this example. Knowledge of whether the mass is permitted to switch on the current time step allows the controller to handle the intermediate linkage delays to stabilize the pendulum without risk of providing a destabilizing control input. The corresponding lower bounds on the achievable performance for controllers of horizon length zero and one are shown in Table 2.5.

Chapter 3

Disturbance Attenuation and ℓ_2-Induced Gain

This chapter presents results for control of switched systems achieving the first of two output performance criteria: that of uniform disturbance attenuation. This is equivalent to bounding the worst-case ℓ_2-induced gain for the system over all possible switching sequences. The primary focus of the chapter will be on achieving uniform strict contractiveness; i.e., that the disturbance-to-output system gain is strictly less than one. The results may be generalized in a straightforward way to consider performance at any level. The chapter begins by developing exact conditions for the existence of a controller which both stabilizes the system and guarantees uniform contractiveness in Section 3.1. A generalization of the induced system gain to provide path-by-path performance measures is given in Section 3.2. Finally, these results are applied to the double-pendulum system of Figure 2.1 to consider disturbance attenuation.

The results on uniform disturbance attenuation initially appeared in [29], and were subsequently published in [30], where the notion of path-by-path performance was also examined.

3.1 Controllers Achieving Uniform Contractiveness

This section develops exact, convex conditions for the existence of a path-dependent controller achieving induced gain level γ. For simplicity, the developments take $\gamma = 1$ and consider contractiveness first; a generalization to any gain level is given at the end of the section.

3.1.1 Disturbance Attenuation Preliminaries for LTV Systems

As in Section 2.1, some results for LTV systems are required first. Specifically, the existence of a characteristic condition for contractiveness (i.e., the KYP lemma) and the necessity of a finite-past dependent solution to this inequality must be developed. Consider the LTV system described by

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t \\ z_t &= C_t x_t + D_t w_t \end{aligned} \tag{3.1}$$

where $w_t \in \mathbb{R}^m$ is an exogenous disturbance signal, and $A_t \in \mathbb{R}^{n\times n}$, $B_t \in \mathbb{R}^{n\times m}$, $C_t \in \mathbb{R}^{l\times n}$, and $D_t \in \mathbb{R}^{l\times m}$ for each $t \geq 0$. The corresponding operator representation of this system is given by

$$\begin{aligned} x &= ZAx + ZBw \\ z &= Cx + Dw. \end{aligned} \tag{3.2}$$

Uniform stability for this system is precisely the same as in Definition 2.1 (for the operator ZA).

Definition 3.1. The LTV system of 3.2 is called *uniformly strictly contractive* if it satisfies

$$\|C\,(I - ZA)^{-1}\,ZB + D\| < 1. \tag{3.3}$$

Remark 3.2. The previous definition is equivalent to the existence of a $\gamma \in (0, 1)$ such that, when $x_0 = 0$, the input w and output z satisfy $\|z\|^2 \leq \gamma^2\|w\|^2$ for every disturbance w. When this holds for a particular value of γ, the system achieves *attenuation level* γ.

The uniform contractiveness of LTV systems is characterized by the well-known Kalman-Yacubovich-Popov (KYP) lemma; the following is an operator representation of that result.

Lemma 3.3. *The system of* (3.2) *is both uniformly exponentially stable and uniformly strictly contractive if and only if there exists a block-diagonal operator* $Y \succ 0$ *such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{3.4}$$

The previous condition holds if and only if there exists a block-diagonal operator $X \succ 0$ *such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{3.5}$$

Proof. A direct proof in the operator setting is given in [24]. The two inequalities (3.4) and (3.5) are shown to be equivalent through two uses of the Schur complement formula and the relationship $X^{-1} = Y$. □

Any solution to (3.4) is sufficient to demonstrate stability and contractivness; the developments that follow are needed to show that a finite-past dependent solution must exist.

Solutions to (3.4) are closely related to solutions of the associated Riccati difference equation. For any symmetric operator Y, define

$$\mathcal{V}(Y) := I - DD^* - CYC^*.$$

For any Y such that $\mathcal{V}(Y)$ is invertible, define

$$\begin{aligned} \mathcal{R}(Y) := {} & (ZA)Y(ZA)^* + (ZB)(ZB)^* \\ & + \left[(ZA)YC^* + (ZB)D^*\right] \mathcal{V}(Y)^{-1} \left[CY(ZA)^* + D(ZB)^*\right]. \end{aligned} \tag{3.6}$$

Much like the sequence $Y^{(M)}$ appearing in the proof of Lemma 2.6, the following Riccati iteration produces a solution to (3.4).

Lemma 3.4. *The system of* (3.2) *is uniformly exponentially stable and uniformly strictly contractive if and only if there exist positive constants ϵ_2, δ_2, η_2 such that, for every $\epsilon \in [0, \epsilon_2]$, the sequence*

$$Y^{(\epsilon,0)} = \epsilon I; \quad Y^{(\epsilon,M+1)} = \mathcal{R}(Y^{(\epsilon,M)}) + \epsilon I \tag{3.7}$$

satisfies, for $M \geq 0$,

$$\mathcal{V}(Y^{(\epsilon,M)}) \geq \eta_2 I; \quad \epsilon I \preceq Y^{(\epsilon,M)} \preceq Y^{(\epsilon,M+1)} \preceq \delta_2 I. \tag{3.8}$$

Then the limit $Y^{(\epsilon)} := \lim_{M\to\infty} Y^{(\epsilon,M)}$ exists and satisfies (3.4).

Before proving the previous claim, a technical lemma demonstrating the properties of the Riccati operator $\mathcal{R}$ is needed.

Lemma 3.5. *Define the operator*

$$\mathcal{A}(Y) := (ZA) + [(ZA)YC^* + (ZB)D^*]\,\mathcal{V}^{-1}C. \tag{3.9}$$

Then any operators Y_1 and Y_2 satisfy the following:

$$\begin{aligned} \mathcal{R}(Y_1) - \mathcal{R}(Y_2) &= \mathcal{A}(Y_2)\,[Y_1 - Y_2]\,\mathcal{A}(Y_2)^* \\ &\qquad + \mathcal{A}(Y_2)\,[Y_1 - Y_2]\,C^*\mathcal{V}(Y_2)^{-1}C\,[Y_1 - Y_2]\,\mathcal{A}(Y_2)^* \end{aligned} \tag{3.10a}$$

$$= \mathcal{A}(Y_1)\,[Y_1 - Y_2]\,\mathcal{A}(Y_2)^* \tag{3.10b}$$

Proof. This proof generalizes that of Lemma 2.6 in [52] to consider block-diagonal operators.

From [42], Lemma 11, the Riccati operator can be rewritten as

$$\mathcal{R}(Y) = \bar{A}Y\bar{A}^* + \bar{Q} + \bar{A}YC^*\mathcal{V}^{-1}CY\bar{A}^*,$$

where

$$\bar{A} = ZA + (ZB)D^*\left(I - DD^*\right)^{-1}C; \text{ and } \bar{Q} = (ZB)(ZB)^* + (ZB)D^*\left(I - DD^*\right)^{-1}D(ZB)^*.$$

Apply [21], Lemma 3.1, to this equation to obtain (3.10a). Using the definitions, it is straightforward to verify that

$$\mathcal{V}(Y_1)^{-1} - \mathcal{V}(Y_2)^{-1} - \mathcal{V}(Y_1)^{-1}C\left(Y_1 - Y_2\right)C^*\mathcal{V}(Y_2)^{-1} = 0.$$

which leads to the equations

$$\begin{aligned} \mathcal{A}(Y_2) &= \mathcal{A}(Y_1)\left(I - \left[Y_1 - Y_2\right]C^*\mathcal{V}(Y_2)^{-1}C\right) \\ I &= \left(I - \left[Y_1 - Y_2\right]C^*\mathcal{V}(Y_2)^{-1}C\right)\left(I + \left[Y_1 - Y_2\right]C^*\mathcal{V}(Y_1)^{-1}C\right). \end{aligned}$$

These equations and (3.10a) produce (3.10b). □

An important consequence of (3.10a) is that the Riccati operator preserves order; that is, if $Y_1 \succeq Y_2 \succeq 0$, then $\mathcal{R}(Y_1) - \mathcal{R}(Y_2) \succeq 0$.

Proof of Lemma 3.4. First suppose that the conditions of (3.8) hold. It follows from (3.7) that

$$Y_k^{(\epsilon,M)} = Y_k^{(\epsilon,M+1)} \implies Y_{k+1}^{(\epsilon,M+1)} = Y_{k+1}^{(\epsilon,M+2)}.$$

as well as that $Y_0^{(\epsilon,0)} = \epsilon I$ for all $M \geq 0$. Then an induction argument shows that for $M \geq k$, the block $Y_k^{(\epsilon,M)}$ is fixed. Let $Y^{(\epsilon)}$ be the block-wise (weak) limit of the sequence. This limit satisfies

$Y^{(\epsilon)} = \mathcal{R}(Y^{(\epsilon)}) + \epsilon I$; rearranging terms produces $\mathcal{R}(Y^{(\epsilon)}) - Y^{(\epsilon)} \preceq -\epsilon I$, from which a Schur complement produces (3.4).

Now suppose the system is uniformly stable and contractive, and let Y be the necessary solution to (3.4). Choose constants α, β such that $\alpha I \preceq Y \preceq \beta I$ and that $-\alpha I$ bounds the left-hand side of (3.4). Taking the Schur complement of the lower-right block $CYC^* + DD^* - I$ produces

$$\mathcal{V}(Y) \succeq \alpha I; \quad \text{and} \quad \mathcal{R}(Y) + \alpha I \preceq Y.$$

Choose $\epsilon \in [0, \alpha]$ and construct $Y^{(\epsilon,M)}$ as in (3.7). The sequence satisfies

$$\epsilon I \preceq Y^{(\epsilon,0)} \preceq \alpha I \preceq Y \preceq \beta I$$

and so, using the monotonicity of the Riccati operator,

$$\epsilon I \preceq Y^{(\epsilon,M+1)} = \mathcal{R}(Y^{(\epsilon,M)}) + \epsilon I \preceq \mathcal{R}(Y) + \alpha I = Y \preceq \beta I$$

and also $\mathcal{V}(Y^{(\epsilon,M)}) \succeq \mathcal{V}(Y) \succeq \alpha I$. Finally, consider that

$$Y^{(\epsilon,0)} = \epsilon I \preceq \epsilon I + \mathcal{R}(Y^{(\epsilon,0)}) = Y^{(\epsilon,1)};$$

so by induction the sequence is monotone for all $M \geq 0$. Then the choice $\epsilon_2 = \alpha$, $\delta_2 = \beta$, and $\eta_2 = \alpha$ satisfies the conditions of (3.8). □

One interpretation of the sequence $Y^{(\epsilon,M)}$ is that it captures the evolution of the forward-iterating Riccati difference equation; the block $Y_t^{(\epsilon,M)}$ (for $t > M$) is the value of a Riccati iteration with initial condition ϵI at time $t_0 = t - M$. Each block of $Y^{(\epsilon,M)}$ also depends on exactly M past parameters; it remains only to show that some element of this sequence satisfies (3.4).

Lemma 3.6. *Suppose that the system of* (3.2) *is both uniformly exponentially stable and uniformly strictly contractive, and construct* $Y^{(\epsilon,M)}$ *as in* (3.7). *Then the following are true:*

(a) There exist constants $c_\epsilon \geq 1$ *and* $\lambda_\epsilon \in (0,1)$ *such that, for all* $M \geq M_0 \geq 0$,

$$\|\mathcal{A}(Y^{(\epsilon,M)}) \cdot \ldots \mathcal{A}(Y^{(\epsilon,M_0)})\| \leq c_\epsilon \lambda_\epsilon^{M-M_0}. \tag{3.11}$$

(b) There exists an M *such that* $Y^{(\epsilon,M)}$ *satisfies* (3.4).

Proof. To prove (a), fix ϵ and $\bar{\epsilon} \in (\epsilon, \epsilon_2)$; also define $Y^{(M)} := Y^{(\bar{\epsilon},M)} - Y^{(\epsilon,M)}$. Since $\mathcal{V}(Y^{(\bar{\epsilon},M)}) \succeq 0$, applying (3.10a) shows

$$\begin{aligned} Y^{(M+1)} &= \mathcal{R}(Y^{(\bar{\epsilon},M)}) - \mathcal{R}(Y^{(\epsilon,M)}) + (\bar{\epsilon} - \epsilon)I \\ &\geq \mathcal{A}(Y^{(\epsilon,M)}) Y^{(M)} \mathcal{A}(Y^{(\epsilon,M)}) + (\bar{\epsilon} - \epsilon)I. \end{aligned}$$

Since $(\bar{\epsilon} - \epsilon)I \preceq Y^{(M)} \preceq (\delta_2 - \epsilon)$ and $\bar{\epsilon} - \epsilon > 0$, rearranging the above provides a Lyapunov inequality demonstrating the stability of the operators $\mathcal{A}(Y^{(\epsilon,M)})$. A proof similar to Lemma 2.5 shows that the corresponding c_ϵ, λ_ϵ are functions only of the bounds on $Y^{(M)}$.

To prove (b), first pick M such that $c_\epsilon^2 \lambda_\epsilon^{2M} < \epsilon/(\delta_2 - \epsilon)$. Then applying (3.10b) repeatedly shows

$$\begin{aligned} \mathcal{R}(Y^{(\epsilon,M)}) - Y^{(\epsilon,M)} &= \mathcal{R}(Y^{(\epsilon,M)}) - \mathcal{R}(Y^{(\epsilon,M-1)}) - \epsilon I \\ &= \mathcal{A}(Y^{(\epsilon,M)})[Y^{(\epsilon,M)} - Y^{(\epsilon,M-1)}]\mathcal{A}(Y^{(\epsilon,M-1)})^* - \epsilon I \\ &= \mathcal{A}(Y^{(\epsilon,M)}) \ldots \mathcal{A}(Y^{(\epsilon,1)})[Y^{(\epsilon,1)} - Y^{(\epsilon,0)}]\mathcal{A}(Y^{(\epsilon,0)})^* \ldots \mathcal{A}(Y^{(\epsilon,M-1)})^* - \epsilon I \\ &\preceq (\delta_2 - \epsilon) c_\epsilon^2 \lambda_\epsilon^{2M} I - \epsilon I. \end{aligned}$$

By choice of M, the last expression is negative definite. Then a Schur complement provides (3.4). □

3.1.2 Disturbance Attenuation for Switched Systems

The finite-past dependent solution to the KYP inequality can now be applied to switched systems. Consider the switched version of (3.1),

$$\begin{aligned} x_{t+1} &= A_{\theta(t)} x_t + B_{\theta(t)} w_t \\ z_t &= C_{\theta(t)} x_t + D_{\theta(t)} w_t. \end{aligned} \tag{3.12}$$

where the system parameters are drawn from a collection $\mathcal{M} = \{(A_i, B_i, C_i, D_i), i \in [N]\}$. The corresponding operator representation is given by

$$\begin{aligned} x &= ZA_\theta x + ZB_\theta w \\ z &= Cx + Dw. \end{aligned} \tag{3.13}$$

The extension of uniform stability to this system is that of Definition 2.8.

Definition 3.7. A switched system is uniformly strictly contractive if, for every admissible switching sequence θ, the corresponding LTV system is uniformly strictly contractive in the sense of Definition 3.1.

As in the previous chapter, the dual form of the KYP inequality (3.5) is used to state the result for switched systems.

Theorem 3.8. *For $H \geq 0$ and $L \geq 0$, the system of* (3.12) *is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an integer $M \geq 0$ and matrices $X_j \succ 0$ for*

$j \in [N]^{M+L+H}$ *such that, for all admissible* $i_{(-L:H)}$ *and* ϕ *as in* (1.3),

$$\begin{bmatrix} A_{\phi(i_{(-L:H)})} & B_{\phi(i_{(-L:H)})} \\ C_{\phi(i_{(-L:H)})} & D_{\phi(i_{(-L:H)})} \end{bmatrix}^* \begin{bmatrix} X_{i_{(-L-M+1:H)}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\phi(i_{(-L:H)})} & B_{\phi(i_{(-L:H)})} \\ C_{\phi(i_{(-L:H)})} & D_{\phi(i_{(-L:H)})} \end{bmatrix} - \begin{bmatrix} X_{i_{(-L-M:H-1)}} & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{3.14}$$

Proof. The proof of this theorem uses the same logic as that of Theorem 2.9. The same constructive method, along with Lemma 3.3, is used to demonstrate sufficiency, while the finite-past solution required by Lemma 3.6 is used to demonstrate necessity. □

The previous result is now applied to closed-loop systems; Consider the switched system given by

$$\begin{aligned} x_{t+1} &= A_{\theta(t)}x_t + B_{1,\theta(t)}w_t + B_{2,\theta(t)}u_t \\ z_t &= = C_{1,\theta(t)}x_t + D_{11,\theta(t)}w_t + D_{12,\theta(t)}u_t \\ y_t &= = C_{2,\theta(t)}x_t + D_{21,\theta(t)}w_t. \end{aligned} \tag{3.15}$$

Once again, the system is connected in feedback with a controller of the form in (2.11) (repeated here for convenience)

$$\begin{aligned} \hat{x}_{t+1} &= \hat{A}_t\hat{x}_t + \hat{B}_t y_t \\ u_t &= \hat{C}_t\hat{x}_t + \hat{D}_t y_t. \end{aligned} \tag{3.16}$$

Following the standard approach of [37], define

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B}_{1,i} = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix}; \quad \tilde{B}_{2,i} = \begin{bmatrix} 0 & B_{2,i} \\ I & 0 \end{bmatrix};$$

$$\tilde{C}_{1,i} = \begin{bmatrix} C_{1,i} & 0 \end{bmatrix}; \quad \tilde{C}_{2,i} = \begin{bmatrix} 0 & I \\ C_{2,i} & 0 \end{bmatrix}; \quad \tilde{D}_{12,i} = \begin{bmatrix} 0 & D_{12,i} \end{bmatrix}; \quad \tilde{D}_{21,i} = \begin{bmatrix} 0 \\ D_{21,i} \end{bmatrix}$$

and also

$$K_t = \begin{bmatrix} \hat{A}_t & \hat{B}_t \\ \hat{C}_t & \hat{D}_t \end{bmatrix}$$

with the same path-dependent controller gains as in the stabilization case. Then the corresponding closed-loop parameters are

$$A_C(i_{(-L:H)}) = \tilde{A}_{i_0} + \tilde{B}_{2,i_0} K_{i_{(-L:H)}} \tilde{C}_{2,i_0}; \qquad B_C(i_{(-L:H)}) = \tilde{B}_{1,i_0} + \tilde{B}_{2,i_0} K_{i_{(-L:H)}} \tilde{D}_{21,i_0};$$
$$C_C(i_{(-L:H)}) = \tilde{C}_{1,i_0} + \tilde{D}_{12,i_0} K_{i_{(-L:H)}} \tilde{C}_{2,i_0}; \qquad D_C(i_{(-L:J)}) = D_{11,i_0} + \tilde{D}_{12,i_0} K_{i_{(-L:H)}} \tilde{D}_{21,i_0}.$$

This produces the closed-loop system

$$\begin{aligned} x_C(t+1) &= A_C(\theta_{(t-L:t+H)}) x_C(t) + B_C(\theta_{(t-L:t+H)}) w_t \\ z(t) &= C_C(\theta_{(t-L:t+H)}) x_C(t) + D_C(\theta_{(t-L:t+H)}) w_t. \end{aligned} \tag{3.17}$$

Theorem 3.9. *For $H \geq 0$ an $L \geq 0$, the system of* (3.17) *is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$,*

$$\begin{bmatrix} A_C(i_{(-L:H)}) & B_C(i_{(-L:H)}) \\ C_C(i_{(-L:H)}) & D_C(i_{(-L:H)}) \end{bmatrix}^* \begin{bmatrix} X_{i_{(-L-M+1:H)}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_C(i_{(-L:H)}) & B_C(i_{(-L:H)}) \\ C_C(i_{(-L:H)}) & D_C(i_{(-L:H)}) \end{bmatrix}^* - \begin{bmatrix} X_{i_{(-L-M:H-1)}} & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{3.18}$$

Proof. Apply Theorem 3.8 to the closed loop system. □

The linearization of the previous conditions, including eliminating the explicit controller variable and partitioning the X_j and their inverses, proceeds by applying Lemmas 1.2 and Lemma 1.3 as is done in [37]. The loss of distinction between M and L discussed in Remark 2.13 also occurs here. As a generalization based on Remark 3.2, note that the system achieves attenuation level γ if and only if the scaled system with parameters $\{(A_i, \gamma^{-1/2}B_i, \gamma^{-1/2}C_i, \gamma^{-1}D_i)\}$ is uniformly strictly contractive. Separating this scalar from the system parameters at the end produces the following result.

Theorem 3.10. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (3.17) *is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that, for all admissible $i_{(-\bar{L}:H)}$,*

$$\begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_{i_0}R_{i_{(-\bar{L}:H-1)}}A_{i_0}^* - R_{i_{(-\bar{L}+1:H)}} & A_{i_0}R_{i_{(-\bar{L}:H-1)}}C_{1,i_0}^* & B_{1,i_0} \\ C_{1,i_0}R_{i_{(-\bar{L}:H-1)}}A_{i_0}^* & C_{1,i_0}R_{i_{(-\bar{L}:H-1)}}C_{1,i_0}^* - \gamma I & D_{11,i_0} \\ B_{1,i_0}^* & D_{11,i_0}^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{3.19a}$$

$$\begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_{i_0}^*S_{i_{(-\bar{L}+1:H)}}A_{i_0} - S_{i_{(-\bar{L}:H-1)}} & A_{i_0}^*S_{i_{(-\bar{L}+1:H)}}B_{1,i_0} & C_{1,i_0}^* \\ B_{1,i_0}^*S_{i_{(-\bar{L}+1:H)}}A_{i_0} & B_{1,i_0}^*S_{i_{(-\bar{L}_1:H)}}B_{1,i_0} - \gamma I & D_{11,i_0}^* \\ C_{1,i_0} & D_{11,i_0} & -\gamma I \end{bmatrix} \begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{3.19b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0 \tag{3.19c}$$

where

$$N_{F,i_0} = N\left(\begin{bmatrix} B_{2,i_0}^* & D_{12,i_0}^* \end{bmatrix}\right); \quad \textit{and} \quad N_{G,i_0} = N\left(\begin{bmatrix} C_{2,i_0} & D_{21,i_0} \end{bmatrix}\right).$$

Furthermore, given solutions to the inequalities of (3.19)*, a controller may be chosen with memory at most* $\bar{L}$*.*

Proof. The method of proof is identical to that of Theorem 2.15, making use of (3.18). □

A useful property of these results is that they are convex in the attenuation level γ; the feasibility problem can therefore be modified by adding the objective of minimizing γ.

3.2 Path-by-Path ℓ_2-Induced Gain

Theorem 3.10 considered *uniform* disturbance attenuation, setting a single gain γ that the system must achieve over all admissible switching sequences. Depending on the system parameters, the value of this uniform gain may be sensitive to a "worst-case" switching sequence with performance far worse than others; other sequences may provide better attenuation. With this in mind, this section considers a measure of *path-by-path* disturbance attenuation which can exploit the differences between such switching sequences. This can be thought of as a generalization of the observation made by Remark 3.2 to allow for a time-varying value of γ.

Definition 3.11. Let $\Gamma = \left\{\gamma_i : i \in [N]^{L+H+1}\right\}$ be a collection of positive parameters. Then the system of (3.12) achieves *path-by-path attenuation levels* Γ if, for every admissible switching sequence θ, it satisfies

$$\sum_{t=0}^{\infty} \|z_t\|_2^2 \leq \sum_{t=0}^{\infty} \gamma^2_{\theta_{(t-L:t+H)}} \|w_t\|_2^2. \tag{3.20}$$

If a system achieves path-by-path attenuation levels Γ, then it also achieves uniform attenuation $\hat{\gamma}$ for any $\hat{\gamma}$ which is an upper bound on Γ. The individual values of the γ_i need not be unique; they could be mode-dependent or depend on paths shorter than those of the controller dependence. The path-indexed gains can be used to scale the closed loop system gains as they appear in (3.18) by considering

$$\left\{(A_C(i_{(-L:H)}), \gamma^{-1/2}_{i_{(-L:H)}} B_i, \gamma^{-1/2}_{i_{(-L:H)}} C_i, \gamma^{-1}_{i_{(-L:H)}} D_i)\right\}.$$

The closed loop system will satisfy attenuation levels Γ if and only if this scaled system is uniformly contractive. Carrying these indexed gains through the elimination procedure leads to the following generalization of Theorem 3.10.

Theorem 3.12. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (3.17) *is uniformly exponentially stable and satisfies attenuation levels Γ if and only if there exist an integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that, for all admissible $i_{(-\bar{L}:H)}$,*

$$\begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_{i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* - R_{i_{(-\bar{L}+1:H)}} & A_{i_0} R_{i_{(-\bar{L}:H-1)}} C_{1,i_0}^* & B_{1,i_0} \\ C_{1,i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* & C_{1,i_0} R_{i_{(-\bar{L}:H-1)}} C_{1,i_0}^* - \gamma_{i_{(-\bar{L}:H)}} I & D_{11,i_0} \\ B_{1,i_0}^* & D_{11,i_0}^* & -\gamma_{i_{(-\bar{L}:H)}} I \end{bmatrix} \times \begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{3.21a}$$

$$\begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} A_{i_0} - S_{i_{(-\bar{L}:H-1)}} & A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} B_{1,i_0} & C_{1,i_0}^* \\ B_{1,i_0}^* S_{i_{(-\bar{L}+1:H)}} A_{i_0} & B_{1,i_0}^* S_{i_{(-\bar{L}_1:H)}} B_{1,i_0} - \gamma_{i_{(-\bar{L}:H)}} I & D_{11,i_0}^* \\ C_{1,i_0} & D_{11,i_0} & -\gamma_{i_{(-\bar{L}:H)}} I \end{bmatrix} \times \begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{3.21b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0 \tag{3.21c}$$

where $\gamma_{i_{(-\bar{L}:H)}} \in \Gamma$ and

$$N_{F,i_0} = N\left(\begin{bmatrix} B_{2,i_0}^* & D_{12,i_0}^* \end{bmatrix}\right); \quad \textit{and} \quad N_{G,i_0} = N\left(\begin{bmatrix} C_{2,i_0} & D_{21,i_0} \end{bmatrix}\right).$$

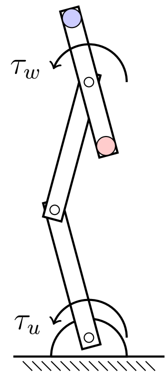

Figure 3.1: The barbell-double pendulum system with moveable mass, subjected to a disturbance about the upper hinge (connecting the barbell to the middle link).

Furthermore, given solutions to the inequalities of (3.21)*, a controller may be chosen with memory at most* $\bar{L}$*.*

As was the case for uniform attenuation levels, the preceding LMIs are convex in the individual gains γ_i. This permits the feasibility problem to be optimized over the values of these gains. In particular, selecting a family of weightings $\{\lambda_i\}$ leads to a Pareto-optimal performance level by minimizing the value of $\sum_i \lambda_i \gamma_i$ subject to the feasibility of (3.21).

3.3 Disturbance Attenuation for the Switching Pendulum

This chapter concludes with a return to the pendulum system used in Example 2.47 (shown again in Figure 3.1 for convenience) with the addition of a disturbance input at the top of the system.

Example 3.13. Consider again the double-pendulum system with attached barbell shown in Figure 3.1. As in the previous chapter, each link has length 1 m and mass 1kg, as well as the 0.1kg moveable mass which switches between the two endpoints of the barbell. The control effort is a torque τ_u applied at the bottom hinge, while a disturbance input τ_w is applied at the topmost hinge. The nonlinear dynamics are again linearized and discretized with time interval $t = 0.05\text{s}$.

$\bar{L}$ / H	0	1	2
0	$+\infty$	$+\infty$	1.04
1	$+\infty$	1.84	0.92
2	$+\infty$	1.40	·
3	$+\infty$	·	

Table 3.1: A table of minimal H_∞ gains for controllers of varying memory and horizon length for the pendulum system of Figure 3.1.

Table 3.1 displays the minimum uniform attenuation levels achieved by minimizing the value of γ subject to the feasibility of the conditions of (3.19). Notice that when $H = 0$ the system is not stabilizable; consequently the system has no finite achievable gain for such controllers. While the given controller knowledge is *sufficient* to provide the closed-loop attenuation level presented in the table, this does not demonstrate necessity to achieve those levels. Indeed, a controller of memory $L < \bar{L}$ may exist which achieves these gains. However, the corresponding closed-loop Lyapunov memory must be greater than zero for such controllers. At this time a converse result which determines the minimum achievable closed-loop gain from an infeasible result is not known.

Chapter 4

Windowed Output Regulation

This chapter presents results for control of switched systems achieving the second of two performance criteria: that of average output variance over a finite-horizon window. The length of this window may vary independently of the controller horizon; a comparison of the information available to the controller and that used for the performance measure is shown in Figure 4.1. Whenever the controller preview horizon is longer than the performance horizon, this problem reduces to an LTV windowed variance problem where the set of possible parameters is finite. This problem is a particular focus of model predictive control methods and has been studied in many contexts (see, e.g., [8, 45, 46, 49, 54, 55, 57, 81] and references) both by online optimization methods and explicit solutions to such optimization problems. The results presented here can be thought of as an offline computation of suitable controller gains in this case. They are equally suited to the case when the controller horizon is shorter than the performance window; in this situation any online optimization would have to optimize over possible future switching modes, a problem which does not easily reduce to a simple quadratic cost-to-go even if the terminal cost is assumed to be quadratic.

The chapter begins by developing existence conditions for controllers that both stabilize the system and achieve a uniform windowed variance performance in Section 4.1. The performance measure is generalized to path-by-path performance where the variance may depend on the switch-

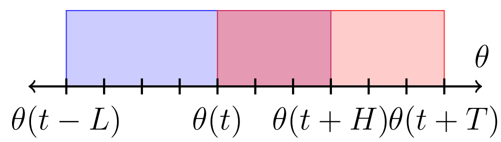

Figure 4.1: A comparison of the switching window used for the performance measure (in red) with the information known to the controller (in blue), along with the overlapping region.

ing sequence in Section 4.2. Numerical examples of both uniform and path-by-path variance regulation are given in Section 4.3.

The results presented here on uniform output variance appeared first in [31]; they were subsequently published, along with path-by-path windowed variance, as part of [30].

4.1 Disturbance Attenuation for Switched Systems

This section develops convex conditions for a path-dependent controller achieving a uniform average windowed variance. The case of path-by-path performance will be considered in the following section.

4.1.1 Windowed Variance Preliminaries for LTV Systems

Once more, the development of controller existence conditions begins by considering LTV systems. As in the previous two chapters, a convex condition for both stability and performance must be developed for which a finite-past dependent solution is necessary. Consider again the LTV system described by

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t w_t \\ z_t &= C_t x_t + D_t w_t. \end{aligned} \tag{4.1}$$

and the corresponding operator representation

$$
\begin{aligned}
x &= ZAx + ZBw \\
z &= Cx + Dw.
\end{aligned}
\tag{4.2}
$$

In contrast to Chapter 3, where the disturbance input w was taken as any bounded signal, the disturbance here will be taken to be an independent, identically distributed sequence of $\mathbb{R}^m$-valued Gaussian random variables. That is, for all $s, t \geq 0$, the input signal w satisfies

$$
\mathbb{E}\left[w_t\right] = 0; \quad \text{and} \quad \mathbb{E}\left[w_t w_s^*\right] = \begin{cases} I & \text{for } t = s \\ 0 & \text{for } t \neq s \end{cases}. \tag{4.3}
$$

Uniform stability of the system is again the same as in Definition 2.1.

Definition 4.1. Let $T \geq 0$. The system of (4.2) satisfies *T-step uniform performance* level γ for $\gamma > 0$ if, for $x_0 = 0$ and w as in (4.3), the output satisfies, for all $t \geq 0$,

$$
\frac{1}{T+1} \sum_{s=t}^{t+T} \mathbb{E}\left[\|z_s\|^2\right] < \gamma. \tag{4.4}
$$

The expression of the performance measure above is difficult to work with directly, so an equivalent representation is developed below. For a system which is uniformly exponentially stable, there exists a unique solution $Y_0 \succ 0$ to the Lyapunov equation

$$
(ZA)Y_0(ZA)^* - Y_0 = -(ZB)(ZB)^*. \tag{4.5}
$$

The block structure of this solution is given by

$$
[Y_0]_t = (B_t B_t^*) + \sum_{s=0}^{t-1} \left(A_t \cdot \ldots A_{s+1}\right) B_s B_s^* \left(A_t \cdot \ldots A_{s+1}\right)^*.
$$

Remark 4.2. It is immediate from the definition of Y_0 that any block-diagonal operator X which satisfies

$$(ZA)X(ZA)^* - X \preceq -(ZB)(ZB)^*$$

will necessarily satisfy $X \succeq Y_0$.

Using the system dynamics of (4.2) to express z_t in terms of w and the properties of w in (4.3), the expectation on output can be rewritten as

$$\mathbb{E}\left[\|z_t\|^2\right] = \text{Tr}\left(\left[CY_0C^* + DD^*\right]_t\right). \tag{4.6}$$

The output variance over the entire performance window may be simplified by considering the *windowed trace* operator for block-diagonal operators

$$\text{Tr}_{(t,T)}(X) := \frac{1}{T+1}\sum_{s=t}^{t+T} \text{Tr}\,(X_s). \tag{4.7}$$

Then the T-step performance condition of (4.4) can be equivalently written as

$$\text{Tr}_{(t,T)}\left(CY_0C^* + DD^*\right) < \gamma^2, \quad \forall t \geq 0.$$

Remark 4.3. Like the trace operator for matrices, the windowed trace preserves order; that is, if $X \succ Y$, then $\text{Tr}_{(t,T)}(X) \geq \text{Tr}_{(t,T)}(Y)$ for all t and T.

This leads to the following conditions for stability and windowed performance.

Lemma 4.4. *The system of* (4.2) *is uniformly exponentially stable and satisfies T-step uniform performance level* γ *if there exists a* $Y \succ 0$ *such that*

$$(ZA)Y(ZA)^* - Y \prec -(ZB)(ZB)^* \tag{4.8a}$$

$$\textit{Tr}_{(t,T)}\left(CYC^* + DD^*\right) < \gamma^2, \quad \forall t \geq 0. \tag{4.8b}$$

Proof. Suppose that a solution to (4.8) exists. Then (4.8a), along with Lemma 2.3, demonstrates uniform exponential stability. Knowing that the system is stable, construct Y_0 as the solution to (4.5). As noted in Remark 4.2, it must be that $Y_0 \preceq Y$; by Remark 4.3, it is also the case that

$$\text{Tr}_{(t,T)}\left(CY_0C^* + DD^*\right) \leq \text{Tr}_{(t,T)}\left(CYC^* + DD^*\right) < \gamma^2.$$

Then applying (4.6) shows that the system satisfies the performance condition. □

The previous lemma shows the sufficiency of these conditions; the following shows the necessity of a finite-past solution to them.

Lemma 4.5. *If the system of* (4.2) *is both uniformly exponentially stable and satisfies T-step uniform performance level* γ*, then there exists a solution* $Y \succ 0$ *to the conditions of* (4.8) *whose blocks depend on a finite number of past parameters.*

Proof. Let $\epsilon > 0$ and consider the sequence of operators $Y^{(\epsilon,0)} = \epsilon I$ and

$$Y^{(\epsilon,M+1)} = (ZA)Y^{(\epsilon,M)}(ZA)^* + (ZB)(ZB)^* + \epsilon I.$$

Inspecting the individual blocks of $Y^{(\epsilon,M)}$ shows that they each depend on at most M past parameters (compare this to the iteration in (3.7) or (2.6)). First observe that

$$Y^{(\epsilon,0)} = \epsilon I \preceq \epsilon I + (ZA)Y^{(\epsilon,M)}(ZA)^* + (ZB)(ZB)^* = Y^{(\epsilon,1)}$$

and that $Y \succeq X$ implies that $(ZA)Y(ZA)^* \succeq (ZA)X(ZA)^*$; together these facts show that the sequence $Y^{(\epsilon,M)}$ is monotone increasing. Notice that $Y_0^{(\epsilon,M)} = \epsilon I$ for all M; also notice that, whenever $Y_t^{(\epsilon,M)} = Y_t^{(\epsilon,M+1)}$, then $Y_{t+1}^{(\epsilon,M+1)} = Y_{t+1}^{(\epsilon,M+2)}$. These two observations and an induction argument show that, for $M \geq t$, $Y_t^{(\epsilon,M)} = Y_t^{(\epsilon,M+1)}$. Let $Y^{(\epsilon)}$ be the block-wise (weak) limit of this sequence. This limit satisfies

$$Y^{(\epsilon)} = (ZA)Y^{(\epsilon)}(ZA) + (ZB)(ZB)^* + \epsilon I.$$

The limit $Y^{(\epsilon)}$ can be decomposed such that $Y^{(\epsilon)} = Y_0 + \epsilon\tilde{Y}$, where Y_0 is the solution to (4.5) and $\tilde{Y}$ is the solution to

$$(ZA)\tilde{Y}(ZA) - \tilde{Y} = -I.$$

Since both Y_0 and $\tilde{Y}$ are bounded operators, there must be a $\beta > 0$ such that $Y^{(\epsilon)} \leq \beta I$. The linearity of the trace operator further shows that

$$\text{Tr}_{(t,T)}\left(CY^{(\epsilon)}C^* + DD^*\right) = \text{Tr}_{(t,T)}\left(CY_0C^* + DD^*\right) + \epsilon\text{Tr}_{(t,T)}\left(C\tilde{Y}C^*\right).$$

Since C and $\tilde{Y}$ are all bounded operators, $\text{Tr}_{(t,T)}(C\tilde{Y}C^*)$ is also bounded and a sufficiently small choice of ϵ will make this term as small as desired.

The system satisfies T-step performance level γ, so by (4.6), $\text{Tr}_{(t,T)}\left(CY_0C^* + DD^*\right) < \gamma^2$; choose ϵ sufficiently small such that $\text{Tr}_{(t,T)}\left(CY^{(\epsilon)}C^* + DD^*\right) < \gamma^2$ also. Now the uniform stability of the system guarantees the existence of c and λ as in Definition 2.1. Choose M large enough that $c^2\lambda^{2M} < \epsilon/(\beta - \alpha)$. Then

$$\begin{aligned}(ZA)Y^{(\epsilon,M)}(ZA)^* - Y^{(\epsilon,M)} &= (ZA)\left[Y^{(\epsilon,M)} - Y^{(\epsilon,M-1)}\right](ZA)^* - \epsilon I \\ &= (ZA)^M\left[Y^{(\epsilon,1)} - Y^{(\epsilon,0)}\right]\left((ZA)^*\right)^M - \epsilon I \\ &= (c^2\lambda^{2M})(\beta - \epsilon)I - \epsilon I \prec 0\end{aligned}$$

using the bound $Y^{(\epsilon,M)} \preceq Y^{(\epsilon)} \preceq \beta I$. This same inequality shows

$$\text{Tr}_{(t,T)}\left(CY^{(\epsilon)}C^* + DD^*\right) \leq \text{Tr}_{(t,T)}\left(CY^{(\epsilon)}C^* + DD^*\right) < \gamma^2.$$

Therefore $Y^{(\epsilon,M)}$ satisfies the conditions of (4.8) and is finite-past dependent. □

4.1.2 Windowed Variance for Switched Systems

The finite-past dependent solution from Theorem 4.5 can now be applied to switched systems. Once more, consider the switched version of (4.1),

$$\begin{aligned} x_{t+1} &= A_{\theta(t)}x_t + B_{\theta(t)}w_t \\ z_t &= C_{\theta(t)}x_t + D_{\theta(t)}w_t. \end{aligned} \tag{4.9}$$

and corresponding operator representation

$$\begin{aligned} x &= ZA_\theta x + ZB_\theta w \\ z &= Cx + Dw. \end{aligned} \tag{4.10}$$

The T-step performance measure is extended to all admissible switching sequences.

Definition 4.6. Let $T \geq 0$. The system of (4.10) satisfies T-step uniform performance level γ for $\gamma > 0$ if, for all admissible θ, the corresponding LTV system satisfies Definition 4.1.

A switched version of the conditions of Lemma 4.4 can now be stated.

Theorem 4.7. *For $H \geq 0$ and $L \geq 0$, the system of* (4.10) *is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists an integer $M \geq 0$ and matrices $Y_j \succ 0$ for $j \in [N]^{L+M+H}$ such that, for all admissible $i_{(-L-M:H)}$ and $\hat{\imath}_{(-L-M:H+T)}$ and ϕ as in* (1.3)*,*

$$A_{\phi(i_{(-L:H)})}Y_{i_{(-L-M:H-1)}}A^*_{\phi(i_{(-L:H)})} - Y_{i_{(-L-M+1:H)}} \prec -B_{\phi(i_{(-L:H)})}B^*_{\phi(i_{(-L:H)})} \tag{4.11a}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} Tr\left(C_{\phi(\hat{\imath}_{(t-L:t+H)})}Y_{\hat{\imath}_{(t-L-M:t+H-1)}}C^*_{\phi(\hat{\imath}_{(t-L:t+H)})} + D_{\phi(\hat{\imath}_{(t-L:t+H)})}D^*_{\phi(\hat{\imath}_{(t-L:t+H)})}\right) < \gamma^2. \tag{4.11b}$$

Proof. This proof follows the same blueprint as used for Theorems 2.9 and 3.8. Sufficiency comes from an application of Lemma 4.4; necessity comes from the finite-past solution required by Lemma 4.5. □

This result can now be applied to a closed-loop system. Consider the switched system given by

$$\begin{aligned} x_{t+1} &= A_{\theta(t)}x_t + B_{1,\theta(t)}w_t + B_{2,\theta(t)}u_t \\ z_t &= C_{1,\theta(t)}x_t + D_{11,\theta(t)}w_t + D_{12,\theta(t)}u_t \\ y_t &= C_{2,\theta(t)}x_t + D_{21,\theta(t)}w_t. \end{aligned} \tag{4.12}$$

Once again, the system is connected in feedback with a controller of the form

$$\begin{aligned} \hat{x}_{t+1} &= \hat{A}_t\hat{x}_t + \hat{B}_t y_t \\ u_t &= \hat{C}_t\hat{x}_t + \hat{D}_t y_t. \end{aligned} \tag{4.13}$$

As in the previous chapter, define

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B}_{1,i} = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix}; \quad \tilde{B}_{2,i} = \begin{bmatrix} 0 & B_{2,i} \\ I & 0 \end{bmatrix};$$
$$\tilde{C}_{1,i} = \begin{bmatrix} C_{1,i} & 0 \end{bmatrix}; \quad \tilde{C}_{2,i} = \begin{bmatrix} 0 & I \\ C_{2,i} & 0 \end{bmatrix}; \quad \tilde{D}_{12,i} = \begin{bmatrix} 0 & D_{12,i} \end{bmatrix}; \quad \tilde{D}_{21,i} = \begin{bmatrix} 0 \\ D_{21,i} \end{bmatrix}$$

and also

$$K_t = \begin{bmatrix} \hat{A}_t & \hat{B}_t \\ \hat{C}_t & \hat{D}_t \end{bmatrix}$$

Then the corresponding closed-loop parameters are

$$A_C(i_{(-L:H)}) = \tilde{A}_{i_0} + \tilde{B}_{2,i_0} K_{i_{(-L:H)}} \tilde{C}_{2,i_0}; \qquad B_C(i_{(-L:H)}) = \tilde{B}_{1,i_0} + \tilde{B}_{2,i_0} K_{i_{(-L:H)}} \tilde{D}_{21,i_0};$$
$$C_C(i_{(-L:H)}) = \tilde{C}_{1,i_0} + \tilde{D}_{12,i_0} K_{i_{(-L:H)}} \tilde{C}_{2,i_0}; \qquad D_C(i_{(-L:J)}) = D_{11,i_0} + \tilde{D}_{12,i_0} K_{i_{(-L:H)}} \tilde{D}_{21,i_0}.$$

This produces the closed-loop system

$$\begin{aligned} x_C(t+1) &= A_C(\theta_{(t-L:t+H)}) x_C(t) + B_C(\theta_{(t-L:t+H)}) w_t \\ z(t) &= C_C(\theta_{(t-L:t+H)}) x_C(t) + D_C(\theta_{(t-L:t+H)}) w_t. \end{aligned} \tag{4.14}$$

Theorem 4.8. *For $H \geq 0$ and $L \geq 0$, the system of* (4.10) *is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists an integer $M \geq 0$ and matrices $Y_j \succ 0$ for $j \in [N]^{L+M+H}$ such that, for all admissible $i_{(-L-M:H)}$ and $\hat{\imath}_{(-L-M:H+T)}$ and ϕ as in* (1.3),

$$A_C(i_{(-L:H)}) Y_{i_{(-L-M:H-1)}} A_C(i_{(-L:H)})^* - Y_{i_{(-L-M+1:H)}} \prec -B_C(i_{(-L:H)}) B_C(i_{(-L:H)})^* \tag{4.15a}$$

$$\begin{aligned} \frac{1}{T+1} \sum_{t=0}^{T} \textbf{\textit{Tr}} \Big(& C_C(\hat{\imath}_{(t-L:t+H)}) Y_{\hat{\imath}_{(t-L-M:t+H-1)}} C_C(\hat{\imath}_{(t-L:t+H)})^* \\ & + D_C(\hat{\imath}_{(t-L:t+H)}) D_C(\hat{\imath}_{(t-L:t+H)})^* \Big) < \gamma^2. \end{aligned} \tag{4.15b}$$

Proof. This is a direct application of Theorem 4.7 to the closed-loop system of (4.14). □

Equivalent, convex conditions to these can be derived using the change of variables similar to that of [51, 75]. Introduce the slack variables $Z_{\hat{\imath}}$ for $\hat{\imath} \in [N]^{L+M+H+1}$ such that

$$C_C(\hat{\imath}_{(t-L:t+H)}) Y_{\hat{\imath}_{(t-L-M:t+H-1)}} C_C(\hat{\imath}_{(t-L:t+H)})^* + D_C(\hat{\imath}_{(t-L:t+H)}) D_C(\hat{\imath}_{(t-L:t+H)})^* \preceq Z_{\hat{\imath}_{(-L-M:H)}}. \tag{4.16}$$

These $Z_{\hat{\imath}}$ can be chosen in such a way that

$$\frac{1}{T+1}\sum_{t=0}^{T} \operatorname{Tr} Z_{\hat{\imath}_{(t-L-M:t+H)}} < \gamma^2. \tag{4.17}$$

Applying a Schur complement to (4.15a) and (4.16), along with (4.17), produces the conditions

$$\begin{bmatrix} -Y^{-1}_{i_{(-L-M:H-1)}} & A_C(i_{(-L:H)})^* & 0 \\ A_C(i_{(-L:H)}) & -Y_{i_{(-L-M+1:H)}} & B_C(i_{(-L:H)}) \\ 0 & B_C(i_{(-L:H)})^* & -I \end{bmatrix} \prec 0 \tag{4.18a}$$

$$\begin{bmatrix} -Y_{\hat{\imath}_{(-L-M:H-1)}} & C_C(\hat{\imath}_{(-L:H)})^* & 0 \\ C_C(\hat{\imath}) & -Z_{\hat{\imath}_{(-L-M:H)}} & D_C(\hat{\imath}_{(-L:H)}) \\ 0 & D_C(\hat{\imath}_{(-L:H)}) & -I \end{bmatrix} \prec 0 \tag{4.18b}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \operatorname{Tr} Z_{\hat{\imath}_{(t-L-M:t+H)}} < \gamma^2. \tag{4.18c}$$

Partition the matrices Y_j and Y_j^{-1} as

$$Y_j = \begin{bmatrix} R_j & T_j \\ T_j^* & \cdot \end{bmatrix}; \quad Y_j^{-1} = \begin{bmatrix} S_j & U_j \\ U_j^* & \cdot \end{bmatrix} \tag{4.19}$$

where $R_j, S_j \in \mathbb{R}^{n\times n}$ and the remaining blocks are compatibly partitioned. Next, define

$$\begin{aligned} W_i = &\begin{bmatrix} S_{i_{(-L-M+1:H)}} A_{i_0} R_{i_{(-L-M:H-1)}} & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} U_{i_{(-L-M+1:H)}} & S_{i_{(-L-M+1:H)}} B_{2,i_0} \\ 0 & I \end{bmatrix} K_{i_{(-L:H)}} \begin{bmatrix} T^*_{i_{(-L-M:H-1)}} & 0 \\ C_{2,i_0} R_{i_{(L-M:H-1)}} & I \end{bmatrix}. \end{aligned} \tag{4.20}$$

for every $i_{(-L-M:H)}$.

Remark 4.9. This change of variable is only reversible when there is one K_i for each W_i; that is, when $M = 0$. Just as noted by Remark 2.13, this removes the distinction between controller memory L and Lyapunov memory M.

Let

$$M_i = \begin{bmatrix} I & S_{i_{(-L-M+1:H)}} \\ 0 & U^*_{i_{(-L-M+1:H)}} \end{bmatrix}; \quad \tilde{M}_i = \begin{bmatrix} I & R_{i_{(-L-M:H-1)}} \\ 0 & T^*_{i_{(-L-M:H-1)}} \end{bmatrix}.$$

Then applying the congruence transformation $M_i \oplus \tilde{M}_i \oplus I$ to (4.18a) and $M_i \oplus I \oplus I$ to (4.18b), along with simplifying algebra, produces the following Theorem.

Theorem 4.10. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (4.14) *is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists and integer $\bar{L} \geq 0$, matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$, and matrices $Z_{\hat{\jmath}}$, $W_{\hat{\jmath}}$ for $\hat{\jmath} \in [N]^{\bar{L}+H+1}$ such that, for all admissible $i_{(-\bar{L}:H)}$ and $\hat{\imath}_{(-\bar{L}:H+T)}$,*

$$H_i + F^*_{i_0} W_i G_{i_0} + G^*_{i_0} W^*_i F_{i_0} \prec 0 \tag{4.21a}$$

$$\hat{H}_i + \hat{F}^*_{i_0} W_i \hat{G}_{i_0} + \hat{G}^*_{i_0} W^*_i \hat{F}_{i_0} \prec 0 \tag{4.21b}$$

$$\frac{1}{T+1} \sum_{t=0}^{T} \mathit{Tr}\, Z_{\hat{\imath}_{(t-\bar{L}:t+H)}} < \gamma^2 \tag{4.21c}$$

where

$$F_{i_0} = \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & B^*_{2,i_0} & 0 & 0 \end{bmatrix}; \quad \hat{F}_{i_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & D^*_{12,i_0} & 0 \end{bmatrix}$$

$$G_{i_0} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_{2,i_0} & 0 & 0 & 0 & D_{21,i_0} \end{bmatrix}; \quad \hat{G}_{i_0} = \begin{bmatrix} 0 & I & 0 & 0 \\ C_{2,i_0} & 0 & 0 & D_{21,i_0} \end{bmatrix}$$

$$H_i = \begin{bmatrix} -S_{i_{(-\bar{L}:H-1)}} & -I & A_{i_0}^* & A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} & 0 \\ \cdot & -R_{i_{(-\bar{L}:H-1)}} & R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* & 0 & 0 \\ \cdot & \cdot & -R_{i_{(-\bar{L}+1:H)}} & -I & B_{1,i_0} \\ \cdot & \cdot & \cdot & -S_{i_{(-\bar{L}+1:H)}} & S_{i_{(-\bar{L}+1:H)}} B_{1,i_0} \\ \cdot & \cdot & \cdot & \cdot & -I \end{bmatrix}$$

$$\hat{H}_i = \begin{bmatrix} -S_{i_{(-\bar{L}:H-1)}} & -I & C_{1,i_0}^* & 0 \\ \cdot & -R_{i_{(-\bar{L}:H-1)}} & -R_{i_{(-\bar{L}:H-1)}} C_{1,i_0}^* & 0 \\ \cdot & \cdot & -Z_i & D_{11,i_0} \\ \cdot & \cdot & \cdot & -I \end{bmatrix}.$$

Furthermore, given solutions to the inequalities of (4.21)*, a controller may be constructed with memory at most* $\bar{L}$*.*

Proof. The proof proceeds by manipulating the equations of (4.18) and making use of the change of variables of (4.20). In particular, a solution to (4.21) allows the $K_{i_{(-\bar{L}:H)}}$ can be constructed algebraically from (4.20). □

4.2 Path-by-Path Windowed Output Variance

The performance measure described in Definition 4.1 depends entirely on the system parameters over the time interval $[t, t+T]$; depending on how the switching signal evolves in time, this may lead to either an advantageous or disadvantageous switching path for minimizing variance. Using this motivation, this performance metric is extended to consider path-by-path performance where the gain level depends on this switching path.

Definition 4.11. Let $T \geq 0$, and let $\Gamma = \left\{\gamma_i : i \in [N]^{T+1}\right\}$ be a collection of positive parameters. The system of (4.9) satisfies T-step *path-by-path* performance levels Γ if, for $x_0 = 0$ and w as in

(4.3), the output satisfies,

$$\frac{1}{T+1}\sum_{s=t}^{t+T}\mathbb{E}\left[\|z_s\|^2\|\right] < \gamma^2_{\theta_{(t:t+T)}} \tag{4.22}$$

for all admissible sequences θ and $t \geq 0$.

The generalization of closed-loop performance to satisfy path-by-path performance follows with the natural alteration of (4.15b) to make use of path-dependent performance levels in place of a uniform gain. Carrying this distinction through the change of variables produces the following path-by-path performance result.

Theorem 4.12. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (4.14) *is uniformly exponentially stable and satisfies T-step performance levels Γ if and only if there exists and integer $\bar{L} \geq 0$, matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$, and matrices $Z_{\hat{\jmath}}$, $W_{\hat{\jmath}}$ for $\hat{\jmath} \in [N]^{\bar{L}+H+1}$ such that, for all admissible $i_{(-\bar{L}:H)}$ and $\hat{\imath}_{(-\bar{L}:H+T)}$,*

$$H_i + F^*_{i_0}W_iG_{i_0} + G^*_{i_0}W^*_iF_{i_0} \prec 0 \tag{4.23a}$$

$$\hat{H}_i + \hat{F}^*_{i_0}W_i\hat{G}_{i_0} + \hat{G}^*_{i_0}W^*_i\hat{F}_{i_0} \prec 0 \tag{4.23b}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathit{Tr}\, Z_{\hat{\imath}_{(t-\bar{L}:t+H)}} < \gamma^2_{i_{(0:T)}}. \tag{4.23c}$$

where F_{i_0}, $\hat{F}_{i_0}$, G_{i_0}, $\hat{G}_{i_0}$, H_i, and $\hat{H}_i$ are all defined as in Theorem 4.10. Furthermore, given solutions to the inequalities of (4.23)*, a controller may be constructed with memory at most $\bar{L}$.*

Proof. The proof of this result is identical to that of Theorem 4.10. □

As was the case with path-by-path disturbance attenuation, the gains $\gamma_{i_{(0:T)}}$ may be incorporated into an objective function with weightings $\{\lambda_{i_{(0:T)}}\}$ to find an optimal path-by-path controller

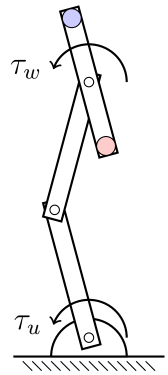

Figure 4.2: The barbell-double pendulum system with moveable mass, subjected to a disturbance about the upper hinge (connecting the barbell to the middle link).

4.3 Numerical Examples of Windowed Variance

In this section two numerical examples are provided to demonstrate control for windowed variance. The first reconsiders the pendulum system of previous chapters, while the second provides a demonstration of switching output maps instead of switching system dynamics.

Example 4.13. Consider again the double-pendulum system with attached barbell shown in Figure 4.2. As in the previous chapters, each link has length 1 m and mass 1kg, as well as the 0.1kg moveable mass which switches between the two endpoints of the barbell. The control effort is a torque τ_u applied at the bottom hinge, while a disturbance input τ_w (with the properties given in (4.3)) is applied at the topmost hinge. The nonlinear dynamics are again linearized and discretized with time interval $t = 0.05$s.

The conditions for T-step uniform performance of Theorem 4.10 are applied to the pendulum system for a performance horizon $T = 3$ and controller parameters $\bar{L} = H = 1$. Minimizing the value of the uniform performance level γ such that the conditions of (4.21) are feasible produces a gain of $\gamma = 2.09$. In contrast, when the path-by-path performance conditions of Theorem 4.12 are applied and the path-by-path gains equally weighted, the resulting minimum gains are given by

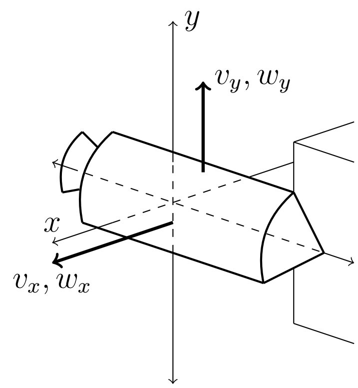

Figure 4.3: A two-degrees-of-freedom spacecraft which travels along a straight trajectory past an obstacle.

$$\gamma_{111} = 1.98, \quad \gamma_{112} = 1.82, \quad \gamma_{121} = 1.95, \quad \gamma_{122} = 2.23$$
$$\gamma_{211} = 2.36, \quad \gamma_{212} = 2.29, \quad \gamma_{221} = 2.69, \quad \gamma_{222} = 2.90.$$

These results demonstrate the utility of path-by-path performance for this system; over the path 112 the variance level is 50% less than the uniform bound; this comes at the expense of higher variance along more difficult paths.

Example 4.14. The small spacecraft of Fig. 4.3 flies on a straight reference trajectory, subject to disturbances w_x, w_y pushing the craft out of alignment. The ship has lateral thrusters v_x, v_y which respond as first-order systems to leaky control efforts u_x, u_y. The resulting system dynamics are

$$\ddot{x} = v_x + w_x; \quad \dot{v_x} = -0.5v_x + u_x + 0.1u_y$$
$$\ddot{y} = v_y + w_y; \quad \dot{v_y} = -0.5v_y + 0.1u_x + u_y.$$

During flight, the craft passes near obstacles in either the x- or y-direction; while near an obstacle, deviations in the direction of the obstacle present a greater risk to the safety of the craft. To this end, the controlled output switches between three modes given by

$$z_1 = \begin{bmatrix} x & y & 0.5u_x & 0.5u_y \end{bmatrix}; \; z_2 = \begin{bmatrix} 5x & 0.5y & 0.5u_x & 0.5u_y \end{bmatrix}; \; z_3 = \begin{bmatrix} 0.5x & 5y & 0.5u_x & 0.5u_y \end{bmatrix}$$

$$
\begin{array}{llll}
\gamma_{111} = 1.65 & \gamma_{211} = 1.48 & \gamma_{311} = 1.48 & \gamma_{121} = 1.85 \\
\gamma_{221} = 1.80 & \gamma_{131} = 1.85 & \gamma_{331} = 1.80 & \gamma_{112} = 1.61 \\
\gamma_{212} = 1.44 & \gamma_{312} = 1.44 & \gamma_{122} = 1.71 & \gamma_{222} = 2.32 \\
\gamma_{113} = 1.61 & \gamma_{213} = 1.44 & \gamma_{313} = 1.44 & \gamma_{133} = 1.71 \\
\gamma_{333} = 2.33. & & &
\end{array}
$$

Figure 4.4: The optimal path-by-path H_∞ gains for controllers with $L = H = 1$ for the system of Example 4.14

where mode 1 represents unobstructed flight, while modes 2 and 3 represent flight near an obstacle in the x- or y-direction respectively. The ship is permitted observation of its x- and y-position. The dynamics are discretized with time interval $t = 0.1$ s. The switching dynamics allow the system to switch between the unobstructed mode and either obstructed modes, or to remain in the current mode (but not to switch directly from one obstructed mode to the other).

This example features a mode-dependent performance measure but mode-independent dynamics. The switching in the controlled output z designates either the $x-$ or $y-$position as "critical" when an obstacle is present (a similar approach is taken in [34]). Once again, the performance window of $T = 3$ is chosen, along with controller parameters $\bar{L} = H = 1$. When the conditions of Theorem 4.10 are considered, the optimal uniform gain is $\gamma = 1.75$. When path-by-path performance levels are allowed, the resulting path-by-path gains are given in Figure 4.4.

The largest path-by-path gains are found for the paths 222 and 333, which correspond to sustained operation near an obstacle (in either direction). This matches the intuition that, near obstacles, deviations are punished more severely and the corresponding random disturbances cause a larger variance in the output. These paths suffer compared to the uniform performance bound, while those away from obstacles obtain improved performance. An unequal weighting of performance to emphasize minimal gains near obstacles or on the most likely paths could be used to further refine performance.

Chapter 5

Extensions to Non-regular Switching Languages

All of the switched linear systems considered thus far have used admissible switching sequences developed by directed graphs. These are also called *regular languages* in automaton theory. Such languages are a natural source for discrete switching logic and appear in many engineering applications. A notable example of such systems are Markvoian jump linear systems, where the switching signal is stochastic rather than nondeterministic. Removing the transition probabilities from any such system produces a switched linear system whose switching language is regular.

Nevertheless, the controller synthesis conditions in Chapters 2 to 4 can be extendedto more general switching languages; this extension is given here, but it not a primary focus of this dissertation. Extensions to non-regular switching languages were considered for memory-only controllers in [52] for disturbance attenuation and in [50, 51] for windowed variance. The results presented here appeared previously in [30].

In place of a directed graph which generates admissible switching sequences, consider the set of admissible sequences to be any set $\Theta \subset [N]^\infty$. No other structure is assumed of the set Θ; in particular the cardinality of Θ may be finite or countably infinite. Of interest for path-dependent controllers are the finite-length subsequences which appear in each admissible switching path. In order to handle negative indices in such paths, introduce the dummy mode 0 and the convention that $\theta(t) = 0$ for all $\theta \in \Theta$ and $t < 0$.

Definition 5.1. A finite-length switching path $i_{(-L:H)}$ occurs in Θ when there exists a $\theta \in \Theta$ and $t \geq 0$ such that $\theta(t-L:t+H) = i_{(-L:H)}$. The set of all such paths which occur in Θ is given by

$$\mathcal{L}_{(L,H)}(\Theta) := \left\{\theta_{(t-L:t+H)} : \theta \in \Theta, t \geq 0\right\}.$$

By construction, some elements of $\mathcal{L}_{(L,H)}(\Theta)$ will contain the dummy mode 0. As these paths may only appear at the beginning of an admissible switching path, any controller gain assigned to such a path would be useful only at system start-up. To reduce the number of controller gains required, some of these paths may be replaced by suitable paths from $\mathcal{L}_{(L,H)}(\Theta)$ which do not contain the dummy mode.

Definition 5.2. The set $\mathcal{M}_{(L,H)}(\Theta) \subset \mathcal{L}_{(L,H)}(\Theta)$ is the smallest subset which satisfies the following properties:

1. For all $\theta \in \Theta$ and $t \geq L$, $\theta_{(t-L:t+H)} \in \mathcal{M}_{(L,H)}(\Theta)$.
2. For all $j \in \mathcal{L}_{(0,H)}(\Theta)$, there exists an $i \in \{0,\ldots,N\}^L$ such that, for every $\theta \in \Theta$ and $0 \leq t \leq L$,

$$(i_{t-L},\ldots,i_{-1},\theta(0),\ldots,\theta(t+H)) \in \mathcal{M}_{(L,H)}(\Theta).$$

The set $\mathcal{M}_{(L,H)}(\Theta)$ is constructed to have fewer elements than $\mathcal{L}_{(L,H)}$ precisely by removing some dummy-led subsequences. The first bullet point above is precisely those elements of $\mathcal{L}_{(L,H)}(\Theta)$ which do not contain 0; the second requires that each admissible sequence θ be well-defined for $t \geq -L$ by left-extending either with the dummy mode 0 or possibly with another element of $\mathcal{L}_{(L,H)}(\Theta)$ which does not contain 0. In this way, the second condition is comparable to the left-extension done in the proof of Theorem 2.9. The set $\mathcal{M}_{(L,H)}(\Theta)$ will serve as the index set for the inequalities in the results that follow. As a prototypical example, consider the non-regular analogue to Theorem 2.9.

Theorem 5.3. *For $H \geq 0$ and $L \geq 0$, the system of* (2.7) *is uniformly exponentially stable if and only if there exist an integer $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{M+L+H}$ such that,*

$$A^*_{\phi\left(i_{(-L:H)}\right)} X_{i_{(-L-M+1:H)}} A_{\phi\left(i_{(-L:H)}\right)} - X_{i_{(-L-M:H-1)}} \prec 0 \tag{5.1}$$

for all $i \in \mathcal{M}_{(L,H)}(\Theta)$ and ϕ as in (1.3).

Proof. The proof of sufficiency is constructive and follows the same logic as in the proof of Theorem 2.9. The properties of $\mathcal{M}_{(L,H)}(\Theta)$ ensure that the left-extension needed to handle the blocks for $0 \leq t \leq L$ may be done successfully. For necessity, considering the block inequalities generated by each admissible switching sequence $\theta \in \Theta$ produces an inequality for every element of $\mathcal{L}_{(L,H)}(\Theta) \supset \mathcal{M}_{(L,H)}(\Theta)$. □

The development of controller synthesis conditions for stabilization and each performance measure proceeds identically to the case of regular switching languages, with only the modification that all sequences in $\mathcal{M}_{(L,H)}(\Theta)$ must be considered. The non-regular analogues of Theorems 2.15, 3.10, and 4.10 are presented below.

Theorem 5.4. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (2.12) *is uniformly exponentially stable if and only if there exist and integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that, for all $i_{(-\bar{L}:H)} \in \mathcal{M}_{(L,H)}(\Theta)$,*

$$N(B^*_{i_0})^* \left(A_{i_0} R_{i_{(-\bar{L}:H-1)}} A^*_{i_0} - R_{i_{(-\bar{L}+1:H)}}\right) N(B^*_{i_0}) \prec 0 \tag{5.2a}$$

$$N(C_{i_0})^* \left(A^*_{i_0} S_{i_{(-\bar{L}+1:H)}} A_{i_0} - S_{i_{(-\bar{L}:H-1)}}\right) N(C_{i_0}) \prec 0 \tag{5.2b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0. \tag{5.2c}$$

Furthermore, given solutions to the inequalities of (5.2)*, a controller may be chosen with memory at most* $\bar{L}$*.*

Theorem 5.5. *Let* $H \geq 0$*. There exists a path-dependent controller with horizon* H *such that* (3.17) *is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an integer* $\bar{L} \geq 0$ *and matrices* $R_j \succ 0$*,* $S_j \succ 0$ *for* $j \in [N]^{\bar{L}+H}$ *such that, for all* $i_{(-\bar{L}:H)} \in \mathcal{M}_{(L,H)}(\Theta)$*,*

$$\begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_{i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* - R_{i_{(-\bar{L}+1:H)}} & A_{i_0} R_{i_{(-\bar{L}:H-1)}} C_{1,i_0}^* & B_{1,i_0} \\ C_{1,i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* & C_{1,i_0} R_{i_{(-\bar{L}:H-1)}} C_{1,i_0}^* - \gamma I & D_{11,i_0} \\ B_{1,i_0}^* & D_{11,i_0}^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{5.3a}$$

$$\begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} A_{i_0} - S_{i_{(-\bar{L}:H-1)}} & A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} B_{1,i_0} & C_{1,i_0}^* \\ B_{1,i_0}^* S_{i_{(-\bar{L}+1:H)}} A_{i_0} & B_{1,i_0}^* S_{i_{(-\bar{L}_1:H)}} B_{1,i_0} - \gamma I & D_{11,i_0}^* \\ C_{1,i_0} & D_{11,i_0} & -\gamma I \end{bmatrix} \begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{5.3b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H)}} & I \\ I & S_{i_{(-\bar{L}:H)}} \end{bmatrix} \succeq 0 \tag{5.3c}$$

where

$$N_{F,i_0} = N\left(\begin{bmatrix} B_{2,i_0}^* & D_{12,i_0}^* \end{bmatrix}\right); \quad \textit{and} \quad N_{G,i_0} = N\left(\begin{bmatrix} C_{2,i_0} & D_{21,i_0} \end{bmatrix}\right).$$

Furthermore, given solutions to the inequalities of (5.3)*, a controller may be chosen with memory at most* $\bar{L}$*.*

To handle the windowed performance condition of Chapter 4, consider the set of all possible switching sequences for the performance window

$$\mathcal{W}_T := \left\{\theta_{(t:t+T)} : \theta \in \Theta, t \geq 0\right\}.$$

Theorem 5.6. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (4.14) *is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists and integer $\bar{L} \geq 0$, matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$, and matrices $Z_{\hat{\jmath}}$, $W_{\hat{\jmath}}$ for $\hat{\jmath} \in [N]^{\bar{L}+H+1}$ such that, for all $i_{(-\bar{L}:H)} \in \mathcal{M}_{(L,H)}(\Theta)$ and all $\hat{\imath}_{(-\bar{L}:H+T)} \in \mathcal{M}_{(-\bar{L}:H)}$ such that $\hat{\imath}_{(0:T)} \in \mathcal{W}_T(\Theta)$,*

$$H_i + F_{i_0}^* W_i G_{i_0} + G_{i_0}^* W_i^* F_{i_0} \prec 0 \tag{5.4a}$$

$$\hat{H}_i + \hat{F}_{i_0}^* W_i \hat{G}_{i_0} + \hat{G}_{i_0}^* W_i^* \hat{F}_{i_0} \prec 0 \tag{5.4b}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathit{Tr}\, Z_{\hat{\imath}_{(t-\bar{L}:t+H)}} < \gamma^2. \tag{5.4c}$$

where

$$F_{i_0} = \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & B_{2,i_0}^* & 0 & 0 \end{bmatrix}; \quad \hat{F}_{i_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & D_{12,i_0}^* & 0 \end{bmatrix}$$

$$G_{i_0} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_{2,i_0} & 0 & 0 & 0 & D_{21,i_0} \end{bmatrix}; \quad \hat{G}_{i_0} = \begin{bmatrix} 0 & I & 0 & 0 \\ C_{2,i_0} & 0 & 0 & D_{21,i_0} \end{bmatrix}$$

$$H_i = \begin{bmatrix} -S_{i_{(-\bar{L}:H-1)}} & -I & A_{i_0}^* & A_{i_0}^* S_{i_{(-\bar{L}+1:H)}} & 0 \\ \cdot & -R_{i_{(-\bar{L}:H-1)}} & R_{i_{(-\bar{L}:H-1)}} A_{i_0}^* & 0 & 0 \\ \cdot & \cdot & -R_{i_{(-\bar{L}+1:H)}} & -I & B_{1,i_0} \\ \cdot & \cdot & \cdot & -S_{i_{(-\bar{L}+1:H)}} & S_{i_{(-\bar{L}+1:H)}} B_{1,i_0} \\ \cdot & \cdot & \cdot & \cdot & -I \end{bmatrix}$$

$$\hat{H}_i = \begin{bmatrix} -S_{i_{(-\bar{L}:H-1)}} & -I & C^*_{1,i_0} & 0 \\ \cdot & -R_{i_{(-\bar{L}:H-1)}} & -R_{i_{(-\bar{L}:H-1)}} C^*_{1,i_0} & 0 \\ \cdot & \cdot & -Z_i & D_{11,i_0} \\ \cdot & \cdot & \cdot & -I \end{bmatrix}.$$

Furthermore, given solutions to the inequalities of (5.4)*, a controller may be constructed with memory at most* $\bar{L}$*.*

Chapter 6

Positive Switched Linear Systems

This chapter addresses a special case of switched linear systems; namely, those whose state vectors remain component-wise non-negative throughout the evolution of the system. Such a restriction appears naturally in engineering contexts such as biological or chemical processes, manufacturing, or thermal management in which the natural states of the system cannot meaningfully take negative values. For this reason, there is much study of the particular case of positive systems including those with nonlinear dynamics ([6, 7]), and in both discrete- and continuous-time linear dynamics ([60, 71, 72, 76]). Previous authors have also considered positive system gain ([80]), distributed control ([73]), and switched positive systems using both common ([36]) and modal ([9]) Lyapunov functions. Presented here are path-dependent controller synthesis conditions, which are known to outperform modal or common controllers. These results were previously presented as part of [28].

While the stabilization results of Chapter 2 certainly apply to positive systems, the resulting controller synthesis conditions make no guarantees that the closed-loop system will remain positive. The results of this chapter explicitly guarantee this property as part of the synthesis conditions. These conditions have the further property of being stated as linear programming problems (LPs) instead of SDPs. This reduces the computational difficulty of solving the controller synthesis conditions. The structure of these LPs remains the same as the more general stabilization and performance results of previous chapters; it is this structure that will be exploited in Chapter 7. In the context of this dissertation, these positive system results may be taken as an intermediate step towards decomposing SDPs.

6.1 Preliminaries for Positive Linear Time-Varying Systems

As was the case for previous controller synthesis conditions, this chapter begins with preliminary results for positive time-varying systems. Consider the LTV system described by

$$x_{t+1} = A_t x_t. \tag{6.1}$$

A block-diagonal representation of this system is given by

$$x = ZAx. \tag{6.2}$$

Definition 6.1. The system of (6.1) is called *positive* if, for every $x_0 \geq 0$, the system satisfies $x_t \geq 0$ for all $t \geq 0$.

This definition is equivalent to taking the positive orthant as an invariant set. A well-known equivalent condition to positivity (for discrete-time systems) is given below.

Lemma 6.2. *The system of* (6.2) *is positive if and only if* $ZA \geq 0$*. This holds if and only if* $A_t \geq 0$ *for all* $t \geq 0$*.*

The definition of stability for these systems is the same as in previous chapters, and is repeated here for convenience.

Definition 6.3. The system of (6.2) is called *uniformly exponentially stable* if there exist constants

$c \geq 1$ and $\lambda \in (0, 1)$ such that, for all $k \geq 0$,

$$\|(ZA)^k\| \leq c\lambda^k. \tag{6.3}$$

The next lemma generalizes a well-known Lyapunov stability result (see, eg., [10, 73]) to block-diagonal operators, while also demonstrating the necessity of a finite-past dependent solution.

Lemma 6.4. *The positive system of* (6.2) *is uniformly exponentially stable if and only if there exist an integer* $M \geq 0$ *and a vector seqeunce* $d = (d_1, d_2, \dots)$ *with each* $d_t > 0$ *such that*

$$(ZA - I)\, d < 0 \tag{6.4}$$

where each d_t *depends on at most* M *past parameters.*

Proof. To show sufficiency, suppose such a sequence d satisfying (6.4) exists. Then $V(x) := d^* x$ serves as a copositive Lyapunov function for the operator $(ZA)^*$; this operator is stable if and only if ZA is stable.

Next, suppose that (6.2) is stable. Choose M large enough that $\|(ZA)^M\| < \frac{1}{\sqrt{n}}$. Now choose

$$d := \sum_{k=0}^{M-1} (ZA)^k \mathbf{1}.$$

By construction $d > 0$; the individual elements of d are given by

$$d_t = \mathbf{1} + A_{t-1}\mathbf{1} + \cdots + (A_{t-1} \cdot \ldots A_{t-M})\, \mathbf{1}$$

which depends on M past parameters (for $t \geq M$; the first $M - 1$ blocks depend on fewer parameters). Substitution into (6.4) gives

$$(ZA - I)\, d = (ZA)^M \mathbf{1} - \mathbf{1}.$$

By choice of M, $\|(ZA)^M \mathbf{1}\| < 1$, so the desired inequality is satisfied. □

6.2 Stabilization of Positive Switched Systems

Having demonstrated a finite-past dependent solution to (6.4), switched systems may now be considered in the same spirit as previous chapters. Consider the switched system

$$x_{t+1} = A_{\theta(t)} x_t \tag{6.5}$$

and the corresponding operator representation

$$x = Z A_\theta x. \tag{6.6}$$

A positive switched linear system must satisfy Definition 6.1 for every admissible switching sequence. To be considered uniformly stable, it must also satisfy stability for every switching sequence.

Corollary 6.5. *The switched system of* (6.6) *is positive if and only if* $A_i \geq 0$ *for every* $i \in [N]$.

Definition 6.6. The switched system of (6.6) is uniformly exponentially stable if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that, for every admissible switching sequence θ, the corresponding LTV system is stable in the sense of Definition 6.3.

The finite-past solution demonstrated by Lemma 6.4 can now be applied to switched systems.

Theorem 6.7. *For* $H \geq 0$ *and* $L \geq 0$*, the system of* (6.6) *is uniformly exponentially stable if and only if there exist an integer* $M \geq 0$ *and vectors* $d_j > 0$ *for* $j \in [N]^{L+M+H}$ *such that, for all admissible* $i_{(-L:H)}$ *and* ϕ *as in* (1.3),

$$A_{\phi(i_{(-L:H)})} d_{i_{(-L-M:H-1)}} - d_{i_{(-L-M+1:H)}} < 0. \tag{6.7}$$

Proof. The proof is like that of Theorem 2.9. Sufficiency is demonstrated by using the individual d_i to construct a suitable sequence d satisfying (6.4) for each sequence θ. Necessity comes from the finite-past solution required by Lemma 6.4 and the observation that, for each $t \geq 0$, the past parameters $A_{t-1}, \ldots, A_{t-M}$ are determined exactly by the switching sequence. □

Closed-loop stabilization can now be considered. Take the switched system with input

$$\begin{aligned} x_{t+1} &= A_{\theta(t)} x_t + B_{\theta(t)} u_t \\ y_t &= C_{\theta(t)} x_t. \end{aligned} \tag{6.8}$$

Combine this system with a static output feedback controller of the form

$$u_t = K_{\theta_{(t-L:t+H)}} y_t \tag{6.9}$$

to form the closed-loop system

$$x_{t+1} = \left(A_{\theta(t)} + B_{\theta(t)} K_{\theta_{(t-L:t+H)}} C_{\theta(t)} \right) x_t. \tag{6.10}$$

The controller gains in (6.9) should be chosen such that the closed-loop system is positive; this is a weaker requirement than having the controller gains be entry-wise nonnegative. Combining this requirement with Theorem 6.7 produces the following result.

Theorem 6.8. *For $H \geq 0$ and $L \geq 0$, the system of* (6.10) *is both positive and uniformly exponentially stable if and only if, for all admissible $i_{(-L:H)}$,*

$$A_{i_0} + B_{i_0} K_{i_{(-L:H)}} C_{i_0} \geq 0 \tag{6.11}$$

and if there exist an integer $M \geq 0$ and $d_j > 0$ for $j \in [N]^{L+M+H}$ such that, for all admissible

$i_{(-L-M:H)}$,

$$\left(A_{i_0} + B_{i_0} K_{i_{(-L:H)}} C_{i_0}\right) d_{i_{(-L-M:H-1)}} - d_{i_{-L-M+1:H)}} < 0. \tag{6.12}$$

Proof. The positivity of the closed-loop system is equivalent to (6.11). Applying Theorem 6.7 to (6.10) produces (6.12). □

A change of variables is required to produce controller existence conditions which are linear in all variables. These developments adapt the state-feedback technique of [72] to multiple (switched) controller gains and generalize the SVD approach of [60] to consider multiple inputs and outputs.

Theorem 6.9. *Let $H \geq 0$. There exists a path-dependent controller with horizon H such that* (6.10) *is both positive and uniformly exponentially stable if and only if there exist an integer $\bar{L} \geq 0$, vectors $d_j > 0$ for $j \in [N]^{\bar{L}+H}$,, and collections of vectors $z_j^{(k)}$ for $j \in [N]^{\bar{L}+H+1}$ and $k \in [p]$ such that, for all admissible $i_{(-\bar{L}:H)}$,*

$$A_{i_0} d_{i_{(\bar{L}:H-1)}} + B_{i_0} \sum_{k=1}^{n} z^{(k)}_{i_{(-\bar{L}:H)}} < d_{i_{(-\bar{L}+1:H)}} \tag{6.13a}$$

$$[A_{i_0}]_{i,j} \left[d_{i_{(-barL:H-1)}}\right]_j + [B_{i_0}]_i \, z^{(j)}_{i_{(-\bar{L}:H)}} \geq 0 \tag{6.13b}$$

where $[A_{i_0}]_{i,j}$ is the (i,j)-th entry of A_{i_0}; $\left[d_{i_{(-\bar{L}:H-1)}}\right]_j$ is the j-th entry of $d_{i_{(-\bar{L}:H-1)}}$; and $[B_{i_0}]_i$ is the i-th row of B_{i_0}.

Furthermore, given solutions to (6.13)*, a controller may be chosen with memory at most $\bar{L}$ by setting*

$$\left[K_{i_{(-\bar{L}:H)}}\right]_j := \frac{1}{\sigma^{(j)}_{i_0} \left[p_{i_{(-\bar{L}:H-1)}}\right]_j} z^{(j)}_{i_{(-\bar{L}:H)}} \tag{6.14}$$

where $\left[K_{i_{(-\bar{L}:H)}}\right]_j$ is the j-th column of $K_{i_{(-\bar{L}:H)}}$, $C_{i_0} V^*_{i_0} =$ diag $\left(\sigma^{(1)}_{i_0}, \ldots, \sigma^{(p)}_{i_0}\right) = \Sigma_{i_0}$ gives the singular values of C_{i_0}, and $p_{i_{(-\bar{L}:H-1)}} := V_{i_0} d_{i_{(-\bar{L}:H-1)}}$. If any of the singular values of $C_{i_0})$ are zero, then the corresponding column $\left[K_{i_{(-\bar{L}:H)}}\right]_j$ can be freely chosen.

Remark 6.10. Just as was the case with the change of variable used in Chapter 4 (or the elimination lemma in Chapters 2 and 3), the distinction between controller memory L and Lyapunov memory M is destroyed by this linearization. This means that the existence of a controller is only guaranteed for $L = \bar{L}$ in the proof below.

Proof. First consider the necessity of the conditions. Suppose that controller gains $K_{i_{(-L:H)}}$ exist such that (6.10) is both positive and stable. Applying Theorem 6.8 and substituting $\bar{L} := L + M$, there must be vectors $d_{i_{-\bar{L}:H-1)}} > 0$ satisfying

$$\left(A_{i_0} + B_{i_0} K_{i_{(-\bar{L}:H)}} C_{i_0}\right) d_{i_{(-\bar{L}:H-1)}} < d_{i_{(-\bar{L}+1:H)}}.$$

Using the definitions of V_{i_0} and $p_{i_{(-\bar{L}:H-1)}}$, obtain the relationship

$$C_{i_0} d_{i_{(-\bar{L}:H-1)}} = \Sigma_{i_0} p_{i_{(-\bar{L}:H-1)}}.$$

Then

$$K_{i_{(-L:H)}} C_{i_0} d_{i_{(-\bar{L}:H-1)}} = \sum_{k=1}^{p} \left[K_{i_{(-L:H)}}\right]_k \sigma_{i_0}^{(k)} \left[p_{i_{(-\bar{L}:H-1)}}\right]_k.$$

The substitution

$$z_{i_{(-\bar{L}:H)}}^{(k)} := \left[K_{i_{(-L:H)}}\right]_k \sigma_{i_0}^{(k)} \left[p_{i_{(-\bar{L}:H-1)}}\right]_k$$

then provides (6.13a). Notice in this definition that, if $\sigma_{i_0}^{(k)} = 0$, then the choice of $\left[K_{i_{(-L:H)}}\right]_k$ is not felt on the controller existence conditions.

The closed-loop system is also positive, so it must be that

$$\left[A_{i_0} + B_{i_0} K_{i_{(-\bar{L}:H)}} C_{i_0}\right]_{i,j} \geq 0$$

for each (i, j)-th entry. This can be expressed equivalently as

$$[A_{i0}]_{i,j} + [B_{i0}]_j \, K_{i_{(-L:H)}} \, [C_{i0}]_i \geq 0.$$

Multiply this inequality by $\left[d_{i_{(-\bar{L}:H-1)}}\right]_j > 0$, and notice that

$$\left[d_{i_{(-\bar{L}:H-1)}}\right]_j = \left[V_{i0}^*\right]_j \, p_{i_{(-\bar{L}:H-1)}}.$$

Consequently,

$$K_{i_{(-L:H)}} \, [C_{i0}]_j \left[d_{i_{(-\bar{L}:H-1)}}\right]_j = \left[K_{i_{(-L:H)}}\right]_j \sigma_{i0}^{(j)} \left[p_{i_{(-\bar{L}:H-1)}}\right]_j = z_{i_{-\bar{L}:H}}^{(j)}.$$

This provides exactly (6.13b).

Sufficiency may be demonstrated by direct substitution of the definition of $K_{i_{-\bar{L}:H)}}$ in (6.14) to the closed-loop system. As noted in the necessity proof, if any of the singular values of C_{i_0} are zero, then the corresponding row of $K_{i_{(-\bar{L}:H)}}$ would not affect the stability LP. As a result, this row may be freely chosen (in particular, a choice of zeros would suffice). □

Chapter 7

Distributed Solutions to Switching Feasibility Problems

This chapter considers the computational difficulties presented by the controller synthesis conditions presented in this dissertation. As previously discussed, the controller synthesis SDPs (or LPs, in the case of Chapter 6), grow combinatorially in both the number of operating modes and the length of the controller information window. This results in significant time and memory requirements to analyze even modest-sized problems; the barbell pendulum system of Examples 2.47, 2.48, 3.13, and 4.13 has only two operating modes, but solving a single controller synthesis LMI requires minutes even when the controller memory is limited to three modes ($L + H \leq 2$) using centralized computing resources such as Matlab.

In order to make these analysis and controller synthesis conditions tractable for larger-scale problems, an effective decomposition for distributed computation is required. The approach taken here uses the sparsity structure of the individual constraints appearing in each controller synthesis SDP. Each inequality appearing in, e.g., the stability condition (2.9) of Theorem 2.9 contains exactly two of decision variables X_j; this represents the only coupling between constraints in this problem. At the same time, the set of constraints involving any particular decision variable is also bounded by the number of switching modes (or fewer, if a switching constraint is used). The path-dependent nature of the controller synthesis conditions increase this sparsity; even when the underlying switching is unconstrained, sparsity develops as path-dependence increases.

This sparse coupling structure is used in the work of Pakazad et. al [65, 64, 66, 67] to develop a decomposition of sparsely-coupled optimization problems taking advantage of this structure. The general approach presented there is adapted here to the particular needs of controller synthesis and the particular structure of induced switching graphs. As a means of developing understanding of the problem structure, the results for positive switched systems are considered first; these appear as LPs, which are simpler to express and manipulate. As the switching structure and corresponding coupling between inequalities is identical to the general SDP conditions, this intuition can be exploited to provide a decomposition approach for the remaining results in this dissertation.

7.1 Distributed Interior-Points Preliminaries

This section provides a summary of the tools and approach presented in [65, 66] for the decomposition of sparsely-coupled LPs; subsequent sections will consider how these results may be applied to positive switched system analysis and controller synthesis.

7.1.1 Primal-Dual Interior Point Method for LPs

Consider first a primal-dual interior points method for solving LPs as given in [14] and repeated in [65]. The standard form of the optimization problem is given by

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0,\ i = 1, \ldots, m \end{aligned} \tag{7.1}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}$ and each $g_i : \mathbb{R}^n \mapsto \mathbb{R}$.

Remark 7.1. The most general form of (7.1) will also include equality constraints of the form $A_i x = b_i$. Since the analysis and controller synthesis conditions considered in this work have no equality constraints, they have been omitted from the developments that follow.

Whenever a strictly feasible point exists, the KKT optimality conditions for this problem give the optimal point $\bar{x}$, $\bar{\lambda}$ as the solution to

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0, \tag{7.2a}$$

$$\lambda_i \geq 0, i = 1, \ldots, m \tag{7.2b}$$

$$g_i(x) \leq 0, i = 1, \ldots, m \tag{7.2c}$$

$$-\lambda_i g_i(x) = 0, i = 1, \ldots, m. \tag{7.2d}$$

The primal-dual interior-points method solves these conditions first by perturbing (7.2d) to the modified form

$$-\lambda_i g_i(x) = \frac{1}{t}.$$

The perturbation parameter t for each iteration is chosen based on the current iterates below. These conditions are then linearized, and a step direction is found using the current iterates x and λ by solving the equations

$$\left(\nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 g_i(x)\right) \Delta x + \sum_{i=1}^{m} \nabla g_i(x) \Delta\lambda_i = -\nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) =: -r_D \tag{7.3a}$$

$$-\lambda_i \Delta g_i(x)^* \Delta x - g_i(x) \Delta\lambda_i = \lambda_i g_i(x) + 1/t =: -\left[r_C\right]_i \tag{7.3b}$$

Letting $G_d := \text{diag}\left(g_1(x), \ldots, g_m(x)\right)$ and $Dg(x) = \left[\nabla g_1(x) \ldots \nabla g_m(x)\right]^*$, the step direction $\Delta\lambda$ can be found (from solving (7.3b)) to be

$$\Delta\lambda = -G_d(x)^{-1} \left(\text{diag}(\lambda) Dg(x) \Delta x - r_C\right). \tag{7.4}$$

This and the substitutions $r = r_D + Dg(x)^* G_d(x)^{-1} r_C$ and

$$H = \nabla^2 f(x) + \sum_{i=1}^{m} \lambda \nabla^2 g(x) - \sum_{i=1}^{m} \frac{\lambda_i}{g_i(x)} \nabla g_i(x) \nabla g_i(x)^* \tag{7.5}$$

allows (7.3a) to be written as

$$H\Delta x = -r. \tag{7.6}$$

Then the optimal step direction Δx is found by inverting H, and the corresponding $\Delta\lambda$ by substitution into (7.4).

Once the optimal step directions are found, a step size is chosen to maintain feasibility after the update. This is done in two phases: first, the step size is scaled to maintain dual feasibility by selecting

$$\alpha_0 = \min\left\{1, \min_i \left\{\frac{-\lambda_i}{\Delta\lambda_i} : \Delta\lambda_i < 0\right\}\right\}. \tag{7.7}$$

This choice ensures that $\lambda_i + \alpha_0 \Delta\lambda_i \geq 0$ for every i. Next, the feasibility of the primal constraints is considered; if it is the case that $g_i(x + \alpha_0 \Delta x) > 0$, then pick $\alpha_1 = \beta\alpha_0$ for some $\beta \in (0, 1)$ (say, $\beta = 0.9$). This scaling is repeated until a suitable α_k is found which maintains primal feasibility.

Once a suitable step direction and size are chosen, the iterates can be updated and a new perturbation parameter t. One choice is to use the average *surrogate duality gap* given by

$$\eta := \frac{1}{m} \sum_{i=1}^{m} -\lambda_i g_i(x)$$

and to choose $t = \mu/\eta$ for some factor $\mu > 1$ (to ensure that t grows larger with each iteration). This results in the barrier value $1/t$ going to zero, approaching an optimal solution for the problem.

7.1.2 Parametric Solutions to Primal-Dual Step Directions

The optimal step direction Δx expressed in (7.6) is the same as the optimality condition for the unconstrained quadratic program

$$\min_{\Delta x} \ \frac{1}{2}\Delta x^* H \Delta x + r^* \Delta x. \tag{7.8}$$

This program can be solved *parametrically* in terms of some subset of Δx. To see this, consider the partitioned version of (7.8) given by

$$\min_{\Delta x_1, \Delta x_2} \ \frac{1}{2}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}^* \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}^* \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}. \tag{7.9}$$

This is algebraically equivalent to

$$\min_{\Delta x_1, \Delta x_2} \ \frac{1}{2}\Delta x_1^* H_{11} \Delta x_1 + \frac{1}{2}\Delta x_2^* H_{22} \Delta x_2 + \Delta x_1^* H_{12} \Delta x_2 + \Delta x_1^* r_1 + \Delta x_2^* r_2. \tag{7.10}$$

The optimal value for Δx_1 is the solution to

$$H_{11}\Delta x_1 + H_{12}\Delta x_2 + r_1 = 0. \tag{7.11}$$

Solving this equation, and substituting the solution back into (7.10), provides a parameterized optimal point as a quadratic function of Δx_2:

$$\begin{aligned} &\min_{\Delta x_1, \Delta x_2} \frac{1}{2}\Delta x_1^* H_{11} \Delta x_1 + \frac{1}{2}\Delta x_2^* H_{22} \Delta x_2 + \Delta x_1^* H_{12} \Delta x_2 + \Delta x_1^* r_1 + \Delta x_2^* r_2 \\ &= \min_{\Delta x_2} \frac{1}{2}\Delta x_2^* \left(H_{22} - H_{12}^* H_{11}^{-1} H_{12}\right) \Delta x_2 + \Delta x_2^* \left(r_2 - H_{12}^* H_{11}^{-1} r_1\right) + c \\ &= m(\Delta x_2) \end{aligned}$$

This means that the optimal value can be expressed as a quadratic function of Δx_2, of the form

$$m(\Delta x_2) = \frac{1}{2}\Delta x_2^* Q \Delta x_2 + q^* \Delta x_2 + c. \tag{7.12}$$

with $Q = H_{22} - H_{12}^* H_{11}^{-1} H_{12}$ and $q = r_2 - H_{12}^* H_{11}^{-1} r_1$. This function may be transmitted as a message to another computational agent and incorporated into a larger optimization over Δx_2; the minimizing value can be returned to find the optimal value of Δx_1 locally.

7.1.3 Descriptions of Subproblem Sparsity

The interior-points method and parametric step direction computation can now be applied to sparsely-coupled optimization problems. Consider a global optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & f_1(x) + \dots f_{\mathcal{N}}(x) \\ \text{subject to} \quad & G_i(x) \leq 0, \;\; i = 1, \dots, \mathcal{N}. \end{aligned} \tag{7.13}$$

in which each $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ and each $G_i : \mathbb{R}^n \mapsto \mathbb{R}^m$. This problem can be thought of as a collection of $\mathcal{N}$ subproblems, coupled through their dependence on a common decision variable x. Suppose that each subproblem depends only on a small number of indices of x. To make this dependence explicit, first construct lower-dimension versions of each f_i and G_i. Let $J_i \subset [n]$ be the indices of x on which subproblem i depends; that is, those used either by f_i or by G_i. A corresponding projection matrix $E_{J_i} : \mathbb{R}^n \mapsto \mathbb{R}^{|J_i|}$ may be chosen and lower-dimension functions $\bar{f}_i$, $\bar{G}_i$ such that

$$\bar{f}_i(E_{J_i}) = f_i(x); \quad \text{and} \quad \bar{G}_i(E_{J_i}) = G_i(x).$$

This produces the modified optimization problem

$$\begin{aligned} \text{minimize} \quad & \bar{f}_1(E_{J_1}x) + \ldots \bar{f}_{\mathcal{N}}(E_{J_{\mathcal{N}}}x) \\ \text{subject to} \quad & \bar{G}_i(E_{J_i}x) \leq 0, \;\; i = 1, \ldots, \mathcal{N}. \end{aligned} \tag{7.14}$$

The relationships between individual decision variables can demonstrated by constructing a *sparsity graph* for this problem. The vertex set of this graph is $[n]$, and each vertex corresponds to a single decision variable. An edge (i, j) between two vertices occurs if and only if those two variables appear in one or more subproblems. This choice results in each J_i inducing a complete subgraph on the sparsity graph.

Any decomposition which assigns these subproblems to different computational agents must satisfy two basic criteria:

- If an agent is assigned subproblem i, it must have knowledge of all decision variables in J_i (in order to compute the corresponding $\bar{f}_i$ and $\bar{g}_i$).
- If two agents share knowledge of a decision variable x_i, they must be able to communicate about that variable, either directly or through an intermediary.

These two properties can be guaranteed by constructing a *clique tree* or tree decomposition of the sparsity graph.

A clique tree begins by considering a chordal graph; these are graphs in which any cycle of length four or longer contains a chord, and are also called *triangular graphs*. When a graph is not chordal, one may construct a *chordal embedding* by adding additional, ”virtual” edges. These edges do not represent additional constraints or relationships in the underlying optimization problem. This process is called a *chordal completion* and is not unique; the general problem of finding a chordal embedding with the fewest added edges is known to be NP-hard [82]. A simple, greedy algorithm from [17] is used in [65] and repeated here as Algorithm 7.1.

Algorithm 7.1 Greedy Chordal Completion Algorithm

Take graph $\mathcal{G}(V, E)$
Let $V_r = V$, $E_r = E$, and $C = \emptyset$
while V_r is not empty **do**
  Let $new = true$.
  Consider subgraph $\mathcal{G}_r(V_r, E_r)$.
  Choose $i \in V_r$ with fewest neighbors in G_r.
  Add edges to E_t such that i and its neighbors are a complete subgraph.
  Let $C_i = \{i\} \cup \{\text{neighbors of } i\}$.
  for $C_j \in C$ **do**
    if $C_i \subseteq C_j$ **then** $new = false$
      Break.
  if new **then**
    Add C_i to C.
  $V_r = V_r \setminus \{i\}$
  $E_r = E_r \setminus \{(i, j) : i \text{ just removed}\}$.

A chordal graph (or one which has been chordally completed) contains one or more maximal, complete subgraphs which are called *cliques* of the graph. In general, these cliques are not disjoint. A corresponding clique tree uses as vertices the cliques C of the chordal graph, and has an edge set satisfying the property that each vertex of the chordal graph induces a subtree (i.e., a connected subgraph) in the clique tree. The clique tree is also not unique; one construction is to construct a maximum weight spanning tree based on the intersections between cliques. This is done in Algorithm 7.2.

Algorithm 7.2 Clique Tree as Maximum Weight Spanning Tree

Take chordal graph $\mathcal{G}(V_C, E_C)$ with cliques $C = \{C_1, \dots, C_Q\}$.
Construct weighted graph $\mathcal{W}(V_W, E_W)$ with $V_W = C$ and edge $(i, j) \in E_W$ having weight $|C_i \cap C_j|$.
Let $V_T = \{1\}$ and $E_T = \emptyset$.
while $V_T \neq V_W$ **do**
  Select edges $E = \{(i, j) \in E_W : i \in V_T \text{ and } j \notin V_T\}$.
  Pick edge $e = (\bar{\imath}, \bar{\jmath}) \in E$ with maximal weight.
  Add $\bar{\jmath}$ to V_T.
  Add e to E_T.

The resulting clique tree represents the computation graph for this problem; each node represents an agent with knowledge of the decision variables specified by its clique. Each of the subproblems of (7.14) is assigned to an appropriate agent with knowledge of all pertinent decision variables. This is always possible because each subproblem index set J_i induced a complete subgraph on the sparsity graph; in the chordal completion these will either be cliques or completely contained in cliques. Finally, a root vertex of the tree is computed to determine the direction of computation within the tree (from leaf to root). For each agent, its knowledge of the decision variables is partitioned into two disjoint sets: the variables shared with its parent agent, and the variables shared only with its children (or itself, if at a leaf).

Computation within the tree begins at the leaf agents. Each such agent considers the subproblem or subproblems assigned to it and follows the interior-points step direction calculation of Section 7.1.1. This step direction calculation is solved parametrically as shown in Section 7.1.2, eliminating variables known only to the leaf agent and parametrizing the optimal value in terms of shared variables. The resulting quadratic form of (7.12) is passed to the parent agent.

Any agent with children begins by waiting for its children to communicate their parametric solutions. These functions, along with the agents own subproblems, produce a corresponding quadratic optimization problem. If the agent is not the root, then it again solves the problem parametrically and passes a message upwards. Each level of this upward pass eliminates variables known to the agent and its subtree, leaving an optimal value expressed only in terms of variables known to the parent. Once the root agent has received messages from all of its children, it may solve the optimization problem completely, providing the optimal step direction for every decision variable it knows. This optimal direction is passed downward to child agents according to their shared knowledge. Each child agent can then reconstruct the remaining portion of the step direction according to (7.11), and, if needed, pass the corresponding shared knowledge down to their children. This downward pass ends with the leaves computing the optimal step direction for their variables and all agents having consistent information about step direction.

The calculation of suitable step size is done similarly with an upward-downward pass; each

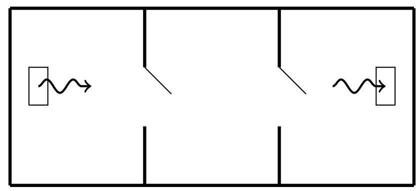

Figure 7.1: A small building with three rooms. Heat may be supplied to the room on the left, while it is radiated away from the room on the right. The doors between the three rooms may be either open or closed.

agent computes the largest suitable step size for their subproblems, and compares it with the step size passed from their children (if any); ultimately, the root agent may broadcast the correct step size to all agents. This communication strategy ensures that all agents maintain consistency through a single iteration of the interior-points method without knowledge of the entire decision variable or the need to compute and invert the corresponding large-dimension matrices.

7.2 Application to Positive Switched System Design

The approach detailed in the previous section can now be tailored to the needs of switched system analysis and synthesis. To illustrate each step of the process, a simple thermal management problem is presented in Example 7.2 below.

Example 7.2. We consider the temperature regulation of the small building pictured in Fig. 7.1. Controlled heat supply comes from a vent in the left-most room, while heat vents to the outside from the right-most room. Heat transfer between rooms (or between the right-most room and outside) is proportional to the temperature difference between rooms. The rooms are separated by two doors and the coefficient of heat transfer between rooms depends on whether the doors are open or closed. We take as the system state the temperatures of each room, and take the ambient temperature to be zero; this ensures the state remains non-negative.

We choose the coefficient of heat transfer between rooms to be 0.5 when a door is shut, and

0.6 when it is open; the transfer rate from the right-most room to the outside is 0.4. We then have four possible operating modes corresponding to the status of the two doors, with continuous-time dynamics given by

$$\dot{x}_t = A_i x_t + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T u_t \tag{7.15}$$

in which the four operating modes are given by

$$A_{OO} = \begin{bmatrix} -0.6 & 0.6 & 0 \\ 0.6 & -1.2 & 0.6 \\ 0 & 0.6 & -1 \end{bmatrix}, A_{CO} = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -1.1 & 0.6 \\ 0 & 0.6 & -1 \end{bmatrix}$$

$$A_{OC} = \begin{bmatrix} -0.6 & 0.6 & 0 \\ 0.6 & -1.1 & 0.5 \\ 0 & 0.5 & -0.9 \end{bmatrix}, A_{CC} = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 0.5 & -0.9 \end{bmatrix}. \tag{7.16}$$

For our example, we take the true discretization of this model over a time interval of one second.

At each time step we allow at most one door to change status. As a result, the system may either remain in the current mode or transition to a mode with one door changed. The switching graph for this system is then given precisely by the graph of Fig. 7.2, in which modes one through four are replaced by the modes OO, OC, CO, and CC, respectively.

First consider the stability analysis conditions of Theorem 6.7. These inequality constraints are well-suited to the sparely-coupled framework in that each inequality (or subproblem) uses exactly two of the d_j, and each decision vector d_j appears in a small number of inequalities. Adding a global slack variable ϵ and introducing an upper bound β on the components of each d_j, a single

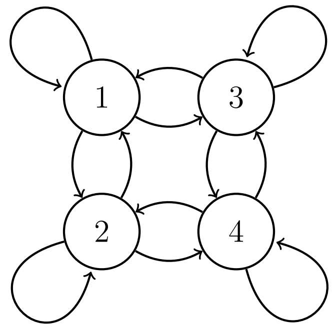

Figure 7.2: A constrained switching graph with four modes. Any walk through this graph (including the self-edges) generates an admissible switching sequence.

subproblem of the form in (6.7) may be written compatible with (7.14)

$$f_i(d, \epsilon) = \epsilon \tag{7.17a}$$

$$G_i(d, \epsilon) = \begin{bmatrix} A_{\phi(i_{(-L:H)})} d_{i_{(-L-M:H-1)}} - d_{i_{(-L-M+1:H)}} - \epsilon \mathbf{1} \\ -d_{i_{(-L-M:H-1)}} - \epsilon \mathbf{1} \\ -d_{i_{(-L-M+1:H)}} - \epsilon \mathbf{1} \\ d_{i_{(-L-M:H-1)}} - \beta \mathbf{1} \\ d_{i_{(-L-M+1:H)}} - \beta \mathbf{1} \end{bmatrix} \tag{7.17b}$$

This formulation ensures that a feasible point may always be found (for some value of ϵ, and feasibility of (6.7) occurs exactly when the minimum ϵ is negative. The upper bounds on each d_j are placed to prevent the minimum from being unbounded below.

To demonstrate the construction of the computation graph, let $L = H = 0$ and $M = 1$ in the conditions of (6.7), and consider the induced switching graph presented in Figure 7.3. Each vertex in this lifted graph corresponds to a single decision variable, and each directed edge represents a subproblem. Therefore, this induced graph is precisely the sparsity graph discussed in Section

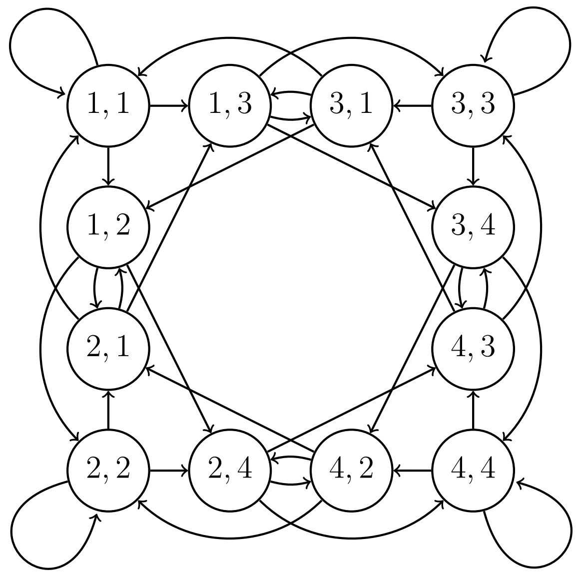

Figure 7.3: The induced switching graph for the switching graph of Fig. 7.2 and path length two.

7.1.3. This graph may be treated directly to construct a suitable computation graph.

The chordal completion of the graph in Figure 7.3 is augmented first by removing the direction of each edge, as well as all self-loops. This transforms the graph into one showing the relationship between variables without concern as to the specific form of each subproblem (any variable is trivally related to itself, so self-edges provide no information and can be discarded). Next, Algorithm 7.1 is applied to find a chordal completion of this graph. The results of this process are shown in Figure 7.4. The chordal completion has the following six cliques:

$$C_1 = \{(1,1),(1,2),(2,1),(1,3),(3,1)\} \tag{7.18a}$$

$$C_2 = \{(1,2),(2,1),(2,2),(2,4),(4,2)\} \tag{7.18b}$$

$$C_3 = \{(1,3),(3,1),(3,3),(3,4),(4,3)\} \tag{7.18c}$$

$$C_4 = \{(2,4),(4,2),(3,4),(4,3),(4,4)\} \tag{7.18d}$$

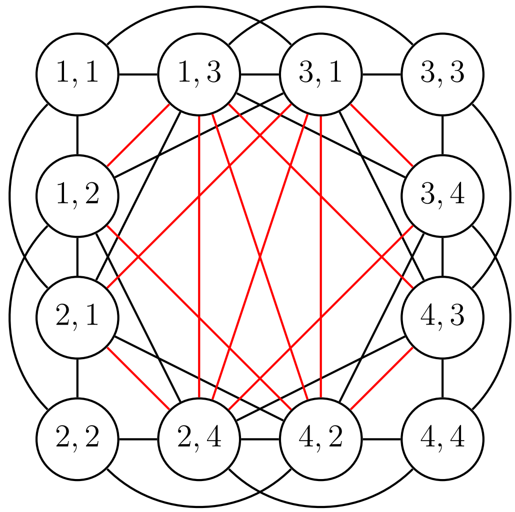

Figure 7.4: The chordal completion of the switching graph of Figure 7.3. Edges shown in red correspond to the edges added by Algorithm 7.1, while those in black are a consequence of the constraints of (6.7).

$$C_5 = \{(1,2),(2,1),(1,3),(3,1),(2,4),(4,2)\} \tag{7.18e}$$

$$C_6 = \{(1,3),(3,1),(2,4),(4,2),(3,4),(4,3)\} \tag{7.18f}$$

Using these cliques, a six-vertex clique tree is found, as shown on the left of Figure 7.5.

Remark 7.3. Before computation may proceed, a root vertex must be chosen from the vertices of the tree in Figure 7.5. Such a choice should minimize tree depth (such that the steps in an upward-downward iteration are small), and should also balance the tree (so that the root is not left waiting for the larger half to finish). The natural choices are the middle two vertices, but either choice produces a tree which is unbalanced. To remedy this situation, cliques 5 and 6 are merged to produce a computation tree with five agents, shown on the right of Figure 7.5.

The observation made by Remark 7.3 is a key limitation of the current software implementation of this approach. While clique tree construction can be done in an automated way for switching systems of any size, determining whether the resulting cliques should be merged to improve performance remains a human-in-the-loop activity. Such mergers may be done to improve symmetry

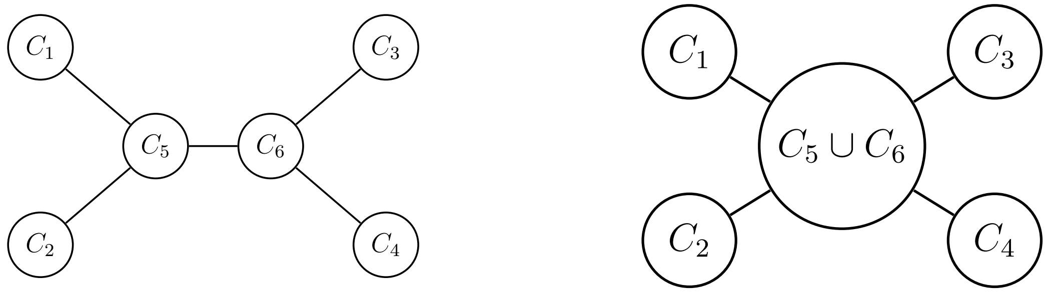

Figure 7.5: On the left, the tree decomposition generated from the chordal completion shown in Fig. 7.3. Each vertex corresponds to one of the maximal cliques listed in (7.18). On the right, the modified computation tree with the center two cliques merged (per Remark 7.3. This produces a balanced computation for better performance.

of the computation tree or to reduce the overall depth (leaf to root) and increase the breadth (number of leaves) for the tree. While this is manageable for small-scale problems with only a few computation agents, this will prove untractable as the system scale grows. To overcome this, a more quantifiable measure of tree balance (either from the root or overall) is needed.

The only remaining task is subproblem assignment. Each subproblem must be assigned to an agent with knowledge of all the variables appearing in the subproblem. This can be done algorithmically using a priority queue based on possible assignments; see Algorithm 7.3.

Algorithm 7.3 Subproblem Assignment Algorithm

Take the list of subproblems $S = \{S_i, i = 1, \ldots, \mathcal{N}\}$, list of cliques/computation agents $C = \{C_j : j = 1, \ldots, Q\}$
Let possible assignments $P_i = \emptyset$ for $i = 1, \ldots, \mathcal{N}$ and assignment lists $L_j = \emptyset$ for $j = 1, \ldots, Q$.
for all S_i in S **do**
  for all C_j in C **do**
    if All variables in S_i are known to C_j **then**
      Add C_j to P_i
Let $S_u = S$
while S_u is not empty **do**
  Sort S_u by possible assignments $|P_i|$ and select the smallest list $P_{\bar{\imath}}$ (breaking ties arbitrarily)
  Sort the elements of $P_{\bar{\imath}}$ by current assignments $|L_j|$ and select the smallest (breaking ties by chosing parents over children)
  Add $S_{\bar{\imath}}$ to L_j.
  Remove S_i from S_u.

The problem of Example 7.2 has precisely 36 subproblems and five computational agents to assign them to. This assignment algorithm gives seven subproblems to each of the leaves and eight subproblems to the root; the slightly larger set-up computations for the root agent may be accomplished while the leaves are computing their optimal step directions and corresponding messages, minimizing idle time for both leaves and root.

If the problem of controller synthesis is considered using the conditions of Theorem 6.9, then the closed-loop induced switching graph remains the same as in Figure 7.3, so the decomposition and assignment of subproblems is identical to the case of stability analysis. What changes is the set of variables required by each subproblem. In addition to the coupling decision variables d_j which are shared among subproblems, each subproblem has the additional collection of vectors $z_i^{(k)}$ for each subproblem. These variables are private to the subproblem, and may always be eliminated before passing information through the communication graph. To separate out this information from the coupling variables, each subproblem is first considered individually; a parameterized solution for the optimal step direction is determined by eliminating just the local variables, and the resulting quadratic form is used to compute the step direction for the d_j. For a smaller-scale problem where each computation agent may handle only a few subproblems, this may be done by the agent itself without delay. However, for larger problems it is more efficient to distribute this preprocessing step. This adds a single level to the computation by establishing gangs of computation agents; the resulting computation tree has the structure:

- The leaves of the computation graph are each assigned a single subproblem; they eliminate the subproblem-specific variables and return a parameterized optimal solution in terms of the coupling variables d_j.
- The parents of these leaves correspond to the computation graph of Figure 7.5 and handle communication regarding the coupling variables. Their communication pattern is the same as in the stability case.

7.3 Implementation with Python and MPI

The problem decomposition and corresponding distributed interior-points algorithm for solving the feasibility problems presented are implemented in an open-source framework to permit maximum portability, scalability, and maintainability. The basic language for this approach is Python 3.5.2; Python is chosen for a number of reasons:

- Python 3 is widely available and pre-installed on many computers and computing clusters. This minimizes the need for special configuration or installation to make use of the code.

- Python has access to robust and efficient linear algebra libraries provided through the 'numpy' package. In particular, Python/numpy make use of the highly-optimized BLAS library which provides many linear algebra operations. A locally-optimized version of BLAS will perform at nearly identical speeds for both Python and compiled languages such as C/C++.

- The object-oriented nature of Python allows the distributed problem structure to be represented in the code. This enforces necessary memory partitions without regard to the underlying memory and communication structure of the system.

- Python and numpy provide easily human-readable code which provides understanding of the algorithm while reading. This makes maintaining and updating the toolbox easier.

Communication between computation agents is accomplished using 'mpipy', a Python-based implementation of the MPI interface. This allows multi-threaded processes to send and receive information from other threads, including broadcast and gather operations (one-to-many and many-to-one) like the signal to begin or end the optimization iteration. It also permits blocking operations to synchronize between threads (to ensure consistency). This interface is language- and platform-independent, so the precise hardware structure used on the computer or computing cluster need not be known in order to perform the algorithm. Another advantage of using MPI is that it allows the process manager to handle the assignment of individual threads to physical cores of the system.

On larger computing clusters where the topology of the network mesh or distribution of memory resources is sparser, this helps minimize any unnecessary communication delays. This also means that the number of physical cores need not always match the number of computation agents for a particular problem. Rather than letting cores idle while their associated process waits to receive a message, these cores can be tasked with other work to maximize efficiency. In particular, when memory is a more restricted resource than time, individual processes may be run on a single core in active memory, then have their local memory placed in slower storage such as disk while another thread computes. Though this is far slower than a computing cluster would be, it allows portions of the problem to be transferred into and out of memory as needed to make the problem tractable on limited hardware.

An implementation of the stability problem of Theorem 6.7 has been used to consider the system of Example 7.2. Three different computational approaches are used to solve the feasibility problem of this system: a centralized solver 'linprog' available in Matlab 9.0 (2016a), a "pseudo-parallel" implementation of the distributed approach of this Chapter, in which the computations required by each agent are done in serial within Matlab, and a parallel implementation of this algorithm performed using the Windows Subsystem for Linux (kernel version 4.4.0-17134-Microsoft x86 64) and Ubuntu 16.04.5 along with Python 3.5.2 and mpirun version 1.10.2. All three cases were executed on a Lenovo Y700 with an Intel i5-6300 CPU with 4 cores and 8 GB of total system memory. These three approaches produces identical numerical results, demonstrating the accuracy of the parallel approach. The total run time of each approach was roughly equivalent due to the small size of the toy example problem; this demonstrates that the distributed approach is not inherently slower than a centralized solver for small problems. As the problem size grows, limitations on memory availability are expected to be the dominant factor for both the centralized and pseudo-parallel approaches, while the parallel approach can take advantage of memory caching to overcome these limitations.

A companion implementation of the controller synthesis conditions was developed as a parallel package. While the specific form of each subproblem must be changed along with the corresponding code, the framework for computational agents remains unchanged (as the message-passing structure is not changed for the problem). Specifically, each subproblem is permitted first to eliminate its local variables before communicating the parameterized solution to the computational agent. In this implementation all subproblem computations are done on the same thread in series rather than providing a cluster of threads for each agent; the limit of four physical nodes on the test laptop prevents any gained efficiency from subdividing between threads as there are not enough physical cores to truly compute in parallel. An intermediate-scale computing cluster with on the order of 10-20 computing nodes is needed to demonstrate true parallel behavior for this problem.

7.4 Sparsity Decomposition for SDPs and Switched Controllers

The controller synthesis conditions presented in Chapters 2 to 4 have a sparse coupling structure which is identical to that of the positive system conditions in Chapter 6. This means that the sparsity-based decomposition used in Section 7.1.3 applies identically to the SDP conditions developed here. The distributed solution of sparsely-coupled SDPs has also been considered by Pakazad [64, 67]. These references make use of the same graph-based decomposition as in the LP case, making use of the KKT conditions for an SDP optimization problem. A key difference between these results and the controller synthesis conditions found here is that the results of [64] are given in terms of the standard primal form of the SDP; i.e.,

$$\begin{aligned} \text{minimize} \quad & W \cdot X \\ \text{subject to} \quad & Q_i \cdot X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0. \end{aligned}$$

In contrast, the controller synthesis conditions found in this dissertation are more naturally expressed using the standard dual form

$$\begin{aligned} \text{minimize} \quad & c^* x \\ \text{subject to} \quad & F_i(x) = F_{i,0} + \sum_{k=1}^{n} x F_{i,k} \preceq 0. \end{aligned}$$

This distinction changes only the details of the expressions for the optimal step direction, and not their basic intuition. The corresponding (perturbed) KKT conditions for optimality can be expressed and reduced to a quadratic optimization problem in the same way as the LP case. This approach uses an identical decomposition and communication structure as the existing LP code. The development of a convex toolbox which can handle any of the controller synthesis conditions of this dissertation is of ongoing research interest.

Chapter 8

Concluding Remarks

This dissertation has presented a framework for controller synthesis and design for switched linear systems. These results provide exact conditions under which a suitable controller may be constructed which stabilizes the system, or additionally achieves a specified level of system performance. All of these results are given in the form of sequences of nested semidefinite programming problems which may be examined in order until a suitable controller is found. The dissertation also considers the computational aspects of switched controller design and makes use of the sparsity inherent in the problem to propose a distributed computation approach to the controller search.

There are two major areas of prospective future research which follow from the work presented here. The first lies in exploration of converse results for the performance conditions presented, e.g., in Chapter 3. The estimation of worst-case system gain for switched systems is still an open problem, which these results only address in a semidecidable way. A framework like the minimum achievable decay results presented is Section 2.2 is the most-desired form of these results. In the LTI case the gain of the system is closely related to the structured singular value of the system parameters; this is a generalization of both the spectral radius and greatest singular value. A corresponding notion of joint structured singular value has not yet been studied widely, providing an opportunity for future work.

A related interest to the previous is the consideration of performance measures for positive switched systems with an LP analogue to the KYP lemma. Results of this type are known for LTI systems, so an extension of the methods of this dissertation to this domain offers an avenue for further investigation.

The second area of continued work lies in the improvement and broadening of the software tools for systematically examining switched systems in parallel. Of particular interest will be continued investigation of the properties of the induced switching graph. These are a particular case of a class known as *de Bruijn graphs*, whose properties are better understood in the unconstrained case (when all possible switching sequences are admissible). The general problems of chordal completion and clique tree construction are known to be hard in general; taking advantage of the structure of switching graphs to speed up or further parallelize this calculation provides an interesting research avenue in graph theory and combinatorics.

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