On the Critical Total Power for Asymptotic k-Connectivity in Wireless Networks

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***Abstract*— In this paper, we investigate the minimum total power (termed as *critical total power*) required to ensure asymptotic k-connectivity in heterogeneous wireless networks where nodes may transmit using different levels of power. We show that under the assumption that wireless nodes form a homogeneous Poisson point process with density λ on a unit square region $[0,1]^2$ and the Toroidal model [17], the critical total power required for maintaining k-connectivity is $\Theta(\frac{\Gamma(c/2+k)}{(k-1)!}\lambda^{1-c/2})$ with probability approaching one as λ goes to infinity, where c is the path loss exponent. Compared with the result that all nodes use a *common* critical transmission power for maintaining k-connectivity [18], [25], we show that the critical total power can be reduced by an order of $(\log \lambda)^{c/2}$ by allowing node to optimally choose different levels of transmission power. These results are not subject to any specific power/topology control algorithm, but rather a fundamental property in wireless networks.**

keywords– **Stochastic processes/queuing theory, Graph theory, Combinatorics**

I. Introduction

A wireless ad hoc network is a collection of wireless mobile hosts which communicate with each other without the support of fixed infrastructure or centralized administration. It has gained tremendous attentions in recent years because of its wide applications in civilian and military fields, and its capability of building mobile wireless networks without the need for pre-existing infrastructures. One important issue in such a network is how to minimize power consumption while maintaining network connectivity. Minimizing power not only saves energy, but also reduces MAC-level collision and hence increases the network capacity. However, this has to be performed subject to maintaining network connectivity. As a matter of fact, in order to enable robust communications in the presence of mobility and node failures, it is important that the networks are k-connected.

The research on reducing power consumption while maintaining (k-)connectivity has been approached independently along two thrusts. In one thrust, researchers aim to determine critical conditions on network parameters (such as the transmission range [19], [15], [17], [18], [22], [25], the number of neighbors [25], [26], the minimum total power required [1], [4], [8], [20], or the node failure probability [23]) to ensure network (k-)connectivity with high probability. Of particular interest is how these critical conditions scale as the number of wireless devices increases. Take the transmission radius as an example. Consider a wireless network on a unit disk on which n nodes are uniformly and randomly placed. Let r_n denote as the critical (minimum) common transmission radius required by all nodes to ensure k-connectivity in such a network. Penrose showed in [18] that under the Torus convention assumption,

$$P(n\pi r_n^2 - \log n - (k-1)\log\log n + \log(k-1)! \le \tau) = \exp(-e^{-\tau}). \quad (1)$$

Wan and Yi further extended the results by considering boundary effects in [25]. Take the minimum total power of all the nodes required to maintain asymptotic (k-)connectivity (termed as *critical total power*) as another example. Both Blough *et al.* [1] and Gomez *et al.* [8] studied the critical total power for 1-connectivity, based on results on the asymptotic total weight for weighted minimal spanning trees [24], [27]. Rengarajan *et al.* [20] gave the expectation of the (lower and upper) bounds on the critical total power for 1-connectivity. Clementi *et al.* [4] studied the problem of assigning transmission ranges for wireless nodes so as to minimize the total power consumption in the special case of path loss exponent $c = 2$ such that any pair of nodes are within h hops.

In the other thrust, researchers aim to devise distributed algorithms in which each node chooses its own transmission power in order to minimize the total transmission power of all wireless nodes, while maintaining (k-)connectivity. This problem is, in general, NP hard in the Euclidean plane [5], and many researchers have developed localized heuristics [21], [12], [14], [13], or efficient algorithms with bounded approximation ratios [11], [3], [9], [2].

In this paper, we address the power consumption issue along the first thrust, and investigate the *critical total power* required for maintaining asymptotic k-*connectivity* in a random wireless network on a unit square $S = [0,1]^2$. Instead of imposing the uniform assumption that all the nodes are subject to the same common minimum power, we consider the *heterogeneous* case and allow each node to choose its own transmission power. Specifically, let $W_{t,i}$ be the critical transmission power node i uses, and $R_{t,i}$ the corresponding transmission range of node i under the power model $W_{t,i} = R_{t,i}^c$, where $2 \leq c \leq 4$ is the path loss exponent. Then the critical total power of all the nodes is $W_c = \sum W_{t,i} = \sum R_{t,i}^c$, where the summation is taken over all the nodes in the network. Under the assumption that wireless nodes are distributed on a unit square $S = [0,1]^2$ according to a homogeneous Poisson point process with density λ and with the use of the Toroidal model (Torus convention) [17], we show that the critical total power $W_c = \sum R_{t,i}^c$ for maintaining k-connectivity is $\Theta(\frac{\Gamma(c/2+k)}{(k-1)!}\lambda^{1-c/2})$ with probability approaching 1 as $\lambda \to \infty$.

The result is obtained by deriving a lower bound and an upper bound on the critical total power. The lower bound is derived based on the necessary condition that every node must be able to reach its kth nearest neighbor in order to maintain strong k-connectivity. The upper bound is derived based on an assertion (which is also proved in the paper) that the resulting network is strongly k-connected, if every node can reach at least k nodes in each of its four quadrants as long as there are at least k nodes in that quadrant. (By "each of its four quadrants", we assume that every node has its own coordinate system which is obtained by shifting the origin of the $[0,1]^2$ plane to its own location.) In the case that there are less than k nodes in a quadrant, the transmission power of the node should be sufficiently large to reach all of them.

Our work differs from (and is perhaps superior to) existing works in several aspects. Although several existing works [1], [8], [20], [4] studied the similar problem, none of them studied the critical total power for k-*connectivity* ($k > 1$). In particular, Blough *et al.* [1] and Gomez *et al.* [8] derived the critical total power only for 1-connectivity. As the proof is based on the results on the asymptotic total weight for weighted minimal spanning trees, the result cannot be easily generalized to the case of k-connectivity for $k > 1$. The work reported in [20] gives the expectation of (lower bound and upper bound of) the total power consumption (for 1-connectivity), while the results in this paper are obtained in the asymptotic sense. Obtaining asymptotic results is significantly more challenging than obtaining expectations. Clementi *et al.* [4] showed that given the upper bound on the number of hops h, the total power incurred by the n nodes that are independently, uniformly distributed in a unit square region is $\Theta(n^{1/h})$ with high probability. Their result only applies to the path loss exponent $c = 2$ and cannot be readily generalized to the case of $c \neq 2$.

Our results are derived under the *heterogeneity* assumption that different nodes may use different levels of transmission power, and hence are more general than those derived under the uniform metric assumptions [7], [15], [18], [22], [25], [26]. Our results suggest that the power saved using optimal, non-uniform transmission ranges is in an order of $(\log \lambda)^{c/2}$ as compared to that using optimal uniform transmission ranges. In a rescaled network where the node density is kept fixed and the size of the square region goes to infinity, our results indicate that the average power of each node is bounded if we allow each node chooses its own transmission power to maintain (k-)connectivity, while the average power of each node is unbounded if all nodes have to choose a common power to maintain (k-)connectivity. These results are not determined by a specific algorithm, but rather a fundamental property in wireless networks.

The rest of the paper is organized as follows. In Section II, we state the system model, formulate the problem, and present preliminary material that will be used in subsequent sections. We then derive in Sections III–IV respectively, the lower and upper bounds on the critical total power. Following that, we compare our result with that derived under the uniform metric assumption and discuss the issue on the transmission power model in Section V. Finally, we conclude the paper in Section VI with a list of future research directions.

II. Preliminaries

In this section we present the system model, and introduce notations that will be used throughout the paper. We also define two frequently-used random variables: $R_{\lambda,k}(\alpha)$ and $R_{\lambda,k}(d,\alpha)$ (to be defined in Subsection II-C), derive their probability distributions and prove two lemmas that will be used in subsequent sections. Finally we present, for the completeness of the paper, Palm theory on Poisson point process.

A. System model

We assume nodes are distributed on a unit square $S=[0,1]^2$ according to a (homogeneous) Poisson point process $\mathcal{P}_\lambda$ with density λ. It is well accepted that n nodes whose locations are independent random variables, each with a uniform distribution on S, are essentially a Poisson point process with density n if the network size is large ([10], page 39). In addition, we assume the Toroidal model (Torus convention) [17] to eliminate the boundary effects. In the Toroidal model, the Euclidean metric $d(i,j)=|X_i-X_j|$ is replaced with $d(i,j)=\min_{z\in\{0,1\}^2}|X_i-X_j-z|$, where X_i is the coordinate of node i. Under the Toroidal model assumption, each node can view the original plane $[0,1]^2$ as the plane $[-\frac{1}{2},\frac{1}{2}]^2$ in a coordinate system centered at itself. Toroidal model is also widely used when analyzing properties of large scale networks ([17], [10] (page 22)).

Let R_i denote the (fixed) transmission range of node i. Different nodes may use different transmission power and hence have different transmission ranges. Node i can directly transmit to node j if and only if $d(i,j)\le R_i$. We further assume that the transmission power of node i is $W_i=R_i^c$, where $2\le c\le 4$ is the path loss exponent (although our analysis applies to any $c>0$). Hence the total power of all nodes is

$$W=\sum_{i\in\mathcal{P}_\lambda} W_i=\sum_{i\in\mathcal{P}_\lambda} R_i^c. \tag{2}$$

The network can be viewed as a directed graph where each wireless node is a vertex and a directed edge exists from vertex i to j if and only if node i can directly transmit to node j. The network is said to be k-connected if and only if the corresponding directed graph is strongly k-connected, i.e., there exists a directed path from any vertex i to any other vertex j even if we remove any $k-1$ nodes from the network. The *critical total power W_c for k-connectivity* is defined as the minimum total power of all nodes required to ensure strong k-connectivity in the formed directed graph. As we are mostly interested in k-connectivity in this paper, the critical total power W_c is henceforth by default for k-connectivity.

let $W_{t,i}$ be the *critical* transmission power node i uses, and $R_{t,i}$ the corresponding transmission range of node i, then $W_c=\sum W_{t,i}=\sum R_{t,i}^c$. We are interested in deriving the asymptotic bound on the critical total power W_c as $\lambda\to+\infty$.

B. Notations

Table I gives the notations used throughout this paper. Several comments are in order:

- We envision a (homogeneous) Poisson point process $\mathcal{P}_\lambda$ on a unit square area $S=[0,1]^2$. This is often related to a binomial point process $\mathcal{X}_n$, i.e., n independent, uniformly distributed random 2-dimensional vectors on S. We use X_i to denote node i's location (coordinate).
- We use C_j to represent a (constant) function independent of λ. Unless specified, C_j only depends on the path loss exponent c and sometimes k, both of which are assumed to be constant in this paper. We may explicitly express c as the parameter of C_i when we need to use the function of C_i with a different parameter (such as $2c$).
- Let $f(X)$ be a function on a random variable X (which can be a vector). By probability theory, the expectation of $f(X)$ is simply the integral of $f(X)$ over the probability space of X, i.e., $E[f(X)]=\int f(X)dP$. The expectation, $E_G[f(X)]$, of a function $f(X)$ under restriction G is the integral of $f(X)$ over the subset G of the probability space, i.e., $E_G[f(X)]=\int_G f(X)dP=\int \mathbf{1}_G f(X)dP$, where $\mathbf{1}_G$ is the indicator function of G. With this definition, by the law of total probability, $E[f(X)]=E_G[f(X)]+E_{\bar{G}}[f(X)]$, where $\bar{G}$ denotes the complement set of G; and by the law of conditional probability, $E_G[f(X)]=E[f(X)|G]P(G)$, where $P(G)$ is the probability that G occurs.
- We define $\mathcal{B}_X(r)$ as the ball (disk in a 2-dimensional space) centered at X with radius r, and $\mathcal{C}_X(\theta,\theta+\alpha)$ as the cone centered at X, with starting angle θ, ending angle $\theta+\alpha$, where $0\le\theta,\alpha\le 2\pi$. The degree of cone $\mathcal{C}_X(\theta,\theta+\alpha)$ is α. We use $\mathcal{C}^*_X(r,\theta,\theta+\alpha)$ to denote the region $C_X(\theta,\theta+\alpha)\cap\mathcal{B}_X(r)$.
- We write $g(\lambda)\approx_\lambda h(\lambda)$ if $g(\lambda)/h(\lambda)\to 1$ as $\lambda\to\infty$, $g(\lambda)=o(h(\lambda))$ if $g(\lambda)/h(\lambda)\to 0$ as $\lambda\to\infty$, and $g(\lambda)=O(h(\lambda))$ if $g(\lambda)\le C\cdot h(\lambda)$ as $\lambda\to\infty$ for some constant C (which may depend on the path loss exponent c).

C. $R_{\lambda,k}(\alpha)$ and $R_{\lambda,k}(d,\alpha)$ and their probability distributions

In an infinite region $\mathbb{R}^2$ with the Poisson point process $\mathcal{P}_\lambda$, we define $R_{\lambda,k}(\alpha)$ as a random variable that represents the distance from a node at X to its kth nearest neighbor in a cone centered at X and with degree α, i.e., $\mathcal{C}_X(\theta,\theta+\alpha)$. (For notational convenience, we may also use $R_\lambda(\alpha)$ to represent $R_{\lambda,k}(\alpha)$ when the neighbor referred to is clear from the context.) Clearly the distribution of $R_{\lambda,k}(\alpha)$ is independent of the choices

TABLE I

NOTATIONS USED

$\mathbb{R}$	Real line, $(-\infty, +\infty)$
S	$[0,1]^2$
$\mathcal{X}_n$	A binomial process (n independent, uniformly distributed random 2-vectors)
$\mathcal{P}_\lambda$	A homogeneous Poisson point process with density λ; $\{X_1, X_2, \cdots X_{N_\lambda}\}$
X_i	Node i's coordinate/location
C_j	(Constant) function that does not depend on λ
$\bar{G}$	The complement set of G
$\mathbf{1}_G$	The indicator function of G
$E[f(X)]$	Expectation of $f(X)$, i.e., $E[f(X)] = \int f(X)dP$
$E_G[f(X)]$	Expectation of $f(X)$ with the restriction G, i.e., $E_G[f(X)] = \int_G f(X)dP$
$\mathcal{B}_X(r)$	Ball of radius r centered at location X
$\mathcal{C}_X(\alpha,\beta)$	Cone that is centered at X and with the starting angle α and the ending angle β
$\mathcal{C}^*_X(r,\alpha,\beta)$	$\mathcal{B}_X(r) \cap \mathcal{C}_X(\alpha,\beta)$
$R_{\lambda,k}(\alpha)(=R_\lambda(\alpha))$	Random variable for the distance from a point X to the kth nearest node in $\mathcal{C}_X(\theta,\theta+\alpha)$
$R_{\lambda,k}(d,\alpha)(=R_\lambda(d,\alpha))$	Random variable for the distance from a point X to the kth nearest node in $\mathcal{C}^*_X(d,\theta,\theta+\alpha)$
$\Gamma(s)$	Gamma function, i.e., $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$
$F_{\Gamma(s)}(x)$	c.d.f. of the Gamma distribution function, i.e., $F_{\Gamma(s)}(x) = (\Gamma(s))^{-1}\int_0^x t^{s-1}e^{-t}dt$
$\approx_\lambda$	$g(\lambda) \approx_\lambda h(\lambda)$ is interpreted as $g(\lambda)/h(\lambda) \to 1$ as $\lambda \to \infty$

of X and θ. $P(R_{\lambda,k}(\alpha) > r)$ is the probability that at most $k-1$ points in the Poisson point process $\mathcal{P}_\lambda$ fall in $\mathcal{C}^*_X(r,\theta,\theta+\alpha)$, and can be expressed as $\exp(-\lambda\alpha r^2/2)\sum_{i=0}^{k-1}\frac{(\lambda\alpha r^2/2)^i}{i!}$. The cumulative distribution function (c.d.f.) $F_{R_{\lambda,k}(\alpha)}$ and the probability density function (p.d.f.) $f_{R_{\lambda,k}(\alpha)}$ of $R_{\lambda,k}(\alpha)$ can then be expressed as

$$F_{R_{\lambda,k}(\alpha)}(r) = P(R_{\lambda,k}(\alpha) \le r) = \begin{cases} 1 - e^{-\lambda\alpha r^2/2}\sum_{i=0}^{k-1}\frac{(\lambda\alpha r^2/2)^i}{i!}, & \text{if } r \ge 0, \\ 0, & \text{otherwise;} \end{cases} \quad (3)$$

$$f_{R_{\lambda,k}(\alpha)}(r) = \begin{cases} \frac{(\lambda\alpha r^2/2)^{k-1}\lambda\alpha r}{(k-1)!}e^{-\lambda\alpha r^2/2}, & \text{if } r \ge 0, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Also, the expectation of $R^c_{\lambda,k}(\alpha)$ (for $c > 0$) can be calculated as

$$\begin{aligned} E[R^c_{\lambda,k}(\alpha)] &= \int_0^\infty f_{R_{\lambda,k}(\alpha)}(r)r^c dr \\ &= \int_0^\infty \frac{(\lambda\alpha r^2/2)^{k-1}\lambda\alpha r}{(k-1)!}e^{-\lambda\alpha r^2/2}r^c dr \\ &\quad \text{(changing variable } t = \lambda\alpha r^2/2) \\ &= \int_0^\infty e^{-t}\left(\frac{2t}{\lambda\alpha}\right)^{\frac{c}{2}}\frac{t^{k-1}}{(k-1)!}dt \\ &= \frac{\Gamma(c/2+k)}{(k-1)!}\left(\frac{2}{\lambda\alpha}\right)^{c/2}, \end{aligned} \quad (5)$$

where the Γ function is defined as $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$.

Another closely related random variable $R_{\lambda,k}(d,\alpha)$ (for $d > 0$) is defined as

$$R_{\lambda,k}(d,\alpha) = \begin{cases} R_{\lambda,k}(\alpha), & \text{if } R_{\lambda,k}(\alpha) \le d, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

$R_{\lambda,k}(d,\alpha)$ can be interpreted as the distance from a node at X to the kth nearest neighbor in a cone centered at X, with degree α, and within radius d, i.e., $\mathcal{C}^*_X(d,\theta,\theta+\alpha)$, where θ is a fixed value. In the case that there are less than k nodes in the cone within radius d, $R_{\lambda,k}(d,\alpha)$ is defined to be 0. Thus, $R^c_{\lambda,k}(d,\alpha)$ is a restriction of $R^c_{\lambda,k}(\alpha)$, under the sub probability space that there are at least k nodes in $\mathcal{C}^*_X(d,\theta,\theta+\alpha)$. The expectation of $R^c_{\lambda,k}(d,\alpha)$ can be similarly calculated as

$$\begin{aligned} E[R^c_{\lambda,k}(d,\alpha)] &= \int_0^d f_{R_{\lambda,k}(\alpha)}(r)r^c dr \\ &= \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!}\left(\frac{2}{\lambda\alpha}\right)^{c/2}F_{\Gamma(c/2+k)}(\lambda\alpha d^2/2), \end{aligned} \quad (7)$$

where $F_{\Gamma(c/2+k)}$ is the c.d.f. of the Gamma distribution with parameter $c/2+k$. With fixed values of $\alpha, d, c, k > 0$, $F_{\Gamma(c/2+k)}(\lambda\alpha d^2/2) \to 1$ as $\lambda \to \infty$. Hence, we obtain the following lemma (that will be used in subsequent sections).

Lemma 1 *For fixed values of $d > 0, c > 0, \alpha > 0$ and k positive integer,*

$$E[R^c_{\lambda,k}(d,\alpha)] \approx_\lambda E[R^c_{\lambda,k}(\alpha)] = \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!}\left(\frac{2}{\lambda\alpha}\right)^{c/2}. \quad (8)$$

Let A, B be two given nodes in the Poisson point process $\mathcal{P}_\lambda$ in the square region S, and $R_{A,\lambda,k}(\alpha)(R_{B,\lambda,k}(\beta))$ be the distance from A (B) to its kth nearest neighbor in a cone of degree $\alpha > 0$. Specific choices of the cones and the locations of nodes A and B are not important in the following lemma.

Lemma 2

$$E[R^c_{A,\lambda,k}(\alpha)R^c_{B,\lambda,k}(\alpha)] \le C_0\lambda^{c(-1+\delta_1)} \tag{9}$$

for some $C_0 > 0$ and any given $\delta_1 > 0$, if λ is sufficiently large, where C_0 only depends on c and α but not on λ.

Proof. For notational convenience, we denote $R_{A,\lambda,k}(\alpha)$ and $R_{B,\lambda,k}(\alpha)$ respectively as R_A and R_B in the following derivation. For any given $\delta_1 > 0$, We can choose $\epsilon > 0$ such that $\alpha\epsilon^2/2 = \lambda^{-1+\delta_1}$. We first note that

$$\begin{aligned} & P(R_AR_B \le \epsilon^2) \\ & \quad \ge \quad P(R_A \le \epsilon \text{ and } R_B \le \epsilon) \\ & \quad = \quad 1 - P(R_A > \epsilon \text{ or } R_B > \epsilon) \\ & \quad \ge \quad 1 - (P(R_A > \epsilon) + P(R_B > \epsilon)) \\ & \quad \ge \quad 1 - 2\exp(-\lambda\alpha\epsilon^2/2)\sum_{i=0}^{k-1}\frac{(\lambda\alpha\epsilon^2/2)^i}{i!} \\ & \quad = \quad 1 - 2\exp(-\lambda^{\delta_1})\sum_{i=0}^{k-1}\frac{(\lambda^{\delta_1})^i}{i!}. \end{aligned} \tag{10}$$

Thus,

$$P(R_AR_B > \epsilon^2) \quad \le \quad 2\exp(-\lambda^{\delta_1})\sum_{i=0}^{k-1}\frac{(\lambda^{\delta_1})^i}{i!}. \tag{11}$$

Now $E[R^c_AR^c_B]$ can be expressed as

$$\begin{aligned} & E[R^c_AR^c_B] \\ & \quad = \quad E[R^c_AR^c_B|R_AR_B \le \epsilon^2]P(R_AR_B \le \epsilon^2) \\ & \quad\quad\quad +E[R^c_AR^c_B|R_AR_B > \epsilon^2]P(R_AR_B > \epsilon^2) \\ & \quad \le \quad E[\epsilon^{2c}|R_AR_B \le \epsilon^2]P(R_AR_B \le \epsilon^2) \\ & \quad\quad\quad +E[1|R_AR_B > \epsilon^2]P(R_AR_B > \epsilon^2) \\ & \quad \le \quad \epsilon^{2c} + 2\exp(-\lambda^{\delta_1})\sum_{i=0}^{k-1}\frac{(\lambda^{\delta_1})^i}{i!} \\ & \quad \le \quad \left(2\lambda^{-1+\delta_1}/\alpha\right)^c + C_1\lambda^{c(-1+\delta_1)} \\ & \quad = \quad C_0\lambda^{c(-1+\delta_1)}, \end{aligned} \tag{12}$$

where the second inequality from the fact that $P(R_AR_B \le \epsilon^2) \le 1$ and Eq. (11). The third inequality results from the choice of ϵ ($\alpha\epsilon^2/2 = \lambda^{-1+\delta_1}$) and the fact that $e^{\lambda^{\delta_1}}$ grows much faster than any polynomial function of λ. The choice of C_1, C_0 is independent of λ and δ_1 if δ_1 is fixed and λ is sufficiently large. □

D. Palm theory on Poisson point process

As Palm theory on the Poisson point process is used in multiple places in the paper, for the completeness of the paper, we state the theorem ([16], Theorem 1.6) below.

Theorem 1 *(Palm theory for Poisson processes) Let $\lambda > 0$. Suppose $j \in N$, and $h(\mathcal{Y}, \mathcal{X})$ is a bounded measurable function defined on all pairs of the form $(\mathcal{Y}, \mathcal{X})$ with $\mathcal{X}$ being a finite subset of $\mathbb{R}^d$ and $\mathcal{Y}$ a subset of $\mathcal{X}$, satisfying $h(\mathcal{Y}, \mathcal{X}) = 0$ except when $\mathcal{Y}$ has j elements. Then*

$$E[\sum_{\mathcal{Y}\subseteq\mathcal{P}_\lambda} h(\mathcal{Y}, \mathcal{P}_\lambda)] = \frac{\lambda^j}{j!}Eh(\mathcal{X}_j, \mathcal{X}_j \cup \mathcal{P}_\lambda), \tag{13}$$

where the sum on the left-hand side is over all subsets $\mathcal{Y}$ of the random Poisson point set $\mathcal{P}_\lambda$, and on the right-hand side the set $\mathcal{X}_j$ is a binomial process with j nodes, independent of $\mathcal{P}_\lambda$.

III. Lower bound on the critical total power

In this section, we derive the lower bound on the critical total power W_c to maintain network k-connectivity.

Theorem 2 *For any given $\delta > 0$, $P(W_c \ge (1-\delta)C_2\lambda^{1-\frac{c}{2}}) \to 1$ as $\lambda \to \infty$, where $C_2 = \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!}\pi^{-\frac{c}{2}}$.*

The proof of Theorem 2 will be given through two propositions. Clearly, in order to maintain strong k-connectivity, every node must be able to reach at least k other nodes. Thus a lower bound on the critical total power is the summation of power incurred by each node such that each node can exactly reach its kth nearest neighbor. Specifically, let X_i be the location of node i, r_i the distance from X_i to node i's kth nearest neighbor, $W_i = r_i^c$, and N_λ the number of nodes in the Poisson point process $\mathcal{P}_\lambda$ in $[0,1]^2$. Then the total power $W_L = \sum_{i=1}^{N_\lambda} W_i = \sum_{i=1}^{N_\lambda} r_i^c$ serves as a lower bound on the critical total power required to maintain k-connectivity. In what follows, we estimate W_L. First, we derive the expectation of W_L.

Proposition 1

$$E[W_L] \approx_\lambda \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!}\pi^{-\frac{c}{2}}\lambda^{1-\frac{c}{2}}. \tag{14}$$

Proof. By Palm theory for the Poisson point process,

$$E[W_L] = E[\sum_{i=1}^{N_\lambda} r_i^c] = \lambda E[r_0^c], \tag{15}$$

where the last expectation is taken over the probability space where node 0 is randomly placed with a uniform distribution on S, together with a set of nodes distributed according to a Poisson point process $\mathcal{P}_\lambda$ and independent of X_0. Under the Toroidal model assumption, node 0 views all the nodes in $\mathcal{P}_\lambda$ as if they reside in $[-\frac{1}{2}, \frac{1}{2}]^2$ of a coordinate system with the origin at X_0. Thus the distribution of r_0 is independent of the choice of X_0. Let s be the distance from X_0 to node 0's kth nearest neighbor in $\mathcal{P}_\lambda$ in $\mathcal{B}_{X_0}(1/2)$ if there are at least k nodes in $\mathcal{B}_{X_0}(1/2)$; and 0 otherwise. Then s has the same distribution as $R_{\lambda,k}(\frac{1}{2}, 2\pi)$. In addition, if $s > 0$ (which means there are at least k nodes in $\mathcal{B}_{X_0}(1/2)$), then $r_0 = s$. Thus $s \le r_0$ and $E[s^c] \le E[r_0^c]$. Also, since $r_0 < 1$,

$$\begin{aligned} E[r_0^c] &= E[r_0^c|s>0]P(s>0) + E[r_0^c|s=0]P(s=0) \\ &= E[s^c|s>0]P(s>0) + E[r_0^c|s=0]P(s=0) \\ &\le E[s^c] + P(s=0). \end{aligned} \tag{16}$$

Since $P(s=0) = e^{-\lambda\pi/4}\sum_{i=0}^{k-1}\frac{(\lambda\pi/4)^i}{i!} = o(\lambda^{-c/2})$ as $\lambda \to \infty$ and $E[s^c] \approx_\lambda \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!}(\lambda\pi)^{-\frac{c}{2}}$ (by Lemma 1), we obtain

$$\begin{aligned} E[r_0^c] &\approx_\lambda \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!}(\lambda\pi)^{-\frac{c}{2}}, \\ E[W_L] &= \lambda E[r_0^c] \approx_\lambda \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!\pi^{\frac{c}{2}}}\lambda^{1-\frac{c}{2}}. \end{aligned} \tag{17}$$

□

As has been shown in Lemma 1, the restriction on the distance to the kth nearest neighbor in a fixed cone (such as in one quadrant) can be ignored when the node density λ approaches infinity. In all the subsequent discussion, we ignore this restriction and assume, whenever desirable, the distance to the kth nearest neighbor can go to infinity (although with a small probability).

In order to bound $|W_L - E[W_L]|$, we need to derive the second moment of W_L (so that Chebyshov's inequality can be applied).

Proposition 2

$$E[W_L^2] \le E[W_L]^2 + C_3\lambda^{1-c+\delta_0} \text{ as } \lambda \to \infty, \tag{18}$$

where $\delta_0 > 0$ is arbitrary but fixed and C_3 is a constant independent of λ.

Proof.

$$\begin{aligned} E[W_L^2] &= E[(\sum_{i=1}^{N_\lambda} W_i)^2] \\ &= E[\sum_{i=1}^{N_\lambda} W_i^2] + 2E[\sum_{1\le i<j\le N_\lambda} W_iW_j] \end{aligned} \tag{19}$$

Since $W_i^2 = r_i^{2c}$, by Proposition 1 we obtain

$$E[\sum_{i=1}^{N_\lambda} W_i^2] = E[\sum_{i=1}^{N_\lambda} r_i^{2c}] \approx_\lambda \frac{\Gamma(c+k)}{(k-1)!}\pi^{-c}\lambda^{1-c}. \tag{20}$$

For the second term of Eq. (19), we apply Palm theory for the Poisson point process again and obtain

$$2E[\sum_{1\le i<j\le N_\lambda} W_iW_j] = \lambda^2 E[W_AW_B], \tag{21}$$

where the last expectation is taken over the probability space where A and B are uniformly and randomly distributed on S, together with a set of nodes distributed according to a Poisson point process $\mathcal{P}_\lambda$.

We first evaluate $E[W_AW_B]$ conditioning on the locations, X_A and X_B, of nodes A and B.

$$E[W_AW_B] = E[E[W_AW_B|X_A, X_B]]. \tag{22}$$

Given the location X_A and X_B, let $|X_A - X_B| \equiv d$. Let G_A be the event that there are at least k nodes in $\mathcal{B}_{X_A}(d/2)$, G_B the event that there are at least k nodes in $\mathcal{B}_{X_B}(d/2)$, and $G = G_A \cap G_B$. Then,

$$\begin{aligned} &E[W_AW_B|X_A, X_B] \\ &\quad = E_G[W_AW_B|X_A, X_B] + E_{\bar{G}}[W_AW_B|X_A, X_B]. \end{aligned} \tag{23}$$

The first term of Eq. (23) can be expressed as

$$\begin{aligned} E_G[W_AW_B|X_A, X_B] &= E_G[r_A^c r_B^c|X_A, X_B] \\ &= E[r_A^c r_B^c \mathbf{1}_G|X_A, X_B] \\ &= E[r_A^c r_B^c \mathbf{1}_{G_A}\mathbf{1}_{G_B}|X_A, X_B] \\ &= E[\tilde{r}_A^c \tilde{r}_B^c|X_A, X_B], \end{aligned} \tag{24}$$

where

$$\tilde{r}_A = r_A\mathbf{1}_{G_A} = \begin{cases} r_A, & \text{if } r_A \le d/2, \\ 0, & \text{otherwise;} \end{cases}$$
$$\tilde{r}_B = r_B\mathbf{1}_{G_B} = \begin{cases} r_B, & \text{if } r_B \le d/2, \\ 0, & \text{otherwise.} \end{cases}$$

Given the locations X_A and X_B, $\tilde{r}_A$ and $\tilde{r}_B$ are completely determined by the node distribution in $\mathcal{B}_{X_A}(d/2)$ and that in $\mathcal{B}_{X_B}(d/2)$ respectively. Since the two regions $\mathcal{B}_{X_A}(d/2)$ and $\mathcal{B}_{X_B}(d/2)$ are disjoint, $\tilde{r}_A$ and $\tilde{r}_B$ are independent. Hence we can evaluate their expectations separately:

$$\begin{aligned} &E_G[W_AW_B|X_A, X_B] \\ &\quad = E[\tilde{r}_A^c|X_A, X_B]E[\tilde{r}_B^c|X_A, X_B]. \end{aligned} \tag{25}$$

Note that the expectation of $\tilde{r}_A^c$ conditioned on X_A and X_B, $E_G[W_AW_B|X_A, X_B]$, is taken over the probability space of a Poisson point process $\mathcal{P}_\lambda$ on S. For each instance (realization) of $\mathcal{P}_\lambda$ on S, we can define $\hat{r}_A$ to be the kth nearest neighbor distance of node A with node B removed from S. Then $\tilde{r}_A \le \hat{r}_A$. Clearly, $\hat{r}_A$ is independent of node B's location. $\hat{r}_A$ is also independent of node A's location because of the homogeneous Poisson point process assumption and the Toroidal model assumption. Thus $E[\tilde{r}_A^c|X_A, X_B] \le E[\hat{r}_A^c|X_A, X_B] = E[\hat{r}_A^c]$. Finally, $\hat{r}_A$ is just the distance between node A (which is uniformly and randomly placed on S) and its kth nearest neighbor from $\mathcal{P}_\lambda$ on S. Thus $E[\hat{r}_A^c] = E[r_0^c]$, where $E[r_0^c]$ is given in Eq. (15). Therefore, $E[\tilde{r}_A^c|X_A, X_B] \le E[r_0^c]$. Similarly, $E[\tilde{r}_B^c|X_A, X_B] \le$

$E[r_0^c]$. Since $E[W_L] = \lambda E[r_0^c]$ by Eq. (15), we obtain that

$$E[E_G[W_A W_B|X_A, X_B]] \leq E[E[r_0^c]^2] = E[r_0^c]^2 = (E[W_L]/\lambda)^2 \quad (26)$$

It remains to evaluate the second term $E_{\bar{G}}[W_A W_B|X_A, X_B]$ in Eq. (23). Since $\bar{G} = \bar{G}_A \cup \bar{G}_B$, we have

$$\begin{aligned} & E_{\bar{G}}[W_A W_B|X_A, X_B] \\ & \leq E_{\bar{G}_A}[W_A W_B|X_A, X_B] + E_{\bar{G}_B}[W_A W_B|X_A, X_B] \\ & = E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B] + E_{\bar{G}_B}[r_A^c r_B^c|X_A, X_B] \\ & = 2E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B], \end{aligned} \quad (27)$$

where the last equality is by symmetry.

The basic idea to bound $E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B]$ is that if the distance, d, between nodes A and B is large, $\bar{G}_A$ occurs with low probability, and that the probability that the distance d is small is low. Specifically, consider the restriction of $|X_A - X_B| = d > \epsilon$ where ϵ is chosen such that $\pi\epsilon^2 = \lambda^{-1+\delta_1}$ for any fixed $\delta_1 > 0$.

$$\begin{aligned} & E_{\{d>\epsilon\}}[E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B]] \\ \leq \; & E_{\{d>\epsilon\}}[E_{\bar{G}_A}[1|X_A, X_B]] \\ \leq \; & P(\bar{G}_A \cap \{d > \epsilon\}) \\ \leq \; & P(\text{There are less than } k \text{ nodes in } \mathcal{B}_A(\epsilon/2)) \\ = \; & \exp(-\lambda\pi(\epsilon/2)^2) \sum_{i=0}^{k-1} \frac{(\lambda\pi(\epsilon/2)^2)^i}{i!} \\ = \; & \exp(-\lambda^{\delta_1}/4) \sum_{i=0}^{k-1} \frac{(\lambda^{\delta_1}/4)^i}{i!} \\ \leq \; & C_4 \lambda^{-(1+c)}, \end{aligned} \quad (28)$$

for some $C_4 > 0$ when λ is sufficiently large. Note that the last inequality results from $\exp(\lambda^{\delta_1}/4)$ grows much faster than any polynomial function of λ as $\lambda \to \infty$.

Next by Lemma 2 (with $\alpha = 2\pi$), for any given $\delta_1 > 0$, if λ is sufficiently large, there exists some constant $C_5 > 0$ such that

$$E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B] \leq E[r_A^c r_B^c|X_A, X_B] \leq C_5 \lambda^{c(-1+\delta_1)} \quad (29)$$

Therefore,

$$\begin{aligned} & E_{\{d\leq\epsilon\}}[E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B]] \\ \leq \; & E_{\{d\leq\epsilon\}}[C_5 \lambda^{c(-1+\delta_1)}] \\ = \; & P(d \leq \epsilon) \cdot C_5 \lambda^{c(-1+\delta_1)} \\ = \; & \pi\epsilon^2 \cdot C_5 \lambda^{c(-1+\delta_1)} \\ = \; & \lambda^{-1+\delta_1} \cdot C_5 \lambda^{c(-1+\delta_1)} \\ = \; & C_5 \lambda^{-1-c+\delta_1(1+c)} \end{aligned} \quad (30)$$

By setting $\delta_1 = \delta_0/(c+1)$, we obtain

$$E_{\{d\leq\epsilon\}}[E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B]] \leq C_5 \lambda^{-1-c+\delta_0}. \quad (31)$$

Combining Eqs. (28) and (31), we obtain

$$E[E_{\bar{G}_A}[r_A^c r_B^c|X_A, X_B]] \leq C_6 \lambda^{-1-c+\delta_0}. \quad (32)$$

Combining Eqs. (22), (23), (26) and (32), we obtain

$$E[W_A W_B] \leq (E[W_L]/\lambda)^2 + C_6 \lambda^{-(c+1)+\delta_0}. \quad (33)$$

Finally combining Eqs. (19)-(21) and (33), we obtain Eq. (18). □

We are now in a position to prove Theorem 2.

Proof of Theorem 2. By Chebyshov's inequality, for any given $\delta' > 0$, when $\lambda \to \infty$,

$$\begin{aligned} & P(|W_L - E[W_L]| \geq \delta' E[W_L]) \\ \leq \; & \frac{Var(W_L)}{\delta'^2 E[W_L]^2} \\ = \; & \frac{E[W_L^2] - E[W_L]^2}{\delta'^2 E[W_L]^2} \\ \leq \; & \frac{C_3 \lambda^{1-c+\delta_0}}{\delta'^2 E[W_L]^2} \\ \approx_\lambda \; & \frac{C_3 \lambda^{1-c+\delta_0}}{\delta'^2 \frac{\Gamma^2(c/2+k)}{((k-1)!)^2 \pi^c} \lambda^{2-c}}, \end{aligned} \quad (34)$$

where the last equation tends to 0 as λ goes to infinity if we choose $\delta_0 < 1$. Hence $P(W_L \geq (1-\delta')E[W_L]) \to 1$ as $\lambda \to \infty$. Since $W_c \geq W_L$, we have $P(W_c \geq (1 - \delta')E[W_L]) \to 1$ as $\lambda \to \infty$. By Proposition 1, $E[W_L] \geq (1 - \delta') \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!\pi^{\frac{c}{2}}} \lambda^{1-\frac{c}{2}}$ for sufficiently large values of λ. Consequently we have

$$P\left(W_c \geq (1-\delta')^2 \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!\pi^{\frac{c}{2}}} \lambda^{1-\frac{c}{2}}\right) \to 1, \quad (35)$$

as $\lambda \to \infty$. Given any $\delta > 0$, we can find $\delta' > 0$ such that $(1-\delta')^2 > (1-\delta)$, and hence as $\lambda \to \infty$,

$$P\left(W_c \geq (1-\delta) \frac{\Gamma(\frac{c}{2}+k)}{(k-1)!\pi^{\frac{c}{2}}} \lambda^{1-\frac{c}{2}}\right) \to 1, \quad (36)$$

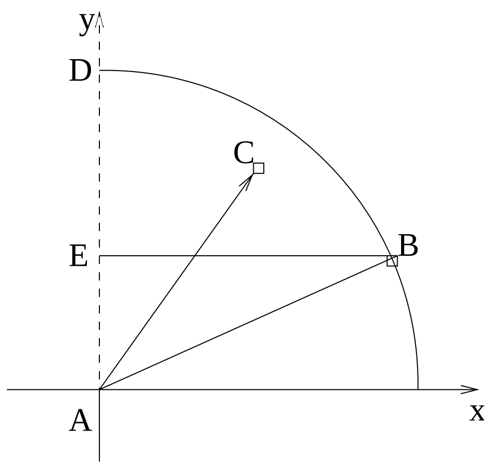
for any given $\delta > 0$, which completes the proof. □

IV. Upper Bound on the Critical Total Power

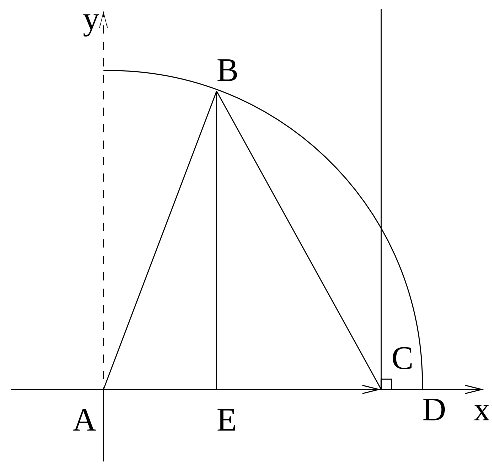
In this section, we derive an upper bound on the critical total power required to maintain k-connectivity. As will be shown later in this section, the upper bound turns out to be of the same order as the lower bound, not only in terms of λ but also in terms of k.

Given the coordinates of all nodes in the plane $[0,1]^2$, each node can define its own coordinate system by only shifting the origin of the $[0,1]^2$ plane to its own location. We use (x_i, y_i) to represent the coordinate of a node i in the original coordinate system (i.e., the plane $[0,1]^2$), and define the p-norm distance d_p between two nodes A and B as

$$d_p(A, B) = (|x_A - x_B|^p + |y_A - y_B|^p)^{1/p}. \quad (37)$$



(a) Case (i) $y_B - y_A < x_B - x_A$



(b) Case (ii) $y_B - y_A \geq x_B - x_A$

Fig. 1. Illustration for Lemma 3

If $p = \infty$, $d_\infty(A, B) = \max(|x_A - x_B|, |y_A - y_B|)$. Clearly p-norm distance does not change under the conversion from the original plane to a new coordinate system with a new origin. Throughout this paper, we use 2-norm distance as the "distance" unless otherwise specified, and $|AB|$ to represent $d_2(A, B)$. We first prove a geometric result on strong 1-connectivity.

Lemma 3 *Given the locations of all nodes on the plane* $[0, 1]^2$, *if each node chooses its power level to reach at least one neighbor in each of the four quadrants in its own coordinate system as long as there exist one or more nodes in that quadrant, the resulting network is strongly (1-)connected. (To eliminate the ambiguity in which quadrant the axis lines belong to, we assign the positive* x*-axis to the first quadrant, the positive* y*-axis to the second quadrant, and so on.)*

Proof. We prove the lemma by contradiction. If the resulting network is not strongly connected, there exists at least a pair of nodes (i, j) such that there exists no (directed) path from node i to node j. Among all the pairs, we choose the one *with the smallest* ∞*-norm distance*. In case of a tie, we choose the pair with the smallest 2-norm distance. Let the chosen pair be nodes (A, B). It suffices to find a pair of disconnected nodes (Y, Z) such that $d_\infty(Y, Z) < d_\infty(A, B)$, or $d_\infty(Y, Z) = d_\infty(A, B)$ and $d_2(Y, Z) < d_2(A, B)$.

Without loss of generality, we assume that there is no directed path from A to B, and node B is in the first quadrant in node A's coordinate system, i.e., $x_A < x_B, y_A \leq y_B$ (note that the first quadrant includes the positive x-axis but not the positive y-axis). Since there exists at least one node B in the first quadrant of node A's coordinate system, node A's power must be able to reach at least one other node C in the first quadrant of its coordinate system. Clearly $d_2(A, C) < d_2(A, B)$ since node A's power is not sufficient to reach node B. In addition, there exists no path from node C to node B; otherwise there would be a path from node A to node B. Now we consider two possible cases.

a) **Case (i)** $y_B - y_A < x_B - x_A$ **(Fig. 1 (a))***:* In this case $d_\infty(A, B) = x_B - x_A \equiv a$ and $|y_A - y_B| < a$. Let D be the intersection point of the cycle centered at A with radius $d_2(A, B)$ and the positive y-axis in node A's coordinate system. Let E be the intersection point of the y-axis in A's coordinate system and a horizontal line through node B. Then $|BE| = a$. As $y_C - y_A \leq |AC| < |AB|$ and $y_B - y_A = |AE|$, we have $y_C - y_B < |AB| - |AE| \leq |BE| = a$. On the other hand, $y_C \geq y_A$ and hence $y_C - y_B \geq y_A - y_B > -a$. Therefore $|y_C - y_B| < a$,

Similarly, $x_C > x_A$, and hence $x_C - x_B > x_A - x_B = -a$. In addition, as $x_C - x_A \leq |AC| < |AB|$ and $x_B - x_A = |BE|$, we have $x_C - x_B < |AB| - |BE| \leq |AE| \leq a$. Therefore $|x_C - x_B| < a$. As such, we conclude $d_\infty(B, C) = \max(|x_C - x_B|, |y_C - y_B|) < d_\infty(A, B)$, which violates the assumption on the pair of nodes (A, B).

b) **Case (ii)** $y_B - y_A \geq x_B - x_A$**(Fig. 1 (b))***:* In this case $d_\infty(A, B) = y_B - y_A \equiv a \geq |x_B - x_A|$. Let D be the intersection point of the cycle centered at A with radius $d_2(A, B)$ and the positive x-axis in node

A's coordinate system. Let E be the intersection point of the x-axis in A's coordinate system and a vertical line through node B. Then $|BE| = a$. As $x_C > x_A$, we have $x_C - x_B > x_A - x_B \geq -a$. Also, since $x_C - x_A \leq |AC| < |AB|$ and $x_B - x_A = |AE|$, we have $x_C - x_B < |AB| - |AE| \leq |BE| = a$. Therefore $|x_C - x_B| < a$.

Since $y_C - y_A < |AB|$ and $y_B - y_A = |BE|$, we have $y_C - y_B < |AB| - |BE| \leq |AE| \leq |BE| = a$. Also, since $y_C \geq y_A$, we have $y_C - y_B \geq y_A - y_B = -a$. Therefore, $|y_C - y_B| \leq a$. As such, we conclude $d_\infty(B, C) \leq a = d_\infty(A, B)$ with equality held if and only if $y_C = y_A$. If $y_C \neq y_A$, we reach the contradiction.

Now assume $y_C = y_A$. By the way nodes A and B are selected, we have $x_C > x_B$ because otherwise $d_\infty(B, C) = d_\infty(A, B)$ and $d_2(B, C) < d_2(A, B)$, which violates the assumption on the pair of nodes (A, B). Now we obtain a disconnected pair of nodes (C, B) that also has the smallest ∞-distance among all the disconnected node pairs, node B is in the second quadrant in node C's coordinate system, and

$$|x_C - x_B| < |y_C - y_B| \tag{38}$$

(as $\angle ACB > \pi/4$). Now we carry out the above analysis on the node pair (C, B). As the positive y-axis belongs to the second quadrant and by Eq. (38), we can only go to case (i). That is, we can find a pair of nodes (G, B) such that there exists no directed path from G to B and $d_\infty(G, B) < d_\infty(C, B) = d_\infty(A, B)$. This violates the assumption on the pair of nodes (A, B), and completes the proof. □

The above proof is primarily based on the distance metrics without use of the Toroidal model. However, it can be easily extended to the distance metrics under the Toroidal model by the following two observations. (i) Under the Toroidal model, each node views all other nodes on the plane $[-\frac{1}{2}, \frac{1}{2}]^2$ of its own coordinate system, and thus the (p-norm) distance between two nodes A, B under the Toroidal model is the same as the (p-norm) distance without use of the Toroidal model, in node A's coordinate system if node B (and all other nodes) are properly mapped to the plane $[-\frac{1}{2}, \frac{1}{2}]^2$ of node A's coordinate system. With this observation and the above proof, if A, B are the pair of nodes holding the extremal property under the Toroidal model, we can find a pair of nodes Y, Z having a smaller ∞-norm distance or the same ∞-norm distance but a smaller 2-norm distance (all) without the Toroidal model in node A's coordinate system than nodes A, B; (ii) Distance of any pair of nodes under the Toroidal model is always not larger than the distance without the Toroidal model (no matter under whose coordinate system). Now the found nodes Y, Z must have a smaller ∞-norm distance or the same ∞-norm distance but smaller 2-norm distance (all) under the Toroidal model than A, B do.

Lemma 3 can be easily extended to accommodate the case of strong k-connectivity as follows.

Lemma 4 *Given the locations of all nodes on the plane $[0, 1]^2$, if each node chooses its power level to reach at least k neighbors in each of the four quadrants in its own coordinate system, as long as there exist k or more nodes in that quadrant (in the case that there are less than k nodes in a quadrant, the transmission power of the node is chosen to reach all of the nodes in that quadrant), the resulting network is strongly k-connected.*

Proof. After removing any $k-1$ nodes from the network, each node can still reach at least one neighbor in each of its four quadrants, as long as that quadrant still contains some nodes. By Lemma 3, the remaining network is strongly connected. Therefore, the original network is at least strongly k-connected. □

Since the above simple topology control mechanism ensures strong k-connectivity in the underlying graph, the total power incurred based on this mechanism provides an upper bound on the critical total power required for k-connectivity. In what follows, we derive an upper bound on the critical total power based on the above topology control algorithm.

Let $W_U = \sum_{i=1}^{N_\lambda} W'_i$, where W'_i is the power consumed by node i under the topology control mechanism introduced in Lemma 4, and the summation is taken over all the points generated by a Poisson point process with density λ on $[0, 1]^2$. Clearly $W_c \leq W_U$. We have the following major result.

Theorem 3 $P(W_c \leq (1+\delta)C_7(c)\lambda^{1-c/2}) \to 1$ *as* $\lambda \to \infty$*, for any* $\delta > 0$*, where*

$$C_7(c) = \frac{4\Gamma(\frac{c}{2} + k)}{(k-1)!}\left(\frac{4}{\pi}\right)^{\frac{c}{2}}. \tag{39}$$

The proof of Theorem 3 will be given through two propositions and one lemma. First we evaluate the expectation of W_U.

Proposition 3 $E[W_U] \leq C_7(c)\lambda^{1-\frac{c}{2}}$ *as* $\lambda \to \infty$*, where* $C_7(c)$ *is given in Eq. (39).*

Proof. By Palm theory for the Poisson point process, we have

$$E[W_U] = E[\sum_{i=1}^{N_\lambda} W'_i] = \lambda E[W'_1], \tag{40}$$

where the last expectation is taken over the probability space where node 1 is randomly placed with a uniform distribution on the region S, together with a set of nodes that are distributed according to a Poisson point process $\mathcal{P}_\lambda$ and independent of X_1.

Let R_{1_i}, $1 \le i \le 4$, be the distance from node 1 to its kth nearest neighbor in the ith quadrant of node 1's coordinate system, and $R_1 = \max\{R_{1_i}, 1 \le i \le 4\}$. The power required for node 1 is then $W'_1 = R_1^c$. Since R_{1_i}'s are independent and have the same distribution as $R_{\lambda,k}(\pi/2)$[1] under the Poisson point process assumption, the expectation of W'_1 can be expressed as

$$\begin{aligned} E[W'_1] &= E[R_1^c] \\ &\le E[R_{1_1}^c + R_{1_2}^c + R_{1_3}^c + R_{1_4}^c] \\ &\approx_\lambda 4E[R_{\lambda,k}^c(\pi/2)] \\ &= \frac{4\Gamma(c/2+k)}{(k-1)!}\left(\frac{4}{\lambda\pi}\right)^{c/2}, \end{aligned} \tag{41}$$

where the last equality results from Eq. (5). Thus, by Eq. (40), we have

$$E[W_U] = \lambda E[W'_1] \le C_7(c)\lambda^{1-\frac{c}{2}}. \tag{42}$$

□

In order to bound $|W_U - E[W_U]|$, we need to estimate the second moment of W_U.

Proposition 4

$$E[W_U^2] \le E[W_U]^2 + C_8\lambda^{1-c+\delta_0} \text{ as } \lambda \to \infty \tag{43}$$

for any given $\delta_0 > 0$ and some constant $C_8 > 0$ that is independent of λ.

Proof.

$$\begin{aligned} E[W_U^2] &= E[\sum_{i=1}^{N_\lambda} W'_i]^2 \\ &= E[\sum_{i=1}^{N_\lambda} W_i'^2] + 2E[\sum_{1\le i<j\le N_\lambda} W'_iW'_j]. \end{aligned} \tag{44}$$

Since $W_i'^2 = R_i^{2c}$, by Proposition 3 we have

$$E[\sum_{i=1}^{N_\lambda} W_i'^2] = E[\sum_{i=}^{N_\lambda} R_i'^{2c}] = C_7(2c)\lambda^{1-c}. \tag{45}$$

It remains to determine the second term of Eq. (44). Applying Palm theory for the Poisson point process, we have

$$2E[\sum_{1\le i<j\le N_\lambda} W'_iW'_j] = \lambda^2 E[W'_AW'_B], \tag{46}$$

[1]More precisely, R_{1_i} is slightly different from $R_{\lambda,k}(\pi/2)$. By carrying out a proof similar to that in Proposition 1, we can show that the ratio of the expectations derived using $R_{\lambda,k}(\pi/2)$ to that using the precise version of R_{1_i} tends to 1 as $\lambda \to \infty$.

where the last expectation is taken over the probability space where nodes A and B are uniformly randomly distributed in the region S, together with a set of nodes that are distributed as a Poisson point process with density λ and is independent of the locations of nodes A and B.

First we evaluate $E[W'_AW'_B]$ conditioning on the locations, X_A and X_B, of nodes A and B, i.e.,

$$E[W'_AW'_B] = E[E[W'_AW'_B|X_A, X_B]]. \tag{47}$$

Given the locations X_A, X_B, let $d = |X_A - X_B|$. For each $i \in \{1,2,3,4\}$, let T_{A_i} be the event that at least k nodes from $\mathcal{P}_\lambda$ fall in node A's ith quadrant within radius $d/2$, and T_{B_i} the event that at least k nodes from $\mathcal{P}_\lambda$ fall in node B's ith quadrant within radius $d/2$. Let $T_A = \cap_{i=1}^4 T_{A_i}$, $T_B = \cap_{i=1}^4 T_{B_i}$, and $T = T_A \cap T_B$. That is, T denotes the event that at least k nodes in the Poisson point process $\mathcal{P}_\lambda$ fall in each of the four quadrants within radius $d/2$ in node A's coordinate system and in each of the four quadrants within radius $d/2$ in node B's coordinate system. By the law of total probability,

$$\begin{aligned} &E[W'_AW'_B|X_A, X_B] \\ &\quad = E_T[W'_AW'_B|X_A, X_B] + E_{\bar{T}}[W'_AW'_B|X_A, X_B]. \end{aligned} \tag{48}$$

The first term in the above Eq. (48) can be written as

$$\begin{aligned} &E_T[W'_AW'_B|X_A, X_B] \\ &= E_T[R_A^cR_B^c|X_A, X_B] \\ &= E[\mathbf{1}_T R_A^cR_B^c|X_A, X_B] \\ &= E[\mathbf{1}_{T_A}\mathbf{1}_{T_B} R_A^cR_B^c|X_A, X_B] \\ &= E[\tilde{R}_A^c\tilde{R}_B^c|X_A, X_B], \end{aligned} \tag{49}$$

where $\tilde{R}_A = R_A\mathbf{1}_{T_A} = R_A\mathbf{1}_{\{R_A\le d/2\}}$, and $\tilde{R}_B = R_B\mathbf{1}_{T_B} = R_B\mathbf{1}_{\{R_B\le d/2\}}$. Now, clearly $\tilde{R}_A$ and $\tilde{R}_B$ are independent because they depend on the node distributions in two disjoint regions, $\mathcal{B}_{X_A}(d/2)$ and $\mathcal{B}_{X_B}(d/2)$, respectively. Therefore, we can evaluate their expectations separately, i.e.,

$$E_T[W'_AW'_B|X_A, X_B] = E[\tilde{R}_A^c|X_A, X_B]E[\tilde{R}_B^c|X_A, X_B]. \tag{50}$$

By a similar argument to that in Proposition 2, we obtain

$$\begin{aligned} E[\tilde{R}_A^c|X_A, X_B] &\le E[R_1^c] = E[W_U]/\lambda, \\ E[\tilde{R}_B^c|X_A, X_B] &\le E[R_1^c] = E[W_U]/\lambda. \end{aligned} \tag{51}$$

Thus

$$E_T[W'_AW'_B|X_A, X_B] \le (E[W_U]/\lambda)^2. \tag{52}$$

Combining Eqs. (47), (48) and (52), we obtain

$$E[W'_AW'_B] \le (E[W_U]/\lambda)^2 + E[E_{\bar{T}}[W'_AW'_B|X_A, X_B]] \tag{53}$$

Now it remains to determine the second term of Eq. (53), which we denote as I_2, i.e.,

$$I_2 \equiv E[E_{\bar{T}}[W'_AW'_B|X_A, X_B]]$$

$$= \quad E[E_{\bar{T}}[R_A^c R_B^c | X_A, X_B]]. \quad (54)$$

Since

$$\bar{T} = (\cup_{l=1}^4 \bar{T}_{A_l}) \cup (\cup_{l=1}^4 \bar{T}_{B_l}), \quad (55)$$

it follows that

$$\begin{aligned} I_2 &\leq E \sum_{l=1}^4 E_{\bar{T}_{A_l}}[R_A^c R_B^c | X_A, X_B] \\ &\quad + E \sum_{l=1}^4 E_{\bar{T}_{B_l}}[R_A^c R_B^c | X_A, X_B] \\ &= 2E \sum_{l=1}^4 E_{\bar{T}_{A_l}}[R_A^c R_B^c | X_A, X_B], \end{aligned} \quad (56)$$

where the last equality is by symmetry.

Since $R_A = \max_{1 \leq i \leq 4} R_{A_i}$, $R_B = \max_{1 \leq j \leq 4} R_{B_j}$, where R_{A_i} (R_{B_j}) is the distance from node A (B) to node A's (B's) kth nearest neighbor in the ith (jth) quadrant of A's (B's) coordinate system, we have $R_A^c \leq \sum_{i=1}^4 R_{A_i}^c$ and $R_B^c \leq \sum_{j=1}^4 R_{B_j}^c$. Hence,

$$I_2 \leq 2E\left[\sum_{l=1}^4 \sum_{i=1}^4 \sum_{j=1}^4 E_{\bar{T}_{A_l}}[R_{A_i}^c R_{B_j}^c | X_A, X_B]\right]. \quad (57)$$

There are a total of 64 possible combinations of (l, i, j) in Eq. (57). We show in the following lemma that each of the 64 terms is at most of the order $\lambda^{-(1+c)+\delta_0}$.

Lemma 5 *For any* $(l, i, j) \in \{1, 2, 3, 4\}^3$,

$$E[E_{\bar{T}_{A_l}}[R_{A_i}^c R_{B_j}^c | X_A, X_B]] \leq C_9 \lambda^{-(1+c)+\delta_0}, \quad (58)$$

for any $\delta_0 > 0$ *and some constant* C_9 *if* λ *is sufficiently large, where* C_9 *only depends on* c, k *and not on* λ.

The proof of Lemma 5 pretty much follows that in Proposition 2. On one hand, if the distance d between nodes A and B is large, the probability that T_{A_l} occurs is low. On the other hand, the probability that the distance d is small is low. The proof is given in Appendix.

Combining Lemma 5 with Eqs. (53), (54) and (57), we obtain

$$E[W'_A W'_B] \leq (E[W_U]/\lambda)^2 + 128 C_9 \lambda^{-(1+c)+\delta_0}. \quad (59)$$

With Eqs. (44)–(46) and (59), we obtain Eq. (43). □

With Propositions 3 and 4, We can prove Theorem 3 in a similar manner to Theorem 2. Due to the space limit, the proof is omitted.

V. DISCUSSIONS

Interpretation of derived results: By Theorem 2 and 3, we reach the following corollary.

Corollary 1 *As* $\lambda \to \infty$,

$$W_c = \Theta\left(\frac{\Gamma(c/2+k)}{(k-1)!}\lambda^{1-\frac{c}{2}}\right) \quad (60)$$

with probability approaching 1, where W_c *is the critical total power required for maintaining* k*-connectivity.*

In general, the path loss exponent is $2 \leq c \leq 4$, although our proof applies to any $c > 0$. In the case of $c > 2$, Corollary 1 indicates that if the density is sufficiently large, the increase in the density reduces the critical total power, and in addition, the critical total power decreases as the path loss exponent increases.

Comparing with the critical total power derived under the uniform metric assumption (given in Eq. (1) and a similar equation in [25]), we conclude that the critical total power can be reduced by a factor of $\Theta((\log \lambda)^{c/2})$ by allowing nodes to optimally choose different levels of transmission power. This is not subject to any specific algorithm, but rather a fundamental property in wireless networks.

Legitimacy of the system model: We claim that the assumption of a unit area region is an abstraction of the real world. The unit area is not necessarily 1 meter^2, but instead can be used to model a L^2meter^2 area. That is, we can rescale the unit area to a square area with side length L and network density λ_0. In this rescaled network, every pair of nodes have a small chance to be very close to each other. A one-to-one correspondence between the values in the unit-area network and those in the rescaled network can be made and is given in Table II.

Consider the average power consumed by each node. In the unit-area network, the average power consumed by each node is of order $\lambda^{-c/2}$ (the constant that contains k is ignored). In the rescaled network, since each edge is rescaled by a factor of L, the power consumption should be multiplied by a factor of L^c. However, if we consider the side length L to be one unit, the node density in the corresponding unit-area square becomes $\lambda = \lambda_0 L^2$. Hence the average power consumption (in the rescaled network) is now $\lambda^{-c/2} L^c = (\lambda_0 L^2)^{-c/2} L^c = \lambda_0^{-c/2}$, which only depends on the density λ_0 in the rescaled network and not on the side length L of the area. On the other hand, if we assume a common critical transmission power among nodes for k-connectivity in the rescaled network, each node has to consume power in the order

TABLE II

ONE-TO-ONE CORRESPONDENCE BETWEEN THE VALUES IN THE UNIT-AREA SQUARE AND THOSE IN THE $L \times L$ SQUARE.

in the unit-area square	in the $L \times L$ square
1	L
r	Lr
$\lambda = \lambda_0 L^2$	λ_0

of $\lambda_0^{-c/2}(\log \lambda_0 L^2)^{c/2}$,[2] which tends to infinity if λ_0 is fixed and $L \to \infty$.

VI. CONCLUSION

We have shown in this paper that in a heterogeneous wireless network in which wireless nodes are distributed in a unit square region $[0,1]^2$ according to a Poisson point process with density λ and nodes may transmit with different levels of power, the critical total power required to maintain k-connectivity is $\Theta(\frac{\Gamma(c/2+k)}{(k-1)!}\lambda^{1-c/2})$ with probability approaching 1, where c is the path loss exponent. This result is obtained by deriving a lower bound and an upper bound on the critical total power. By comparing the result against those obtained when all nodes use the uniform critical transmission power for k-connectivity [18], [25], we conclude that with the use of (optimal) power control, the critical total power can be reduced by an factor of $\Theta((\log \lambda)^{c/2})$, irregardless of the power/topology control algorithm used.

In this paper we assume Torus convention to eliminate the need to consider boundary effects. As has been pointed out in [6], boundary effects may affect the (uniform) critical transmission range for k-connectivity. We will investigate whether boundary phenomena will affect the critical total power required for maintaining k-connectivity in heterogeneous networks. This is a subject of future research.

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[2] By Eq. (1), each node needs power $r_\lambda^c = \Theta((\frac{\log \lambda}{\lambda})^{c/2})$ (ignoring the less significant terms) in the unit-area network. Rescaling to the large network with side length L, each node needs power $r_\lambda^c L^c = \Theta(\lambda_0^{-c/2}(\log(\lambda_0 L^2))^{c/2})$. Although the Eq. (1) (comes from [18]) assumes n independently randomly placed nodes while our results are based on Poisson point processes, they are comparable through Poissonization and De-Poissonization techniques (see [16]).

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APPENDIX

First consider the restriction of $|X_A - X_B| = d > \epsilon$ where ϵ is chosen such that $\pi\epsilon^2 = \lambda^{-1+\delta_1}$ for any fixed $\delta_1 > 0$.

$$\begin{aligned} & E_{\{d>\epsilon\}}[E_{\bar{T}_{A_l}}[R^c_{A_i} R^c_{B_j} | X_A, X_B]] \\ \leq\ & E_{\{d>\epsilon\}}[E_{\bar{T}_{A_l}}[1 | X_A, X_B]] \\ \leq\ & P(\bar{T}_{A_l} \cap \{d > \epsilon\}) \\ \leq\ & P(\text{There are less than } k \text{ nodes in } \mathcal{C}^*_A(\epsilon/2, (l-1)\pi/2, l\pi/2)) \\ =\ & \exp(-\lambda\pi(\epsilon/2)^2/4) \sum_{i=0}^{k-1} \frac{(\lambda\pi(\epsilon/2)^2/4)^i}{i!} \\ =\ & \exp(-\lambda^{\delta_1}/16) \sum_{i=0}^{k-1} \frac{(\lambda^{\delta_1}/16)^i}{i!} \\ \leq\ & C_{10}\lambda^{-(1+c)}, \end{aligned} \tag{61}$$

for some $C_{10} > 0$ when λ is sufficiently large. Note that the last inequality results from $\exp(\lambda^{\delta_1}/16)$ grows much faster than any polynomial function of λ as $\lambda \to \infty$.

Next by Lemma 2 (with $\alpha = \pi/2$), for any given $\delta_1 > 0$, if λ is sufficiently large, there exists some constant $C_{11} > 0$ such that

$$\begin{aligned} & E_{\bar{T}_{A_l}}[R^c_{A_i} R^c_{B_j} | X_A, X_B] \\ \leq\ & E[R^c_{A_i} R^c_{B_j} | X_A, X_B] \leq C_{11}\lambda^{c(-1+\delta_1)} \end{aligned} \tag{62}$$

Therefore,

$$\begin{aligned} & E_{\{d\leq\epsilon\}}[E_{\bar{T}_{A_l}}[R^c_{A_i} R^c_{B_j} | X_A, X_B]] \\ \leq\ & E_{\{d\leq\epsilon\}}[C_{11}\lambda^{c(-1+\delta_1)}] \\ =\ & P(d \leq \epsilon) \cdot C_{11}\lambda^{c(-1+\delta_1)} \\ =\ & \pi\epsilon^2 \cdot C_{11}\lambda^{c(-1+\delta_1)} \\ =\ & \lambda^{-1+\delta_1} \cdot C_{11}\lambda^{c(-1+\delta_1)} \\ =\ & C_{11}\lambda^{-1-c+\delta_1(1+c)} \end{aligned} \tag{63}$$

By setting $\delta_1 = \delta_0/(c+1)$, we obtain

$$E_{\{d\leq\epsilon\}}[E_{\bar{T}_{A_l}}[R^c_{A_i} R^c_{B_j} | X_A, X_B]] \leq C_{11}\lambda^{-1-c+\delta_0}. \tag{64}$$

Combining Eqs. (61) and (64), we obtain

$$E[E_{\bar{T}_{A_l}}[R^c_{A_i} R^c_{B_j} | X_A, X_B]] \leq C_9\lambda^{-1-c+\delta_0}. \tag{65}$$

□