POLYNOMIALS IN ALGEBRAIC COMBINATORICS

BY

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DISSERTATION

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Abstract

A long-standing theme in algebraic combinatorics is to study bases of the rings of symmetric functions, quasisymmetric functions, and polynomials. Classically, these bases are homogeneous functions, however, the introduction of K-theoretic combinatorics has led to increased interest in finding inhomogeneous deformations of classical bases.

Joint with A. Yong and N. Tokcan, we introduce the notion of saturated Newton polytope (SNP), a property of polynomials, and study its prevalence in algebraic combinatorics. We find that many, but not all, of the families that arise in other contexts of algebraic combinatorics are SNP. We introduce a family of polytopes called the Schubitopes and connect it to the Newton polytopes of the Schubert polynomials and the key polynomials.

Semistandard skyline fillings are a combinatorial model that arises from specializing the combinatorics of Macdonald polynomials. We define a set-valued extension which allows us to define inhomogeneous deformations of the Demazure atoms, key polynomials, and quasisymmetric Schur functions. We prove that these deformations act in many ways like their homogeneous counterparts.

We then continue the work on set-valued skyline fillings. Joint with O. Pechenik and D. Searles, we provide deformations of the quasikey polynomials and the fundamental particles. This allows us to lift the quasisymmetric Grothendieck polynomials from the ring of quasisymmetric polynomials to the ring of polynomials and give expansions between the different bases under consideration that are analogous to the homogeneous case.

We end with some conjectures on the structure constants of equivariant Schubert calculus in Type B and C, including a generalization of the Horn inequalities to this setting.

Dedicated to all my friends and family who never stopped believing in me.

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LIST OF SYMBOLS

Partitions, Compositions, and Permutations

λ	A partition.
α	A composition.
γ	A weak composition.
$\ell(\gamma)$	The length of γ.
$\lvert\gamma\rvert$	The size of γ.
$\mathbf{x}^\gamma$	The monomial $x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_\ell^{\gamma_\ell}$.
$\lambda \vdash n$	λ is a partition of n.
$\mathsf{Par}(n)$	The set of partitions of n.
$\mathsf{Comp}(n)$	The set of compositions of n.
$\mathsf{WC}(n)$	The set of weak compositions of n.
Par	The set of all partitions.
Comp	The set of all compositions.
WC	The set of all weak compositions.
λ'	The conjugate of λ.
$\subseteq$	Partition containment.
$\leq_D$	Dominance order on partitions, compositions, or permutations.
$\alpha \preceq \beta$	α is a refinement of β.

γ^*	The weak composition formed by reversing the order of the parts of γ.
γ^+	The composition formed by dropping zeros from γ.
$\lambda(\gamma)$	The partition formed by sorting the parts of γ in weakly decreasing order.
$\mathsf{PermutWC}(\lambda)$	The set of all weak compositions γ such that $\lambda(\gamma^+) = \lambda$.
$\mathsf{PermutC}(\lambda)$	The set of all compositions α such that $\lambda(\alpha) = \lambda$.
$\mathsf{Expand}(\alpha)$	The set of all weak compositions γ such that $\gamma^+ = \alpha$.
S_m	The group of permutations on m letters.
S_∞	The group of permutations of $\mathbb{Z}_{>0}$ that fix all but finitely many elements.
w	A permutation.
s_i	The elementary transposition swapping i and $i+1$.
$\mathsf{D}(w)$	The Rothe diagram of w.
$c(w)$	The Lehmer code of w.
w_0	The longest permutation in S_n.
$\leq_B$	(Strong) Bruhat order on permutations.
$\mathsf{Koh}(w)$	The set of all Kohnert diagrams of w.
$w(\gamma)$	The result of applying the permutation w to γ.
w_γ	The permutation such that $w_\gamma(\lambda_\gamma) = \gamma$.
$\ll$	The order $\lambda(\gamma) = \lambda(\delta)$ and $w_\gamma \leq_B w_\delta$.
$\leqslant$	Lexicographic order on weak compositions.

Polynomial Families and Polynomial Rings

m_λ	Monomial symmetric function.
e_λ	Elementary symmetric function.
h_λ	Homogeneous symmetric function.
p_λ	Power sum symmetric function.

s_λ	Schur function.
$\hat{s}_\lambda$	Dual Schur function.
M_α	Monomial quasisymmetric function.
F_α	Fundamental quasisymmetric function.
S_α	Quasisymmetric Schur function.
$\mathfrak{m}_\gamma$	Monomial function.
$\mathfrak{S}_w$	Schubert polynomial.
$\mathtt{Atom}_\gamma$	Demazure atom.
$\mathtt{Key}_\gamma$	Key polynomial.
$\mathtt{Slide}_\gamma$	Slide polynomial.
$\mathtt{QKey}_\gamma$	Quasikey polynomial.
$\mathtt{Par}_\gamma$	Fundamental particle.
Ks_λ	Symmetric Grothendieck function.
KF_α	Multifundamental quasisymmetric function.
KS_α	Quasisymmetric Grothendieck function.
$K\mathfrak{S}_w$	Grothendieck function.
$K\mathtt{Atom}_\gamma$	Lascoux atom.
$K\mathtt{Key}_\gamma$	Lascoux polynomial.
$K\mathtt{Slide}_\gamma$	Glide polynomial.
$K\mathtt{QKey}_\gamma$	QuasiLascoux polynomial.
$K\mathtt{Par}_\gamma$	Kaon.
P_λ	Symmetric Macdonald polynomial.
J_λ	Integral form of the Macdonald polynomial.
H_λ	Modified Macdonald polynomial.
$\widetilde{H}_\lambda$	Transformed, modified Macdonald polynomial.
E_γ	Nonsymmetric Macdonald polynomial.

Sym	The ring of symmetric functions.
QSym	The ring of quasisymmetric functions.
Pol	The ring of polynomials in arbitrarily many variables.

Polynomial Operators, Coefficients, and Polytopes

$[f_\alpha]g_\beta$	The coefficient of f_α in g_β.
$\mathsf{K}_{\lambda,\mu}$	Kostka coefficient.
$\mathsf{LR}^{\nu}_{\lambda,\mu}$	Littlewood-Richardson coefficient.
∂_i	$\frac{1-s_i}{x_i - x_{i+1}}$.
π_i	$\partial_i x_i$.
$\hat{\pi}_i$	$\pi_i - 1$.
$\tilde{\partial}_i$	$\partial_i(1+\beta x_{i+1})$.
τ_i	$\pi_i(1+\beta x_{i+1})$.
$\hat{\tau}_i$	$\tau_i - 1$.
$\mathsf{Newton}(f)$	The Newton polytope of f.
$\mathcal{P}_\lambda$	The permutahedron of λ.
$\mathcal{S}_{\mathsf{D}}$	The Schubitope associated to D.

Tableaux and Fillings

T	A tableau.
F	A filling.
$c(T)$	The content of T.
$\mathbf{x}^T$	The monomial of T.

$\mathsf{read}(T)$	The reading word of T.
$\mathsf{ex}(T)$	The excess of T.
$\mathsf{SSYT}(\lambda)$	Semistandard Young tableaux of shape λ.
$\mathsf{RT}(\lambda)$	Reverse tableaux of shape λ.
$\mathsf{SSCT}(\alpha)$	Semistandard composition tableaux of shape α.
$\mathsf{SkyFill}(\gamma, \mathbf{b})$	Semistandard skyline fillings of shape γ and basement $\mathbf{b}$.
$\mathsf{SetSSYT}(\lambda)$	Semistandard set-valued Young tableaux of shape λ.
$\mathsf{SetRT}(\lambda)$	Set-valued reverse tableaux of shape λ.
$\mathsf{SetSSCT}(\lambda)$	Semistandard set-valued composition tableaux of shape α.
$\mathsf{SetSkyFill}(\gamma, \mathbf{b})$	Semistandard set-valued skyline fillings of shape γ and basement $\mathbf{b}$.
$\mathbf{b}_i$	The basement $\mathbf{b} = (1, 2, \ldots, n)$.

CHAPTER 1

Introduction

We are interested in three nested rings:

$$\mathsf{Sym}_m \subseteq \mathsf{QSym}_m \subseteq \mathsf{Pol}_m$$

where Sym_m (resp. QSym_m and Pol_m) is the ring of symmetric polynomials (resp. quasisymmetric polynomials and polynomials) in m variables. A common theme in algebraic combinatorics is to study different bases of these rings and the relationships between those bases. In this dissertation, there are three types of relationships of interest:

Definition 1.1 (Inhomogeneous deformation). *A family of polynomials $\{F_\alpha\}_{\alpha\in A}$ for some indexing set A is an inhomogeneous deformation of a homogeneous family $\{f_\alpha\}_{\alpha\in A}$ if*

$$F_\alpha = f_\alpha + \textit{higher order terms.}$$

Definition 1.2 (Lift). *A family of polynomials $\{g_\beta\}_{\beta\in B}$ is a lift of a family of polynomials $\{f_\alpha\}_{\alpha\in A}$ if there exists an inclusion $\iota : A \hookrightarrow B$ and for all α, we have $g_{\iota(\alpha)} = f_\alpha$.*

Definition 1.3 (Combinatorial expansion). *A family of polynomials $\{g_\beta\}_{\beta\in B}$ expands combinatorially in a basis $\{f_\alpha\}_{\alpha\in A}$ if for all g_β,*

$$g_\beta = \sum_{\alpha\in A} c_{\beta,\alpha} f_\alpha$$

with $c_{\beta,\alpha} \in \mathbb{Z}_{\geq 0}$.

The coefficient of f_α in g_β, $c_{\beta,\alpha}$ in the definition above, is denoted $[f_\alpha]g_\beta$. When there is a combinatorial expansion, we are interested in a combinatorial rule to describe the coefficients

$c_{\beta,\alpha}$. Specifically, we seek a counting rule that describes $c_{\beta,\alpha}$ as the number of an explicit set of objects thereby giving a manifestly nonnegative description of the coefficients. A particular type of combinatorial expansion is called a refinement.

Definition 1.4 (Refinement)**.** *A family of polynomials* $\{g_\beta\}_{\beta\in B}$ *is a refinement of* $\{f_\alpha\}_{\alpha\in A}$ *if for each* α*,*

$$f_\alpha = \sum_{\beta\in B} c_{\alpha,\beta} g_\beta$$

with the following conditions on $c_{\alpha,\beta}$*:*

1. $c_{\alpha,\beta} \in \{0,1\}$ *and*
2. *for fixed* β_0*,* $c_{\alpha,\beta_0} = 1$ *for exactly one* α*.*

In other words, $\{g_\beta\}_{\beta\in B}$ is a refinement of $\{f_\alpha\}_{\alpha\in A}$ if we can partition B such that each f_α is the sum of the g_β's in some block of the partition.

This chapter first defines the objects that will serve as indexing sets to the families of polynomials under consideration (Section 1.1). We then introduce the bases of Sym_m (Section 1.2), QSym_m (Section 1.3), and Pol_m (Section 1.4) that will be needed throughout this dissertation. Section 1.5 discusses the combinatorics of Macdonald polynomials and its use in different bases of Pol_m whereas Section 1.6 introduces K-theoretic combinatorics.

1.1 Partitions, Compositions, and Permutations

Our main sources for the next two sections are [Man01, Sta99a].

A **partition** is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0.$$

The λ_i are the **parts** of λ and the number of parts is the **length**, $\ell(\lambda)$. The **size** of λ is

$$|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i.$$

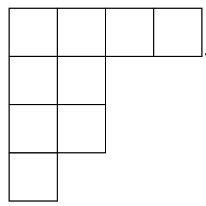
If $|\lambda| = n$, we say that λ is a partition of n, denoted $\lambda \vdash n$. We also denote λ by $1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots$ where $m_i(\lambda)$ is the number of times i appears as a part of λ. For example, $\lambda = (4,2,2,1) = 4^1 2^2 1^1$ is a partition of 9 with 4 parts.

The set of all partitions of n is denoted $\mathsf{Par}(n)$ while the set of all partitions (of any size) is denoted Par. The set $\mathsf{Par}(n)$ is a lattice under the partial order **dominance order**:

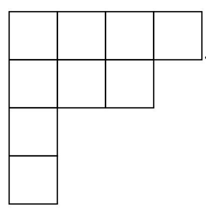
Definition 1.5 (Dominance Order)**.** *For partitions $\lambda, \mu \in \mathsf{Par}(n)$, we say λ* **dominates** *μ, denoted $\lambda \geq_D \mu$, if for all k,*

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i.$$

For a partition λ, we define its **Young diagram** as the diagram with λ_i boxes in row i, reading the rows from top to bottom. Thus for $\lambda = (4, 2, 2, 1)$, the Young diagram is



Given a partition λ, the **conjugate** of λ, denoted λ', is the partition formed by transposing a Young diagram. Formally $\lambda'_i = \#\{k : \lambda_k \geq i\}$. For example, for $\lambda = (4, 2, 2, 1)$ depicted above, $\lambda' = (4, 3, 1, 1)$ pictured below:



A composition (resp. weak composition) is a sequence of positive (resp. nonnegative) integers, and the parts, size and length of a composition are defined as above. We denote the set of all compositions of n by $\mathsf{Comp}(n)$ and of any size by Comp. Likewise $\mathsf{WC}(n)$ is the set of all weak compositions of n and WC is the set of all weak compositions.

For a weak composition γ, we define γ^* to be the weak composition formed by reversing the order of the parts of γ and γ^+ to be the composition formed by removing parts of size 0 from γ. Likewise, we define $\lambda(\gamma)$ to be the partition formed by sorting the parts of γ in weakly decreasing order and removing any parts of size 0. Thus, we define $\mathsf{PermutWC}(\lambda)$ to be the set of all weak compositions γ such that $\lambda(\gamma) = \lambda$ and $\mathsf{PermutC}(\lambda)$ to be the set of all compositions α such that $\lambda(\alpha) = \lambda$. Furthermore, we define $\mathsf{Expand}(\alpha)$ to be the set of all weak compositions γ such that $\gamma^+ = \alpha$. For example, for $\gamma = (4, 1, 0, 2)$, $\gamma^* = (2, 0, 1, 4)$,

$\gamma^+ = (4, 1, 2)$, and $\lambda(\gamma) = (4, 2, 1)$. Finally, let $\mathbf{x} = \{x_1, x_2, \ldots\}$ and then

$$\mathbf{x}^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_\ell^{\gamma_\ell}.$$

Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$ and S_∞ be the group of permutations of $\mathbb{N}$ that fix all but finitely many elements. Generally, we will write permutations in one-line notation. For example $w = 51432$ is the permutation where $w(1) = 5, w(2) = 1, w(3) = 4, w(4) = 3$, and $w(5) = 2$. An **inversion** of a permutation w is a pair (i, j) with $i < j$ and $w(i) > w(j)$ and the **length** of w, $\ell(w)$, is the number of inversions. Thus for $w = 51432$, the inversions are

$$\{(1, 2), (1, 3), (1, 4), (1, 5), (3, 4), (3, 5), (4, 5)\}$$

and $\ell(w) = 7$.

The **simple transposition** s_i is the permutation that switches the numbers in positions i and $i + 1$ while fixing the remaining letters, and it is well-known that the set $\{s_i\}_{i=1}^{n-1}$ generates S_n. Therefore, every permutation in S_n can be written as a product $s_{i_1} s_{i_2} \cdots s_{i_\ell}$. Such a product, read left to right, is called a **decomposition** of w. The minimum number of factors in a decomposition of w is $\ell(w)$ and a decomposition that uses $\ell(w)$ factors is **reduced**. A **reduced word** records a reduced decomposition by including just the indices of the transpositions. For example, for $w = 51432$, $s_4 s_3 s_2 s_1 s_4 s_3 s_4$ is a reduced decomposition, while 4321434 is a reduced word as seen by the following transformations

$$12345 \stackrel{s_4}{\Rightarrow} 12354 \stackrel{s_3}{\Rightarrow} 12534 \stackrel{s_2}{\Rightarrow} 15234 \stackrel{s_1}{\Rightarrow} 51234 \stackrel{s_4}{\Rightarrow} 51243 \stackrel{s_3}{\Rightarrow} 51423 \stackrel{s_4}{\Rightarrow} 51432.$$

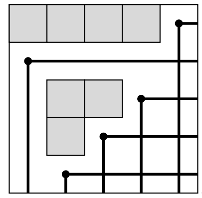
The following two relations hold for the simple transpositions:

$$\begin{aligned} s_i s_j &= s_j s_i \qquad && |i - j| \geq 2 \qquad && \text{(commutation relation)} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && && \text{(braid relation).} \end{aligned}$$

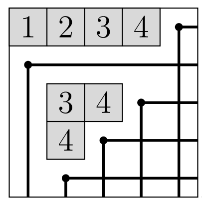
Furthermore, given any two reduced decompositions d_1 and d_2 of a permutation w, there is a sequence of these moves that transforms d_1 into d_2. **(Strong) Bruhat order** $u \leq_B w$ is the ordering on permutations obtained as the closure of the relation $w \leq_B wt_{ij}$ if $\ell(wt_{ij}) = \ell(w) + 1$ where t_{ij} is the transposition interchanging i and j.

Given a permutation $w \in S_n$, we form the **Rothe diagram** of w, denoted $\mathsf{D}(w)$, as follows. We use matrix coordinates to describe the position of squares on a $n \times n$ grid. Mark dots on the squares $(i, w(i))$ and cross off all squares to the right and below these dots. Then, $\mathsf{D}(w)$

is the remaining squares. For example, for $w = 51432$, the diagram $\mathsf{D}(w)$ is



The number of boxes in $\mathsf{D}(w)$ is $\ell(w)$, and from $\mathsf{D}(w)$, we can recover a reduced word of w as follows. Number the boxes of $\mathsf{D}(w)$ from left to right where the numbers in row i start at i. Continuing our example, we have



Then read the reduced word by reading top to bottom, right to left: 4321434. We call this word the **canonical reduced word** of w.

The **Lehmer code** of w, denoted $c(w)$, is the weak composition where the ith position is the number of boxes in row i of $\mathsf{D}(w)$. In the example above, $c(w) = (4, 0, 2, 1, 0)$. It is a simple exercise to show that a permutation is uniquely identified by its code [Man01, Proposition 2.1.2].

Given a permutation w and weak composition γ, let $w(\gamma) = (\gamma_{w^{-1}(1)}, \gamma_{w^{-1}(2)}, \ldots, \gamma_{w^{-1}(\ell)})$. For example, if $w = 3142$ and $\gamma = (4, 2, 2, 1)$, then $w(\gamma) = (2, 1, 4, 2)$. If $\lambda(\gamma) = \lambda$, let w_γ be the shortest (in terms of length) permutation such that $w_\gamma(\lambda) = \gamma$.

1.2 Symmetric Functions

The group S_m acts on a polynomial in m variables $f(x_1, x_2, \ldots, x_m)$ by

$$w \cdot f(x_1, x_2, \ldots, x_m) = f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(m)}).$$

The polynomial f is **symmetric** if $w \cdot f = f$ for all $w \in S_m$, or equivalently f is symmetric if it is invariant under permuting x_i and x_{i+1} for all i. Then, Sym_m^n is the vector space of homogeneous symmetric polynomials of degree n in m variables. For example,

$$f(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \in \mathsf{Sym}_3^3.$$

The ring of symmetric polynomials in m variables is

$$\mathsf{Sym}_m = \bigoplus_{n=0}^{\infty} \mathsf{Sym}_m^n.$$

For each $m \geq 0$, the map

$$r_m : \mathsf{Sym}_{m+1}^n \to \mathsf{Sym}_m^n$$

that sets $x_{n+1} = 0$ is a surjective homomorphism. Then, we define

$$\mathsf{Sym}^n = \varprojlim_m \mathsf{Sym}_m^n,$$

the vector space of homogeneous symmetric functions of degree n. Finally, the ring of symmetric functions is

$$\mathsf{Sym} = \bigoplus_{n=0}^{\infty} \mathsf{Sym}^n,$$

or symmetric formal power series of bounded degree.

1.2.1 Bases of Symmetric Functions

In this section we introduce four bases of Sym^n, indexed by $\{\lambda \in \mathsf{Par}(n)\}$.

Definition 1.6. *The monomial symmetric function m_λ is*

$$m_\lambda = \sum_{\gamma \in \mathsf{PermutWC}(\lambda)} \mathbf{x}^\gamma.$$

For example,

$$\begin{aligned}
m_{(1)} &= x_1 + x_2 + x_3 + \dots \\
m_{(1,1)} &= x_1x_2 + x_1x_3 + x_2x_3 + \dots \\
m_{(3,1)} &= x_1^3x_2 + x_1x_2^3 + x_1^3x_3 + x_1x_3^3 + x_2^3x_3 + x_2x_3^3 + \dots.
\end{aligned}$$

By definition, if f is a symmetric function,

$$[\mathbf{x}^{w(\lambda)}]f = [\mathbf{x}^{\lambda}]f$$

for all $w \in S_\infty$. Thus f can be expressed as a finite sum of the monomial symmetric functions. Furthermore, since each monomial appears in exactly one m_λ, $\{m_\lambda : \lambda \in \mathsf{Par}(n)\}$ is a basis of Sym^n.

Definition 1.7. *The elementary symmetric function e_λ is defined*

$$e_k = m_{1^k} = \sum_{\substack{(i_1,\ldots,i_k)\in\mathbb{Z}^k_{>0} \\ i_1<i_2<\ldots<i_k}} x_{i_1}x_{i_2}\cdots x_{i_k}$$

and then

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_\ell}.$$

For example,

$$\begin{aligned}
e_{(1)} &= x_1 + x_2 + x_3 + \ldots && = m_{(1)} \\
e_{(1,1)} &= (x_1 + x_2 + x_3 + \ldots)^2 && = m_{(2)} + 2m_{(1,1)} \\
e_{(3)} &= x_1x_2x_3 + \ldots && = m_{(1,1,1)} \\
e_{(3,1)} &= (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + \ldots)(x_1 + x_2 + x_3 + \ldots) && = m_{(2,1,1)} + 4m_{(1,1,1,1).}
\end{aligned}$$

The fundamental theorem of symmetric functions states that $\{e_\lambda : \lambda \vdash n\}$ is a basis of Sym^n [Sta99a, Theorem 7.4.4].

Definition 1.8. *The homogeneous symmetric function h_λ is defined*

$$h_k = \sum_{\substack{(i_1,\ldots,i_k)\in\mathbb{Z}^k_{>0} \\ i_1\le i_2\le\ldots\le i_k}} x_{i_1}x_{i_2}\cdots x_{i_k} = \sum_{\lambda\in\mathsf{Par}(k)} m_\lambda$$

and then

$$h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_\ell}.$$

For example,

$$\begin{aligned}
h_{(1)} &= x_1 + x_2 + x_3 + \dots &&= m_{(1)} \\
h_{(1,1)} &= (x_1 + x_2 + x_3 + \dots)^2 &&= m_{(2)} + 2m_{(1,1)} \\
h_{(3)} &= x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + \dots &&= m_{(3)} + m_{(2,1)} + m_{(1,1,1)} \\
h_{(3,1)} &= h_{(3)} h_{(1)} &&= m_{(4)} + 2m_{(3,1)} + 2m_{(2,2)} + 3m_{(2,1,1)} + 4m_{(1,1,1,1)}.
\end{aligned}$$

Definition 1.9. *The power sum symmetric function p_λ is defined*

$$p_k = m_k = \sum_{i=1}^{\infty} x_i^k$$

and then

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

For example,

$$\begin{aligned}
p_{(1)} &= x_1 + x_2 + x_3 + \dots &&= m_{(1)} \\
p_{(1,1)} &= (x_1 + x_2 + x_3 + \dots)^2 &&= m_{(2)} + 2m_{(1,1)} \\
p_{(3)} &= x_1^3 + x_2^3 + x_3^3 + \dots &&= m_{(3)} \\
p_{(3,1)} &= (x_1^3 + x_2^3 + x_3^3 + \dots)(x_1 + x_2 + x_3 + \dots) &&= m_{(4)} + m_{(3,1)}.
\end{aligned}$$

1.2.2 The Schur Basis

The Schur basis of symmetric functions provides a critical link between symmetric functions and other branches of mathematics such as Schubert calculus and representation theory of GL_n and S_n; see, e.g., the textbook [Ful97]. We will describe the connection between Schur functions and Schubert calculus in Chapter 5 where we include some new conjectures on Schubert calculus.

A **Young tableau** is a filling of each box of the Young diagram with positive integers.

For example,

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&5\\ \hline 2&5\\ \cline{1-2} 5&6\\ \cline{1-2}\end{array}.$$

A Young tableau is **semistandard** if the rows are weakly increasing and the columns are strictly increasing, as in the tableau above. Furthermore a semistandard Young tableau is **standard** if it uses the numbers $1, \ldots, n$ exactly once. For a fixed partition λ, we denote the set of semistandard Young tableau of shape λ by $\mathsf{SSYT}(\lambda)$.

The **content** of a tableau T is the weak composition $c(T) = (c_1, c_2, \ldots, c_m)$ where c_i is the number of i's in T. To a tableau T, we associate a monomial $\mathbf{x}^T = \mathbf{x}^{c(T)} = x_1^{c_1} x_2^{c_2} \cdots x_m^{c_m}$. In the example above, the content is $(2, 2, 0, 0, 3, 1)$ and the monomial is $x_1^2 x_2^2 x_5^3 x_6$.

Definition 1.10. *The Schur function s_λ is*

$$s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^T.$$

For example,

$$\mathsf{SSYT}((1)) = \left\{ \begin{array}{|c|}\hline 1\\ \hline\end{array}, \begin{array}{|c|}\hline 2\\ \hline\end{array}, \begin{array}{|c|}\hline 3\\ \hline\end{array}, \ldots \right\}$$

$$\mathsf{SSYT}((1,1)) = \left\{ \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}, \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}, \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}, \ldots \right\}$$

$$\mathsf{SSYT}((3)) = \left\{ \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline\end{array}, \right.$$
$$\left. \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&3\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 2&2&2\\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 2&2&3\\ \hline\end{array}, \ldots \right\}$$

and thus

$$\begin{aligned}
s_{(1)} &= x_1 + x_2 + x_3 + \ldots &&= m_{(1)}\\
s_{(1,1)} &= x_1x_2 + x_1x_3 + x_2x_3 + \ldots &&= m_{(1,1)}\\
s_{(3)} &= x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2^3 + \ldots &&= m_{(3)} + m_{(2,1)} + m_{(1,1,1)}.
\end{aligned}$$

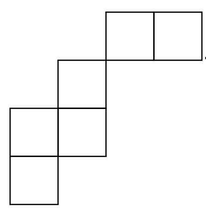
The Schur functions expand combinatorially in the monomial basis

$$s_\lambda = \sum_\mu \mathsf{K}_{\lambda,\mu} m_\mu \tag{1.1}$$

where $\mathsf{K}_{\lambda,\mu}$ is the number of semistandard Young tableaux of shape λ and content μ. The numbers $\mathsf{K}_{\lambda,\mu}$ are called the **Kostka coefficients**. The vanishing of $\mathsf{K}_{\lambda,\mu}$ is governed by dominance order [Man01, Exercise 1.2.11] as

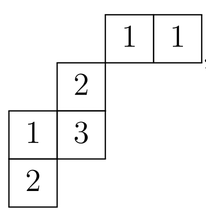
$$\mathsf{K}_{\lambda,\mu} \neq 0 \text{ if and only if } \mu \leq_D \lambda.$$

A partition λ **contains** a partition μ, denoted $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all i. In this case, we represent the skew-shape λ/μ by removing the boxes of μ from the Young diagram of λ when the northwest corners of λ and μ coincide. For example, $(2,1) \subseteq (4,2,2,1)$ and the skew-shape $(4,2,2,1)/(2,1)$ is depicted



A **tableau** on a skew-shape is a filling of the remaining boxes with positive integers, and again such a tableau is **semistandard** if the rows are weakly increasing and the columns are strictly decreasing.

The **reading word** of a tableau T, denoted $\mathsf{read}(T)$ is the word formed by reading the entries of T by rows top to bottom, right to left. For example, for the semistandard skew tableau

$T =$ 

$\mathsf{read}(T) = 112312$. A word w is **lattice** if for all i, all initial segments of w contain at least as many i's as $i+1$'s. In this example, $\mathsf{read}(T)$ is lattice but the word 112332 is not lattice because the initial segment 11233 contains two 3's but only one 2. A tableau is **lattice** if $\mathsf{read}(T)$ is lattice.

The Schur functions have nonnegative **structure constants**, i.e.

$$s_\lambda s_\mu = \sum_{\nu \in \mathsf{Par}(|\lambda|+|\mu|)} \mathsf{LR}^{\nu}_{\lambda,\mu} s_\nu$$

with $\mathsf{LR}^{\nu}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$.

Theorem 1.11. *The* **Littlewood-Richardson coefficient** $\mathsf{LR}^{\nu}_{\lambda,\mu}$ *is equal to the number of lattice, semistandard Young tableaux of skew-shape* ν/λ *and content* μ*.*

Theorem 1.11 is known as the **Littlewood-Richardson rule** for the Schur functions s_λ. For example, let $\lambda = (3,1), \mu = (4,2,1)$, and $\nu = (6,3,1,1)$. Then $\mathsf{LR}^{\nu}_{\lambda,\mu} = 2$ and the two witnessing fillings are

$$\begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 1 & 2 & & & \\ 2 & & & & & \\ 3 & & & & & \end{array} \quad \text{and} \quad \begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 2 & 2 & & & \\ 1 & & & & & \\ 3 & & & & & \end{array}.$$

1.3 Quasisymmetric Functions

The ring of symmetric polynomials in m variables $\mathsf{Sym_m}$ embeds inside the ring of quasisymmetric polynomials in m variables, denoted $\mathsf{QSym_m}$. A polynomial $f(x_1, x_2, \ldots, x_m)$ is **quasisymmetric** if for all increasing sequences of positive integers $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ and for all compositions with k parts $(\alpha_1, \alpha_2, \ldots, \alpha_k)$,

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}]f = [x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}]f.$$

Equivalently, consider the following action of s_i on a monomial:

$$s_i \curvearrowright \mathbf{x}^\gamma = \begin{cases} \mathbf{x}^{s_i(\gamma)} & \text{if } \gamma_i = 0 \text{ or } \gamma_{i+1} = 0 \\ \mathbf{x}^\gamma & \text{if } \gamma_i \neq 0 \text{ and } \gamma_{i+1} \neq 0 \end{cases}. \tag{1.2}$$

This extends to an action on polynomials and $f(x_1, \ldots, x_m)$ is quasisymmetric if and only if it is invariant under the action of $\hat{s}_i$ for $i = 1, \ldots, m-1$. We define QSym^n_m to be the vector space of homogeneous quasisymmetric polynomials of degree n in m variables. For example,

$$f(x_1, x_2, x_3) = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 \in \mathsf{QSym}^3_3.$$

The ring of quasisymmetric polynomials in m variables is

$$\mathsf{QSym}_m = \bigoplus_{n=0}^{\infty} \mathsf{QSym}_m^n.$$

For each $m \geq 0$, the map

$$r_m : \mathsf{QSym}_{m+1}^n \to \mathsf{QSym}_m^n$$

that sets $x_{n+1} = 0$ is a surjective homomorphism. Thus, we can define

$$\mathsf{QSym}^n = \varprojlim_m \mathsf{QSym}_m^n,$$

the vector space of homogeneous quasisymmetric functions of degree n. Finally, the ring of quasisymmetric functions is

$$\mathsf{QSym} = \bigoplus_{n=0}^{\infty} \mathsf{QSym}^n,$$

or quasisymmetric formal power series of bounded degree.

The study of quasisymmetric functions dates back to the thesis of R. P. Stanley [Sta71] and subsequent work by I. Gessel in [Ges84]. We define two bases of QSym^n, both indexed by the set of compositions of n.

Definition 1.12. *The monomial quasisymmetric function M_α is*

$$M_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} \mathbf{x}^\gamma.$$

For example,

$$\begin{aligned}
M_{(1)} &= x_1 + x_2 + x_3 + \dots \\
M_{(1,1)} &= x_1x_2 + x_1x_3 + x_2x_3 + \dots \\
M_{(3)} &= x_1^3 + x_2^3 + x_3^3 + \dots \\
M_{(3,1)} &= x_1^3x_2 + x_1^3x_3 + x_2^3x_3 + \dots \\
M_{(1,3)} &= x_1x_2^3 + x_1x_3^3 + x_2x_3^3 + \dots .
\end{aligned}$$

Recall from Section 1.1 that given a composition α, the partition $\lambda(\alpha)$ is the partition formed by sorting the parts of α in weakly decreasing order and $\mathsf{PermutC}(\lambda)$ is the set of all

α such that $\lambda(\alpha) = \lambda$. Since by the definitions of both,

$$m_\lambda = \sum_{\alpha \in \mathsf{PermutC}(\lambda)} M_\alpha,$$

$\{M_\alpha\}$ is a quasisymmetric refinement of $\{m_\lambda\}$.

Compositions of n are in bijection with subsets of $[n-1] = \{1, 2, \ldots, n-1\}$ where

$$\mathsf{set}((\alpha_1, \alpha_2, \ldots, \alpha_\ell)) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \ldots + \alpha_{\ell-1}\}.$$

For example $\mathsf{set}((1,2,2,1)) = \{1,3,5\}$.

Definition 1.13 (Gessel, pg. 291 [Ges84])**.** *For $|\alpha| = n$, Gessel's fundamental quasisymmetric function F_α is*

$$F_\alpha = \sum_{\substack{(i_1,\ldots,i_n) \in \mathbb{Z}^n_{>0} \\ i_1 \leq i_2 \leq \ldots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in \mathsf{set}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For example,

$$\begin{aligned} F_{(1)} &= x_1 + x_2 + x_3 + \ldots &&= M_{(1)} \\ F_{(1,1)} &= x_1x_2 + x_1x_3 + x_2x_3 + \ldots &&= M_{(1,1)} \\ F_{(3)} &= x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2x_3 + \ldots &&= M_{(3)} + M_{(2,1)} + M_{(1,2)} + M_{(1,1,1)} \\ F_{(1,1,2)} &= x_1x_2x_3^2 + x_1x_2x_3x_4 + x_1x_3x_4^2 + x_1x_3x_4x_5 + \ldots &&= M_{(1,1,2)} + M_{(1,1,1,1)}. \end{aligned}$$

The composition β is a **refinement** of the composition α, denoted $\alpha \succeq \beta$, if α can be obtained by summing consecutive parts of β. For example, $(4,1,3,2) \succeq (3,1,1,1,2,1,1)$. The fundamental quasisymmetric basis expands combinatorially in the monomial quasisymmetric basis [Ges84, Equation 2]:

$$F_\alpha = \sum_{\substack{\beta \in \mathsf{Comp}(|\alpha|) \\ \alpha \succeq \beta}} M_\beta. \tag{1.3}$$

1.4 Polynomials

Finally, the ring of quasisymmetric polynomials in m variables $\mathsf{QSym_m}$ embeds inside of $\mathsf{Pol_m}$, the ring of polynomials in m variables. We denote the ring of polynomials in arbitrarily many variables as Pol. Bases of Pol are typically indexed by weak compositions.

Definition 1.14. *The monomial basis of* Pol *is*

$$\mathfrak{m}_\gamma = \mathbf{x}^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_\ell^{\gamma_\ell}.$$

Recall that for a weak composition γ, the composition γ^+ is formed by removing the parts of size 0 from γ. It is clear $\{\mathfrak{m}_\gamma\}$ is a polynomial refinement of $\{M_\alpha\}$ (and therefore $\{m_\lambda\}$) as

$$M_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} \mathfrak{m}_\gamma.$$

The **Schubert polynomials** are a basis of Pol that generalize the Schur functions and were introduced by A. Lascoux and M.-P. Schützenberger [LS82a]. As seen in Chapter 5, each Schur function represents the cohomology of a Schubert variety in the Grassmannian. Analogously, each Schubert polynomial represents the cohomology of a Schubert variety in the flag manifold. Recall from Section 1.1 that permutations are uniquely determined by their Lehmer code and thus Schubert polynomials are often indexed by permutations.

A. Lascoux and M.-P. Schützenberger define the Schubert polynomials recursively using **divided difference operators**. Let

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}$$

and let $w_0 \in S_n$ be the longest permutation in S_n, i.e. $w_0 = n \ldots 21$.

Definition 1.15 (Lascoux-Schützenberger [LS82a])**.** *The Schubert polynomial* $\mathfrak{S}_{w_0}$ *is*

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For $w \neq w_0$*, there exists* i *such that* $w(i) < w(i+1)$*, and the Schubert polynomial* $\mathfrak{S}_w$ *is*

$$\mathfrak{S}_w = \partial_i \mathfrak{S}_{w s_i}.$$

For example,

$$\begin{aligned}
\mathfrak{S}_{54321} &= x_1^4x_2^3x_3^2x_4 \\
\partial_4\mathfrak{S}_{54321} =& \mathfrak{S}_{54312} = x_1^4x_2^3x_3^2 \\
\partial_3\mathfrak{S}_{54312} =& \mathfrak{S}_{54132} = x_1^4x_2^3x_3 + x_1^4x_2^3x_4 \\
\partial_2\mathfrak{S}_{54132} =& \mathfrak{S}_{51432} = x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_2^2x_4 + x_1^4x_2x_3x_4 + x_1^4x_3^2x_4.
\end{aligned}$$

It is straightforward to check that the divided difference operators satisfy the commutation and braid relations of the simple transpositions. We can therefore define $\partial_w = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}$ where $i_1i_2\cdots i_\ell$ is any reduced word of w. The Schubert polynomial is then

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}x_1^{n-1}x_2^{n-2}\cdots x_{n-1}.$$

For more details see [Man01, Definition 2.3.4] and the discussion immediately preceding it.

We now discuss some properties of the Schubert polynomials. The Schubert polynomials are stable, i.e. for $w \in S_n$ and $\iota : S_n \hookrightarrow S_{n+1}$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$, and thus we can consider w as a member of S_∞. Furthermore, the Schubert polynomials expand combinatorially in the monomial basis.

Moreover, the Schubert polynomials are a lift of the Schur functions from Sym to Pol. A **descent** of a permutation is i such that $w(i) > w(i+1)$. A permutation is said to be **Grassmannian** if it has at most one descent. For example, $w = 51432$ has descents at 1,3, and 4 and so is not Grassmannian, but $w = 1347256$ is Grassmannian because the only descent is at 4. Equivalently, a permutation w is Grassmannian if and only if $c(w)^*$ is a partition. In this case,

$$\mathfrak{S}_w = s_{c(w)^*}(x_1, \ldots, x_k) \tag{1.4}$$

where k is the position of the descent of w.

Finally, products of Schubert polynomials expand combinatorially in the Schubert basis. For $u \in S_m$ and $v \in S_n$,

$$\mathfrak{S}_u\mathfrak{S}_v = \sum_{w \in S_{m+n}} c_{u,v}^w \mathfrak{S}_w.$$

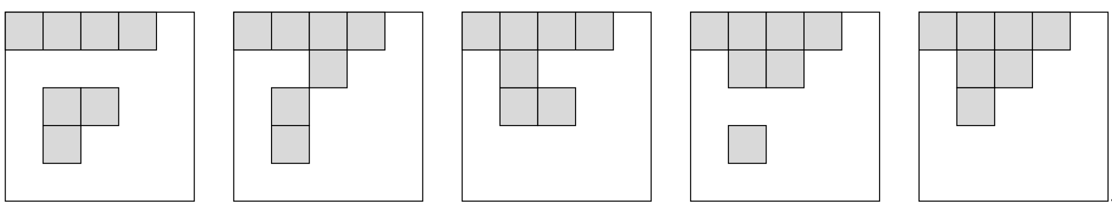
While $\{c_{u,v}^w\}$ are known to be nonnegative for geometric reasons, giving a counting rule for $\{c_{u,v}^w\}$ is a well-known open problem [Sta99b, Problem 11].

There are many combinatorial rules that describe the monomial expansion of the Schubert polynomials, including reduced words and compatible sequences [BJS93], RC graphs [BB93], strand diagrams [FK96], and many others. The first combinatorial rule was conjectured by A. Kohnert [Koh91]. A proof was given by R. Winkel [Win03] and a new proof has been given by S. Assaf [Ass17].

A diagram is a subset of boxes in the $n \times n$ grid, and a Kohnert move in a diagram moves the rightmost box of any row up to the next available row (jumping boxes if necessary). Let $\mathsf{Koh}(w)$ be the set of all diagrams generated by Kohnert moves from $\mathsf{D}(w)$ where diagrams are not included with multiplicity even though there might be multiple sequences of moves to a particular diagram. The **content** of a diagram is $c(\mathsf{D}) = (c_1, c_2, \ldots, c_n)$ where c_i is the number of boxes in row i of D. We then have

$$\mathfrak{S}_w = \sum_{\mathsf{D} \in \mathsf{Koh}(w)} \mathbf{x}^{c(\mathsf{D})} \tag{1.5}$$

where (1.5) is [Win03, Theorem 2] and [Ass17, Corollary 6.8]. Continuing our example with $w = 51432$, we have the following diagrams giving us the Schubert polynomial above:


.

None of our results will rely on Kohnert's rule, however, we include it here due to its connection to the combinatorics of Chapter 2.

1.5 Combinatorics of Macdonald Polynomials

The Macdonald polynomials are families of symmetric polynomials introduced by I. G. Macdonald [Mac88]. For a more detailed reference, see the book [Mac95b].

Recall from Section 1.1 that for a partition λ, the number of parts of size i of λ is $m_i(\lambda)$. Then define

$$z_\lambda = \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!.$$

The standard inner product on symmetric functions is

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$$

where

$$\delta_{\lambda,\mu} = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}.$$

The Schur functions are the unique family that satisfy

1. Upper triangular with respect to the monomial basis, i.e. $f_\lambda = m_\lambda + \sum_{\substack{\mu \in \mathsf{Par}(|\lambda|) \\ \mu <_D \lambda}} c_{\lambda,\mu} m_\mu$ and
2. Orthogonal with respect to the inner product, i.e. $\langle f_\lambda, f_\mu \rangle = 0$ if $\lambda \neq \mu$.

I. G. Macdonald defined a q, t-analogue of this inner product:

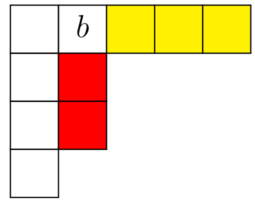
$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \left(\prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \right).$$

Definition 1.16 (Macdonald, Equation 4.7 [Mac95b])**.** *The Macdonald polynomials P_λ are the the unique family satisfying*

1. *Upper triangular with respect to the monomial basis, i.e. $f_\lambda = m_\lambda + \sum_{\substack{\mu \in \mathsf{Par}(|\lambda|) \\ \mu <_D \lambda}} c_{\lambda,\mu} m_\mu$ and*
2. *Orthogonal with respect to the q, t-inner product, i.e. $\langle f_\lambda, f_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$.*

I. G. Macdonald proved such a family exists [Mac88, Theorem 2.3]. From this definition it is clear that the Macdonald polynomials are a q, t-generalization of the Schur functions since when $q = t$, we recover them.

In addition to the original Macdonald polynomials $\{P_\lambda\}$, a few transformations of the Macdonald polynomials are often studied. For a partition λ, suppose $b = (i, j)$ is a box of its Young diagram. Then, $\mathsf{arm}(b) = \#\{k > i : \lambda_k \geq j\}$, or the number strictly boxes below b and $\mathsf{leg}(b) = \lambda_i - j$, or the number of boxes strictly right of b. For example, in the diagram below, the box b is in position (1,2). The arm of b is depicted in red and thus $\mathsf{arm}(b) = 2$, while the leg of b is depicted in yellow and $\mathsf{leg(b)} = 3$.



Definition 1.17 (Macdonald, Equation 8.3 [Mac95b])**.** *The symmetric function J_λ is the integral form of the symmetric Macdonald polynomial P_λ defined*

$$J_\lambda = P_\lambda \prod_{b\in\lambda} 1 - q^{\mathsf{arm}(b)} t^{\mathsf{leg}(b)+1}.$$

The **dual Schur functions** $\hat{s}_\mu$ are the symmetric functions that satisfy

$$\langle s_\lambda, \hat{s}_\mu \rangle_t = \delta_{\lambda,\mu}$$

where the inner product is defined

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda,\mu} z_\lambda \left(\prod_{i=1}^{\ell(\lambda)} \frac{1}{1-t^{\lambda_i}} \right).$$

Then the q,t**-Kostka coefficients** are

$$\mathsf{K}_{\lambda,\mu}(q,t) = [\hat{s}_\mu] J_\lambda.$$

Definition 1.18 (Macdonald; see Proposition 2.4 [Hai99])**.** *The modified Macdonald polynomials H_λ are defined*

$$H_\lambda = \sum_\mu \mathsf{K}_{\lambda,\mu}(q,t) s_\mu$$

and the transformed Macdonald polynomials $\widetilde{H}_\lambda$ are

$$\widetilde{H}_\lambda = t^{\nu(\lambda)} H_\mu \left(\mathbf{x}, q, \frac{1}{t} \right) = \sum_\mu \tilde{\mathsf{K}}_{\lambda,\mu}(q,t) s_\mu$$

where $\nu(\lambda) = \sum_{i=1}^{\ell(\lambda)} \lambda_i (i-1)$ *and* $\tilde{\mathsf{K}}_{\lambda,\mu} = t^{\nu(\lambda)} \mathsf{K}_{\lambda,\mu}(q, \frac{1}{t})$.

J. Haglund, M. Haiman, and N. Loehr [HHL05] define a **filling** F as an assignment of the boxes of λ with positive integers. The content of F, denoted $c(F)$ is the weak composition counting the number of i's in F and $\mathbf{x}^F = \mathbf{x}^{c(F)}$. A **descent** of F is a box that is filled with an integer strictly greater than the integer immediately to its left. Let $\mathsf{Des}(F)$ be the set of descents of F.

Two cells are said to be **attacking** if they are in the same column, or if they are in adjacent columns and the left box is strictly above the right box. The two kinds of attacking cells are

$$\begin{array}{|c|c|c|c|}\hline & a & & \\ \hline & & & \\ \hline & b & \\ \cline{1-3} & \\ \cline{1-2} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline & b & & \\ \hline & & & \\ \hline & & a \\ \cline{1-3} & \\ \cline{1-2} \end{array}.$$

A pair of attacking cells is an **inversion** if $a > b$ and let $\mathsf{Inv}(F)$ be the set of inversions of F. Then, given the definitions of $\mathsf{arm}(b)$ and $\mathsf{leg}(b)$ above

$$\mathsf{maj}(F) = \sum_{b \in \mathsf{Des}(F)} (\mathsf{leg}(b) + 1)$$

and

$$\mathsf{inv}(F) = |\mathsf{Inv}(F)| - \sum_{b \in \mathsf{Des}(F)} \mathsf{arm}(b).$$

For example, for the filling

$$\begin{array}{|c|c|c|c|}\hline 4 & 4 & 2 & 6 \\ \hline 2 & 3 & 2 & 1 \\ \hline 6 & 4 & 4 \\ \cline{1-3} 6 & 3 \\ \cline{1-2} \end{array}$$

there are two descents: the 6 in the first row and the 3 in the second. Therefore

$$\mathsf{maj}(F) = (1 + 0) + (1 + 2) = 4.$$

There are 8 inversions: (1) the 4 and 2 in column one, (2-3) the first 4 in column 2 with both 3's in column two, (4) the second 4 in column 2 with the second 3 in column 2, (5) the 6 and 1 in column four, (6-7) the 2 in column 1 with the second 4 and 3 in column 2, and

(8) the 3 in column 2 and the 4 in column 3. Thus

$$\mathsf{inv}(F) = 8 - (1+2) = 5.$$

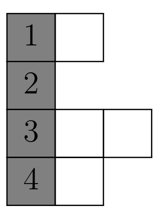
Theorem 1.19 (Haglund-Haiman-Loehr, Theorem 2.2 [HHL05])**.**

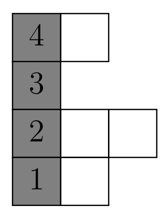
$$\widetilde{H}_\lambda = \sum_F q^{\mathsf{inv}(F)} t^{\mathsf{maj}(F)} \mathbf{x}^F$$

where the sum is over all fillings F of λ.

In addition to the symmetric Macdonald polynomials, the **nonsymmetric Macdonald polyomials**, introduced and studied by I. Cherednik [Che95], I. G. Macdonald [Mac95a], and E. Opdam [Opd95], provide a refinement of the symmetric Macdonald polynomials; see [HHL08, Proposition 5.3.1]. The nonsymmetric Macdonald polynomials $E_\gamma(\mathbf{x}; q, t)$ are indexed by weak compositions and are elements of $\mathbb{Q}[x_1, \ldots](q, t)$. In [HHL08], J. Haglund, M. Haiman, and N. Loehr give a combinatorial rule for the nonsymmetric Macdonald polynomials involving skyline fillings that has similarities to their rule in the symmetric case.

The **skyline diagram** for γ with **basement** $\mathbf{b} = (b_1, \ldots, b_k)$ consists of k left-justified rows with γ_i boxes in row i, plus an additional column 0 containing the value b_i in row i. Let $\mathbf{b}_i = (1, 2, \ldots, \ell(\gamma))$ and $\mathbf{b}_i^* = (\ell(\gamma), \ell(\gamma)-1, \ldots, 1)$. Then the skyline diagram $\gamma = (1, 0, 2, 1)$ and basements $\mathbf{b}_i$ (left) and $\mathbf{b}_i^*$ (right) are shown below.





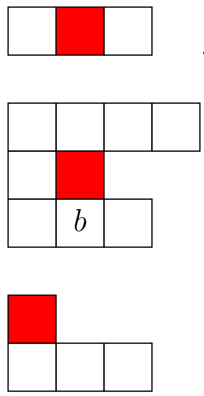
A **skyline filling** is an assignment of positive integers to the boxes of the skyline diagram. Given a filling F, the **content** of F is the weak composition $c(F) = (c_1, c_2, \ldots, c_\ell)$ where c_i is the number of i's in F, excluding any i's in the basement. Then, the **monomial** $\mathbf{x}^F = \mathbf{x}^{c(F)} = x_1^{c_1} x_2^{c_2} \cdots x_\ell^{c_\ell}$, and the **size** of F, denoted $|F|$, is $|c(F)|$. A **descent** of a filling F is a box filled with a number strictly greater than the number to its left and $\mathsf{Des}(F)$ is the set of descents of F.

The **leg** of a box b, denoted $\mathsf{leg}(b)$, is the number of boxes strictly right of b. For $b = (i, j)$, $\mathsf{arm}(b)$ is the number of boxes that are either

- (i', j) with $i' < i$ and $\gamma_{i'} \leq \gamma_i$, or

- $(i', j-1)$ with $i' > i$ and $\gamma_{i'} < \gamma_i$.

When $j = 1$, it is possible for boxes in the arm of b to be in the basement of the skyline diagram. For example, the arm of the box b below is denoted in red:



The two types of **attacking cells** in the nonsymmetric case are

Attacking cells can include cells in the basement. A filling is **non-attacking** if $a \neq b$ for all pairs of attacking cells. Furthermore, a pair of attacking cells is an **inversion** if $a > b$ and we denote the set of inversions of F by $\mathsf{Inv}(F)$. For example, the filling below on the left is attacking because the 2's attack each other but the filling on the right is non-attacking. On the right hand filling, every pair of numbers in the basement and the first column is an inversion. Likewise the 2 in the third column and the 1 in the second is an inversion.

Finally, maj, inv, and coinv are defined:

$$\mathsf{maj}(F) = \sum_{b \in \mathsf{Des}(F)} (\mathsf{leg}(b) + 1)$$

$$\mathsf{inv}(F) = |\mathsf{Inv}(F)| - |\{i < j : \gamma_i \le \gamma_j\}| - \sum_{b \in \mathsf{Des}(F)} \mathsf{arm}(b)$$

$$\mathsf{coinv}(F) = \left(\sum_{b \in F} \mathsf{arm}(\mathsf{b}) \right) - \mathsf{inv}(F).$$

Theorem 1.20 (Haglund-Haiman-Loehr; Theorem 3.5.1 [HHL08])**.**

$$E_\gamma(\mathbf{x}; q, t) = \sum_F \mathbf{x}^F q^{\mathsf{maj}(F)} t^{\mathsf{coinv}(F)} \prod_{\substack{b \in F \\ F(b) \ne F(d(b))}} \frac{1-t}{1 - q^{\mathsf{leg}(b)+1} t^{\mathsf{arm}(b)+1}}$$

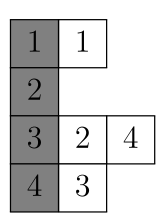
where the sum is over non-attacking fillings F of γ and $d(b)$ is the box immediately left of b.

We will take this combinatorial description as our definition. An alternate description of E_γ (Corollary 3.6.4 [HHL08]) involves **triples**. Triples consist of three boxes on two rows $i < j$. As pictured, there are two types of triples depending on the relative lengths of the rows.

Type A $\qquad\qquad$ Type B

$\gamma_i \ge \gamma_j \qquad\qquad \gamma_i < \gamma_j$

A triple is a **inversion triple** if $c < b < a$, $a \le c < b$, or $b < a \le c$, and otherwise is a **coinversion triple.** For example, in the filling below, the boxes $(3,0), (3,1)$, and $(4,1)$ form a coinversion triple:



In this formulation, the t-statistic provides a weight on the coinversion triples instead of a weight on the inversions.

1.5.1 Specializations of Macdonald Polynomials

Many polynomial families of interest are specializations of either the symmetric or nonsymmetric Macdonald polynomials. We have already seen that the Schur functions are the $q = t$ specializations of P_λ. Furthermore, the **Hall-Littlewood polynomials** [LS78a] are the $q = 0$ specialization of P_λ whereas the **Jack polynomials** [KS96] are the specialization formed by setting $t = q^\alpha$ and letting $q \to 1$.

Of particular interest for us are two specializations of the nonsymmetric Macdonald polynomials: the **Demazure atoms** at $q = t = 0$ and the **key polynomials** at $q = t = \infty$. Just as the Schur polynomials are the characters of the irreducible polynomial representations of GL_n, the key polynomials are the characters of Demazure modules of type A [Dem74]. These polynomials inherit their combinatorics from the Macdonald polynomials; however, they were originally studied by A. Lascoux and M.-P. Schützenberger [LS90] and V. Reiner and M. Shimozono [RS95] using their definition in terms of divided difference operators. Let

$$\pi_i = \partial_i x_i \text{ and } \tilde{\pi}_i = \pi_i - 1.$$

Since π_i and $\tilde{\pi}_i$ satisfy the braid and commutation relations, for a permutation w we can define π_w (resp. $\tilde{\pi}_w$) as $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_\ell}$ (resp. $\tilde{\pi}_{i_1}\tilde{\pi}_{i_2}\cdots\tilde{\pi}_{i_\ell}$) where $i_1 i_2 \cdots i_\ell$ is any reduced word of w.

Definition 1.21 (Lascoux-Schützenberger, [LS90])**.** *The Demazure atom* $\texttt{Atom}_\gamma$ *and the key polynomial* $\texttt{Key}_\gamma$ *are*

$$\texttt{Atom}_\gamma = \tilde{\pi}_{w_\gamma}\mathbf{x}^{\lambda(\gamma)} \qquad\qquad \texttt{Key}_\gamma = \pi_{w_\gamma}\mathbf{x}^{\lambda(\gamma)}.$$

In [LS89], A. Lascoux and M.-P. Schützenberger prove that the Schubert polynomials combinatorially expand in the key polynomials.

Theorem 1.22 (Lascoux-Schützenberger, Theorem 5 [LS89])**.**

$$\mathfrak{S}_w = \sum_T \texttt{Key}_{c(T)}$$

where the sum is over a particular tableaux T, the specifics we do not need here.

Combinatorially, the Demazure atoms and key polynomials are the generating functions for **semistandard** skyline fillings.

Definition 1.23 (Mason, Section 3 [Mas09]). *A skyline filling is semistandard if*

(M1) entries do not repeat in a column,

(M2) rows are weakly decreasing (including the basement), and

(M3) every triple (including those with basement boxes) is an inversion triple.

Let $\mathsf{SkyFill}(\gamma, \mathbf{b})$ be the set of semistandard skyline fillings of shape γ and basement $\mathbf{b}$.

Theorem 1.24 (Mason, Theorem 1.1 [Mas09]). *The Demazure atom* $\mathtt{Atom}_\gamma$ *is*

$$\mathtt{Atom}_\gamma = E_\gamma(\mathbf{x}; 0, 0) = \sum_{F \in \mathsf{SkyFill}(\gamma, \mathbf{b}_i)} \mathbf{x}^F.$$

As an example, the semistandard skyline fillings with basement $\mathbf{b}_i$ and the Demazure atom for the rearrangements of $(2, 1, 0)$ are in Figure 1.1.

Theorem 1.25 (Mason, Theorem 1.2 [Mas09]). *The key polynomial is*

$$\mathtt{Key}_\gamma = \sum_{F \in \mathsf{SkyFill}(\gamma^*, \mathbf{b}_i^*)} \mathbf{x}^F.$$

For example,

$$\mathsf{SkyFill}((2,0,1), \mathbf{b}_i^*) = \left\{ \begin{array}{ccc} 3 & 3 & 3 \\ 2 & & \\ 1 & 1 & \end{array}, \begin{array}{ccc} 3 & 3 & 2 \\ 2 & & \\ 1 & 1 & \end{array}, \begin{array}{ccc} 3 & 3 & 1 \\ 2 & & \\ 1 & 1 & \end{array}, \begin{array}{ccc} 3 & 2 & 2 \\ 2 & & \\ 1 & 1 & \end{array}, \begin{array}{ccc} 3 & 2 & 1 \\ 2 & & \\ 1 & 1 & \end{array} \right\}$$

$$\mathtt{Key}_{(1,0,2)} = x_1 x_3^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2.$$

For two compositions γ and δ we write

$$\gamma \ll \delta \ \text{ if } \lambda(\gamma) = \lambda(\delta) \text{ and } w_\gamma \leq_B w_\delta$$

where $\lambda(\gamma)$, w_γ, and $\leq_B$ are defined in Section 1.1.

$$\mathsf{SkyFill}((2,1,0),\mathbf{b}_i) = \left\{ \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \right\} \qquad \mathsf{SkyFill}((0,2,1),\mathbf{b}_i) = \left\{ \begin{array}{lll} 1 & & \\ 2 & 2 & 2 \\ 3 & 3 & \end{array}, \begin{array}{lll} 1 & & \\ 2 & 2 & 1 \\ 3 & 3 & \end{array} \right\}$$

$$\mathsf{SkyFill}((2,0,1),\mathbf{b}_i) = \left\{ \begin{array}{lll} 1 & 1 & 1 \\ 2 & & \\ 3 & 3 & \end{array} \right\} \qquad \mathsf{SkyFill}((1,0,2),\mathbf{b}_i) = \left\{ \begin{array}{lll} 1 & 1 & \\ 2 & & \\ 3 & 3 & 3 \end{array}, \begin{array}{lll} 1 & 1 & \\ 2 & & \\ 3 & 3 & 2 \end{array} \right\}$$

$$\mathsf{SkyFill}((1,2,0),\mathbf{b}_i) = \left\{ \begin{array}{lll} 1 & 1 & \\ 2 & 2 & 2 \\ 3 & & \end{array} \right\} \qquad \mathsf{SkyFill}((0,1,2),\mathbf{b}_i) = \left\{ \begin{array}{lll} 1 & & \\ 2 & 2 & \\ 3 & 3 & 3 \end{array} \right\}$$

and thus

$$\begin{aligned} \mathtt{Atom}_{(2,1,0)} &= x_1^2 x_2 & \qquad \mathtt{Atom}_{(0,2,1)} &= x_2^2 x_3 + x_1 x_2 x_3 \\ \mathtt{Atom}_{(2,0,1)} &= x_1^2 x_3 & \qquad \mathtt{Atom}_{(1,0,2)} &= x_1 x_3^2 + x_1 x_2 x_3 \\ \mathtt{Atom}_{(1,2,0)} &= x_1 x_2^2 & \qquad \mathtt{Atom}_{(0,1,2)} &= x_2 x_3^2 \end{aligned}$$

Figure 1.1: The set $\mathsf{SkyFill}(\gamma, \mathbf{b}_i)$ and the Demazure atom $\mathtt{Atom}_\gamma$ for γ a rearrangement of $(2,1,0)$.

Theorem 1.26.

$$\mathtt{Key}_\gamma = \sum_{\substack{\delta \in \mathsf{WC}(|\gamma|) \\ \delta \ll \gamma}} \mathtt{Atom}_\delta.$$

In the literature, this decomposition can be found in Section 1 of [Mas09] and a proof is given [Pun16, Lemma 3.5].

In [Mas08], S. Mason gives a combinatorial proof that the Demazure atoms are a polynomial refinement of the Schur functions s_λ:

Theorem 1.27 (Mason, (1.1) [Mas08])**.**

$$s_\lambda = \sum_{\gamma \in \mathsf{PermutWC}(\lambda)} \mathtt{Atom}_\gamma.$$

Theorems 1.26 and 1.27 together show that the key polynomials are a lift of the Schur functions from Sym_m to Pol_m where Sym_m is symmetric polynomials in m variables. That is, when γ is weakly increasing with m parts, then

$$\mathtt{Key}_\gamma = s_{\gamma^*}(x_1, x_2, \ldots, x_m). \tag{1.6}$$

This is because when γ is weakly increasing, $w_\gamma = w_0$, and so we have $\delta \ll \gamma$ for all weak compositions δ such that $\lambda(\delta) = \lambda(\gamma)$ and $\ell(\delta) \leq m$. Thus, the decompositions in Theorems 1.26 and 1.27 are identical.

Recall that a function is quasisymmetric if for all i, it is invariant under the action of s_i defined in (1.2). Suppose F is a filling that contains i's but no $i+1$'s. Let F_i be the filling formed by replacing all i's with $i+1$'s. A model for fillings produces a quasisymmetric function if for every F above, F_i is a valid filling. This will hold if the rules for determining a valid filling consist only of inequalities between the chosen entries in the boxes.

Thus, the Schur function is quasisymmetric (as it is in fact symmetric) since the rules only require the rows to be weakly increasing and the columns to be strictly increasing. The Demazure atom is not quasisymmetric as the basement is fixed at $b_i = i$ and thus is a rule that is not an inequality on the chosen entries.

The **quasisymmetric Schur functions** were introduced by J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg in [HLMvW11a]. Combinatorially, S_α is generated by **semistandard composition tableaux** of shape α which are fillings of α (with no basement) such that

1. entries do not repeat in a column,
2. rows are weakly decreasing,
3. every triple is an inversion triple, and
4. the leftmost column is weakly increasing from top-to-bottom.

Let $\mathsf{SSCT}(\alpha)$ be the set of all semistandard composition tableau of shape α.

Definition 1.28 (Haglund et al., Definitions 4.3 and 5.1 [HLMvW11a])**.** *The quasisymmetric Schur function S_α is*

$$S_\alpha = \sum_{T \in \mathsf{SSCT}(\alpha)} \mathbf{x}^T.$$

For example,

$$\mathsf{SSCT}((1,2)) = \left\{ \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 3 & 3 \\ \hline \end{array}, \dots \right\}$$

$$S_{(1,2)} = x_1x_2^2 + x_1x_3^2 + x_1x_2x_3 + x_2x_3^2 + \dots.$$

In essence, semistandard composition tableau are semistandard skyline fillings with the basement omitted and the first column weakly increasing to account for being able to insert rows of length 0 into a skyline filling. Then, it is clear from the combinatorics that

$$S_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} \mathtt{Atom}_\gamma, \tag{1.7}$$

and thus the Demazure atoms provide a polynomial refinement of the quasisymmetric Schur functions. (In fact, this decomposition was the original definition of S_α [HLMvW11a, Definition 5.1].) Likewise, just as $\{M_\alpha\}$ is a quasisymmetric refinement of $\{m_\lambda\}$, we have that $\{S_\alpha\}$ is a quasisymmetric refinement of $\{s_\lambda\}$:

$$s_\lambda = \sum_{\alpha \in \mathsf{PermutC}(\lambda)} S_\alpha = \sum_{\gamma \in \mathsf{PermutWC}(\lambda)} \mathtt{Atom}_\gamma. \tag{1.8}$$

Finally, the quasisymmetric Schur functions combinatorially expand in the fundamental quasisymmetric functions.

Proposition 1.29 (Haglund et al., Proposition 5.2, [HLMvW11a]).

$$S_\alpha = \sum_T F_{\mathsf{d}(T)}$$

where the sum is over a specific set of standard reverse tableaux and $\mathsf{d}(T)$ *is the composition corresponding to the set* $\{i : \ i+1$ *does not appear strictly left of* i *in* $T\}$.

1.6 K-Theoretic Combinatorics

In [LS82b], A. Lascoux and M.-P. Schützenberger defined the **Grothendieck polynomials**. These polynomials are K-theoretic analogues of the Schubert polynomials in that they represent the K-theory classes of Schubert varieties of the flag variety the same way the Schubert polynomials represent the cohomology classes of Schubert varieties. To help the reader keep track of the relations between all the families of polynomials under consideration, we denote the K-analogue of each basis by merely prepending a 'K' to the notation for that family. Let

$$\tilde{\partial}_i = \partial_i(1+\beta x_{i+1}).$$

Definition 1.30 (Lascoux-Schützenberger [LS82b]). *The Grothendieck polynomial* $K\mathfrak{S}_{w_0}$ *is*

$$K\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}.$$

For $w \neq w_0$*, there exists* i *such that* $w(i) < w(i+1)$*, and then the Grothendieck polynomial* $K\mathfrak{S}_w$ *is*

$$K\mathfrak{S}_w = \tilde{\partial}_i K\mathfrak{S}_{ws_i}.$$

In the original definition of A. Lascoux and M.-P. Schützenberger, $\beta = -1$. The β notation was introduced by S. Fomin and A. Kirillov who gave formulas for $K\mathfrak{S}_w$ that are analogous to those for the Schubert polynomials [FK96].

The Schubert polynomial $\mathfrak{S}_w$ is a homogeneous polynomial of degree $\ell(w)$. It is clear from the definition of $\tilde{\partial}_i$ that a term of degree $\ell(w)+k$ in the $\mathbf{x}$ variables in $K\mathfrak{S}_w$ will have a coefficient of β^k. Then, setting $\beta = 0$ recovers the lowest degree terms of $K\mathfrak{S}_w$, which equal $\mathfrak{S}_w$. Thus $K\mathfrak{S}_w$ is an inhomogeneous deformation of $\mathfrak{S}_w$.

The Grothendieck polynomials corresponding to Grassmannian permutations are the symmetric Grothendieck polynomials Ks_λ and are the K-analogue of the Schur functions. C. Lenart made important contributions to the combinatorics of Ks_λ, including the expansion of Ks_λ into the Schur functions [Len00]. Building on this work, A. Buch opened up the possibility of set-valued combinatorics with a set-valued tableau rule for Ks_λ [Buc02].

A **set-valued tableau** is a filling of the Young diagram with sets of positive integers. We compare sets with $A \leq B$ (resp. $A < B$) if for every $a \in A, b \in B$, $a \leq b$ (resp. $a < b$). Then, a set-valued tableau is **semistandard** if the rows are weakly increasing and the columns are strictly increasing, where we compare sets. Equivalently, a set-valued tableau is semistandard if for any selection of one entry in each box, the resulting tableau is semistandard.

The content, size, and monomial of a set-valued tableau are unchanged from a single-valued tableau. If T is a set-valued tableau of shape λ, then the **excess** of T is $\mathsf{ex}(T) = |T| - |\lambda|$. Let $\mathsf{SetSSYT}(\lambda)$ be the set of all semistandard set-valued tableaux of shape λ.

Theorem 1.31 (Buch, Theorem 3.1 [Buc02])**.**

$$Ks_\lambda = \sum_{T \in \mathsf{SetSSYT}(\lambda)} \beta^{\mathsf{ex}(T)} \mathbf{x}^T.$$

For example,

$$\mathsf{SetSSYT}((2,1)) = \left\{ \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 12 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 23 \end{ytableau}, \begin{ytableau} 1 & 13 \\ 23 \end{ytableau}, \ldots \right\}$$

$$Ks_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + \beta x_1^2 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + \beta x_1 x_2 x_3^2 + \beta^2 x_1^2 x_2 x_3^2 + \ldots.$$

Since a semistandard set-valued tableau with zero excess is just an ordinary semistandard tableau, setting $\beta = 0$ in Ks_λ yields s_λ. In other words, Ks_λ is an inhomogeneous deformation of s_λ.

In recent years, there has been interest in finding "K-theoretic" analogues of many objects within algebraic combinatorics. For example, a (partial) list as follows was given by O. Pechenik and A. Yong in [PY16]. This interest includes elements of the theory of Young tableaux motivated by the symmetric Grothendieck polynomials [Len00, Buc02, BKS$^+$07, TY09b, BS16, GMP$^+$16, HKP$^+$17, LMS16], but this combinatorics is only part of a broader conversation. For example, K-analogues have been studied for Hopf algebras [LP07, PP16, Pat16], cyclic sieving [Pec14, Rho17, PSV16], homomesy [BPS16], longest increasing sequences of random words [TY11], poset edge densities [RTY16], and plane partitions [DPS17, HPPW16].

Of particular interest later in this thesis is the notion of a **genomic tableau** due to O. Pechenik and A. Yong [PY16]. A genomic tableau is a tableau is a tableau T such that each box is filled with a label i_j and for each i, $\{j | i_j \text{ appears in } T\} = \{1, 2, \dots, k_i\}$ for some k_i. A **family** i of labels is the collection $\{i_j\}_{j=1}^{k_i}$, and the **gene** i_j is the set of all labels for fixed i and j. Furthermore, the genes must partition the labels of a particular family as the entries of T are read from left to right, and no gene can contain two boxes in the same row. The **content** of T is the number of genes of each family. Then, a genomic tableau is **semistandard** if it is semistandard when considering only the families of the labels. For example, the genomic tableau below, where each color corresponds to a gene, is semistandard:

$$\begin{array}{|c|c|c|c|}\hline 1_1 & 1_2 & 2_2 & 3_2 \\ \hline 2_1 & 2_2 & 3_1 \\ \cline{1-3} 3_1 \\ \cline{1-1}\end{array}.$$

A genomic tableau is **lattice** if every tableau formed by selecting one element of each gene is lattice.

The Littlewood-Richardson rule for Ks_λ gives a combinatorial rule for the structure constants $\mathsf{KLR}^{\nu}_{\lambda,\mu}$:

$$Ks_\lambda Ks_\mu = \sum_{\nu} \mathsf{KLR}^{\nu \in \mathsf{Par}}_{\lambda,\mu} Ks_\nu.$$

While the first Littlewood-Richardson rule for Ks_λ was given by A. Buch [Buc02], genomic tableaux were introduced to give an equivariant K-theoretic Littlewood-Richardson rule [PY15], which specializes to the K-theoretic case.

Theorem 1.32 (Pechenik-Yong, Theorem 1.4 [PY16])**.** *The coefficient $\mathsf{KLR}^{\nu}_{\lambda,\mu}$ is the number of lattice, semistandard genomic tableaux of shape ν/λ and content μ.*

Another example of a K-analogue is the multi-fundamental functions of T. Lam and P. Pylyavskyy [LP07]. These are the K-analogues of Gessel's fundamental quasisymmetric functions. Recall from Definition 1.13,

$$F_\alpha = \sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n_{>0} \\ i_1 \le i_2 \le \dots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathsf{set}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

T. Lam and P. Pylyavskyy's definition of the multi-fundamental KF_α replaces the single integer i_j with a subset of integers S_{i_j}.

Definition 1.33 (Lam-Pylyavskyy, Definition 5.4 [LP07]). *The multi-fundamental function KF_α for $|\alpha| = n$ is defined*

$$KF_\alpha = \sum_{\substack{S_{i_1} \leq S_{i_2} \leq \ldots \leq S_{i_n} \\ S_{i_j} < S_{i_{j+1}} \text{ if } j \in \mathsf{set}(\alpha)}} \beta^{\mathsf{ex}(S_{i_1}, \ldots, S_{i_n})} \mathbf{x}^{S_{i_1}} \mathbf{x}^{S_{i_2}} \cdots \mathbf{x}^{S_{i_n}}$$

where each S_i is a finite, nonempty subset of $\mathbb{Z}_{>0}$, $\mathbf{x}^S = \prod_{i \in S} x_i$ and $\mathsf{ex}(S_{i_1}, \ldots, S_{i_n}) = \sum_{j=1}^{n} \left(|S_{i_j}| - 1\right)$.

For example, for $\alpha = (1, 2)$, $\mathsf{set}(\alpha) = \{1\}$ and thus admissible sequences of sets are of the form (S_1, S_2, S_3) where $S_1 < S_2 \leq S_3$. Some possible examples are

$$\{(1, 2, 2), (1, 2, 3), (1, 23, 3), (12, 3, 34), \ldots\}.$$

Again, the lowest degree terms occur when $|S_i| = 1$ for all i. In this case, we recover the definition of F_α and thus KF_α is an inhomogeneous deformation of F_α.

In [Las01], A. Lascoux modifies the divided difference operators used to define the Demazure atoms and key polynomials in order to define the Lascoux atoms and Lascoux polynomials. Let

$$\tau_i = \pi_i(1 + \beta x_{i+1}) \text{ and } \hat{\tau}_i = \tau_i - 1.$$

Definition 1.34 (Lascoux [Las01]). *The Lascoux polynomial $\widehat{K\mathtt{Key}}_\gamma$ is*

$$\widehat{K\mathtt{Key}}_\gamma = \tau_{w_\gamma} \mathbf{x}^{\lambda(\gamma)}$$

while the Lascoux atom $\widehat{K\mathtt{Atom}}_\gamma$ is

$$\widehat{K\mathtt{Atom}}_\gamma = \hat{\tau}_{w_\gamma} \mathbf{x}^{\lambda(\gamma)}.$$

Like with the Grothendieck functions $K\mathfrak{S}_w$, we can see from the operator definition that β is merely tracking the excess degree of each term and thus $\widehat{K\mathtt{Atom}}_\gamma$ is an inhomogeneous deformation of $\mathtt{Atom}_\gamma$ and $\widehat{K\mathtt{Key}}_\gamma$ is an inhomogeneous deformation of $\mathtt{Key}_\gamma$. It is open to prove a combinatorial rule describing these polynomials. A conjectural description for $\widehat{K\mathtt{Key}}_\gamma$ using a generalization of Kohnert's rule was given by C. Ross and A. Yong [RY15].

1.7 Organization

In Chapter 2, we introduce the notion of saturated Newton polytope (SNP), a property describing polynomials. We discuss its prevalence in algebraic combinatorics by evaluating whether the families of polynomials discussed in this chapter, among others, are SNP. We also describe the Newton polytopes of several families in algebraic combinatorics. In Chapter 3, we will give another conjectural description of the Lascoux atoms and polynomials by generalizing the skyline fillings of Section 1.5. We will then use this combinatorial model to define K-analogues of the quasisymmetric Schur functions, and show that these inhomogeneous deformations behave in many ways like their homogeneous counterparts. In Chapter 4, we continue this work on set-valued skyline fillings and inhomogeneous deformations and introduce two new bases of polynomials. We show how these new bases fit into the existing relationships between K-analogues. Finally, in chapter 5, we give some new conjectures on Schubert calculus.

CHAPTER 2

Saturated Newton Polytopes

This chapter is derived from joint work with N. Tokcan and A. Yong that appears on the arXiv [MTY17].

2.1 Introduction

The **Newton polytope** of a polynomial $f = \sum_{\gamma \in \mathbb{Z}^n_{\geq 0}} c_\gamma \mathbf{x}^\gamma \in \mathbb{C}[x_1, \ldots, x_n]$ is the convex hull of its exponent vectors, i.e.,

$$\mathsf{Newton}(f) = \mathrm{conv}(\{\gamma : c_\gamma \neq 0\}) \subseteq \mathbb{R}^n.$$

Definition 2.1. *The polynomial f has saturated Newton polytope (SNP) if $c_\gamma \neq 0$ whenever $\gamma \in \mathsf{Newton}(f)$.*

Example 2.2. Let f be the determinant of a generic $n \times n$ matrix. The exponent vectors correspond to permutation matrices. Then, $\mathsf{Newton}(f)$ is the **Birkhoff polytope** of $n \times n$ doubly stochastic matrices. SNPness says there are no additional lattice points, which is obvious here. (The Birkhoff-von Neumann theorem states all lattice points are vertices.) $\square$

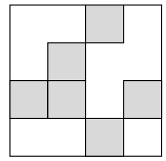
Generally, polynomials are not SNP. Worse still, SNP is not preserved by basic polynomial operations. For example, $f = x_1^2 + x_2x_3 + x_2x_4 + x_3x_4$ is SNP but f^2 is not (it misses $x_1x_2x_3x_4$). Nevertheless, there are a number of families of polynomials in algebraic combinatorics where every member is (conjecturally) SNP. Examples motivating our investigation include:

- The *Schur polynomials* are SNP. This rephrases R. Rado's theorem [Rad52] about permutahedra and dominance order on partitions (cf. Proposition 2.7).

- Classical *resultants* are SNP (Theorem 2.23). Their Newton polytopes were studied by I. M. Gelfand-M. Kapranov-A. Zelevinsky [GKZ90]. Classical *discriminants* are SNP up to quartics — but not quintics (Proposition 2.26).
- *Cycle index polynomials* from Redfield–Pólya theory (Theorem 2.33)
- C. Reutenauer's symmetric polynomials [Reu95] linked to the free Lie algebra and to Witt vectors (Theorem 2.35)
- J. R. Stembridge's symmetric polynomials [Ste91] associated to totally nonnegative matrices (Theorem 2.31)
- R. P. Stanley's symmetric polynomials [Sta84], introduced to enumerate reduced words of permutations (Theorem 2.71)
- Generic (q,t)-evaluation of *symmetric Macdonald polynomials* (Theorem 2.40, Proposition 2.45)
- *Key polynomials* (Conjecture 2.49). We give two conjectural descriptions of the Newton polytopes (Conjectures 2.48 and 2.50). We determine a list of vertices of the Newton polytopes (Theorem 2.51) and conjecture this list is complete (Conjecture 2.52).
- *Schubert polynomials* (Conjecture 2.63). We conjectured (Conjecture 2.76) a description of the Newton polytopes in terms of the Schubitope, which we introduce below.
- Grothendieck and Lascoux polynomials are also conjecturally SNP (Conjectures 2.66 and 2.69).

In more recent developments, some of these conjectures have been fully or partially resolved. In [FMS17], A. Fink, K. Mészáros, and A. St. Dizier prove that the keys and the Schuberts are SNP (Corollary 8). They also prove our conjectural descriptions of their Newton polytopes (Theorem 10). Additionally, the Grothendieck polynomials have been proven to be SNP in two cases: in the symmetric case by L. Escobar and A. Yong [EY17] and in the case that $w = 1\pi$ where π is a dominant permutation by K. Mészáros and A. St. Dizier (Theorem C [MS17]).

A core part of our study concerns the Schubitopes, a new family of polytopes. A **diagram** D is a subset boxes of an $n \times n$ grid as in the diagram below:



Fix $S \subseteq [n] = \{1, 2, \ldots, n\}$ and a column $c \in [n]$. Let $\mathsf{word}_{c,S}(\mathsf{D})$ be formed by reading c from top to bottom and recording

- (if $(r, c) \notin \mathsf{D}$ and $r \in S$,
-) if $(r, c) \in \mathsf{D}$ and $r \notin S$, and
- $\star$ if $(r, c) \in \mathsf{D}$ and $r \in S$.

Let

$$\theta_{\mathsf{D}}^{c}(S) = \#\text{paired ( )'s in } \mathsf{word}_{c,S}(\mathsf{D}) + \#\star\text{'s in } \mathsf{word}_{c,S}(\mathsf{D}).$$

Set $\theta_{\mathsf{D}}(S) = \sum_{c \in [n]} \theta_{\mathsf{D}}^{c}(S)$. For instance, for D above

$$\begin{aligned}
&\theta_{\mathsf{D}}(\{1\}) = 4 && \theta_{\mathsf{D}}(\{1,2\}) = 6 && \theta_{\mathsf{D}}(\{1,2,3\}) = 6\\
&\theta_{\mathsf{D}}(\{2\}) = 4 && \theta_{\mathsf{D}}(\{1,3\}) = 6 && \theta_{\mathsf{D}}(\{1,2,4\}) = 6\\
&\theta_{\mathsf{D}}(\{3\}) = 4 && \theta_{\mathsf{D}}(\{1,4\}) = 5 && \theta_{\mathsf{D}}(\{1,3,4\}) = 6\\
&\theta_{\mathsf{D}}(\{4\}) = 1 && \theta_{\mathsf{D}}(\{2,3\}) = 5 && \theta_{\mathsf{D}}(\{2,3,4\}) = 5.\\
& && \theta_{\mathsf{D}}(\{2,4\}) = 4 && \\
& && \theta_{\mathsf{D}}(\{3,4\}) = 4 &&
\end{aligned}$$

Define the **Schubitope** of D as

$$\mathcal{S}_{\mathsf{D}} = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_{\geq 0}^{n} : \sum_{i=1}^{n} \alpha_i = \#\mathsf{D} \text{ and } \sum_{i \in S} \alpha_i \leq \theta_{\mathsf{D}}(S) \text{ for all } S \subset [n] \right\}.$$

As we conjectured in Conjectures 2.48 and 2.76, the Schubitope for a skyline diagram and for a Rothe diagram respectively are the Newton polytopes of a key and Schubert polynomial [FMS17, Theorem 10]. Fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The **$\lambda$-permutahedron**, denoted $\mathcal{P}_\lambda$, is the convex hull of the S_n-orbit of λ in $\mathbb{R}^n$. The Schubitope is a generalization of the permutahedron (Proposition 2.86). Figure 2.1 depicts $\mathcal{S}_{\mathsf{D}(21543)}$, which is a three-dimensional convex polytope in $\mathbb{R}^4$. Conjecture 2.82 asserts that **Ehrhart polynomials** of

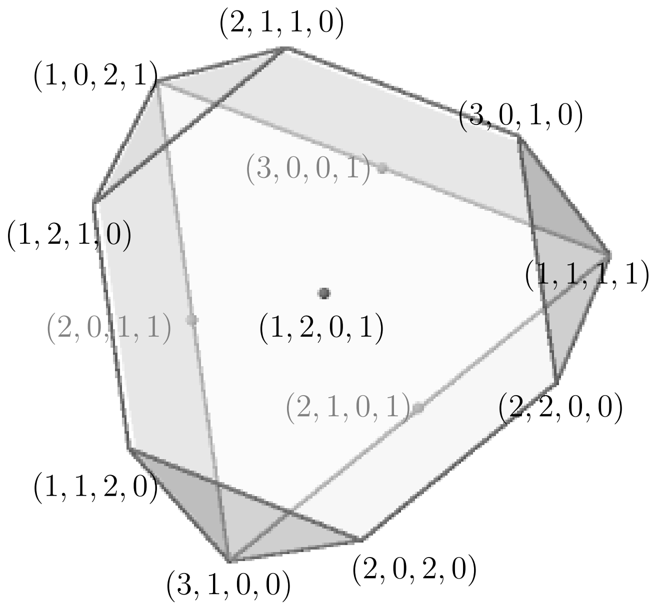


Figure 2.1: The Schubitope $\mathcal{S}_{\mathsf{D}(21543)}$.

Schubitopes $\mathcal{S}_{\mathsf{D}(w)}$ have positive coefficients; cf. [CL17, Conjecture 1.2].

A cornerstone of the theory of symmetric polynomials is the combinatorics of Littlewood-Richardson coefficients. An important special case of these numbers are the Kostka coefficients $\mathsf{K}_{\lambda,\mu}$, see Equation 1.1. Recall that the nonzeroness of $\mathsf{K}_{\lambda,\mu}$ is governed by dominance order which is defined by the linear inequalities (see Definition 1.5). Alternatively, Rado's theorem [Rad52, Theorem 1] states this order characterizes when $\mathcal{P}_\mu \subseteq \mathcal{P}_\lambda$. These two viewpoints on dominance order are connected since $\mathcal{P}_\lambda$ is the Newton polytope of the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$.

For Schubert polynomials, there is no analogous Littlewood-Richardson rule. However, with a parallel in mind, we propose a "dominance order for permutations" *via* Newton polytopes. The inequalities of the Schubitope generalize dominance order; see Proposition 2.86.

2.1.1 Organization

Section 2.2 develops and applies basic results about SNP symmetric polynomials. Section 2.3 turns to flavors of Macdonald polynomials and their specializations, including the key polynomials and Demazure atoms. Section 2.4 concerns quasisymmetric functions. Monomial quasisymmetric and Gessel's fundamental quasisymmetric polynomials are not SNP, but have equal Newton polytopes. The quasisymmetric Schur polynomials are also not SNP, which demonstrates a qualitative difference with Schur polynomials. Section 2.5 discusses Schubert polynomials and a number of variations. We define dominance order for permutations and study its poset-theoretic properties. We connect the Schubitope to work of A. Kohnert [Koh91] and explain a salient contrast (Remark 2.84).

2.2 Symmetric functions

2.2.1 Basic facts about SNP

Recall from Section 1.2 that Sym is the ring of symmetric functions (of finite degree) and Sym_n is the ring of symmetric polynomials in n variables. Then, given $f \in \mathsf{Sym}$, let

$$f_n = f(x_1, \ldots, x_n) \in \mathsf{Sym}_n$$

be the specialization that sets $x_i = 0$ for $i \geq n+1$. Whether f_n is SNP depends on n. For example, if $f = p_{(2)} = \sum_i x_i^2$, $f_1 = x_1^2$ is SNP while $f_2 = x_1^2 + x_2^2$ is not.

Definition 2.3. *The symmetric function $f \in \mathsf{Sym}$ is SNP if $f_m(x_1, \ldots, x_m)$ is SNP for all $m \geq 1$.*

Proposition 2.4 (Stability of SNP)**.** *Let $f \in \mathsf{Sym}$. Then f is SNP if there exists $m \geq \deg(f)$ such that f_m is SNP.*

Proof. We first show that if f_m is SNP, f_n is SNP for any $n \leq m$. Suppose

$$\gamma \in \texttt{Newton}(f_n) \subseteq \texttt{Newton}(f_m).$$

Since f_m is SNP, $\mathbf{x}^\gamma$ is a monomial of f_m. However, since

$$\gamma \in \texttt{Newton}(f_n) \in \mathbb{R}^n,$$

the nonzero positions of γ must be in positions $1, \ldots, n$, and thus $\mathbf{x}^\gamma$ is a monomial of $f_n = f_m(x_1, \ldots, x_n, 0, \ldots, 0)$.

To complete the proof, we now show if f_m for $m \geq \deg(f)$ is SNP, then f_n is SNP for all

$n \geq m$. Suppose $\gamma \in \texttt{Newton}(f_n)$ and thus

$$\gamma = \sum_i c_i \beta^i \tag{2.1}$$

where $\mathbf{x}^{\beta^i}$ is a monomial of f. Since $n \geq m \geq \deg(f)$, there are at most m coordinates where $\gamma_j > 0$, say $j_1, \ldots, j_m$. Furthermore, since each $\beta^i \in \mathbb{Z}^n_{>0}$, if $c_i > 0$, then $\beta^i_j = 0$ for $j \neq j_1, \ldots, j_m$. Choose $w \in S_n$ such that $w(j_c) = c$ for $c = 1, \ldots, m$. Applying w to (2.1) gives

$$w(\gamma) = \sum_i c_i w(\beta^i).$$

Since the nonzero coordinates of $w(\beta^i)$ occur in positions $1, \ldots, m$, the nonzero coordinates of $w(\gamma)$ only occur in these positions as well. Since $f \in \mathsf{Sym}$ and each x^{β^i} is a monomial of f, then each $\mathbf{x}^{w(\beta^i)}$ is a monomial of f_m as well. Consequently, $w(\gamma) \in \texttt{Newton}(f_m)$. Since f_m is SNP, $[\mathbf{x}^{w(\gamma)}]f \neq 0$. Again, $f \in \mathsf{Sym}$ implies $[\mathbf{x}^\gamma]f \neq 0$. Hence, f_n is SNP. $\square$

Remark 2.5. *In the proof of Proposition 2.4, w is chosen so that the nonzero components of the vectors γ and $w(\gamma)$ are in the same relative order. Thus the result extends to the quasisymmetric case of Section 2.4.* $\square$

Remark 2.6. *The stabilization constant* $\deg(f)$ *is tight. Let $\mathsf{f}^\lambda = [m_{1^{|\lambda|}}]s_\lambda$, or equivalently the number of standard Young tableaux of shape λ. Then, let $f = s_\lambda - \mathsf{f}^\lambda m_{1^{|\lambda|}}$. The polynomial f_n is SNP for $n < |\lambda| = \deg(f)$ but not SNP for $n \geq |\lambda|$.*

Proposition 2.7. *Suppose $f \in \mathsf{Sym}_n$ is homogeneous of degree d and thus*

$$f = \sum_{\mu \in \mathsf{Par}(d)} c_\mu s_\mu.$$

Suppose there exists λ with $c_\lambda \neq 0$ such that $c_\mu \neq 0$ only if $\mu \leq_D \lambda$. If $n < \ell(\lambda)$, we have $f = 0$. Otherwise:

(I) $\mathsf{Newton}(f) = \mathcal{P}_\lambda \subset \mathbb{R}^n$.

(II) The vertices of $\mathsf{Newton}(f)$ *are rearrangements of λ.*

(III) If moreover $c_\mu \geq 0$ for all μ, then f has SNP.

Proof. If $\mu \leq_D \lambda$, then $\ell(\mu) \geq \ell(\lambda)$. Thus if $n < \ell(\lambda)$, we have $s_\mu(x_1, \ldots, x_n) \equiv 0$ for all μ such that $c_\mu \neq 0$. Otherwise, suppose $n \geq \ell(\lambda)$.

(I): Recall from (1.1) that $s_\lambda = \sum_{\substack{\mu \in \mathsf{Par}(d) \\ \mu \leq_D \lambda}} \mathsf{K}_{\lambda,\mu} m_\mu$. Then since $f = \sum_{\substack{\mu \in \mathsf{Par}(d) \\ \mu \leq_D \lambda}} c_\mu s_\mu$, we have

$$f = \sum_{\substack{\mu \in \mathsf{Par}(d) \\ \mu \leq_D \lambda}} d_\mu m_\mu.$$

By the definitions of both,

$$\mathsf{Newton}(m_\mu(x_1, \ldots, x_n)) = \mathcal{P}_\mu \subset \mathbb{R}^n. \tag{2.2}$$

Also,

$$\mathsf{Newton}(f + g) = \mathsf{conv}(\mathsf{Newton}(f) \cup \mathsf{Newton}(g)).$$

Hence,

$$\mathsf{Newton}(f) = \mathrm{conv}\left(\bigcup_{\substack{\mu \in \mathsf{Par}(d) \\ \mu \leq_D \lambda}} \mathsf{Newton}(m_\mu)\right) = \mathrm{conv}\left(\bigcup_{\substack{\mu \in \mathsf{Par}(d) \\ \mu \leq_D \lambda}} \mathcal{P}_\mu\right).$$

R. Rado's theorem [Rad52, Theorem 1] states:

$$\mathcal{P}_\mu \subseteq \mathcal{P}_\lambda \iff \mu \leq_D \lambda. \tag{2.3}$$

Thus by (2.3),

$$\mathsf{Newton}(f) = \mathrm{conv}\left(\bigcup_{\mu \leq_D \lambda} \mathcal{P}_\mu\right) = \mathcal{P}_\lambda,$$

proving (I).

(II): In view of (I), it suffices to know this claim for $\mathcal{P}_\lambda$. This is well-known, but we include a proof for completeness.

Since $\mathcal{P}_\lambda$ is the convex hull of the S_n-orbit of λ, any vertex of $\mathcal{P}_\lambda$ is a rearrangement of λ. It remains to show that every such rearrangement β is in fact a vertex. Thus it suffices to show there is no nontrivial convex combination

$$\beta = \sum_{\substack{\gamma \in \mathsf{PermutWC}(\lambda) \\ \gamma \neq \beta}} c_\gamma \gamma. \tag{2.4}$$

Let $\lambda = \Lambda_1^{k_1} \cdots \Lambda_m^{k_m}$ with $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_m$. Since β is a rearrangement of λ, let

$i^1_1, \ldots, i^1_{k_1}$ be the positions in β of the k_1 parts of size Λ_1. Since $\gamma_{i^1_j} \leq \Lambda_1$ for all γ we have that $c_\gamma = 0$ whenever γ satisfies $\gamma_{i^1_j} \neq \Lambda_1$ for any $1 \leq j \leq k_1$.

Let $i^2_1, \ldots, i^2_{k_2}$ be the positions in β of the k_2 parts of size Λ_2. Similarly, $c_\gamma = 0$ whenever γ satisfies $\gamma_{i^2_j} \neq \Lambda_2$ for any $1 \leq j \leq k_2$. Continuing, we see that $c_\gamma = 0$ for all $\gamma \neq \beta$. That is, there is no convex combination (2.4), as desired.

(III): Suppose γ is a lattice point in $\mathsf{Newton}(f) = \mathcal{P}_\lambda \subset \mathbb{R}^n$. By symmetry, $\mathcal{P}_{\lambda(\gamma)} \subseteq \mathcal{P}_\lambda$. Then by (2.3), we have $\lambda(\gamma) \leq_D \lambda$ and so by (1.1), $\mathsf{K}_{\lambda,\lambda(\gamma)} \neq 0$. Since $\mathbf{x}^\gamma$ appears in $m_{\lambda(\gamma)}(x_1, \ldots, x_n)$, then $\mathbf{x}^\gamma$ appears in $f(x_1, \ldots, x_n)$ (here we are using the Schur positivity of f and the fact $\ell(\lambda(\gamma)) \leq n$). Thus f is SNP. $\square$

Example 2.8 (Schur positivity does not imply SNP)*.* Let

$$f = s_{(8,2,2)} + s_{(6,6)}.$$

It is enough to show $f_3 = f(x_1, x_2, x_3)$ is not SNP. Now, in 3 variables, $m_{(8,2,2)}$ and $m_{(6,6)}$ appear in the monomial expansion of f_3. However, $[m_{(7,4,1)}]f_3 = 0$ since $(7, 4, 1)$ is not $\leq_D$-comparable with $(8, 2, 2)$ nor $(6, 6, 0)$. Yet,

$$(7, 4, 1) = \frac{1}{2}(8, 2, 2) + \frac{1}{2}(6, 6, 0) \in \mathsf{Newton}(f_3).$$

Hence f is not SNP. $\square$

Example 2.9 (The Schur positivity assumption in Proposition 2.7(III) is necessary)*.* The function

$$f = s_{(3,1)}(x_1, x_2) - s_{(2,2)}(x_1, x_2) = x_1^3 x_2 + x_1 x_2^3$$

is not SNP. $\square$

Example 2.10 (SNP does not require a unique $\leq_D$-maximal term)*.* The function

$$f = s_{(2,2,2)} + s_{(3,1,1,1)}$$

is SNP but $(2, 2, 2)$ and $(3, 1, 1, 1)$ are $\leq_D$-incomparable. An instance of this from the literature of algebraic combinatorics is found in Example 2.38. $\square$

Proposition 2.11 (Products of Schur polynomials are SNP)**.** *The product*

$$s_{\lambda^{(1)}} s_{\lambda^{(2)}} \cdots s_{\lambda^{(N)}} \in \mathsf{Sym}$$

is SNP for any partitions $\lambda^{(1)}, \ldots, \lambda^{(N)}$.

Proof. We have

$$s_\lambda s_\mu = \sum_\nu \mathsf{LR}_{\lambda,\mu}^{\nu \in \mathsf{Par}(|\lambda|+|\mu|)} s_\nu \in \mathsf{Sym},$$

where $\mathsf{LR}_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ is the Littlewood-Richardson coefficient (see Theorem 1.11). By homogeneity, $\mathsf{LR}_{\lambda,\mu}^{\nu} = 0$ unless $|\nu| = |\lambda| + |\mu|$. Let $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. It suffices to show $\mathsf{LR}_{\lambda,\mu}^{\lambda+\mu} \neq 0$ and $\nu \leq_D \lambda + \mu$ whenever $\mathsf{LR}_{\lambda,\mu}^{\nu} \geq 0$. Actually, we show $s_{\lambda+\mu}$ is the unique $\leq_D$-maximal term in the Schur expansion of $s_\lambda e_{\mu'}$. Since $e_{\mu'} = s_\mu +$ (positive sum of Schur functions), this will suffice. The strengthening holds by an easy induction on the number of nonzero parts of μ and the Pieri rule in the form of [Sta99a, Example 7.15.8]. Iterating this argument shows that $s_{\lambda^{(1)}} \cdots s_{\lambda^{(N)}}$ has unique $\leq_D$ maximal term $s_{\lambda^{(1)}+\cdots+\lambda^{(N)}}$ and hence is SNP. $\square$

A polytope $\mathcal{P}$ possesses the **integer decomposition property** if for all $k \in \mathbb{Z}_{>0}$ and for all lattice points α of $k\mathcal{P}$, there exists lattice points $\alpha_1, \ldots, \alpha_k$ of $\mathcal{P}$ such that

$$\alpha = \alpha_1 + \cdots + \alpha_k,$$

where $k\mathcal{P} = \{k\alpha : \alpha \in \mathcal{P}\}$; see, e. g., [CHHH14].

Example 2.12. The permutahedron $\mathcal{P}_\lambda$ has the integer decomposition property.

Proof. The **Minkowski sum** of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ is

$$\mathcal{P} + \mathcal{Q} = \{p + q : p \in \mathcal{P}, q \in \mathcal{Q}\}.$$

Thus,

$$\mathsf{Newton}(f \cdot g) = \mathsf{Newton}(f) + \mathsf{Newton}(g).$$

Furthermore, it is clear

$$k\mathcal{P} = \underbrace{\mathcal{P} + \cdots + \mathcal{P}}_{k \text{ times}}.$$

Now consider $k\mathcal{P}_\lambda = \mathsf{Newton}(s_\lambda^k)$. By Proposition 2.11, we have s_λ^k is SNP. Thus, for $\alpha \in k\mathcal{P}_\lambda$, we know $\mathbf{x}^\alpha$ is a monomial of s_λ^k. We then have monomials $\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_k}$ of s_λ such that

$$\mathbf{x}^\alpha = \mathbf{x}^{\alpha_1} \cdots \mathbf{x}^{\alpha_k}.$$

Since $\mathsf{Newton}(s_\lambda) = \mathcal{P}_\lambda$, each α_i is a lattice point of $\mathcal{P}_\lambda$ and thus, $\mathcal{P}_\lambda$ has the integer decomposition property. □

We thank J. Hofscheier for pointing out the connection to the integer decomposition property. Recall the power sum symmetric polynomial from Definition 1.9:

$$p_k = \sum_{i=1}^{n} x_i^k \text{ and } p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots .$$

Proposition 2.13. *Let*

$$f = \sum_{\lambda \in \mathsf{Par}(n)} c_\lambda p_\lambda \in \mathsf{Sym}$$

be not identically zero. Assume $c_\lambda \geq 0$ for all λ, and that f is Schur positive. Then f is SNP.

Proof. Recall, (n) indexes the trivial representation of S_n that sends each $\pi \in S_n$ to the 1×1 identity matrix. As $\chi^{(n)}$ is the trace of this matrix, we have $\chi^{(n)}(\mu) = 1$ for all conjugacy classes $\mu \in \mathsf{Par}(n)$.

We have

$$p_\mu = \sum_{\lambda \in \mathsf{Par}(n)} \chi^\lambda(\mu) s_\lambda.$$

Therefore,

$$f = \sum_{\lambda \in \mathsf{Par}(n)} c_\lambda p_\lambda = \left(\sum_{\lambda \in \mathsf{Par}(n)} c_\lambda \right) s_{(n)} + \sum_{\substack{\lambda \in \mathsf{Par}(n) \\ \lambda \neq (n)}} d_\lambda s_\lambda.$$

By hypothesis, each $c_\lambda \geq 0$. Since $f \not\equiv 0$, some $c_\lambda > 0$ and hence $s_{(n)}$ appears. Now, (n) is the (unique) maximum in $(\mathsf{Par}(n), \leq_D)$. Also, since f is Schur positive, each $d_\lambda \geq 0$. Hence the result follows from Proposition 2.7(III). □

2.2.2 Examples and counterexamples

Recall from Section 1.1 that λ' is the conjugate of λ, i.e. the shape obtained by transposing the Young diagram of λ. Then, let

$$\omega : \mathsf{Sym} \to \mathsf{Sym}$$

be the involutive automorphism defined by

$$\omega(s_\lambda) = s_{\lambda'}.$$

Example 2.14 (The map ω does not preserve SNP). Example 2.8 shows $f = s_{(8,2,2)} + s_{(6,6)} \in \mathsf{Sym}$ is not SNP. Now

$$\omega(f) = s_{(3,3,1,1,1,1,1,1)} + s_{(2,2,2,2,2,2)} \in \mathsf{Sym}.$$

To see that $\omega(f)$ is SNP, it suffices to show that any partition ν that is is a linear combination of rearrangements of $\lambda = (3,3,1,1,1,1,1,1)$ and $\mu = (2,2,2,2,2,2)$ satisfies $\nu \leq_D \lambda$ or $\nu \leq_D \mu$. We leave the details to the reader. □

Example 2.15 (Monomial symmetric and forgotten symmetric polynomials). It is immediate from (2.2) and (2.3) that

$$m_\lambda \in \mathsf{Sym} \text{ is SNP } \iff \lambda = 1^n.$$

The **forgotten symmetric functions** are defined by

$$f_\lambda = (-1)^{|\lambda| - \ell(\lambda)} \omega(m_\lambda).$$

Proposition 2.16. *The forgotten symmetric function $f_\lambda \in \mathsf{Sym}$ is SNP if and only if $\lambda = 1^n$.*

Proof. ($\Leftarrow$) If $\lambda = 1^n$ then $m_\lambda = s_{1^n}$ and $f_\lambda = s_{(n,0,0,\ldots,0)}$ which is SNP.

($\Rightarrow$) We use the following formula [Sta99a, Exercise 7.9]:

$$f_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$$

where $a_{\lambda\mu}$ is the number of distinct rearrangements $(\gamma_1, \ldots, \gamma_\ell)$ of $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that

$$\left\{ \sum_{s=1}^{i} \gamma_s : 1 \leq i \leq \ell(\lambda) \right\} \supseteq \left\{ \sum_{t=1}^{j} \mu_t : 1 \leq j \leq \ell(\mu) \right\}. \tag{2.5}$$

Suppose $\lambda \neq 1^n$. If $\mu = 1^n$ then

$$\left\{ \sum_{t=1}^{j} \mu_t : 1 \leq j \leq \ell(\mu) \right\} = \{1, 2, 3, \ldots, n\}.$$

On the other hand, $\ell(\lambda) < n$ and hence the set on the lefthand side of (2.5) has size strictly smaller than n. Thus $a_{\lambda,1^n} = 0$.

Now, $1^n \in \mathcal{P}_\mu$ for all $\mu = (\mu_1, \ldots, \mu_n) \vdash n$. Then since f_λ is m-positive, $1^n \in \mathsf{Newton}(f_\lambda)$ so long as we are working in at least $\deg(f_\lambda)$ many variables. If $\lambda \neq 1^n$ then $a_{\lambda,1^n} = 0$ means 1^n is not an exponent vector of f_λ. Thus, if $\lambda \neq 1^n$, then f_λ is not SNP, the contrapositive of $(\Rightarrow)$. $\square$

Example 2.17 (Elementary and complete homogeneous symmetric polynomials). Recall from Definitions 1.7 and 1.8, the elementary symmetric polynomial e_k is the sum of all degree k monomials with distinct variables whereas the complete homogeneous symmetric polynomial h_k is the sum of all degree k monomials. Then,

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \text{ and } h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots .$$

Proposition 2.18. *Each e_λ and h_λ is SNP.*

Proof. Since $e_k = s_{1^k}$ and $h_k = s_{(k)}$ the claim holds by Proposition 2.11. $\square$

By the Pieri rule,

$$e_{(1)}(x_1, \ldots, x_n)^k = \sum_\lambda \mathsf{f}^\lambda s_\lambda(x_1, \ldots, x_n).$$

In particular $s_{(k,0,\ldots,0)}$ appears on the right hand side. Since $\lambda \leq_D (k)$ for all $\lambda \vdash k$, by Proposition 2.7(I) one recovers that the Minkowski sum of k regular simplices in $\mathbb{R}^n$ is $\mathcal{P}_{(k,0,\ldots,0)}$. Similarly, by the argument of Proposition 2.11, $\mathcal{P}_\lambda = \mathsf{Newton}(e_{\lambda'})$ and hence one recovers that $\mathcal{P}_\lambda$ is a Minkowski sum of hypersimplices. For earlier work see, e.g., [Pos09, Cro10, AM09]. $\square$

Example 2.19 (e-positivity does not imply SNP). A symmetric function $f \in \mathsf{Sym}$ is **e-positive** if $f = \sum_\lambda a_\lambda e_\lambda$ where $a_\lambda \geq 0$ for every λ. Since

$$e_\lambda = \sum_\mu \mathsf{K}_{\mu',\lambda} s_\mu,$$

e-positivity implies Schur positivity. Consider

$$f = e_{(3,3,1,1,1,1,1,1)} + e_{(2,2,2,2,2,2)} \in \mathsf{Sym}.$$

In the monomial expansion, $m_{(8,2,2)}$ and $m_{(6,6)}$ appear. However, $m_{(7,4,1)}$ does not appear. This implies f is not SNP. $\square$

Example 2.20 (More on power sum symmetric polynomials). Recall the power sum symmetric polynomials defined in Definition 1.9 and immediately before Proposition 2.13. Clearly p_k is not SNP if $k > 1$ and $n > 1$. Also, p_λ is not SNP for $n > 1$ whenever $\lambda_i \geq 2$ for all i. This is since $x_1^{|\lambda|}$ and $x_2^{|\lambda|}$ both appear as monomials in p_λ but $x_1^{|\lambda|-1}x_2$ does not. Furthermore:

Proposition 2.21. *The power sum symmetric polynomial $p_\lambda \in \mathsf{Sym}_n$ for $n > \ell(\lambda)$ is SNP if and only if $\lambda = 1^k$.*

Proof. ($\Leftarrow$) If $\lambda = 1^k$, then $p_\lambda = e_\lambda$ which is SNP by Proposition 2.18.

($\Rightarrow$) Suppose $\lambda_1 \geq 2$ and let $\ell = \ell(\lambda)$. Then since $n > \ell$, the monomials $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_\ell^{\lambda_\ell}$ and $x_2^{\lambda_2}\cdots x_\ell^{\lambda_\ell}x_{\ell+1}^{\lambda_1}$ appear in p_λ. Thus,

$$(\lambda_1 - 1, \lambda_2, \ldots, \lambda_\ell, 1) = \frac{\lambda_1 - 1}{\lambda_1}(\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0) + \frac{1}{\lambda_1}(0, \lambda_2, \ldots, \lambda_\ell, \lambda_1) \in \mathsf{Newton}(p_\lambda).$$

However, this point cannot be an exponent vector since it has $\ell + 1$ nonzero components whereas every monomial of p_λ uses at most ℓ distinct variables. $\square$

By Proposition 2.4, this shows that $p_\lambda \in \mathsf{Sym}$ is SNP if and only if $\lambda = 1^k$.

Example 2.22 (The resultant, the Gale-Ryser theorem and $(0,1)$-matrices). Let

$$f = \sum_{i=0}^{m} a_i z^i \text{ and } g = \sum_{i=1}^{n} b_i z^i$$

be two polynomials of degree m and n respectively and with roots

$$\{x_1, \ldots, x_m\} \text{ and } \{y_1, \ldots, y_n\}$$

respectively (not necessarily distinct). The **resultant** is

$$R(f,g) = a_m^n b_n^m \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j).$$

This polynomial is separately symmetric in the $\mathbf{x}$ and $\mathbf{y}$ variables. In [GKZ90] the Newton polytope of $R(f,g)$ is determined; see also the book [GKZ94]. However, we are not aware of the following result appearing explicitly in the literature.

Theorem 2.23. *The resultant $R(f,g)$ is SNP.*

Proof. Consider

$$F = \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j).$$

In fact, $[\mathbf{x}^\alpha \mathbf{y}^\beta]F$ equals the number of $(0,1)$-matrices of dimension $m \times n$ whose row sums are given by α and column sums are given by β; see, e.g., [Sta99a, Proposition 7.4.3]. Let $\mathsf{M}(\alpha, \beta)$ equal the number of these matrices. The Gale-Ryser theorem states

$$\mathsf{M}(\alpha, \beta) > 0 \iff \lambda(\beta) \leq_D \lambda(\alpha)'. \tag{2.6}$$

Call a pair of vectors $(\alpha, \beta) \in \mathbb{Z}_{\geq 0}^{m+n}$ a **GR pair** if it satisfies either of the equivalent conditions in (2.6).

In fact F is SNP. Suppose

$$(\alpha^{(1)}, \beta^{(1)}), (\alpha^{(2)}, \beta^{(2)}), \ldots, (\alpha^{(N)}, \beta^{(N)})$$

are GR pairs and

$$(\alpha, \beta) = \sum_{t=1}^{N} d_i(\alpha^{(t)}, \beta^{(t)})$$

with $d_i \geq 0$ and $\sum_{t=1}^{N} d_i = 1$ be a convex combination. The SNPness of F is equivalent to the claim (α, β) is a GR pair whenever $(\alpha, \beta) \in \mathbb{Z}_{\geq 0}^{m+n}$. The latter claim is immediate from [Bar12, Theorem 3, part 1] which establishes the "approximate log-concavity" of $\mathsf{M}(\alpha, \beta)$. We thank A. Barvinok for pointing out this reference to us.

Now notice that

$$\begin{aligned}
F \text{ is SNP} &\iff \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j^{-1}) \text{ is SNP} \\
&\iff y_1^m y_2^m \cdots y_n^m \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j^{-1}) \text{ is SNP} \\
&\iff \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) \text{ is SNP} \\
&\iff \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j) \text{ is SNP} \\
&\iff R(f, g) \text{ is SNP.}
\end{aligned}$$

The final equivalence is true since $a_0, b_0 \neq 0$ and the previous equivalence holds since the polynomials in the third and fourth lines clearly share the same monomials. This relation between F and $R(f,g)$ appears in [GKZ90] where the authors use it to obtain a formula for the monomials of $R(f,g)$ in terms of counts for $(0,1)$-matrices. $\square$

Conjecture 2.64 claims a generalization of Theorem 2.23; see Example 2.65.

Example 2.24 (Powers of the Vandermonde). The **Vandermonde determinant** is

$$a_{\delta_n} = \prod_{\substack{i,j \in \mathbb{Z}_{>0} \\ 1 \leq i < j \leq n}} (x_i - x_j).$$

(This polynomial is only skew-symmetric.) It is known that

$$\mathsf{Newton}(a_{\delta_n}) = \mathcal{P}_{(n-1,n-2,\ldots,2,1,0)} \subset \mathbb{R}^n;$$

see e.g., [Pos09, Proposition 2.3].

Proposition 2.25. *The Vandermonde determinant a_{δ_n} is SNP if and only if $n \leq 2$.*

The classical **discriminant** is $\Delta_n = \alpha_{\delta_n}^2$. Its Newton polytope was also determined by in work of I. M. Gelfand-M. Kapranov-A. V. Zelevinsky [GKZ90].

Proposition 2.26. *The discriminant Δ_n is SNP if and only if $n \leq 4$.*

Proposition 2.26 is a curious coincidence with the Abel-Ruffini theorem. Our proofs of Propositions 2.25 and 2.26 will use this lemma:

Lemma 2.27. *If $a_{\delta_n}^k$ is not SNP, then $a_{\delta_{n+1}}^k$ is not SNP.*

Proof. Suppose $a_{\delta_n}^k$ is not SNP. There exists a lattice point $\alpha \in \mathsf{Newton}(a_{\delta_n}^k)$ that is not an exponent vector of $a_{\delta_n}^k$. Hence we have a convex combination

$$\alpha = \sum_{i=1}^{N} c_i \beta^i$$

where β^i is an exponent vector. For $\gamma \in \mathbb{R}^n$, let $\gamma' = (\gamma, kn) \in \mathbb{R}^{n+1}$. Since

$$a_{\delta_{n+1}}^k = a_{\delta_n}^k \times \prod_{i=1}^{n} (x_i - x_{n+1})^k,$$

each $(\beta^i)'$ is an exponent vector of $a^k_{\delta_{n+1}}$ and hence α' is a lattice point of $\mathsf{Newton}(a^k_{\delta_{n+1}})$. Since $\mathbf{x}^{\alpha'} = \mathbf{x}^{\alpha} x_{n+1}^{kn}$ and x_{n+1} does not appear in $a^k_{\delta_n}$, if α' is an exponent vector of $a^k_{\delta_{n+1}}$, then α is an exponent vector of $a^k_{\delta_n}$, a contradiction. Thus $a^k_{\delta_{n+1}}$ is not SNP. $\square$

Proof of Propositions 2.25 and 2.26: Clearly, a_{δ_n} is SNP for $n = 1, 2$. One checks that $(1,1,1) \in \mathsf{Newton}(a_{\delta_3})$ but is not an exponent vector of a_{δ_3}.

Separately, one checks Δ_n is SNP for $n \leq 4$. Also Δ_5 is not SNP. In fact, the only lattice points that are not exponent vectors are all 5! rearrangements of $(1, 3, 4, 5, 7)$.

Now apply Lemma 2.27 to complete an induction argument for each of the two propositions being proved. $\square$

Conjecture 2.28. *For all k, there exists N_k such that $a^k_{\delta_n}$ is not SNP for any $n \geq N_k$.*

More precisely, for $1 \leq j \leq 4$ we computed $N_{2j-1} = 3$ and moreover that $(1, 3j-2, 3j-2)$ is a lattice point that is not an exponent vector. Moreover, $N_2 = 5, N_4 = 4, N_6 = 4, N_8 = 3$. For more on (higher) powers of the Vandermonde, see, e.g., [STW94, Bal11]. $\square$

Example 2.29 (q-discriminant). The **q-discriminant** is $\prod_{1 \leq i < j \leq n} (x_i - q x_j)$. At $q = -1$,

$$f_n = \prod_{1 \leq i < j \leq n} (x_i + x_j) \in \mathsf{Sym}_n.$$

It is known that

$$f_n = s_{\rho_n}(x_1, x_2, \ldots, x_n) \quad \text{where } \rho_n = (n-1, n-2, \ldots, 3, 2, 1, 0).$$

Hence f_n is SNP and $\mathsf{Newton}(f_n) = \mathcal{P}_{\rho_n} \subset \mathbb{R}^n$. $\square$

Example 2.30 (Totally nonnegative matrices). Let

$$M = (m_{ij})_{1 \leq i,j \leq n}$$

be an $n \times n$ **totally nonnegative real matrix**. That is, the determinant of every square submatrix of M is nonnegative. Define

$$F_M = \sum_{w \in S_n} \left(\prod_{i=1}^{n} m_{i,w(i)} \right) p_{\lambda(w)},$$

where $\lambda(w)$ is the cycle type of w.

Theorem 2.31. *The symmetric function F_M is SNP.*

Proof. By assumption, $m_{ij} \geq 0$. A theorem of J. R. Stembridge [Ste91] (cf. [Sta99a, Exercise 7.92]) states that F_M is also Schur positive. Now apply Proposition 2.13. $\square$

Example 2.32 (Redfield–Pólya theory). Let G be a subgroup of S_n. The **cycle index polynomial** is

$$Z_G = \frac{1}{|G|} \sum_{g \in G} p_{\lambda(g)},$$

where $\lambda(g)$ is the cycle type of g.

Theorem 2.33. *The cycle index polynomial Z_G has SNP.* $\square$

Proof. It is true that

$$Z_G = \sum_{\lambda \in \mathsf{Par}(n)} c_\lambda s_\lambda,$$

where each $c_\lambda \in \mathbb{Z}_{\geq 0}$; see [Sta99a, pg. 396]. This positivity is known for representation-theoretic reasons (no combinatorial proof is available). Now use Proposition 2.13. $\square$

Example 2.34 (C. Reutenauer's q_λ basis). C. Reutenauer [Reu95] introduced a new basis $\{q_\lambda\}$ of symmetric polynomials, recursively defined by setting

$$\sum_{\lambda \in \mathsf{Par}(n)} q_\lambda = s_{(n)},$$

where $q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots$.

Theorem 2.35. *The symmetric function q_λ has SNP.*

Proof. C. Reutenauer in *loc. cit.* conjectured that $-q_{(n)}$ is Schur positive for $n \geq 2$. Indeed,

$$q_{(1)} = s_{(1)}, q_{(2)} = -s_{(1,1)}, q_{(3)} = -s_{(2,1)}.$$

Reutenauer's conjecture was later established by W. M. Doran IV [IV96]. The proof sets

$$f(n,k) = \sum_{\substack{\lambda \in \mathsf{Par}(n) \\ \min(\lambda_i) \geq k}} q_\lambda a.$$

The argument inducts on n and proceeds by showing that

$$-f(n,k) = s_{(n-1,1)} + \sum_{\substack{i\in\mathbb{Z}_{>0} \\ 2\leq i<k}} (-f(i,i))(-f(n-i,i)).$$

His induction claim is that $-f(n,k)$ is Schur positive for $k \geq 2$. Let us strengthen his induction hypothesis, and assume $-f(n,k)$ is Schur positive with $s_{(n-1,1)}$ as the unique $\leq_D$ maximal term. In the induction step, note each s_α appearing in $-f(i,i)$ has $\alpha_1 \leq i-1$ and each s_β in $-f(n-i,i)$ has $\beta_1 \leq n-i-1$. Thus, by the argument of Proposition 2.11, if s_γ appears in $s_\alpha s_\beta$ then $\gamma_1 \leq n-2$, implying the strengthening we need.

It follows from the above argument and the Littlewood-Richardson rule that if $\lambda = (\lambda_1,\ldots,\lambda_\ell,1^r)$ where each $\lambda_i \geq 2$ then q_λ has a unique $\leq_D$-leading term $s_{(a,b)}$ where $a = |\lambda|-\ell$ and $b=\ell$. Thus, q_λ has SNP by Proposition 2.7(III). □

Example 2.36 (Stanley's chromatic symmetric polynomial). For a graph G, let c_G be **Stanley's chromatic symmetric polynomial** [Sta95]. If $G = K_{1,3}$,

$$c_G(x_1,x_2,\ldots) = m_{(3,1)} + 6m_{(2,1,1)} + 24m_{(1,1,1,1)}$$

is not SNP as it is missing $(2,2)$. □

It is an open problem to determine for which graphs G is c_G Schur positive. Likewise, we can ask for which graphs is c_G SNP.

Conjecture 2.37. *Let G be a graph where c_G is Schur positive. Then c_G is SNP.*

This conjecture has exhaustively checked for all graphs with at most 7 vertices and many graphs on 8 vertices. For graphs with 7 vertices, there are 771 graphs such that c_G is both Schur positive and SNP. Of the 145 graphs such that c_G is not Schur positive, 64 have SNP and 81 do not.

Example 2.38 (Kronecker product of Schur polynomials). The **Kronecker product** is

$$s_\lambda * s_\mu = \sum_{\nu\in\mathsf{Par}(|\lambda|)} \mathsf{Kron}^{\nu}_{\lambda,\mu} s_\nu \in \mathsf{Sym}.$$

$\mathsf{Kron}^{\nu}_{\lambda,\mu}$ is the **Kronecker coefficient**, the multiplicity of the S_n-character χ^ν appearing in $\chi^\lambda \otimes \chi^\mu$. We conjecture that $s_\lambda * s_\mu$ is SNP. We have verified this for all $\lambda,\mu \in \mathsf{Par}(n)$ for

$1 \leq n \leq 7$. Consider

$$s_{(4,2)} * s_{(2,2,1,1)} = s_{(1,1,1,1,1,1)} + s_{(2,1,1,1,1)} + 2s_{(2,2,1,1)} + s_{(3,1,1,1)} + 2s_{(3,2,1)} + s_{(3,3)} + s_{(4,1,1)}.$$

Notice $(3,3)$ and $(4,1,1)$ are both $\leq_D$-maximal. Hence in this case, SNPness cannot be blamed on Proposition 2.7(III); cf. [AAV12, Lemma 3.2] and [Val00]. $\square$

Example 2.39 (Lascoux-Leclerc-Thibon (LLT) polynomials). A. Lascoux, B. Leclerc, and J. Y. Thibon [LLT97] introduced $G_\lambda^{(m)}(\mathbf{x};q)$. Then, $G_\lambda^{(m)}(\mathbf{x};1)$ is a product of Schur polynomials. Hence $G_\lambda^{(m)}(\mathbf{x};1)$ is SNP by Proposition 2.11. However, $G_\lambda^{(m)}(\mathbf{x};q) \in \mathsf{Sym}_n[q]$ is not always SNP. One example is

$$G^{(2)}_{(3,3)}(x_1,x_2;q) = q^3(x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3) + q(x_1^2x_2 + x_2^2x_1).$$

LLT polynomials arise in the study of Macdonald polynomials, the topic of Section 2.3. (Another related topic is **affine Schubert calculus**; see the book [LLM+14].) $\square$

2.3 Macdonald polynomials

2.3.1 Symmetric and nonsymmetric Macdonald polynomials

The definitions of the Macdonald polynomials used in this section can be found in Section 1.5. Recall that the symmetric Macdonald polynomial P_λ has the form

$$P_\lambda(\mathbf{x};q,t) = m_\lambda(\mathbf{x}) + \sum_{\substack{\mu\in\mathsf{Par}(|\lambda|)\\ \mu<_D\lambda}} \mathsf{c}_{\lambda,\mu}(q,t)m_\mu(\mathbf{x}) \tag{2.7}$$

where $\mathsf{c}_{\lambda,\mu}(q,t) \in \mathbb{Q}(q,t)$.

Theorem 2.40. *For any (q_0,t_0) in a Zariski open subset of $\mathbb{C}^2$, the specialization $P_\lambda(\mathbf{x};q_0,t_0)$ is SNP, and $\mathsf{Newton}(P_\lambda(\mathbf{x};q_0,t_0)) = \mathcal{P}_\lambda \subset \mathbb{R}^n$ whenever $n \geq \ell(\lambda)$.*

Lemma 2.41. *For any $(q_0,t_0) \in \mathbb{C}^2$, we have $\mathsf{Newton}(P_\lambda(\mathbf{x};q_0,t_0)) = \mathcal{P}_\lambda \subset \mathbb{R}^n$ whenever $n \geq \ell(\lambda)$.*

Proof. This is by (2.7) and Proposition 2.7(I). Since $n \geq \ell(\lambda)$, we have $s_\lambda(x_1,\ldots,x_n) \not\equiv 0$. $\square$

Lemma 2.42. *Fix $q_0, t_0 \in \mathbb{C}$. Then, $P_\lambda(\mathbf{x}; q_0, t_0)$ is SNP if and only if $\mathsf{c}_{\lambda,\mu}(q_0, t_0) \neq 0$ for all $\mu <_D \lambda$.*

Proof. ($\Rightarrow$) By Lemma 2.41, $\mathsf{Newton}(P_\lambda(\mathbf{x}; q_0, t_0)) = \mathcal{P}_\lambda$. Thus each $\mu <_D \lambda$ appears as a lattice point of $\mathsf{Newton}(P_\lambda(\mathbf{x}; q_0, t_0))$. Since we assume $P_\lambda(\mathbf{x}; q_0, t_0)$ is SNP, we know

$$[\mathbf{x}^\mu] P_\lambda(\mathbf{x}; q_0, t_0) \neq 0.$$

Among the monomial symmetric functions, $\mathbf{x}^\mu$ only appears in m_μ. Hence $\mathsf{c}_{\lambda,\mu}(q_0, t_0) \neq 0$, as desired.

The proof of ($\Leftarrow$) just reverses the above argument, using the fact that

$$\mu \in \mathsf{Newton}(P_\lambda(\mathbf{x}; q_0, t_0)) = \mathcal{P}_\lambda \iff \gamma \in \mathsf{Newton}(P_\lambda(\mathbf{x}; q_0, t_0)) = \mathcal{P}_\lambda$$

for any rearrangement γ of $\mu \in \mathbb{R}^n$. $\square$

Proof of Theorem 2.40: The Newton polytope assertion is by Lemma 2.41. Now

$$P_\lambda(\mathbf{x}; 0, 0) = s_\lambda(\mathbf{x})$$

and m_μ appears in s_λ for every $\mu <_D \lambda$. Hence $\mathsf{c}_{\lambda,\mu}(q, t) \not\equiv 0$. Now choose q, t that is neither a pole nor a root of any of these rational functions (for $\mu <_D \lambda$). Therefore the SNP assertion follows from Lemma 2.42. $\square$

The Hall-Littlewood polynomial is $P_\lambda(\mathbf{x}; t) = P_\lambda(\mathbf{x}, 0, t)$. One has

$$P_\lambda(\mathbf{x}; t) = \sum_\mu \mathsf{K}_{\lambda,\mu}(t) s_\mu(\mathbf{x})$$

where $\mathsf{K}_{\lambda,\mu}(t)$ is the **Kostka-Foulkes polynomial**. It is known that

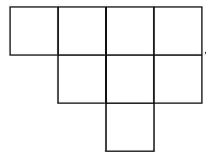
$$\mathsf{K}_{\lambda,\mu}(t) = \sum_{\substack{T \in \mathsf{SSYT}(\lambda) \\ c(T) = \mu}} t^{\mathsf{charge}(T)}$$

where $\mathsf{charge}(T) \in \mathbb{Z}_{\geq 0}$ [LS78b, Theorem 1']. Since these tableaux can only occur if $\mu \leq_D \lambda$, we have $\mathsf{K}_{\lambda,\mu}(t) \not\equiv 0$ if and only if $\mu \leq_D \lambda$. Hence we immediately obtain:

Proposition 2.43. *If $t_0 > 0$ then $P_\lambda(\mathbf{x}; t_0) \in \mathsf{Sym}$ is SNP and whenever $n \geq \ell(\lambda)$*

$$\mathsf{Newton}(P_\lambda(\mathbf{x}; t_0)) = \mathcal{P}_\lambda \subset \mathbb{R}^n.$$

When λ is a partition with distinct parts, we can represent its **shifted diagram** as the diagram where the ith row begins in column i. For example, for $\lambda = (4, 3, 1)$, the shifted diagram of λ is



Then **shifted semistandard tableaux** of λ are fillings of the shifted diagram with numbers from the alphabet

$$1' < 1 < 2' < 2 < \dots$$

such that rows and columns are weakly increasing, primed entries repeat only in columns, and unprimed entries repeat only in rows. Then the **content** of a shifted tableaux T is the weak composition $c(T)$ where $c(T)_i$ is the number of i and i' entries in T. The theory of shifted tableaux can be found in D. Worley's thesis [Wor84].

The **Schur $P-$polynomial** is

$$SP_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$$

where the sum is over shifted semistandard Young tableaux T of a partition λ with distinct parts. The **Schur $Q-$polynomial** is

$$SQ_\lambda(\mathbf{x}) = 2^{\ell(\lambda)} SP_\lambda. \tag{2.8}$$

Proposition 2.44. *The symmetric functions $SP_\lambda(\mathbf{x})$ and $SQ_\lambda(S)$ are SNP and*

$$\mathsf{Newton}(SP_\lambda(\mathbf{x})) = \mathsf{Newton}(SQ_\lambda(\mathbf{x})) = \mathcal{P}_\lambda.$$

Proof. In fact,

$$SP_\lambda(\mathbf{x}) = P_\lambda(\mathbf{x}; t = -1);$$

see [Ste89], and $K_{\lambda,\lambda}(t) = 1$. Furthermore, SP_λ is Schur positive; see, e.g., [Ste89, p. 131–132]. Thus the result follows from Proposition 2.7(III). □

The modified Macdonald polynomial $\widetilde{H}_\lambda(\mathbf{x}; q, t)$ is a certain transformation of $P_\lambda(\mathbf{x}; q, t)$; see Definition 1.18.

Proposition 2.45. *For any $q_0, t_0 > 0$, we have $\widetilde{H}_\lambda(\mathbf{x}; q_0, t_0)$ is SNP and whenever $n \geq |\lambda|$,*

$$\mathsf{Newton}(\widetilde{H}_\lambda(\mathbf{x}; q_0, t_0)) = \mathcal{P}_{|\lambda|} \subset \mathbb{R}^n.$$

Proof. The formula of J. Hagland-M. Haiman-N. Loehr [HHL05] (see Theorem 1.19) states that

$$\widetilde{H}_\lambda(\mathbf{x}; q_0, t_0) = \sum_F \mathbf{x}^F q_0^{\mathsf{inv}(F)} t_0^{\mathsf{maj}(F)},$$

where the sum is over all fillings F of λ. Thus, for $q_0, t_0 > 0$, every monomial of degree $|\lambda|$ appears. □

However, $\widetilde{H}_{(3,1,1)}(x_1, x_2, x_3, x_4, x_5; q, t)$ is not SNP as it misses the monomial $qtx_3x_4^4$.

Example 2.46 (Modified q, t-Kostka polynomials are not SNP). Consider the expansion

$$\widetilde{H}_\lambda(\mathbf{x}; q, t) = \sum_{\mu \in \mathsf{Par}(|\lambda|)} \widetilde{\mathsf{K}}_{\lambda,\mu}(q, t) s_\mu(\mathbf{x})$$

from Definition 1.18. The coefficients $\widetilde{\mathsf{K}}_{\lambda,\mu}(q, t)$ are the (modified) q, t-Kostka coefficients. Now,

$$\widetilde{\mathsf{K}}_{(2,2,2),(3,3)}(q, t) = qt^4 + qt^3 + q^3 + qt^2 + t^3$$

is not SNP as it is missing $q^2 t$. Hence, $\widetilde{\mathsf{K}}_{\lambda,\mu}(q, t)$ need not be SNP. □

Let γ be a weak composition and let $t_{ij}(\gamma)$ swap positions i and j of γ. Furthermore, let

$$m_{ij}(\gamma) = \gamma + e_i - e_j$$

where e_i is the vector with a 1 in position i and 0's elsewhere. S. Sahi [Sah00] defined a partial order on weak compositions $<_S$ whose covering relations are

- if $i < j$ and $\gamma_j > \gamma_i$ then $t_{ij}(\gamma) <_S \gamma$, and
- if $i < j$ and $\gamma_j - \gamma_i > 1$ then $m_{ij}(\gamma) <_S t_{ij}(\gamma)$;

see [HHL08, Section 2.1]. It is part of a definition of the nonsymmetric Macdonald polynomial $E_\gamma(\mathbf{x}; q, t)$ that

$$E_\gamma(\mathbf{x}; q, t) = \mathbf{x}^\gamma + \sum_{\substack{\delta \in \mathsf{WC}(|\gamma|) \\ \delta <_S \gamma}} \mathsf{d}_{\gamma,\delta}(q, t) \mathbf{x}^\delta$$

where $\mathsf{d}_{\gamma,\delta}(q,t) \in \mathbb{Q}(q,t)$ [HHL08]. S. Sahi [Sah00] proved each $\mathsf{d}_{\gamma,\delta}(q,t) \not\equiv 0$. Let $\widehat{\mathcal{P}}_\gamma$ be the convex hull of all weak compositions δ such that $\delta \leq_S \gamma$. Thus $\widehat{\mathcal{P}}_\gamma$ is the Newton polytope of $E_\gamma(\mathbf{x}; q_0, t_0)$ for any generic choice of $(q_0, t_0) \in \mathbb{C}^2$. The conjecture below says $E_\gamma(\mathbf{x}; q, t)$ is "generically SNP".

Conjecture 2.47. *If $\delta \in \widehat{\mathcal{P}}_\gamma$ and $\delta \in \mathbb{Z}^n_{\geq 0}$ then $\delta \leq_S \gamma$.*

Conjecture 2.47 has been checked for $n \leq 7$ and whenever $|\gamma| \leq 7$.

2.3.2 Keys and Demazure atoms

We now investigate SNP for two specializations of $E_\gamma(\mathbf{x}; q, t)$: the key polynomials $\mathtt{Key}_\gamma = E_\gamma(\mathbf{x}; \infty, \infty)$ and the Demazure atoms $\mathtt{Atom}_\gamma = E_\gamma(\mathbf{x}; 0, 0)$ defined in Definition 1.21. As in Section 1.5, define D_γ to be the skyline diagram with a left-justified row of γ_i boxes in row i.

Conjecture 2.48. $\mathcal{S}_{\mathsf{D}_\gamma} = \mathsf{Newton}(\mathtt{Key}_\gamma)$.

We have a combinatorial proof (omitted here) of the "$\supseteq$" part of Conjecture 2.48; see Remark 2.85. This conjecture has been resolved to be true by A. Fink, K. Mészáros, and A. St. Dizier (Theorem 10, [FMS17]).

Conjecture 2.49. *The key polynomial $\mathtt{Key}_\gamma$ has SNP.*

This conjecture has also been resolved as true by A. Fink, K. Mészáros, and A. St. Dizier (Corollary 8, [FMS17]). We have a second conjectural description of $\mathsf{Newton}(\mathtt{Key}_\gamma)$. For any (weak) composition γ, let $\delta <_{\mathtt{Key}} \gamma$ be the partial order with covering relations

- if $i < j$ and $\gamma_j > \gamma_i$ then $t_{ij}(\gamma) <_{\mathtt{Key}} \gamma$, and
- if $i < j$ and $\gamma_j - \gamma_i > 1$ then $m_{ij}(\gamma) <_{\mathtt{Key}} \gamma$.

Observe that $\delta <_{\mathtt{Key}} \gamma$, then $\delta <_S \gamma$. However, the converse fails as

$$m_{12}((0,2)) = (1,1) <_S (2,0) = t_{12}((0,2))$$

but one does not have $(1,1) <_{\mathtt{Key}} (2,0)$.

Conjecture 2.50.

$$\mathtt{Key}_\gamma = \mathbf{x}^\gamma + \sum_{\substack{\delta \in \mathsf{WC}(|\gamma|) \\ \delta <_{\mathtt{Key}} \gamma}} \mathsf{k}_{\gamma,\delta} \mathbf{x}^\delta$$

with $\mathsf{k}_{\gamma,\delta} > 0$ *for all* $\delta <_{\mathtt{Key}} \gamma$*, and thus*

$$\mathsf{Newton}(\mathtt{Key}_\gamma) = \{\delta | \delta \leq_{\mathtt{Key}} \gamma\}.$$

Recall from Section 1.5, that for two weak compositions γ and δ we write

$$\gamma \ll \delta \text{ if } \lambda(\gamma) = \lambda(\delta) \text{ and } w_\gamma \leq_B w_\delta.$$

Theorem 2.51. *If* $\delta \ll \gamma$ *then* δ *is a vertex of* $\mathsf{Newton}(\mathtt{Key}_\gamma)$*.*

Conjecture 2.52. *The converse of Theorem 2.51 holds.*

Our proof of Theorem 2.51 uses the decomposition of $\mathtt{Key}_\gamma$ into Demazure atoms given in Theorem 1.26:

$$\mathtt{Key}_\gamma = \sum_{\substack{\delta \in \mathsf{PermutWC}(\lambda(\gamma)) \\ \delta \ll \gamma}} \mathtt{Atom}_\delta. \tag{2.9}$$

By the way,

Conjecture 2.53. *The Demazure atom* $\mathtt{Atom}_\gamma$ *has SNP.*

Conjectures 2.50, 2.52 and 2.53 have been checked for $|\alpha| \leq 7$ where α has at most three parts of size zero.

Proposition 2.54. *Suppose* $\delta \ll \gamma$*. Let* $\lambda = \lambda(\delta) = \lambda(\gamma)$*. Then*

$$\{\lambda\} \subseteq \mathsf{Newton}(\mathtt{Key}_\delta) \subseteq \mathsf{Newton}(\mathtt{Key}_\gamma) \subseteq \mathcal{P}_\lambda \subseteq \mathbb{R}^n,$$

where n *is the position of the last nonzero part of* γ*.*

Proof. Using (2.9) twice, we have

$$\begin{aligned}
\mathtt{Key}_\gamma &= \sum_{\substack{\kappa \in \mathsf{PermutWC}(\lambda) \\ \kappa \ll \gamma}} \mathtt{Atom}_\kappa \\
&= \sum_{\substack{\kappa \in \mathsf{PermutWC}(\lambda) \\ \kappa \ll \delta}} \mathtt{Atom}_\kappa + \sum_{\substack{\kappa \in \mathsf{PermutWC}(\lambda) \\ \kappa \ll \gamma \\ \kappa \not\ll \delta}} \mathtt{Atom}_\kappa \\
&= \mathtt{Key}_\delta + \sum_{\substack{\kappa \in \mathsf{PermutWC}(\lambda) \\ \kappa \ll \gamma \\ \kappa \not\ll \delta}} \mathtt{Atom}_\kappa.
\end{aligned}$$

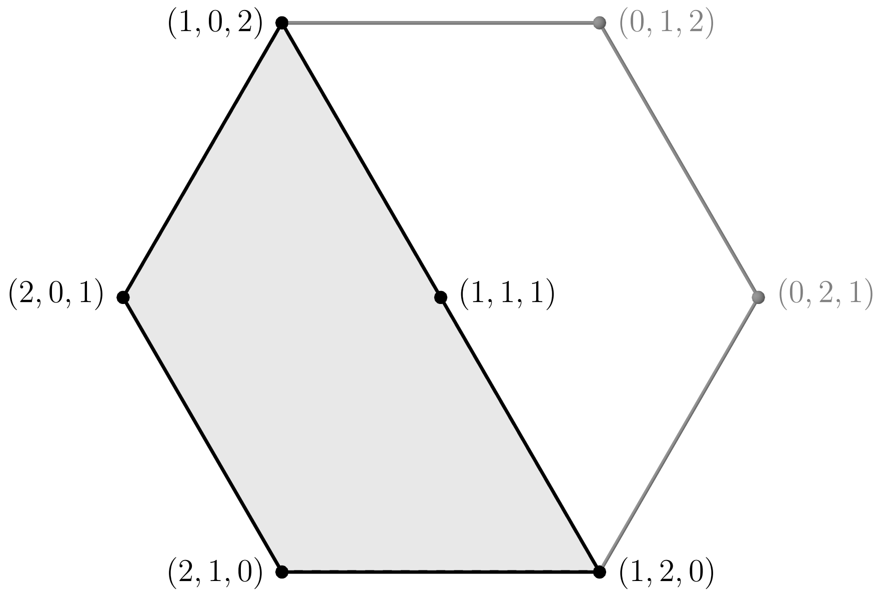


Figure 2.2: The permutahedron for $\lambda = (2, 1, 0)$. The shaded region is $\mathsf{Newton}(\mathtt{Key}_{1,0,2})$. See Proposition 2.54.

Since each $\mathtt{Atom}_\kappa$ is monomial positive [Mas09, Theorem 1.1],

$$\mathsf{Newton}(\mathtt{Key}_\delta) \subseteq \mathsf{Newton}(\mathtt{Key}_\gamma).$$

Now, λ is $\ll$-minimum among rearrangements of λ. By definition $\mathtt{Key}_\lambda = \mathbf{x}^\lambda$. This explains the leftmost containment.

Similarly, λ^* is the $\ll$-maximum among rearrangements of λ in $\mathbb{Z}^n$ and $\mathtt{Key}_{\lambda^*} = s_\lambda$ (see Equation (1.6)). However we know $\mathsf{Newton}(s_\lambda) = \mathcal{P}_\lambda$. $\square$

Figure 2.2 provides an example of this nesting property for the Newton polytopes of $\mathtt{Key}_\gamma$.

Lemma 2.55. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are polytopes such that $\mathcal{P} \subseteq \mathcal{Q}$. If v is a vertex of $\mathcal{Q}$ and $v \in \mathcal{P}$, then v is a vertex of $\mathcal{P}$.*

Proof. The point v is a vertex of $\mathcal{Q}$ if and only if there is a separating hyperplane H, i.e., there exists a vector $\mathbf{c}$ such that $\mathbf{c}^T v \leq \mathbf{c}^T y$ for all $y \in \mathcal{Q}$. Since $\mathcal{P} \subseteq \mathcal{Q}$, H works for $\mathcal{P}$ also. $\square$

Proof of Theorem 2.51: Now,

$$\mathtt{Key}_\gamma = \mathbf{x}^\gamma + (\text{positive sum of monomials});$$

see, e.g., [RS95, Corollary 7]. Hence, γ is in $\mathsf{Newton}(\mathtt{Key}_\gamma)$. By Proposition 2.54,

$$\delta \in \mathsf{Newton}(\mathtt{Key}_\delta) \subseteq \mathsf{Newton}(\mathtt{Key}_\gamma) \text{ if } \delta \ll \gamma.$$

Again applying Proposition 2.54 we have that $\mathsf{Newton}(\mathtt{Key}_\gamma) \subseteq \mathcal{P}_{\lambda(\gamma)}$. Now we are done by combining Proposition 2.7(II) and Lemma 2.55. □

2.4 Quasisymmetric functions

Recall from Section 1.3 that QSym is the ring of quasisymmetric functions, M_α is the monomial quasisymmetric basis (Definition 1.12), and F_α is Gessel's fundamental basis (Definition 1.13).

Example 2.56 (The monomial quasisymmetric function M_α need not be SNP). For example, $M_{(2)} = p_2 = x_1^2 + x_2^2 + \cdots$ does not have SNP. □

Theorem 2.57. *We have* $\mathsf{Newton}(F_\alpha(x_1,\ldots,x_n)) = \mathsf{Newton}(M_\alpha(x_1,\ldots,x_n)) \subset \mathbb{R}^n$. *The vertices of this polytope are* $\{\gamma \in \mathbb{Z}^n_{\geq 0} : \gamma^+ = \alpha\}$.

Proof. Recall from (1.3),

$$F_\alpha = \sum_{\substack{\beta \in \mathsf{Comp}(|\alpha|) \\ \beta \preceq \alpha}} M_\beta.$$

Then, each M_β is a positive sum of monomials and M_α appears in the expansion (1.3). Thus,

$$\mathsf{Newton}(F_\alpha(x_1,\ldots,x_n)) \supseteq \mathsf{Newton}(M_\alpha(x_1,\ldots,x_n)).$$

Now suppose $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k_{>0}$ and $\widehat{\beta} \preceq \beta$ where

$$\widehat{\beta} = (\beta_1, \beta_2, \ldots, \beta_i', \beta_i'', \ldots, \beta_k) \in \mathbb{Z}^{k+1}_{>0}$$

and $\beta_i = \beta_i' + \beta_i''$.

We wish to show

$$\mathsf{Newton}(M_{\widehat{\beta}}(x_1,\ldots,x_n)) \subseteq \mathsf{Newton}(M_\beta(x_1,\ldots,x_n)). \tag{2.10}$$

By induction, this implies the remaining containment

$$\mathsf{Newton}(F_\alpha(x_1,\ldots,x_n)) \subseteq \mathsf{Newton}(M_\alpha(x_1,\ldots,x_n)).$$

Suppose $\mathbf{x}^{\widetilde{\beta}}$ is a monomial of $M_{\hat{\beta}}$ and thus

$$\widetilde{\beta} = (\widetilde{\beta_1},\ldots,\widetilde{\beta_n}) \in \mathbb{Z}^n_{\geq 0}$$

where $(\widetilde{\beta})^+ = \widehat{\beta}$. Then,

$$\widetilde{\beta} = (0,\ldots,0,\beta_1,0,\ldots,0,\beta_2,\ldots,\beta_i',0,\ldots,0,\beta_i'',\ldots,\beta_k,0,\ldots,0)$$

where we are depicting the 0's inserted between components of $\widehat{\beta}$ to obtain $\widetilde{\beta}$.

Now let

$$\beta^\circ = (0,\ldots,0,\beta_1,0,\ldots,0,\beta_2,\ldots,\beta_i,0,\ldots,0,0,\ldots,\beta_k,0,\ldots,0)$$

and

$$\beta^\bullet = (0,\ldots,0,\beta_1,0,\ldots,0,\beta_2,\ldots,0,0,\ldots,0,\beta_i,\ldots,\beta_k,0,\ldots,0).$$

That is β° and $\beta^\bullet$ differ from $\widetilde{\beta}$ only by replacing β_i' and β_i'' by β_i, respectively.

Since $\beta_i',\beta_i'' \geq 0$, we have that

$$\widetilde{\beta} = \frac{\beta_i'}{\beta_i}\beta^\circ + \frac{\beta_i''}{\beta_i}\beta^\bullet$$

is a convex combination. This proves (2.10) and hence the asserted equality of Newton polytopes.

Every monomial of $M_\alpha(x_1,\ldots,x_n)$ is a monomial of $m_\alpha(x_1,\ldots,x_n)$. Therefore,

$$\mathsf{Newton}(M_\alpha(x_1,\ldots,x_n)) \subseteq \mathsf{Newton}(m_\alpha(x_1,\ldots,x_n)).$$

Recall,

$$\mathsf{Newton}(m_\alpha(x_1,\ldots,x_n)) = \mathcal{P}_{\lambda(\alpha)} \subseteq \mathbb{R}^n.$$

The vertices of $\mathcal{P}_{\lambda(\alpha)}$ are all rearrangements of α; cf. Proposition 2.7(II). Thus, every exponent vector of $m_\alpha(x_1,\ldots,x_n)$ is a vertex of $\mathcal{P}_{\lambda(\alpha)}$. Furthermore, the containment argument above shows that every lattice point γ of $\mathsf{Newton}(F_\alpha)$ (coming from a monomial) such that $\gamma \neq \alpha$ is a convex combination of $\{\gamma|\lambda(\gamma) = \lambda(\alpha)\}$, and thus cannot be a vertex of $\mathsf{Newton}(F_\alpha)$.

Since all vertices of $\mathsf{Newton}(F_\alpha)$ come from a monomial of F_α, to obtain the final claim of the theorem we may appeal to Lemma 2.55. $\square$

Example 2.58 (The fundamental quasisymmetric function F_α need not be SNP). One can check that

$$F_{(2,2)} = M_{(2,2)} + M_{(2,1,1)} + M_{(1,1,2)} + M_{(1,1,1,1)}.$$

Thus, $(0,1,2,1) = \frac{1}{2}(0,2,2,0) + \frac{1}{2}(0,0,2,2) \in \mathsf{Newton}(F_{(2,2)})$. However, $(0,1,2,1)$ is not an exponent vector of $F_{(2,2)}$. Hence $F_{(2,2)}$ is not SNP. $\square$

The quasisymmetric Schur function (Definition 1.28) has the form

$$S_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} \mathtt{Atom}_\gamma.$$

Many aspects of quasi-Schur theory are parallel to Schur theory [HLMvW11a]. For instance, consider the transition between the S and M bases of QSym:

$$S_\alpha = \sum_{\beta \in \mathsf{Comp}(|\alpha|)} \overline{\mathsf{K}}_{\alpha,\beta} M_\beta$$

where $\overline{\mathsf{K}}_{\alpha,\beta}$ is the number of semistandard composition tableaux of shape α and content β. Hence $\overline{\mathsf{K}}_{\alpha,\beta}$ is an analogue of the Kostka coefficient. However, there are divergences from the perspective of Newton polytopes as seen in the next three examples:

Example 2.59 (The quasisymmetric Schur function S_α need not be SNP). An example is $S_{(2,1,3)}$. In at least four variables, $x_1x_2^2x_3^2x_4$ does not appear but $x_1^2x_2^2x_3^2$ and $x_2^2x_3^2x_4^2$ both do. Nonetheless, it should be interesting to describe the Newton polytope, and to characterize when S_α is SNP. $\square$

Example 2.60. In the symmetric function case,

$$\mathsf{Newton}(s_\lambda(x_1, \ldots, x_n)) = \mathsf{Newton}(m_\lambda(x_1, \ldots, x_n)) = \mathcal{P}_\lambda \subset \mathbb{R}^n.$$

However,

$$(0,0,2,2) \in \mathsf{Newton}(S_{(1,3)}(x_1,x_2,x_3,x_4)) \quad \text{but} \quad (0,0,2,2) \notin \mathsf{Newton}(M_{(1,3)}(x_1,x_2,x_3,x_4)).$$

Hence $\mathsf{Newton}(S_\alpha(x_1, \ldots, x_n)) \neq \mathsf{Newton}(M_\alpha(x_1, \ldots, x_n))$ in general. $\square$

Example 2.61. We may define a dominance order $\preceq'_D$ on strict compositions by $\alpha \preceq'_D \beta$ if $\mathsf{Newton}(M_\alpha) \subseteq \mathsf{Newton}(M_\beta)$. The above example shows that $\overline{\mathsf{K}}_{\alpha,\beta} > 0$ if and only if $\beta \preceq'_D \alpha$ is not generally true. This is in contrast with (2.3). $\square$

2.5 Schubert polynomials and variations

2.5.1 The Schubert SNP conjectures

A. Lascoux and M.-P. Schützenberger introduced the Schubert polynomials [LS82a], which are defined in Section 1.4 (see Definition 1.15).

Example 2.62 (The operators ∂_i and π_i do not preserve SNP). This polynomial is SNP:

$$f = x_1^4 + x_1^3x_2 + x_1^2x_2^2 + 2x_1x_2^3.$$

However

$$\partial_1(f) = x_1^3 + x_2^3$$

is not SNP.

Since $\pi_i(g) = \partial_i(x_i \cdot g)$, if we set

$$g = x_1^3 + x_1^2x_2 + x_1x_2^2 + 2x_2^3$$

we have $\pi_1(g) = \partial_1(f)$. Hence, π_i does not preserve SNP. $\square$

However, despite this:

Conjecture 2.63. *The Schubert polynomial $\mathfrak{S}_w$ has SNP.*

This conjecture has since been confirmed by A. Fink, K. Mészáros, and A. St. Dizier (Corollary 8, [FMS17]).

The **double Schubert polynomial** $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ is defined by setting

$$\mathfrak{S}_{w_0}(\mathbf{x};\mathbf{y}) = \prod_{\substack{i,j \in mathbbZ_{>0} \\ i+j \leq n}} (x_i - y_j)$$

and recursively determining $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ for $w \neq w_0$ precisely as for $\mathfrak{S}_w(\mathbf{x})$.

Conjecture 2.64. *The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ is SNP.*

We have checked Conjecture 2.64 for $n \leq 5$ (and many other cases). Since $\mathfrak{S}_w(\mathbf{x}; 0) = \mathfrak{S}_w(\mathbf{x})$, Conjecture 2.64 generalizes Conjecture 2.63.

Example 2.65 (Double Schubert polynomials are generalized resultants). A permutation is dominant if its diagram is a partition. Then for w equal to the dominant permutation

$$w(i) = \begin{cases} n+i & i = 1, \ldots, m \\ i-m & i = m+1, \ldots, m+n \end{cases},$$

$$\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i - y_j).$$

One reference is [Man01, Proposition 2.6.7]. This has the same Newton polytope as $R(f, g)$. Thus Conjecture 2.64 is a generalization of Theorem 2.23. □

A. Lascoux and M.-P. Schützenberger also introduced the family of **Grothendieck polynomials** [LS82b]; see Definition 1.30.

Conjecture 2.66. *The Grothendieck polynomial $\mathfrak{G}_w$ has SNP.*

Conjecture 2.66 has been exhaustively checked for $n \leq 7$. Conjecture 2.66 generalizes Conjecture 2.63 since

$$\mathsf{Newton}(\mathfrak{S}_w) = \mathsf{Newton}(\mathfrak{G}_w) \cap \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \alpha_i = \#\mathsf{D}_w \right\}.$$

This conjecture has been proven in the two special cases below.

Theorem 2.67 (Escobar-Yong [EY17]). *The symmetric Grothendieck function Ks_λ is SNP.*

Theorem 2.68 (Mészáros-St. Dizier, Theorem C [MS17]). *For π a dominant permutation, $K\mathfrak{S}_{1\pi}$ has SNP.*

The Lascoux polynomials $\widehat{K\mathtt{Key}}_\gamma$ and Lascoux atoms $\widehat{K\mathtt{Atom}}_\gamma$ (see Definition 1.34) that arise in combinatorial K-theory also seem to be SNP.

Conjecture 2.69. *The Lascoux polynomial $\widehat{K\mathtt{Key}}_\gamma$ has SNP.*

Conjecture 2.70. *The Lascoux atom $\widehat{K\mathtt{Atom}}_\gamma$ has SNP.*

Conjectures 2.69 and 2.70 have been verified for $|\alpha| \leq 7$ where α has at most three parts of size zero.

2.5.2 Stanley polynomials and the stable limit of Conjecture 2.63

For $w \in S_n$, let $1^t \times w \in S_{t+n}$ be the permutation defined

$$1^t \times w(i) = \begin{cases} i & 1 \leq i \leq t \\ n+i & t+1 \leq i \leq t+n \end{cases}.$$

The **Stanley symmetric polynomial** (also known as the **stable Schubert polynomial**) is defined by

$$F_w = \lim_{t \to \infty} \mathfrak{S}_{1^t \times w} \in \mathsf{Sym}.$$

The power series F_w is well-defined and was originally introduced by R. P. Stanley in [Sta84]. The next result is a "stable limit" version of Conjecture 2.76.

Theorem 2.71. *The Stanley symmetric function $F_w \in \mathsf{Sym}$ is SNP.*

Our proof rests on:

Theorem 2.72 (Theorems 3.2, 4.1, [Sta84])**.** *For*

$$F_w = \sum_{\lambda \in \mathsf{Par}(\ell(w))} \mathsf{a}_{w,\lambda} s_\lambda,$$

$\mathsf{a}_{w,\lambda} \geq 0$ *and there exists $\lambda(w)$ and $\mu(w)$ such that if $\mathsf{a}_{w,\lambda} \neq 0$, then $\lambda(w) \leq_D \lambda \leq_D \mu(w)$.*

Proof of Theorem 2.71: Combine Theorem 2.72 and Proposition 2.7(III). □

Corollary 2.73. *Any skew-Schur polynomial $s_{\lambda/\mu}(\mathbf{x})$ has SNP.*

Proof. To every skew shape λ/μ there is a 321-avoiding permutation $w_{\lambda/\mu}$ with the property that $F_{w_{\lambda/\mu}}(\mathbf{x}) = s_{\lambda/\mu}$ [BJS93]. Now apply Theorem 2.71. □

Let

$$S_{\infty,\ell} = \{w \in S_\infty : \ell(w) = \ell\}.$$

Declare

$$u \leq_D v \text{ for } u, v \in S_{\infty,\ell} \text{ if } \mathsf{Newton}(\mathfrak{S}_u) \subseteq \mathsf{Newton}(\mathfrak{S}_v).$$

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0)$, define $w_{\lambda,k} \in S_{\lambda_1+k}$ to be the unique permutation that satisfies

$$w_{\lambda,k}(i) = \lambda_{k-i+1} + i \text{ for } 1 \leq i \leq k$$

and is **Grassmannian**, i.e., it has at most one descent, at position k. Then one has

$$\mathfrak{S}_{w_{\lambda,k}} = s_\lambda(x_1, \ldots, x_k).$$

We now show that $(S_\infty, \leq_D)$ extends $(\mathsf{Par}(n), \leq_D)$:

Proposition 2.74. *Suppose $\lambda, \mu \in \mathsf{Par}(n)$ and let $k = \max\{\ell(\lambda), \ell(\mu)\}$. Then $\lambda \leq_D \mu$ if and only if $w_{\lambda,k} \leq_D w_{\mu,k}$.*

Proof. Since $\mathfrak{S}_{w_{\lambda,k}}(x_1, \ldots, x_k) = s_\lambda(x_1, \ldots, x_k)$ and $\mathfrak{S}_{w_{\mu,k}}(x_1, \ldots, x_k) = s_\mu(x_1, \ldots, x_k)$,

$$\mathcal{P}_\lambda = \mathsf{Newton}(s_\lambda(x_1, \ldots, x_k)) = \mathsf{Newton}(\mathfrak{S}_{w,\lambda}(x_1, \ldots, x_k)) \subseteq \mathbb{R}^k.$$

The same statement holds where we replace λ by μ. Now apply Rado's theorem (2.3). $\square$

Figure 2.3 shows part of $(S_{\infty,2}, \leq_D)$. From this one can see that the poset is not graded, just like dominance order $\leq_D$ on partitions is not graded. Unlike $\leq_D$, it is not a lattice: in Figure 2.3, the elements 231456 and 312456 do not have a unique least upper bound as 142356 and 214356 are incomparable minimal upper bounds.

Theorem 2.75. *Every two elements $u, v \in S_{\infty,\ell}$ have an upper bound under $\leq_D$.*

Proof. Suppose $\{\alpha_i\}$ and $\{\beta_j\}$ are the exponent vectors of $\mathfrak{S}_u$ and $\mathfrak{S}_v$, respectively. It suffices to show there exists $w \in S_{\infty,\ell}$ such that

$$\mathfrak{S}_w = \sum_i \mathbf{x}^{\alpha_i} + \sum_j \mathbf{x}^{\beta_j} + (\text{positive sum of monomials}).$$

We first show that there is a F_w such that each $s_{\lambda(\alpha_i)}$ and $s_{\lambda(\beta_j)}$ appear (possibly with multiplicity). A theorem of S. Fomin-C. Greene [FG98] states that

$$F_w = \sum_{\nu \in \mathsf{Par}(\ell(w))} \mathsf{a}_{w,\nu} s_\nu$$

where $\mathsf{a}_{w,\nu}$ is the number of semistandard tableaux of shape ν such that the top-down, right-to-left reading word is a reduced word for w. Let

$$w = s_1 s_3 s_5 \cdots s_{2\ell-1}.$$

Clearly this decomposition is reduced. All reduced words of w are obtained by permuting the simple transpositions.

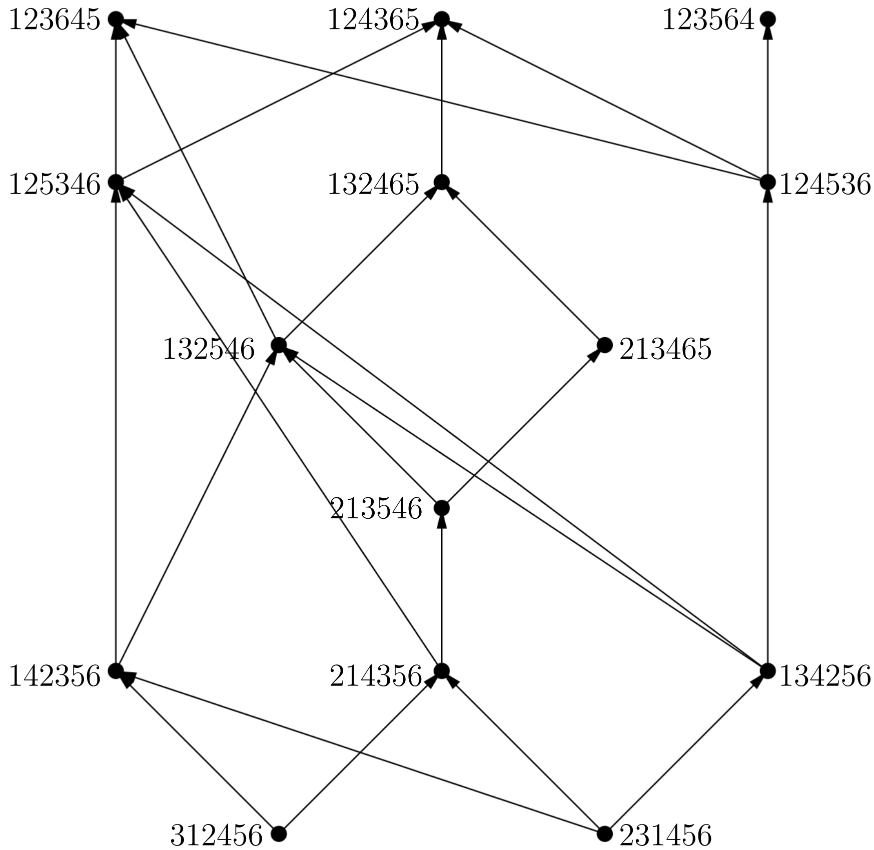


Figure 2.3: The S_6 part of the Hasse diagram of $(S_{\infty,2}, \leq_D)$

Filling *any* shape of size ℓ by successively placing $1, 3, 5, \ldots, 2\ell - 1$ along rows in left to right order gives a semistandard tableaux. Thus every s_μ where $\mu \vdash \ell$ appears in F_w. In particular each $s_{\lambda(\alpha_i)}$ and each $s_{\lambda(\beta_j)}$ appears. Since $\mathbf{x}^{\lambda(\alpha_i)}$ appears in $s_{\lambda(\alpha_i)}$, by symmetry of $s_{\lambda(\alpha_i)}$, $\mathbf{x}^{\alpha_i}$ appears as well. That is, $\mathbf{x}^{\alpha_i}$ appears in F_w. Similarly $\mathbf{x}^{\beta_j}$ appears in F_w.

By definition, for any monomial $\mathbf{x}^\gamma$ appearing in F_w, there is a finite N_γ such that $\mathbf{x}^\gamma$ appears in $\mathfrak{S}_{1^{N_\gamma} \times w}$. It suffices to pick N larger than all N_{α_i} and N_{β_j}. $\square$

2.5.3 Inequalities for $\mathsf{Newton}(\mathfrak{S}_w)$

Let

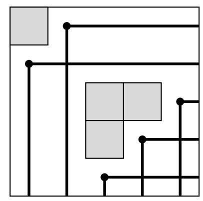
$$\mathsf{D}_w = \{(i,j) : 1 \leq i, j \leq n, w(i) > j \text{ and } w^{-1}(j) > i\}$$

be the **Rothe diagram** of a permutation $w \in S_n$ (see Section 1.1).

Conjecture 2.76. $\mathcal{S}_{\mathsf{D}_w} = \mathsf{Newton}(\mathfrak{S}_w)$.

This has been confirmed by A. Fink, K. Mészáros, and A. St. Dizier (Theorem 10, [FMS17]). Notice that Conjecture 2.76 is equivalent to the assertion that $w \leq_D v$ if and only if $\theta_{\mathsf{D}_w}(S) \leq \theta_{\mathsf{D}_v}(S)$ for all $S \subseteq [n]$.

Example 2.77. Suppose $w = 21543$, the Rothe diagram D_w is given by



One can check that the defining inequalities are

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4$$

$$\alpha_1 \leq 3,\ \alpha_2 \leq 2,\ \alpha_3 \leq 2,\ \alpha_4 \leq 1$$

$$\alpha_1 + \alpha_2 \leq 4,\ \alpha_1 + \alpha_3 \leq 4,\ \alpha_1 + \alpha_4 \leq 4,\ \alpha_2 + \alpha_3 \leq 3,\ \alpha_2 + \alpha_4 \leq 3, \alpha_3 + \alpha_4 \leq 3$$

$$\alpha_1 + \alpha_2 + \alpha_3 \leq 4,\ \alpha_1 + \alpha_2 + \alpha_4 \leq 4,\ \alpha_2 + \alpha_3 + \alpha_4 \leq 3$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 4.$$

together with $\alpha_i \geq 0$ for each i. The polytope is depicted in Section 2.1. □

One can uniquely reconstruct $u \in S_\infty$ with the defining inequalities.

Proposition 2.78. *If $u, v \in S_n$ are of the same length and $\theta_{\mathsf{D}_u}(S) = \theta_{\mathsf{D}_v}(S)$ for all $S = \{i, i+1, \ldots, n\}$ where $1 \leq i \leq n$, then $u = v$.*

Proof. Let
$$c_i(\pi) = \#\{j : (i,j) \in \mathsf{D}_\pi\}.$$

Thus $(c_1(\pi), c_2(\pi), \ldots)$ is the Lehmer code of π. The Lehmer code uniquely determines $\pi \in S_\infty$; see, e.g., [Man01, Proposition 2.1.2]. Hence it suffices to show the codes of u and v are the same. This follows from:

$$\sum_{j=1}^{i} c_j(u) = \ell - \theta_{\mathsf{D}_u}(\{i+1, i+2, \ldots, n\}) = \ell - \theta_{\mathsf{D}_v}(\{i+1, i+2, \ldots, n\}) = \sum_{j=1}^{i} c_j(v),$$

for $i = 1, 2, \dots n-1$. $\square$

The inequalities of $\mathcal{S}_{\mathsf{D}}$ are in general redundant. If

$$\theta_{\mathsf{D}}(S) = \theta_{\mathsf{D}}(T) \text{ and } S \supseteq T \tag{2.11}$$

then the inequality

$$\sum_{i \in T} \alpha_i \leq \theta_{\mathsf{D}}(T)$$

is unnecessary. Similarly, if

$$S = \bigsqcup_i \hat{s}_i \ \text{ and } \ \theta_{\mathsf{D}}(S) = \sum_i \theta_{\mathsf{D}}(\hat{s}_i) \tag{2.12}$$

then the S-inequality is implied by the $\hat{s}_i$ inequalities.

Problem 2.79. *Give the minimal set of inequalities associated to* D_w *(or more generally, any* D*).*

Example 2.80. Continuing Example 2.77, minimal inequalities are

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4$$

$$\alpha_1 \leq 3,\ \alpha_2 \leq 2,\ \alpha_3 \leq 2,\ \alpha_4 \leq 1$$

$$\alpha_1 + \alpha_2 \leq 4,\ \alpha_1 + \alpha_3 \leq 4,\ \alpha_2 + \alpha_3 \leq 3,$$

$$\alpha_2 + \alpha_3 + \alpha_4 \leq 3,$$

combined with positivity. This minimization is obtained using reductions (2.11) and (2.12).

Example 2.81. If $w = 23154$ then using the reductions (2.11) and (2.12) leaves:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3, \qquad \alpha_3 + \alpha_4 \leq 1, \qquad \alpha_1 + \alpha_3 + \alpha_4 \leq 2, \qquad \alpha_2 + \alpha_3 + \alpha_4 \leq 2.$$

However, $\alpha_3 + \alpha_4 \leq 1$ is actually not necessary. Thus reductions (2.11) and (2.12) are not sufficient for determining minimal inequalities. $\square$

Given a polytope P, its **Ehrhart polynomial**, denoted $L_P(t)$, is the polynomial such that for $t \in \mathbb{Z}_{\geq 1}$, $L_P(t)$ equals the number of lattice points in the polytope tP. E. Ehrhart [Ehr62] showed that for a polytope of dimension d in $\mathbb{R}^n$, $L_P(t)$ is in fact a polynomial of degree d. For more see, e.g., [BR07].

w	$\mathfrak{S}_w$	$\dim \mathcal{S}_{\mathsf{D}_w}$	vertices of $\mathcal{S}_{\mathsf{D}_w}$	$L_{\mathcal{S}_{\mathsf{D}_w}}(t)$
1243	$x_1 + x_2 + x_3$	2	$(1,0,0),(0,1,0),(0,0,1)$	$\frac{1}{2}t^2 + \frac{3}{2}t + 1$
1324	$x_1 + x_2$	1	$(1,0),(0,1)$	$t+1$
1342	$x_1x_2 + x_1x_3 + x_2x_3$	2	$(1,1,0),(1,0,1),(0,1,1)$	$\frac{1}{2}t^2 + \frac{3}{2}t + 1$
1423	$x_1^2 + x_1x_2 + x_2^2$	1	$(2,0),(0,2)$	$2t+1$
1432	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3$ $+x_1x_2x_3 + x_2^2x_3$	2	$(2,0,1),(1,2,0),$ $(2,1,0),(0,2,1)$	$\frac{3}{2}t^2 + \frac{5}{2}t + 1$
2143	$x_1^2 + x_1x_2 + x_1x_3$	2	$(2,0,0),(1,1,0),(1,0,1)$	$\frac{1}{2}t^2 + \frac{3}{2}t + 1$
2413	$x_1^2x_2 + x_1x_2^2$	1	$(2,1),(1,2)$	$t+1$
2431	$x_1^2x_2x_3 + x_1x_2^2x_3$	1	$(2,1,1),(1,2,1)$	$t+1$
3142	$x_1^2x_2 + x_1^2x_3$	1	$(2,1,0),(2,0,1)$	$t+1$
4132	$x_1^3x_2 + x_1^3x_3$	1	$(3,1,0),(3,0,1)$	$t+1$

Table 2.1: Additional data about positive dimensional Schubitopes $\mathcal{S}_{\mathsf{D}_w}$ for $w \in S_4$.

Conjecture 2.82. *If $L_{\mathcal{N}(\mathcal{S}_{\mathsf{D}_w})}(t) = c_dt^d + \cdots + c_0$, then $c_i > 0$ for $i = 0, \ldots, d$.*

Conjecture 2.82 also seems true for $\mathcal{S}_{\mathsf{D}}$ where D is arbitrary. We have exhaustively checked this for $n = 4$ and many random cases for $n = 5$.

Table 2.1 gives some data about the positive dimensional Schubitopes $\mathcal{S}_{\mathsf{D}_w}$ for $w \in S_4$.

2.5.4 Relationship of the Schubitope to Kohnert's rule

In Section 1.4, we described Kohnert's combinatorial rule for $\mathfrak{S}_w$ [Koh91]. With this rule in hand, one obtains part of Conjecture 2.63, see Proposition 2.83 below. Even though Conjecture 2.63 has been confirmed, it is open to give a combinatorial proof of the inequalities and so we give one for one direction of the containment below.

Proposition 2.83. $\mathcal{S}_{\mathsf{D}_w} \supseteq \mathsf{Newton}(\mathfrak{K}_w)$.

Proof. Consider a diagram $D \in \mathsf{Koh}(w)$ such that $c(D) = \alpha$. Each Kohnert move preserves the number of boxes. Hence $\sum_{i=1}^n \alpha_i = \#\mathsf{D}_w$ holds.

Now fix a column c and $S \subseteq [n]$. Compare the positions of the boxes of D to the boxes of D_w. Let

$$T_{D,S,c} = \#\text{boxes of } D \text{ in the rows of } S \text{ and column } c.$$

Also, let $U_{D,S,c}$ be the number of pairs (r, r'), **with no coordinate repeated**, such that

$$r \in S, r' \notin S, r < r', (r, c) \notin \mathsf{D}_w \text{ but } (r', c) \in \mathsf{D}_w.$$

Since Kohnert moves only bring boxes in from lower rows into higher rows (i.e., boxes migrate from the south),

$$T_{D,S,c} \leq T_{\mathsf{D}_w,S,c} + U_{\mathsf{D}_w,S,c}.$$

Now it is easy to check that

$$\theta_D^c(S) = T_{\mathsf{D}_w,S,c} + U_{\mathsf{D}_w,S,c}.$$

Since α_i counts the number of boxes in row i of D, we have

$$\sum_{i\in S}\alpha_i = \sum_{c=1}^{n} T_{D,S,c} \leq \sum_{c=1}^{n} T_{\mathsf{D}_w,S,c} + U_{\mathsf{D}_w,S,c} = \sum_{c=1}^{n} \theta_D(S) = \theta_D(S),$$

as required. □

Remark 2.84. *Unlike the computation of each $\theta_D^c(s)$, the Kohnert moves are not column independent. Perhaps surprisingly, Conjecture 2.63 says that the* a priori *coarse upper bound on $\sum_{i\in s}\alpha_i$ captures all monomials appearing in the Schubert polynomial.* □

Remark 2.85. *Kohnert's rule extends to key polynomials (with proof). Hence the same argument (which we omit) establishes the "$\supseteq$" containment of Conjecture 2.48.* □

Fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let D_λ be the Young diagram for λ (in French notation) placed flush left in $n \times n$ (hence row n has λ_1 boxes).

Proposition 2.86 (The Schubitope is a generalized permutahedron)**.** $\mathcal{S}_{\mathsf{D}_\lambda} = \mathcal{P}_\lambda \subset \mathbb{R}^n$.

Lemma 2.87. *If $w(i) < w(i+1)$, then $\mathcal{S}_{\mathsf{D}_w}$ is symmetric about i and $i+1$. That is,*

$$(\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n) \in \mathcal{S}_{\mathsf{D}_w} \iff (\alpha_1, \alpha_2, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n) \in \mathcal{S}_{\mathsf{D}_w}.$$

Proof. Suppose $S \subseteq [n]$ such that $i \in S, i+1 \notin S$. Let S' be the set formed from S by replacing i with $i+1$. Then it suffices to show for any column c,

$$\theta_{\mathsf{D}_w}^c(S) = \theta_{\mathsf{D}_w}^c(S').$$

Since $w(i) < w(i+1)$, if $(i,c) \in \mathsf{D}_w$, then $(i+1,c) \in \mathsf{D}_w$ as well. There are three cases:
Case 1: $((i,c),(i+1,c) \in \mathsf{D}_w)$. In $\mathsf{word}_{c,S}(\mathsf{D}_w)$, rows i and $i+1$ contribute $\star)$ whereas in $\mathsf{word}_{c,S'}(\mathsf{D}_w)$ the contribution is $)\star$. The $)$ does not change whether or not it is paired and thus $\theta_{\mathsf{D}_w}^c(S) = \theta_{\mathsf{D}_w}^c(S')$.

$\textsf{Case 2}$: $((i,c) \notin \mathsf{D}_w, (i+1,c) \in \mathsf{D}_w)$. In $\texttt{word}_{c,S}(\mathsf{D}_w)$, rows i and $i+1$ contribute (). In $\mathsf{word}_{c,S'}(\mathsf{D}_w)$, the contribution is $\star$. Both contribute 1 to $\theta^c_{\mathsf{D}_w}(S)$ and $\theta^c_{\mathsf{D}_w}(S')$ respectively. Hence $\theta^c_{\mathsf{D}_w}(S) = \theta^c_{\mathsf{D}_w}(S')$.
$\textsf{Case 3}$: $((i,c),(i+1,c) \notin \mathsf{D}_w)$. In both $\mathsf{word}_{c,S}(\mathsf{D}_w)$ and $\mathsf{word}_{c,S'}(\mathsf{D}_w)$, rows i and $i+1$ contribute (. The (does not change whether or not it is paired and so $\theta^c_{\mathsf{D}_w}(S) = \theta^c_{\mathsf{D}_w}(S')$. $\square$

Proof of Proposition 2.86: By Proposition 2.7(I),

$$\mathsf{Newton}(s_\lambda(x_1,\ldots,x_n)) = \mathcal{P}_\lambda \subseteq \mathbb{R}^n \tag{2.13}$$

Let $w_{\lambda,n}$ be the Grassmannian permutation associated to λ. This permutation only has descent at position n. Then

$$\mathfrak{S}_{w_{\lambda,n}} = s_\lambda(x_1,\ldots,x_n). \tag{2.14}$$

We next show that

$$\mathcal{S}_{\mathsf{D}_{w_{\lambda,n}}} = \mathsf{Newton}(\mathfrak{S}_{w_{\lambda,n}}). \tag{2.15}$$

The "$\supseteq$" containment of (2.15) is given by Proposition 2.83. In the case at hand, this proposition can be deduced from A. Kohnert's work [Koh91] who proved his conjecture for Grassmannian permutations. Below we will use that in *loc cit.*, A. Kohnert proved the Grassmannian case by giving a weight-preserving bijection $\phi : \mathsf{SSYT}(\lambda,[n]) \to \mathsf{Koh}(w_{\lambda,n})$, where $\mathsf{SSYT}(\lambda,[n])$ is the set of semistandard tableaux of shape λ with fillings using $1,2,\ldots,n$.

We now obtain the other containment of (2.15). Let $(\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathcal{S}_{\mathsf{D}_{w_{\lambda,n}}}$. In fact, $\mathsf{D}_{w_{\lambda,n}}$ differs from D_λ by removing empty columns and left justifying. Hence it is clear from the definition of $\theta_{\mathsf{D}_\lambda}(S)$ that

$$\sum_{i=1}^{t} \alpha_i \leq \sum_{i=1}^{t} \lambda_i \text{ for } t = 1,\ldots,n. \tag{2.16}$$

Lemma 2.87 implies that $\mathsf{D}_{w_{\lambda,n}}$ has an S_n-action by permutation of the coordinates. Hence if $\beta = \lambda(\alpha)$ is the decreasing rearrangement of α, then β also satisfies (2.16), where β replaces α. That is, $\beta \leq_D \lambda$.

Therefore by (1.1), $K_{\lambda,\beta} \neq 0$ and there exists a semistandard tableau of shape λ and content β. By the symmetry of $s_\lambda(x_1,\ldots,x_n)$ and the fact it is the weight-generating series for $\mathsf{SSYT}(\lambda,[n])$, there is a semistandard tableau U of shape λ and content α.

Now apply Kohnert's bijection ϕ to contain $D \in \mathsf{Koh}(w_{\lambda,n})$ with $c(D) = \alpha$, as desired. This completes the proof of (2.15).

Since D_w and D_λ only differ by a column permutation $\mathcal{S}_{\mathsf{D}_\lambda} = \mathcal{S}_{\mathsf{D}_{w_{\lambda,n}}}$. Now combine this with (2.15), (2.14) and (2.13). $\square$

The above result can be also deduced by comparing the inequalities of $\mathcal{S}_{\mathsf{D}_\lambda}$ with those for $\mathcal{P}_\lambda$. However, the above argument has elements that might apply more generally.

CHAPTER 3

Set-Valued Skyline Fillings

This chapter derives from work previously appearing on the arXiv [Mon16] and a condensed version was presented at FPSAC [Mon17].

3.1 Introduction

In this chapter we will introduce **set-valued skyline fillings** and use them to define inhomogeneous deformations of the Demazure atoms and key polynomials (Definition 1.21), as well as the quasisymmetric Schur functions (Definition 1.28). We then give generalizations of results about ordinary skyline fillings to show how our definition provides a K-analogue to Demazure atoms. The combinatorial Lascoux atoms defined here (Definition 3.2) are conjecturally the same as those defined by divided difference operators (Conjecture 3.18).

3.1.1 Definition of Set-Valued Skyline Fillings

Recall from Definition 1.23, a skyline filling is semistandard if

(M1) entries do not repeat in a column,

(M2) rows are weakly decreasing (including the basement), and

(M3) every triple (including those with basement boxes) is an inversion triple.

Furthermore, $\mathsf{SkyFill}(\gamma, \mathbf{b})$ is the set of semistandard skyline fillings of shape γ and basement $\mathbf{b}$ and we denote the basement $b_i = i$ as $\mathbf{b}_i$. The Demazure atom $\mathtt{Atom}_\gamma$ is

$$\mathtt{Atom}_\gamma = \sum_{F \in \mathsf{SkyFill}(\gamma, \mathbf{b}_i)} \mathbf{x}^F;$$

see Theorem 1.24.

A **set-valued filling** is an assignment of non-empty subsets of positive integers to the boxes of the skyline diagram. The maximum entry in each box is the **anchor entry** and all other entries are **free entries**. For a filling F and a box (r, c) of F, we denote the anchor entry of (r, c) by $\mathsf{anc}_F(r, c)$ and the set of free entries of (r, c) by $\mathsf{free}_F(r, c)$. The **content** of F is $c(F) = (c_1, \ldots, c_\ell)$ where c_i is the number of i entries (anchor or free) not in the basement of F. The **size** of F, denoted $|F|$, is $|c(F)|$ and for γ the shape of F, the **excess** of F is $\mathsf{ex}(F) = |F| - |\gamma|$. Finally, the **monomial** of F is $\mathbf{x}^F = \mathbf{x}^{c(F)}$.

Definition 3.1. *A set-valued filling F is semistandard if*

(S1) entries do not repeat in a column,

(S2) rows are weakly decreasing where sets $A \geq B$ if $\min A \geq \max B$,

(S3) every triple of anchor entries is an inversion triple, and

(S4) if $a \in \mathsf{free}_F(r, c)$ then for all $r' \neq r$, either $a > \mathsf{anc}_F(r', c)$, $a < \mathsf{anc}_F(r', c+1)$, or $\mathsf{anc}_F(r', c) > \mathsf{anc}_F(r, c)$.

Let $\mathsf{SetSkyFill}(\gamma, \mathbf{b})$ be the set of semistandard set-valued skyline diagrams of shape γ and basement $\mathbf{b}$. The concept of anchor entries is a key part of this definition, see Remark 3.13. Examples of semistandard set-valued skyline fillings, with their corresponding monomials, are given below where anchor entries are written in bold.

1	**1**		
2			
3	**3**2	**2**	**2**1
4	**4**	**4**31	

$\beta^4 x_1^3 x_2^3 x_3^2 x_4^2$

5	**4**	**3**	
4			
3	**3**2	**2**	**2**1
2	**1**		

$\beta^2 x_1^2 x_2^3 x_3^2 x_4$

Definition 3.2. *The combinatorial Lascoux atom* $K\texttt{Atom}_\gamma$ *is*

$$K\texttt{Atom}_\gamma(x_1, \ldots, x_\ell; \beta) = \sum_{F \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{\mathsf{ex}(F)} \mathbf{x}^F.$$

Note that the definition of semistandard set-valued skyline fillings is different from the one stated previously in [Mon16] and [Mon17]. There are several ways to define (S4) such that the combinatorial Lascoux atom is unchanged. As seen in the proof of Theorem 3.4, the key feature of (S4) is guaranteeing there is only one semistandard skyline filling with fixed anchor and free entries in each column. In this work, we choose the definition given in light of the results of Chapter 4.

Figure 3.1 gives the semistandard skyline fillings for basement $\mathbf{b}_i$ for weak compositions that are rearrangements of $(2, 1, 0)$. Clearly, setting $\beta = 0$ yields $\texttt{Atom}_\gamma$, and thus $K\texttt{Atom}_\gamma$ is a inhomogeneous deformation of $\texttt{Atom}_\gamma$. This mostly shows combinatorial Lascoux atoms form a new (finite) basis of $\mathsf{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$ – this is Proposition 3.10.

Definition 3.3. *The combinatorial Lascoux polynomial* $K\texttt{Key}_\gamma$ *is*

$$K\texttt{Key}_\gamma(x_1, \ldots, x_\ell; \beta) = \sum_{F \in \mathsf{SetSkyFill}(\gamma^*, \mathbf{b}_i^*)} \beta^{\mathsf{ex}(F)} \mathbf{x}^F.$$

Again it is clear that setting $\beta = 0$ yields $\texttt{Key}_\gamma$ and thus $K\texttt{Key}_\gamma$ is an inhomogeneous deformation of $\texttt{Key}_\gamma$. The same argument as the proof of Proposition 3.10 completes the proof that the combinatorial Lascoux polynomials are a (finite) basis of Pol.

Definition 3.2 is our K-analogue of the Demazure atom. We give generalizations of earlier results supporting this view, and the combinatorial Lascoux atoms conjecturally satisfy the natural recurrence for K-theoretic Demazure atoms (see Conjecture 3.18).

3.1.2 Main Results

Recall from Theorem 1.27,

$$s_\lambda = \sum_{\gamma \in \mathsf{PermutWC}(\lambda)} \texttt{Atom}_\gamma,$$

and thus the Demazure atoms are a polynomial refinement of the Schur functions. We generalize this to Ks_λ and $K\texttt{Atom}_\gamma$, the K-analogues of s_λ and $\texttt{Atom}_\gamma$, respectively.

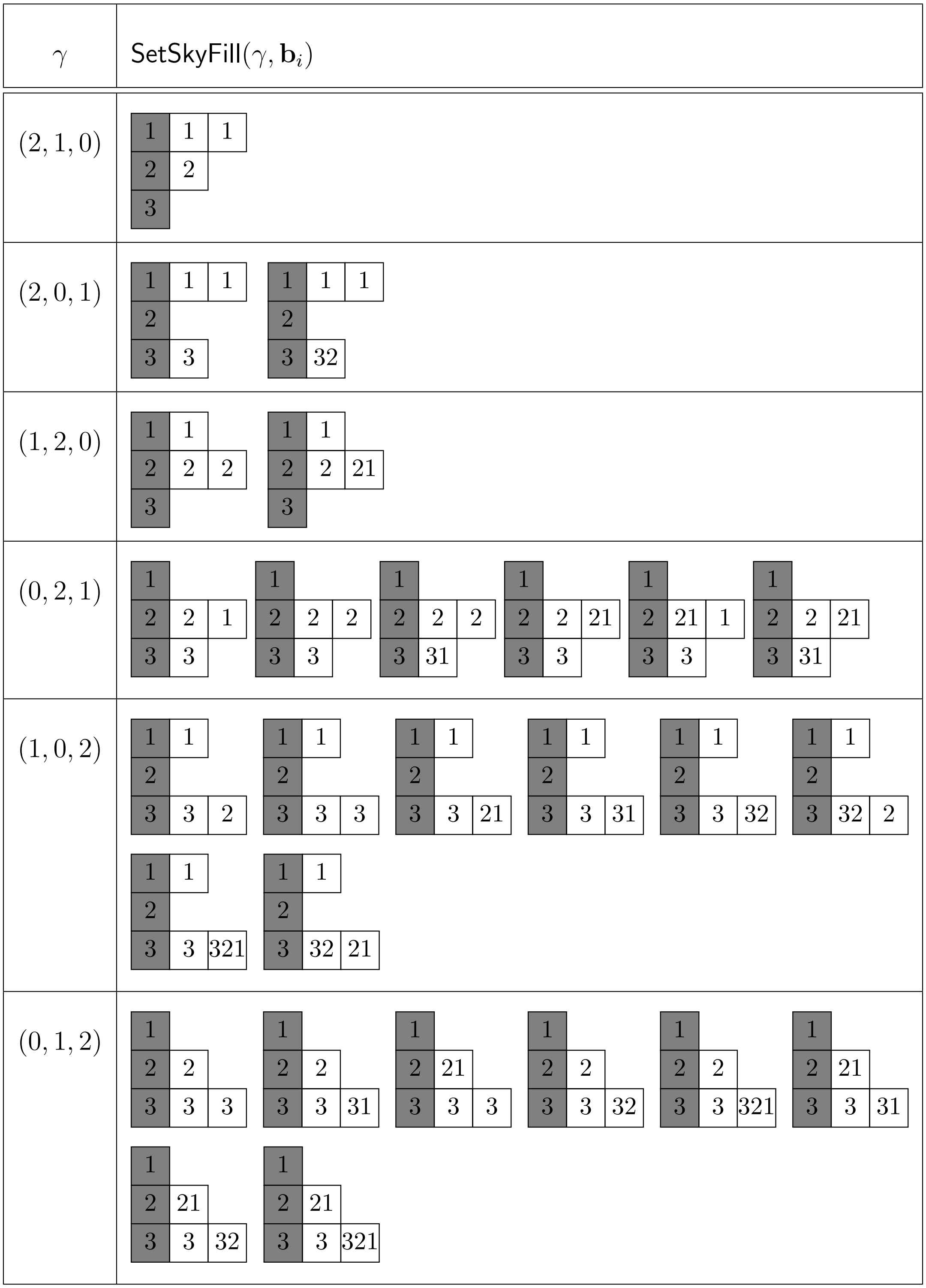


Figure 3.1: $\mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$ for $\gamma \in \mathsf{PermutWC}((2,1))$ and $\ell(\gamma) = 3$.

Theorem 3.4.

$$Ks_\lambda = \sum_{\gamma \in \mathsf{PermutWC}(\gamma)} K\mathtt{Atom}_\gamma.$$

Equation (1.7) gives the decomposition of the quasisymmetric Schur functions into Demazure atoms. The analogous decomposition is:

Definition 3.5. *The quasisymmetric Grothendieck function KS_α is*

$$KS_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} K\mathtt{Atom}_\gamma.$$

By combining Theorem 3.4 and Definition 3.5, we decompose Ks_λ into quasisymmetric Grothendieck functions which generalizes the decomposition in (1.7).

Corollary 3.6.

$$Ks_\lambda = \sum_{\alpha \in \mathsf{PermutC}(\lambda)} KS_\alpha.$$

Theorem 3.7. *As α runs over all compositions, the functions $\{KS_\alpha\}$ form a basis for* QSym.

Theorem 3.7 generalizes [HLMvW11a, Proposition 5.5]. As seen below, the expansion of a power series f into Lascoux atoms allows us to determine if f is quasisymmetric or symmetric. If it is, the expansion allows us to determine if f is KS_α- or Ks_λ-positive, which is often of interest, cf. [LMvW13, Section 1.1].

Proposition 3.8. *Suppose $f = \sum_{\gamma \in \mathsf{WC}} c_\gamma K\mathtt{Atom}_\gamma$. Then*

1. *f is quasisymmetric if and only if for all $\gamma \in \mathsf{WC}$, $c_\gamma = c_\delta$ for all $\delta \in \mathsf{Expand}(\gamma^+)$, and*
2. *f is symmetric if and only if for all $\gamma \in \mathsf{WC}$, $c_\gamma = c_\delta$ for all $\delta \in \mathsf{PermutWC}(\lambda(\gamma))$.*

Furthermore, if f is quasisymmetric, f is KS_α-positive if and only if f is $K\mathtt{Atom}_\gamma$-positive. If f is symmetric, f is Ks_λ-positive if and only if f is $K\mathtt{Atom}_\gamma$-positive.

Section 3.2 further investigates Lascoux atoms while Section 3.3 focuses on quasisymmetric Grothendieck functions. Finally, in Section 3.4, we state further conjectures about Lascoux atoms that continue the analogy with Demazure atoms.

3.2 Combinatorial Lascoux Atoms

We first show that combinatorial Lascoux atoms form a finite basis of $\mathsf{Pol} = \mathbb{Z}[x_1, .x_2, \ldots]$. Let $\leqslant$ be the lexicographic order on monomials and for γ, define

$$\mathsf{S}(\gamma) = \{\delta \in \mathsf{WC} : \ell(\delta) \leq \ell(\gamma) \text{ and } \max(\delta) \leq \max(\gamma)\}.$$

Lemma 3.9. *For a weak composition* γ*,*

$$K\mathtt{Atom}_\gamma = \mathbf{x}^\gamma + \sum_{\substack{\delta \in \mathsf{S}(\gamma) \\ \delta \geqslant \gamma}} c_{\gamma,\delta} \beta^{|\delta| - |\gamma|} \mathbf{x}^\delta.$$

Proof. We first show there is exactly one semistandard set-valued skyline filling with basement $\mathbf{b}_i$ of shape and content γ, namely the filling where row i is filled with γ_i i entries. First, we observe that since we have exactly as many entries as we have boxes, there can be no free entries.

Since the rows are weakly decreasing (S2) and $b_i = i$, for any i_0, the boxes in the first i_0 rows can only have the values $1, \ldots, i_0$. Since the first row can only contain 1's and we have γ_1 boxes in the first row and γ_1 entries with value 1, all 1's must be placed in the first row. Likewise, the second row can only contain 1's and 2's. However all 1's were placed in row 1, and so we have γ_2 boxes in row 2 and exactly γ_2 2's that can be placed in the second row. Thus all the 2's must be placed in the second row. Proceeding in this manner, we see row i must contain all i's. Thus, $\mathbf{x}^\gamma$ appears in $K\mathtt{Atom}_\gamma$ with coefficient 1 because the filling formed by filling row i with all i's for anchor entries and no free entries is the unique element of $\mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$ with content γ.

Now suppose $\delta \leqslant \gamma$ and we will show there is no element of $\mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$ with content δ. Since $\delta \leqslant \gamma$, there exists i_0 such that $\delta_j = \gamma_j$ for $j < i_0$ and $\delta_{i_0} < \gamma_{i_0}$. Thus,

$$\sum_{i=1}^{i_0} \gamma_i > \sum_{i=1}^{i_0} \delta_i.$$

However, $\sum_{i=1}^{i_0} \gamma_i$ is the number of boxes in the first i_0 rows and $\sum_{i=1}^{i_0} \delta_i$ is the number of instances of the numbers $1, \ldots, i_0$. Thus there are more boxes in rows $1, \ldots, i_0$ than instances of the numbers $1, \ldots, i_0$, and so at least one box in rows $1, \ldots, i_0$ must be empty. Then no element of $\mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$ has content δ.

Finally consider $F \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$ of content δ and we will show $\delta \in \mathsf{S}(\gamma)$. Since F has m columns, excluding the basement, and numbers cannot repeat in a column by (S1), each i can appear at most m times in F. Thus $\max(\delta) \leq \max(\gamma)$. Furthermore, since the rows are weakly decreasing by (S2) and thus the entries in row i are bounded by i, F cannot contain an entry larger than $\ell(\gamma)$ and so $\ell(\delta) \leq \ell(\gamma)$. $\square$

Proposition 3.10. *For all $f \in \mathsf{Pol}$, there is a unique expansion $f = \sum_\gamma c_\gamma K\mathtt{Atom}_\gamma$, where all but finitely many $c_\gamma = 0$, i.e. $\{K\mathtt{Atom}_\gamma\}$ forms a finite basis of Pol.*

Proof. We first establish some properties of the set $\mathsf{S}(\gamma)$. Observe $|\mathsf{S}(\gamma)| = (\max(\gamma)+1)^{\ell(\gamma)}$ and thus is finite. Furthermore, if $\delta \in \mathsf{S}(\gamma)$, then $\mathsf{S}(\delta) \subseteq \mathsf{S}(\gamma)$ as $\max(\delta) \leq \max(\gamma)$ and $\ell(\delta) \leq \ell(\gamma)$.

We now consider the expansion of $\mathbf{x}^\gamma$ into Lascoux atoms. By Lemma 3.9,

$$\mathbf{x}^\gamma = K\mathtt{Atom}_\gamma - \sum_{\substack{\delta \in \mathsf{S}(\gamma) \\ \delta \geqslant \gamma}} c_{\gamma,\delta} \beta^{|\delta|-|\gamma|} \mathbf{x}^\delta.$$

Let δ_1 be the lexicographically smallest term such that $c_{\gamma,\delta_1} \neq 0$. Then since $\mathsf{S}(\delta_1) \subseteq \mathsf{S}(\gamma)$,

$$\mathbf{x}^\gamma = K\mathtt{Atom}_\gamma - c_{\gamma,\delta_1}\beta^{|\delta_1|-|\gamma|} K\mathtt{Atom}_{\delta_1} + \sum_{\substack{\delta \in \mathsf{S}(\gamma) \\ \delta \geqslant \delta_1}} (c_{\gamma,\delta_1} c_{\delta_1,\delta} - c_{\gamma,\delta}) \beta^{|\delta|-|\gamma|} \mathbf{x}^\delta.$$

We then iterate this process with the lexicographically smallest term remaining in the sum, and thus after the ith step,

$$\mathbf{x}^\gamma = K\mathtt{Atom}_\gamma + \sum_{j=1}^{i} a_{\gamma,\delta_j} \beta^{|\delta_j|-|\gamma|} K\mathtt{Atom}_{\delta_j} + \sum_{\substack{\delta \in \mathsf{S}(\gamma) \\ \delta \geqslant \delta_i}} b_{\gamma,\delta} \beta^{|\delta|-|\gamma|} \mathbf{x}^\delta.$$

Since we take the lexicographically smallest term remaining at each step, for all i, $\delta_i \in \mathsf{S}(\gamma)$ and $\gamma \leqslant \delta_1 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_i$. Since there are finitely many weak compositions in $\mathsf{S}(\gamma)$, this process must terminate and we have a finite expansion of $\mathbf{x}^\gamma$ into Lascoux atoms.

Since any monomial $\mathbf{x}^\gamma$ has a finite expansion in Lascoux atoms, any $f \in \mathsf{Pol}$ does as well. Finally, suppose

$$0 = \sum_{\gamma \in \mathsf{WC}} c_\gamma K\mathtt{Atom}_\gamma$$

and by setting $\beta = 0$,

$$0 = \sum_{\gamma \in \mathsf{WC}} c_\gamma \mathtt{Atom}_\gamma.$$

Since Demazure atoms form a basis of polynomials, $c_\gamma = 0$ for all γ, and $\{K\mathtt{Atom}_\gamma\}$ is a linearly independent set. $\square$

A **set-valued reverse tableau** is a filling of the shape λ with non-empty sets of positive integers with weakly decreasing rows and strictly decreasing columns. Let $\mathsf{SetRT}(\lambda)$ be the collection of set-valued reverse tableaux of shape λ. Since Ks_λ is symmetric, it is equivalent to consider Ks_λ as the sum over set-valued reverse tableaux. Then, we define the map

$$\hat{\rho} : \bigsqcup_{\gamma \in \mathsf{PermutWC}(\lambda)} \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i) \to \mathsf{SetRT}(\lambda)$$

as follows. First, sort the anchor entries of each column into decreasing order and then place the free entries in the unique box in their column such that the columns remain strictly decreasing and the free entries remain free.

For the inverse $\hat{\rho}^{-1}$, start with an empty skyline diagram with basement $\mathbf{b}_i$. Work by columns left to right, top to bottom and place each anchor entry in the highest row such that weakly decreasing rows is preserved. When all anchor entries have been placed, place the free entries with the smallest anchor entry in their column such that the rows are weakly decreasing and the free entries remain free. In the special case where there are no free entries, $\hat{\rho}$ and $\hat{\rho}^{-1}$ are precisely the bijections ρ and ρ^{-1} given by Mason in [Mas08].

Example 3.11. Given the filling $F = \begin{array}{|c|c|c|c|} \hline 1 & \mathbf{1} \\ \cline{1-2} 2 \\ \cline{1-4} 3 & \mathbf{3}2 & \mathbf{2} & \mathbf{2}1 \\ \hline 4 & \mathbf{4} & \mathbf{4}31 \\ \cline{1-3} 5 & \mathbf{5} \\ \cline{1-2} \end{array}$ we calculate $\hat{\rho}(F) = \begin{array}{|c|c|c|} \hline \mathbf{5} & \mathbf{4}3 & \mathbf{2}1 \\ \hline \mathbf{4} & \mathbf{2}1 \\ \cline{1-2} \mathbf{3}2 \\ \cline{1-1} \mathbf{1} \\ \cline{1-1} \end{array}$.

Theorem 3.12. *The map $\hat{\rho}$ is a bijection and $\hat{\rho}^{-1}$ is its inverse.*

Proof. The anchor entries of a semistandard set-valued skyline filling form an ordinary semistandard skyline filling, and likewise the anchor entries of a set-valued reverse tableau form an ordinary reverse tableau. Since $\hat{\rho}$ and $\hat{\rho}^{-1}$ act exactly on the anchor entries by ρ and ρ^{-1} which are well-defined and mutual inverses [Mas08], $\hat{\rho}$ and $\hat{\rho}^{-1}$ are well-defined and mutual inverses on the anchor entries. Thus, since the anchor entries determine the shape of the resulting filling or tableau, $\hat{\rho}$ and $\hat{\rho}^{-1}$ produce fillings and tableaux of the correct shapes. Thus we only need to show $\hat{\rho}$ and $\hat{\rho}^{-1}$ are well-defined and mutual inverses on the free entries.

The map $\hat{\rho}$ is well-defined on the free entries. Let $F \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$ and we want to show that $T = \hat{\rho}(F) \in \mathsf{SetRT}(\lambda(\gamma))$. By construction the columns of T are strictly decreasing and so we only need to show that the rows of T are weakly decreasing. Suppose to the contrary that row i is not and then there must exist $\alpha \in \mathtt{free}_T(i,j)$ for some j such that $\alpha < \mathtt{anc}_T(i, j+1)$. Since $\alpha \in \mathtt{free}_T(i,j)$, there are exactly i anchor entries of column j bigger than α:

$$\mathtt{anc}_T(1,j) > \mathtt{anc}_T(2,j) > \ldots > \mathtt{anc}_T(i,j) > \alpha > \mathtt{anc}_T(i+1,j) > \ldots > \mathtt{anc}_T(k,j).$$

Since the anchor entries of column j of T are the anchor entries of column j of F, there are exactly i anchor entries bigger than α in column j of F. However, since $\alpha < \mathtt{anc}_T(i, j+1)$, there are at least i anchor entries in column $j+1$ of F bigger than α:

$$\alpha < \mathtt{anc}_T(i, j+1) < \mathtt{anc}_T(i-1, j+1) < \ldots < \mathtt{anc}_T(1, j+1).$$

Since α is a free entry of F, it must be in the box of one of the i anchor entries $\mathtt{anc}_T(k,j)$ for $k = 1, \ldots, i$. However, since the rows of F are weakly decreasing (S2), none of the i anchor entries $\mathtt{anc}_T(k, j+1)$ for $k = 1, \ldots, i$ can appear to the right of the box of α. Thus there are at most $i-1$ anchor entries in column j that can appear to the left of i anchor entries in column $j+1$ that are bigger than α, contradicting that $F \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$.

The map $\hat{\rho}^{-1}$ is well-defined on the free entries. Let $T \in \mathsf{SetRT}(\lambda)$ and we want to show $\hat{\rho}^{-1}(T) = F \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. Since by definition $\hat{\rho}^{-1}$ places free entries with the smallest anchor entry such that rows are weakly decreasing, we only need to show that such a row exists. Thus suppose $\alpha \in \mathtt{free}_T(i,j)$. Since

$$\mathtt{anc}_T(1,j) > \mathtt{anc}_T(2,j) > \ldots > \mathtt{anc}_T(i,j) > \alpha$$

there are i anchor entries in column j of F that are bigger than α, and so α can be placed in any of these i boxes and remain free.

Since the rows of T are weakly decreasing, there are at most $i-1$ anchor entries of column $j+1$ that are bigger than α:

$$\alpha \geq \mathtt{anc}_T(i, j+1) > \mathtt{anc}_T(i+1, j+1) > \ldots.$$

Since these at most $i-1$ entries in column $j+1$ cannot appear to the right of all i possibilities in column j, α can be placed in one of the i boxes in column j where the anchor entry is larger than α.

The maps $\hat{\rho}$ and $\hat{\rho}^{-1}$ are mutual inverses. Since the columns of T are strictly decreasing, there is at most one set-valued reverse tableau of fixed anchor and free entries in each column. Thus $\hat{\rho}\hat{\rho}^{-1}(T) = T$ as both $\hat{\rho}$ and $\hat{\rho}^{-1}$ preserve the anchor and free entries of each column of T.

For the same reason, $\hat{\rho}^{-1}\hat{\rho}(F) = F$. In [Mas08], Mason showed there is at most one semistandard skyline filling (of any shape) with basement $\mathbf{b}_i$ with given entries in each column. Since the anchor entries form a semistandard skyline filling and free entries are required to be with the smallest possible anchor entry, there is at most one set-valued semistandard skyline filling (of any shape) with basement $\mathbf{b}_i$ with given anchor and free entries in each column. $\square$

This proves the decomposition of Ks_λ into Lascoux atoms.

Theorem 3.4.

$$Ks_\lambda = \sum_{\gamma \in \mathsf{PermutWC}(\gamma)} K\texttt{Atom}_\gamma.$$

Remark 3.13. One might expect a semistandard set-valued skyline filling to be a filling such that any selection of one number from each box is a semistandard skyline filling. However, then the left tableau below would not be semistandard as the right tableau violates the (M3) condition in rows 2 and 3:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 32 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Compare this with [KMY08, Section 1.2].

3.3 Quasisymmetric Grothendieck Functions

Recall from Section 1.3 that a function f is quasisymmetric if for any positive integers $\alpha_1, \ldots, \alpha_k$ and strictly increasing sequence of positive integers $i_1 < i_2 < \ldots < i_k$,

$$[x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}]f = [x_1^{\alpha_1} \ldots x_k^{\alpha_k}]f.$$

In [HLMvW11a], the quasisymmetric Schur function was originally defined

$$S_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} \mathtt{Atom}_\gamma.$$

Combinatorially, however, recall from Section 1.5.1 that a semistandard composition tableau is a skyline filling of a composition α with no basement and strictly increasing entries from top to bottom along the first column. Recall from Definition 1.28,

$$S_\alpha = \sum_{T \in \mathsf{SSCT}(\alpha)} \mathbf{x}^T$$

where $\mathsf{SSCT}(\alpha)$ is the collection of all semistandard composition tableaux of shape α.

Definition 3.14. *A semistandard set-valued composition tableau is a filling T of a composition shape α with non-empty subsets of positive integers such that*

(Q1) entries weakly decrease along rows,

(Q2) anchor entries form a semistandard composition tableau, and

(Q3) if $a \in \mathsf{free}_T(r,c)$ then for all $r' \neq r$, $a > \mathsf{anc}_T(r',c)$, $a < \mathsf{anc}_T(r',c+1)$, or $\mathsf{anc}_T(r',c) > \mathsf{anc}_T(r,c)$.

Let $\mathsf{SetSSCT}(\alpha)$ be the collection of semistandard set-valued composition tableaux of shape α. Inserting rows of size 0 allows the anchor entries of the first column to be any increasing sequence, and thus

$$KS_\alpha = \sum_{\gamma \in \mathsf{Expand}(\alpha)} K\mathtt{Atom}_\gamma = \sum_{T \in \mathsf{SetSSCT}(\alpha)} \beta^{\mathsf{ex}(T)} \mathbf{x}^T. \tag{3.1}$$

Proposition 3.15. *The function KS_α is quasisymmetric.*

Proof. Fix a composition α and fix i. It suffices to show

$$\#\{T \in \mathsf{SetSSCT}(\alpha) : T \text{ has content } \beta = (\beta_1, \ldots, \beta_{i-1}, \beta_i, 0, \beta_{i+2}, \ldots, \beta_n)\}$$

equals

$$\#\{T \in \mathsf{SetSSCT}(\alpha) : T \text{ has content } \hat{\beta} = (\beta_1, \ldots, \beta_{i-1}, 0, \beta_i, \beta_{i+2}, \ldots, \beta_n)\}.$$

Suppose $T \in \mathsf{SetSSCT}(\alpha)$ has no $i+1$'s and let T_i be the tableau formed from T by replacing all i's replaced with $i+1$'s. We will show that $T_i \in \mathsf{SetSSCT}(\alpha)$. To show T_i is semistandard we need to show that entries weakly decrease along rows (Q1), anchor entries form a semistandard composition tableau (Q2), and free entries are with the smallest possible anchor entry (Q3).

Free entries stay free and anchor entries stay anchor: If $\mathsf{anc}_T(r,c) = i$, then $i > x$ for all $x \in \mathsf{free}_T(r,c)$. Thus $\mathtt{free}_{T_i}(r,c) = \mathsf{free}_T(r,c)$ and so $i+1 > i > \mathsf{free}_{T_i}(r,c)$ and so $i+1 = \mathsf{anc}_{T_i}(r,c)$ Furthermore, if $i \in \mathsf{free}_T(r,c)$, then $\mathsf{anc}_T(r,c) \geq i+2$ because T has no $i+1$'s. Thus $i+1 < \mathsf{anc}_{T_i}(r,c) = \mathsf{anc}_T(r,c)$ and so $i+1 \in \mathsf{free}_{T_i}(r,c)$.

(Q1) is still valid: Since the rows of T are weakly decreasing and T has no $i+1$'s, replacing the i's with $i+1$'s does not break weakly decreasing.

(Q2) is still valid: For the same reason as above, the anchor entries of the first column of T_i are still strictly increasing and entries still do not repeat in a column. We now show replacing an i with $i+1$ does not turn an inversion triple into a coinversion triple. Suppose we have $b < a \leq c$ with $b = i$. Since T has no $i+1$'s, $a \geq i+2$ and when we replace b with $i+1$, we still have $b < a \leq c$. Furthermore suppose we have $a \leq c < b$ with $b = i$. Then clearly we still have $a \leq c < b$ when b is replaced by $i+1$. Similar arguments work when a or c (or both) is i.

(Q3) is still valid: Finally suppose $i \in \mathsf{free}_T(r,c)$ and thus $i+1 \in \mathsf{free}_{T_i}(r,c)$. We want to show that for all $r' \neq r$, $i+1 > \mathsf{anc}_{T_i}(r',c)$, $i+1 < \mathsf{anc}_{T_i}(r',c+1)$, or $\mathsf{anc}_{T_i}(r',c) > \mathsf{anc}_{T_i}(r,c)$. Thus consider $r' \neq r$. Since $i \in \mathsf{free}_T(r,c)$, we have $i > \mathsf{anc}_T(r',c)$, $i < \mathsf{anc}_T(r',c+1)$, or $\mathsf{anc}_T(r',c) > \mathsf{anc}_{T_i}(r,c)$. In the first case, $i+1 > i > \mathsf{anc}_T(r',c) = \mathsf{anc}_{T_i}(r',c)$. In the second, since T has no $i+1$'s and $i < \mathsf{anc}_T(r',c+1)$, we have $i+1 < \mathsf{anc}_T(r',c+1) = \mathsf{anc}_{T_i}(r',c+1)$. In the last, since T has no $i+1$'s and $\mathsf{anc}_T(r',c) > \mathsf{anc}_T(r,c) > i$,

$$\mathsf{anc}_{T_i}(r'c) = \mathsf{anc}_T(r',c) > \mathsf{anc}_T(r,c) = \mathsf{anc}_{T_i}(r',c).$$

If an anchor i becomes an $i+1$, no free entry will move to its box since if i was the smallest anchor entry possible for a given free entry, replacing all i's with $i+1$'s guarantees $i+1$ will be the smallest anchor entry possible for a given free entry.

A very similar argument can be applied to a tableau with no i's and replacing all $i+1$'s with i's, and thus KS_α is in fact quasisymmetric. □

We now complete the proof of Theorem 3.7 and Proposition 3.8. Recall QSym_n is the ring of quasisymmetric polynomials in n variables.

Proposition 3.16. *The set*

$$\{KS_\alpha(x_1, \ldots, x_n, 0, \ldots) : \alpha \in \mathsf{Comp}, \ell(\alpha) \le n\}$$

forms a basis of QSym_n.

Proof. By definition,

$$KS_\alpha(x_1, \ldots, x_n, 0, \ldots) = \sum_{\substack{\gamma \in \mathsf{Expand}(\alpha) \\ \ell(\gamma) \le n}} K\mathtt{Atom}_\gamma.$$

The lexicographically smallest γ that appears in this sum is α' where α' is the weak composition formed by prepending 0's to α until α' has n parts. Thus by Lemma 3.9, the lexicographically smallest monomial of KS_α is $\mathbf{x}^{\alpha'}$ and every monomial that appears in KS_α is in $\mathsf{S}(\alpha')$.

Then, by the same argument as the proof of Theorem 3.10, for each α', there are finitely many possible terms that can be introduced by subtracting $[\mathbf{x}^{\alpha'}]f \cdot KS_\alpha(x_1, \ldots, x_n, 0, \ldots)$ as only lexicographically larger terms in $\mathsf{S}(\alpha')$. Thus the process of subtracting KS_α with appropriate coefficient for $\mathbf{x}^{\alpha'}$ the lexicographically largest term appearing will terminate and any quasisymmetric polynomial in n variables can be expanded in $\{KS_\alpha\}$.

Finally, the same argument as with the Lascoux atoms specializing $\beta = 0$ shows that the quasisymmetric Grothendieck functions are linearly independent and thus $\{KS_\alpha\}$ as α runs over compositions with at most n parts is a basis for QSym_n. □

Theorem 3.7. *As α runs over all compositions, the functions $\{KS_\alpha\}$ form a basis for* QSym.

Proof. Let f be a quasisymmetric function. By Proposition 3.16, for any n,

$$f(x_1, \ldots, x_n, 0, \ldots) = \sum_{\alpha \in \mathsf{Comp}} c_\alpha KS_\alpha(x_1, \ldots, x_n, 0, \ldots).$$

The expansion of the terms of f of degree at most n is determined by $f(x_1, \ldots, x_n, 0, \ldots)$. Thus as $n \to \infty$, the expansion of f into KS_α stabilizes and $\{KS_\alpha\}$ is a basis for QSym. □

These new bases provide a method for determining when a function is quasisymmetric (resp. symmetric), and then furthermore KS_α-positive (resp. Ks_λ-positive) .

Proposition 3.8. *Suppose* $f = \sum_{\gamma \in \mathsf{WC}} c_\gamma K\mathtt{Atom}_\gamma$. *Then*

1. *f is quasisymmetric if and only if for all $\gamma \in \mathsf{WC}$, $c_\gamma = c_\delta$ for all $\delta \in \mathsf{Expand}(\gamma^+)$, and*

2. *f is symmetric if and only if for all $\gamma \in \mathsf{WC}$, $c_\gamma = c_\delta$ for all $\delta \in \mathsf{PermutWC}(\lambda(\gamma))$.*

Furthermore, if f is quasisymmetric, f is KS_α-positive if and only if f is $K\mathtt{Atom}_\gamma$-positive. If f is symmetric, f is Ks_λ-positive if and only if f is $K\mathtt{Atom}_\gamma$-positive.

Proof. Consider

$$f = \sum_{\alpha \in \mathsf{Comp}} c_\alpha KS_\alpha = \sum_{\alpha \in \mathsf{Comp}} c_\alpha \sum_{\gamma \in \mathsf{Expand}(\alpha)} K\mathtt{Atom}_\gamma.$$

Since $\{KS_\alpha\}$ is a basis of QSym, f is quasisymmetric if and only if it has an expansion in the KS_α's, and as above if and only if $c_\gamma = c_\delta$ for all $\gamma^+ = \delta^+$. Furthermore, in this case, f is KS_α-positive if and only if it is $K\mathtt{Atom}_\gamma$-positive. Likewise, consider

$$f = \sum_{\lambda \in \mathsf{Par}} c_\lambda Ks_\lambda = \sum_{\lambda \in \mathsf{Par}} c_\lambda \sum_{\gamma \in \mathsf{PermutWC}(\lambda)} K\mathtt{Atom}_\gamma.$$

By the same argument, f is symmetric if and only if $c_\gamma = c_\delta$ for all $\lambda(\gamma) = \lambda(\delta)$ and if f is symmetric, f if Ks_λ-positive if and only if it is $K\mathtt{Atom}_\gamma$-positive. $\square$

3.4 Conjectures

We have defined the Lascoux atoms combinatorially in terms of set-valued skyline fillings, but there is also a natural definition based on divided difference operators given in Section 1.6 (see Definition 1.34). Recall for

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}} \qquad \pi_i = \partial_i x_i \qquad \hat{\pi}_i = \pi_i - 1,$$

the Demazure character is $\mathtt{Key}_\gamma = \pi_{w_\gamma} \mathbf{x}^{\lambda(\gamma)}$ and the Demazure atom is $\mathtt{Atom}_\gamma = \hat{\pi}_{w_\gamma} \mathbf{x}^{\lambda(\gamma)}$. The K-theoretic deformations of these operators are:

$$\tilde{\partial}_i = \partial_i (1 + \beta x_{i+1}) \qquad \tau_i = \pi_i (1 + \beta x_{i+1}) \qquad \hat{\tau}_i = \tau_i - 1.$$

Then the Lascoux polynomial is $\widehat{K\mathtt{Key}}_\gamma = \tau_{w_\gamma} \mathbf{x}^{\lambda(\gamma)}$ and the Lascoux atom is $\widehat{K\mathtt{Atom}}_\gamma = \hat{\tau}_{w_\gamma} \mathbf{x}^{\lambda(\gamma)}$. By manipulating the operators above, we obtain the following decomposition of the Lascoux polynomial into Lascoux atoms that matches the Demazure case.

Theorem 3.17.
$$\widehat{K\texttt{Key}}_\gamma = \sum_{\substack{\delta \in \mathsf{PermutWC}(\lambda(\gamma)) \\ \delta \ll \gamma}} \widehat{K\texttt{Atom}}_\delta$$

where recall $\delta \ll \gamma$ if $\lambda(\delta) = \lambda(\gamma)$ and $w_\delta \leq_B w_\gamma$ in (strong) Bruhat order.

Proof. In sections 2 and 3 of [Pun16], A. Pun gives a proof in the case that $\beta = 0$ using relations derived between ∂_i, π_i, and $\hat{\pi}_i$. To extend this proof line by line to the Lascoux case, we only need to show $\hat{\tau}_i\hat{\tau}_i = -\hat{\tau}_i$.

We first show that $\tau_i\tau_i = \tau_i$. To do this, consider $\pi_i x_{i+1} f = \partial_i(x_i x_{i+1} f)$. Since $x_i x_{i+1}$ is symmetric in i and $i+1$,

$$\pi_i x_{i+1} f = \partial_i(x_i x_{i+1} f) = x_i x_{i+1} \partial_i f.$$

Now, from Proposition 3.1 of [Pun16], $\pi_i^2 = \pi$ and $\partial_i \pi_i = 0$. Then

$$\begin{aligned}
\tau_i^2 &= (\pi_i(1+\beta x_{i+1}))(\pi_i(1+\beta x_{i+1})) \\
&= \pi_i^2(1+\beta x_{i+1}) + \pi_i \beta x_{i+1} \pi_i (1+\beta x_{i+1}) \\
&= \pi_i(1+\beta x_{i+1}) + \beta x_i x_{i+1} \partial_i \pi_i (1+\beta x_{i+1}) \\
&= \tau_i + 0.
\end{aligned}$$

Since $\tau_i = 1 + \hat{\tau}_i$ and $\tau_i^2 = \tau_i$,

$$1 + \hat{\tau}_i = (1+\hat{\tau}_i)^2 = 1 + 2\hat{\tau}_i + \hat{\tau}_i^2.$$

Thus
$$\hat{\tau}_i^2 = -\hat{\tau}_i.$$

□

A conjectural combinatorial model for $\widehat{K\texttt{Key}}_\gamma$ using K-Kohnert diagrams was given by C. Ross and A. Yong in [RY15], but there are no proven combinatorial rules for $\widehat{K\texttt{Key}}_\gamma$ or $\widehat{K\texttt{Atom}}_\gamma$. However, we have checked the following conjectures for all weak compositions γ with at most 8 boxes and at most 8 rows.

Conjecture 3.18.

$$\widehat{K\texttt{Atom}}_\gamma = K\texttt{Atom}_\gamma = \sum_{F \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|F|-|\gamma|} \mathbf{x}^F.$$

Conjecture 3.19.

$$\widehat{K\mathtt{Key}}_\gamma = K\mathtt{Key}_\gamma = \sum_{F \in \mathsf{SetSkyFill}(\gamma^*, \mathbf{b}_i^*)} \beta^{|F|-|\gamma|}\mathbf{x}^F.$$

In [HLMvW11b], J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg refine the Littlewood-Richardson rule to give the expansion of $\mathtt{Atom}_\gamma \cdot s_\lambda$ into Demazure atoms. In [PY16], O. Pechenik and A. Yong develop the theory of genomic tableaux to describe multiplication in K-theory; see Section 1.6. We conjecture the natural genomic analogue of the rule of J. Haglund et al. extends to Lascoux atoms.

When δ, γ are weak compositions with $\gamma_i \leq \delta_i$ for all i, a **skew skyline diagram** of shape δ/γ is formed by starting with the skyline diagram of shape δ and given basement and extending the basement into the cells of γ. If n is the largest entry allowed in the filling, a **large basement** is a basement such that all basement entries of the basement are larger than n and decrease from top to bottom. As seen in [HLMvW11b], with a large basement, the exact basement entries do not determine valid skyline fillings and thus we denote them by $*$.

Generalizing the notion of a genomic tableau from Section 1.6, a **genomic skyline filling** is a filling of δ/γ with labels i_j where i is a positive integer and for each i, $\{j | i_j \text{ appears in the filling}\}$ is an initial segment of $\mathbb{N}$, i.e. $\{1, 2, \ldots, k_i\}$ for some k_i. The set of labels $\{i_j\}$ for all j is the family i, while the set of all labels i_j for fixed i and j is the gene i_j. The content of a genomic filling is the weak composition $c(F)$ where c_i is the number of genes of family i. The **column reading word** of a skyline filling reads the entries of the boxes (excluding the basement) in columns from top to bottom, right to left.

Definition 3.20. *A genomic skyline filling is semistandard if*

(G1) at most one entry from a family (resp. gene) appears in a column (resp. row)

(G2) the label families are weakly decreasing along rows,

(G3) every triple with three distinct genes is an inversion triple comparing families, and

(G4) for every i, the genes appear in weakly decreasing order along the reading word.

A word w is **reverse lattice** if for all i, any initial segment of w contains more $i+1$'s than i's. A genomic filling is **reverse lattice** if for any selection of exactly one label per gene, the column reading word is reverse lattice.

Conjecture 3.21.

$$K\texttt{Atom}_\gamma \cdot Ks_\lambda = \sum_{\delta \in \mathsf{WC}} \tilde{a}^{\delta}_{\gamma,\lambda} K\texttt{Atom}_\delta$$

where $\tilde{a}^{\delta}_{\gamma,\lambda}$ is the number of reverse lattice, semistandard genomic skyline fillings of skew-shape δ/γ (using a large basement) with content λ^*.

This has been checked for all products with γ with at most 5 rows and 5 boxes and λ with at most 5 boxes.

Example 3.22. $\tilde{a}^{(3,1,4)}_{(1,0,2),(2,1)} = 2$ and the two witnessing fillings are

$$\begin{array}{cccc} * & 2_1 & 1_1 & \\ 1_1 & & & \\ * & * & 2_1 & 2_2 \end{array} \qquad \begin{array}{cccc} * & 2_1 & 1_1 & \\ 2_1 & & & \\ * & * & 2_1 & 2_2 \end{array} .$$

The following tableau is a semistandard genomic skyline filling of appropriate shape and content but is not lattice: $\begin{array}{cccc} * & 2_2 & 1_1 & \\ 2_1 & & & \\ * & * & 2_2 & 1_1 \end{array}$.

CHAPTER 4

Kaons and QuasiLascoux Polynomials

This chapter derives from joint work with O. Pechenik and D. Searles that is in preparation.

4.1 Introduction and Background

In this chapter, we will introduce two new bases of Pol and study their relationships to

- the combinatorial Lascoux polynomials and atoms (see Chapter 3),
- the multi-fundamental quasisymmetric functions (see Section 1.3),
- the quasisymmetric Grothendieck functions (see Section 3.3),
- the glide polynomials introduced in [PS17],
- the quasikey polynomials introduced in [AS16], and
- the fundamental particles introduced in [Sea17].

We will use set-valued skyline fillings in order to give combinatorial expansions between the different bases.

4.1.1 Homogeneous Bases

In this section, we will introduce a number of bases where each member is a homogeneous polynomial; see Table 4.1 for a summary. We will explain the known lifts (see Table 4.2) and combinatorial expansions (see Table 4.3) between these bases. Figure 4.1 contains all of the relationships of interest to this chapter. Later in this chapter, we will extend these relations to their inhomogeneous deformations.

The **fundamental slide polynomials** were introduced by S. Assaf and D. Searles [AS17]. A weak composition δ is a **slide** of the weak composition γ if it can be obtained from γ by a finite sequence of the following local moves:

(m1) $0p \Rightarrow p0$, (for $p \in \mathbb{Z}_{>0}$);

(m2) $0p \Rightarrow qr$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p$).

Definition 4.1 (Assaf-Searles, Definition 3.6 [AS17])**.** *The fundamental slide polynomial* $\texttt{Slide}_\gamma$ *is*

$$\texttt{Slide}_\gamma = \sum_\delta \mathbf{x}^\delta,$$

where the sum is all slides δ of γ.

For example,

$$\texttt{Slide}_{(0,1,0,2)} = \mathbf{x}^{(0,1,0,2)} + \mathbf{x}^{(1,0,0,2)} + \mathbf{x}^{(0,1,2,0)} + \mathbf{x}^{(1,0,2,0)} + \mathbf{x}^{(1,2,0,0)} + \mathbf{x}^{(0,1,1,1)} +$$

$$\mathbf{x}^{(1,0,1,1)} + \mathbf{x}^{(1,1,0,1)} + \mathbf{x}^{(1,1,1,0)}.$$

The **fundamental particles** were introduced by D. Searles [Sea17] as the generating function for a subset of the slides called **mesonic slides**.

Definition 4.2. *Let γ be a weak composition with nonzero entries in positions $n_1 < \ldots n_\ell$. The weak composition δ is a mesonic slide of γ if it can be obtained from γ by a finite sequence of the local moves (m1) and (m2) that never applies (m1) at positions $n_j - 1$ and n_j for any j.*

Definition 4.3 (Searles, Definition 4.1 [Sea17])**.** *The fundamental particle* $\texttt{Par}_\gamma$ *is*

$$\texttt{Par}_\gamma = \sum_\delta \mathbf{x}^\delta,$$

where the sum is over all mesonic slides of γ.

For example,

$$\texttt{Par}_{(0,1,0,2)} = \mathbf{x}^{(0,1,0,2)} + \mathbf{x}^{(0,1,1,1)}.$$

Name	Symbol	Ring	Definition
Schur function	s_λ	Sym	1.10
Fundamental quasisymmetric function	F_α	QSym	1.13
Quasisymmetric Schur function	S_α	QSym	1.28
Schubert polynomial	$\mathfrak{S}_w$	Pol	1.15
Key polynomial	$\mathtt{Key}_\gamma$	Pol	1.21
Demazure atom	$\mathtt{Atom}_\gamma$	Pol	1.21
Slide polynomial	$\mathtt{Slide}_\gamma$	Pol	4.1
Fundamental particle	$\mathtt{Par}_\gamma$	Pol	4.3
Quasikey polynomial	$\mathtt{QKey}_\gamma$	Pol	4.5

Table 4.1: The homogeneous bases we will consider in this chapter.

The **quasikey polynomials** were introduced in [AS16] and defined in terms of their positive expansion in fundamental slide polynomials. In [Sea17], it was proved that quasikey polynomials expand combinatorially in Demazure atoms, and we take this expansion as our definition.

Definition 4.4. *Given weak compositions γ and δ of n we say that δ* **dominates** *γ, denoted by $\delta \geq_D \gamma$, if for all $i = 1, \ldots, \ell$,*

$$\delta_1 + \cdots + \delta_i \geq_D \gamma_1 + \cdots + \gamma_i.$$

Definition 4.5 (Assaf-Searles, Theorem 3.4 [Sea17])**.** *The quasikey polynomial $\mathtt{QKey}_\gamma$ is*

$$\mathtt{QKey}_\gamma = \sum_{\substack{\delta \in \mathsf{Expand}(\gamma^+) \\ \delta \geq_D \gamma}} \mathtt{Atom}_\delta.$$

For example,

$$\mathtt{QKey}_{(0,1,0,3)} = \mathtt{Atom}_{(0,1,0,3)} + \mathtt{Atom}_{(1,0,0,3)} + \mathtt{Atom}_{(0,1,3,0)} + \mathtt{Atom}_{(1,0,3,0)} + \mathtt{Atom}_{(1,3,0,0)}.$$

Table 4.1 summarizes the homogeneous bases we will use in this chapter. We now ask how each basis of Sym or QSym lifts to a basis of Pol. This is summarized in Table 4.2.

Sections 1.4 and 1.5 show that the Schubert polynomials and the key polynomials are a lift of the Schur functions from Sym to Pol; see (1.4) and (1.6).

Then, for a weak composition γ, let $0^k \times \gamma$ be the composition formed by prepending k zeros to the front of γ.

Basis of Pol	Lifts	Reference
$\mathfrak{S}_w$	s_λ	Equation 1.4
$\mathtt{Key}_\gamma$	s_λ	Equation 1.6
$\mathtt{Slide}_\gamma$	F_α	Proposition 4.6
$\mathtt{QKey}_\gamma$	S_α	Proposition 4.7

Table 4.2: The lifts of the homogeneous bases from Sym and QSym to Pol.

Proposition 4.6 (Assaf-Searles, Lemma 3.8 [AS17])**.** *For a composition α with $\ell(\alpha) = \ell$,*

$$\mathtt{Slide}_{0^n \times \alpha}(x_1, x_2, \ldots, x_{n+\ell}) = F_\alpha(x_1, x_2, \ldots, x_{n+\ell}).$$

Thus, the fundamental slide polynomials lift the fundamental quasisymmetric polynomials from QSym to Pol.

Proposition 4.7 (Assaf-Searles, Proposition 4.10 [AS16])**.** *For a composition α with $\ell(\alpha) = \ell$,*

$$\mathtt{QKey}_\alpha(x_1, x_2, \ldots, x_{n+\ell}) = S_{0^n \times \alpha}(x_1, x_2, \ldots, x_{n+\ell}).$$

Thus, the quasikey polynomials lift the quasisymmetric Schur functions from QSym to Pol.

Finally, we consider the different combinatorial expansions between these different homogeneous families. These expansions are summarized in Table 4.3.

Equation (1.8) gives the combinatorial expansion of the Schur function into quasisymmetric Schur functions, while Proposition 1.29 gives the expansion of quasisymmetric Schur functions into fundamental quasisymmetric functions. Furthermore, Theorem 1.22 gives the combinatorial expansion of the Schubert polynomial into key polynomials.

Now, given a composition γ, define a **left swap** to be the exchange of two parts $\gamma_i \le \gamma_j$ where $i < j$.

Definition 4.8 (Assaf-Searles, Section 3 [AS16])**.** *Let $\mathsf{lswap}(\gamma)$ be the set of weak compositions δ that can be obtained from γ by a (possibly empty) sequence of left swaps starting with γ.*

For example,

$$\mathsf{lswap}((0,1,0,3)) = \{(0,1,0,3),(0,1,3,0),(1,0,0,3),(1,0,3,0),(1,3,0,0),\\ (0,3,0,1),(0,3,1,0),(3,0,0,1),(3,0,1,0),(3,1,0,0)\}.$$

Following (3.3) in [AS16], define $\mathsf{Qlswap}(\gamma)$ to be all $\delta \in \mathsf{lswap}(\gamma)$ such that among all elements $\epsilon \in \mathsf{lswap}(\gamma)$ such that $\epsilon^+ = \delta^+$, then δ is $\geq_D$-minimal. Continuing our example above,

$$\mathsf{Qlswap}((0,1,0,3)) = \{(0,1,0,3),(0,3,0,1)\}.$$

Theorem 4.9 (Assaf-Searles, Theorem 3.7, [AS16])**.**

$$\mathtt{Key}_\gamma = \sum_{\delta \in \mathsf{Qlswap}(\gamma)} \mathtt{QKey}_\delta.$$

Thus, the key polynomials combinatorially expand in the quasikey polynomials. Decomposing further, each quasikey polynomial combinatorially expands in the fundamental slides.

Theorem 4.10 (Assaf-Searles, Theorem 3.4 [AS16])**.**

$$\mathtt{QKey}_\gamma = \sum_{\delta \in \mathsf{WC}(|\gamma|)} \mathsf{C}_{\gamma,\delta}\mathtt{Slide}_\delta$$

where $\mathsf{C}_{\gamma,\delta}$ *are particular nonnegative integers.*

The formula of S. Assaf and D. Searles uses combinatorial objects not needed here, however, we will give a different explicit formula for $\mathsf{C}_{\gamma,\delta}$ in Theorem 4.38.

As seen in chapters 2 and 3, another approach to the study of key polynomials is to consider their expansion, not into slides, but rather into Demazure atoms. The same expansion we have seen with Bruhat order (Theorem 1.26) can also be written in terms of left swaps.

Lemma 4.11 (Searles, Lemma 3.1 [Sea17])**.**

$$\mathtt{Key}_\gamma = \sum_{\delta \in \mathsf{lswap}(\gamma)} \mathtt{Atom}_\delta.$$

We have defined the quasikey polynomials in terms of their Demazure atom expansion and have seen that the quasikey polynomials expand combinatorially in the slide basis as well. However, neither the Demazure atoms nor the fundamental slides expand combinatorially

Basis	Combinatorially Expands In	Reference
s_λ	S_α	Equation 1.8
S_α	F_α	Proposition 1.29
$\mathfrak{S}_w$	$\mathtt{Key}_\gamma$	Theorem 1.22
$\mathtt{Key}_\gamma$	$\mathtt{QKey}_\gamma$	Theorem 4.9
$\mathtt{QKey}_\gamma$	$\mathtt{Atom}_\gamma$	Definition 4.5
$\mathtt{QKey}_\gamma$	$\mathtt{Slide}_\gamma$	Theorem 4.10
$\mathtt{Slide}_\gamma$	$\mathtt{Par}_\gamma$	Proposition 4.12
$\mathtt{Atom}_\gamma$	$\mathtt{Par}_\gamma$	Theorem 4.13

Table 4.3: The combinatorial expansions between the homogeneous bases.

in each other [Sea17, Proposition 4.6]. A common expansion of the Demazure atoms and fundamental slides is provided by the fundamental particle basis.

Proposition 4.12 (Searles, Proposition 4.4 [Sea17])**.**

$$\mathtt{Slide}_\gamma = \sum_{\substack{\delta \in \mathsf{Expand}(\gamma^+) \\ \delta \geq_D \gamma}} \mathtt{Par}_\delta.$$

Theorem 4.13 (Searles, Theorem 4.11 [Sea17])**.**

$$\mathtt{Atom}_\gamma = \sum_{\delta \in \mathsf{WC}(|\gamma|)} \mathsf{D}_{\gamma,\delta} \mathtt{Par}_\delta$$

where $\mathsf{D}_{\gamma,\delta}$ *are nonnegative integers.*

We will describe a set of combinatorial objects counted by $\mathsf{D}_{\gamma,\delta}$ in Theorem 4.29. This theorem generalizes the expansion of D. Searles for a Demazure atom into fundamental particles. The relations among these nine families of polynomials are illustrated in Figure 4.1.

4.1.2 K-theoretic Analogues

We are interested in K-theoretic analogues of the nine homogeneous families described in Section 4.1.1, i.e. we seek inhomogeneous deformations of these polynomials where the relationships between the deformations mirror their homogeneous counterparts.

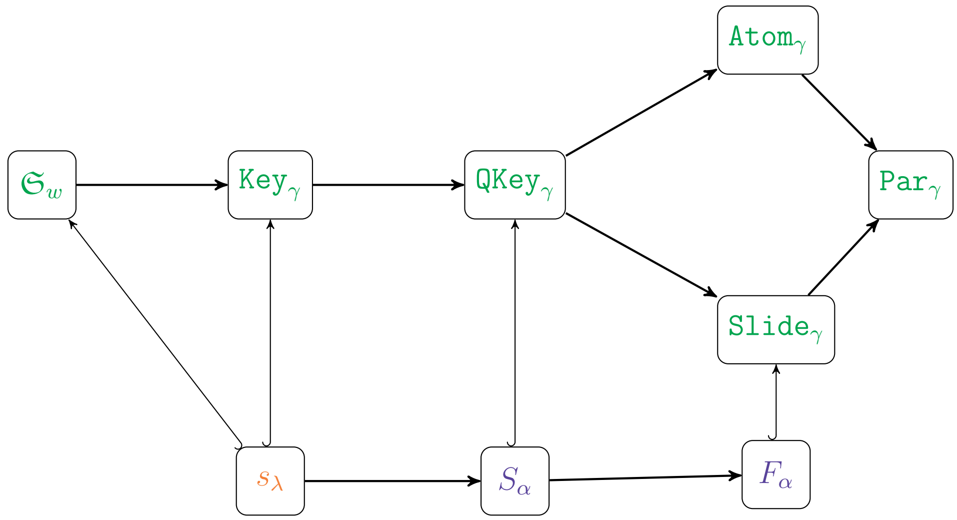


Figure 4.1: The nine homogeneous families of polynomials considered here. Families depicted in orange are bases of Sym, those in purple are bases of QSym, and those in green are bases of Pol. The thinner hooked arrows pointing up denote that the basis at the tail lifts to the basis at the head. The thicker arrows pointing to the right denote that the basis at the tail combinatorially expands in the basis at the head.

The glide polynomials were defined by O. Pechenik and D. Searles [PS17] and provide an inhomogeneous deformation of the slide polynomials. In [PS17], a **weak komposition** is a weak composition where the positive integers may be colored arbitrarily black or red. For a weak komposition b, the **excess** $\mathsf{ex}(b)$ is the number of red entries in b.

Definition 4.14 (Pechenik-Searles, Definition 2.2, [PS17]). *Let γ be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. The weak komposition b is a glide of γ if there exist integers $0 = i_0 < i_1 < \cdots < i_\ell$ such that, for each $1 \leq j \leq \ell$, we have*

(G1) $\gamma_{n_j} = b_{i_{j-1}+1} + \cdots + b_{i_j} - \mathsf{ex}(b_{i_{j-1}+1}, \ldots, b_{i_j})$,

(G2) $i_j \leq n_j$, *and*

(G3) the leftmost nonzero entry among $b_{i_{j-1}+1}, \ldots, b_{i_j}$ is black.

Alternatively, b is a glide of γ if it can be obtained from γ by a finite sequence of the following local moves:

(m1) $0p \Rightarrow p0$, (for $p \in \mathbb{Z}_{>0}$);

(m2) $0p \Rightarrow qr$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p$);

Name	Symbol	Definition	K-analogue of
Symmetric Grothendieck function	Ks_λ	1.31	s_λ
Multi-fundamental quasisymmetric function	KF_α	1.33	F_α
Quasisymmetric Grothendieck function	KS_α	3.5	S_α
Grothendieck polynomial	$K\mathfrak{S}_w$	1.30	$\mathfrak{S}_w$
Lascoux polynomial	$K\mathtt{Key}_\gamma$	1.34 and 3.3	$\mathtt{Key}_\gamma$
Lascoux atom	$K\mathtt{Atom}_\gamma$	1.34 and 3.2	$\mathtt{Atom}_\gamma$
Glide polynomial	$K\mathtt{Slide}_\gamma$	4.15	$\mathtt{Slide}_\gamma$
★			$\mathtt{Par}_\gamma$
■			$\mathtt{QKey}_\gamma$

Table 4.4: The inhomogeneous deformations of the polynomials from Section 4.1.1.

(m3) $0p \Rightarrow qr$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p + 1$).

For example, let $\gamma = (0, 2, 0, 0, 2, 0, 1)$. The weak kompositions

$$(1, 2, 2, 0, 1, 1, 0) \text{ and } (2, 1, 2, 1, 1, 1, 0)$$

are glides of γ.

Definition 4.15 (Pechenik-Searles, Definition 2.5, [PS17]). *The glide polynomial $K\mathtt{Slide}_\gamma$ is*

$$K\mathtt{Slide}_\gamma = \sum_b \beta^{\mathsf{ex}(b)} x_1^{b_1} \cdots x_\ell^{b_\ell},$$

where the sum is over all glides b of γ.

For example.

$$K\mathtt{Slide}_{(0,2,0,1)} = \mathbf{x}^{(0,2,0,1)} + \mathbf{x}^{(0,2,1,0)} + \mathbf{x}^{(1,1,0,1)} + \mathbf{x}^{(1,1,1,0)} + \mathbf{x}^{(2,0,0,1)} + \mathbf{x}^{(2,0,1,0)} + \mathbf{x}^{(2,1,0,0)} +$$

$$\beta\mathbf{x}^{(0,2,1,1)} + \beta\mathbf{x}^{(1,1,1,1)} + \beta\mathbf{x}^{(1,2,0,1)} + \beta\mathbf{x}^{(1,2,1,0)} + \beta\mathbf{x}^{(2,0,1,1)} + 2\beta\mathbf{x}^{(2,1,0,1)} + 2\beta\mathbf{x}^{(2,1,1,0)} +$$

$$\beta^2\mathbf{x}^{(1,2,1,1)} + 2\beta^2\mathbf{x}^{(2,1,1,1)}.$$

As it is clear from the local move definitions of slides and glides, every slide of γ is a glide of γ and thus $K\mathtt{Slide}_\gamma$ is an inhomogeneous deformation of $\mathtt{Slide}_\gamma$. The inhomogeneous deformations of the polynomials from Section 4.1.1 are summarized in Table 4.4.

We now ask for analogous relationships to those described in Section 4.1.1 between these inhomogeneous deformations, starting with lifts of the bases of Sym and QSym to Pol.

Basis of Pol	Lifts	Reference
$K\mathfrak{S}_w$	Ks_λ	Definition 1.30
$K\mathtt{Key}_\gamma$	Ks_λ	Theorems 1.26 and 1.27
$K\mathtt{Slide}_\gamma$	KF_α	Proposition 4.6
■	KS_α	

Table 4.5: The lifts of the inhomogeneous bases from Sym and QSym to Pol.

Basis	Combinatorially Expands In	Reference
Ks_λ	KS_α	Corollary 3.6
KS_α	KF_α	
$K\mathfrak{S}_w$	$K\mathtt{Key}_\gamma$	open
$K\mathtt{Key}_\gamma$	■	
■	$K\mathtt{Atom}_\gamma$	
■	$K\mathtt{Slide}_\gamma$	
$K\mathtt{Slide}_\gamma$	★	
$K\mathtt{Atom}_\gamma$	★	

Table 4.6: The combinatorial expansions between the inhomogeneous bases.

Recall from Section 1.6, the Grothendieck polynomials lift the symmetric Grothendieck polynomials, which correspond to Grassmannian permutations. Furthermore, as in the Demazure case, combining Theorems 3.4 and 3.17 gives that the Lascoux polynomials are a lift of the symmetric Grothendieck polynomials.

O. Pechenik and D. Searles show that the glide polynomials are a lift of the multi-fundamental quasisymmetric polynomials.

Theorem 4.16 (Pechenik-Searles, Theorem 3.5 [PS17])**.** *For a composition α with $\ell(\alpha) = \ell$,*

$$K\mathtt{Slide}_{0^n\times\alpha}(x_1, x_2, \ldots, x_{n+\ell}) = KF_\alpha(x_1, x_2, \ldots, x_{n+\ell}).$$

In terms of combinatorial expansions between the K-theoretic analogues, Corollary 3.6 gave the expansion of Ks_λ in terms of KS_α. The expansion of the Grothendieck polynomials expand in the Lascoux polynomials is still an open problem, and we will establish the other relationships in Table 4.6 in this chapter.

4.1.3 Statement of Results

The main goal of this chapter is to define the bases that play the role of ★ and ■ in Section 4.1.2. We start with ★, a basis that should be an inhomogeneous deformation of the fundamental particles and should simultaneously combinatorially expand the Lascoux atoms and the glide polynomials.

Definition 4.17. *Let γ be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. The weak komposition b is a mesonic glide of γ if, for each $1 \leq j \leq \ell$, we have*

- *(G1′) $\gamma_{n_j} = b_{n_{j-1}+1} + \cdots + b_{n_j} - \mathsf{ex}(b_{n_{j-1}+1}, \ldots, b_{n_j})$,*
- *(G3′) the leftmost nonzero entry among $b_{n_{j-1}+1}, \ldots, b_{n_j}$ is black, and*
- *(G4′) $b_{n_j} \neq 0$.*

Equivalently, a weak komposition b is a mesonic glide of γ if b can be obtained from γ by a finite sequence of the local moves (m1), (m2), and (m3) that never applies (m1) at positions $n_j - 1$ and n_j for any j.

Observe that, in particular, a mesonic glide is a glide that happens to satisfy additional conditions. For example, let $\gamma = (0, 3, 0, 2)$. Then $b = (2, 1, 1, 2)$ is a mesonic glide of γ. On the other hand, while $b' = (3, 1, 0, 2)$ is also a glide of γ, it is not mesonic. To see this fact, observe that γ has nonzero entries in positions $n_1 = 2$ and $n_2 = 4$, while the only (i_1, i_2) satisfying conditions (G1), (G2), and (G3) of Definition 4.14 for b' is $(1, 4)$. Since $i_1 < n_1$, b' is not mesonic. The reader may check that both b and b' can be obtained from γ by a finite sequence of the local moves (m1), (m2), and (m3). However, the reader may also check that b' cannot be obtained without applying (m1) at positions 1 and 2.

Definition 4.18. *The kaon $K\mathtt{Par}_\gamma$ is*

$$K\mathtt{Par}_\gamma = \sum_b \beta^{\mathsf{ex}(b)} \mathbf{x}^b,$$

where the sum is over all mesonic glides of γ.

For example,

$$\begin{aligned}
K\mathtt{Par}_{(0,3,0,2)} &= \mathbf{x}^{(0,3,0,2)} + \mathbf{x}^{(0,3,1,1)} + \mathbf{x}^{(1,2,0,2)} + \mathbf{x}^{(1,2,1,1)} + \mathbf{x}^{(2,1,0,2)} + \mathbf{x}^{(2,1,1,1)} \\
&+ \beta\mathbf{x}^{(0,3,1,2)} + \beta\mathbf{x}^{(0,3,2,1)} + \beta\mathbf{x}^{(1,2,1,2)} + \beta\mathbf{x}^{(1,2,2,1)} + \beta\mathbf{x}^{(1,3,0,2)} + \beta\mathbf{x}^{(1,3,1,1)} \\
&+ \beta\mathbf{x}^{(2,1,1,2)} + \beta\mathbf{x}^{(2,1,2,1)} + \beta\mathbf{x}^{(2,2,0,2)} + \beta\mathbf{x}^{(2,2,1,1)} + \beta\mathbf{x}^{(3,1,0,2)} + \beta\mathbf{x}^{(3,1,1,1)} \\
&+ \beta^2\mathbf{x}^{(1,3,1,2)} + \beta^2\mathbf{x}^{(1,3,2,1)} + \beta^2\mathbf{x}^{(2,2,1,2)} + \beta^2\mathbf{x}^{(2,2,2,1)} + \beta^2\mathbf{x}^{(3,1,1,2)} + \beta^2\mathbf{x}^{(3,1,2,1)}.
\end{aligned}$$

Theorem 4.19. *The kaons satisfy the properties of* ★*, i.e. are*

1. *a basis of* Pol*,*
2. *inhomogeneous deformations of the fundamental particles,*
3. *provide a combinatorial expansion of the Lascoux atoms, and*
4. *provide a combinatorial expansion of the glide polynomials.*

This theorem will be proved in Section 4.2; see Theorem 4.26, Proposition 4.24, Theorem 4.29, and Proposition 4.25.

Recall that ■ is a basis of Pol that is an inhomogeneous deformation of the quasikey polynomials, lifts the quasisymmetric Grothendieck polynomials from QSym to Pol, provides a combinatorial expansion for the Lascoux polynomials, and combinatorially expands in the Lascoux atoms and glide polynomials. We mimic the Demazure atom expansion of the quasikey polynomials to define the **quasiLascoux** polynomials.

Definition 4.20. *The quasiLascoux polynomial* $K\mathtt{QKey}_\gamma$ *is*

$$K\mathtt{QKey}_\gamma = \sum_{\substack{\delta \in \mathsf{Expand}(\gamma^+) \\ \delta \geq_D \gamma}} K\mathtt{Atom}_\delta.$$

Theorem 4.21. *The quasiLascoux polynomials satisfy the properties of* ■*, i.e. are*

1. *a basis of* Pol*,*
2. *inhomogeneous deformations of the quasikey polynomials,*
3. *a lift of the quasisymmetric Grothendieck functions,*
4. *provide a combinatorial expansion of the Lascoux polynomials,*

5. *combinatorially expand in the glide polynomials, and*

6. *combinatorially expand in the Lascoux atoms.*

This theorem will be proved in Sections 4.3 and 4.4; see Theorem 4.35, Equation 4.1, Equation 4.2, Theorem 4.44, and Theorem 4.38. The last claim is our definition of the quasiLascoux polynomials. A consequence of Theorem 4.21 is that the quasisymmetric Grothendieck basis expands combinatorially into the basis of multifundamental quasisymmetric polynomials; see Corollary 4.41. The relations among the nine families of K-theoretic polynomials and their nine families of $\beta = 0$ specializations are illustrated in Figure 4.2.

Except for the Grothendieck polynomials $\{K\mathfrak{G}_w\}$ and their symmetric subset $\{Ks_\lambda\}$, the geometric significance of these deformed polynomials is currently mysterious. While, for example, the glide polynomials seem useful in the study of Grothendieck polynomials [PS17], it is unknown how to interpret any single glide polynomial $K\mathtt{Slide}_\gamma$ as representing a geometric object or datum. We conclude with two conjectures that appear, to us, to suggest some deeper mathematical structure.

Conjecture 4.22. *For weak compositions γ and δ, let*

$$M_{\gamma,\delta}(\beta) = [K\mathtt{Slide}_\delta]K\mathtt{QKey}_\gamma.$$

Then for fixed γ, we have

$$\sum_{\delta \in \mathsf{WC}} M_{\gamma,\delta}(-1) \in \{0, 1\}.$$

For example, for $\gamma = (0, 6, 6, 2)$,

$$\sum_{\delta \in \mathsf{WC}} M_\gamma^\delta = 16\beta^3 + 75\beta^2 + 94\beta + 36.$$

Substituting in $\beta = -1$ gives us 1. This has been checked for all γ with at most 3 zeros and $|\gamma| \leq 7$.

Conjecture 4.23. *For weak compositions γ and δ, let*

$$N_{\gamma,\delta}(\beta) = [K\mathtt{Par}_\delta]K\mathtt{Atom}_\gamma.$$

Then for fixed γ, we have

$$\sum_{\delta \in \mathsf{WC}} N_{\gamma,\delta}(-1) \in \{0, 1\}.$$

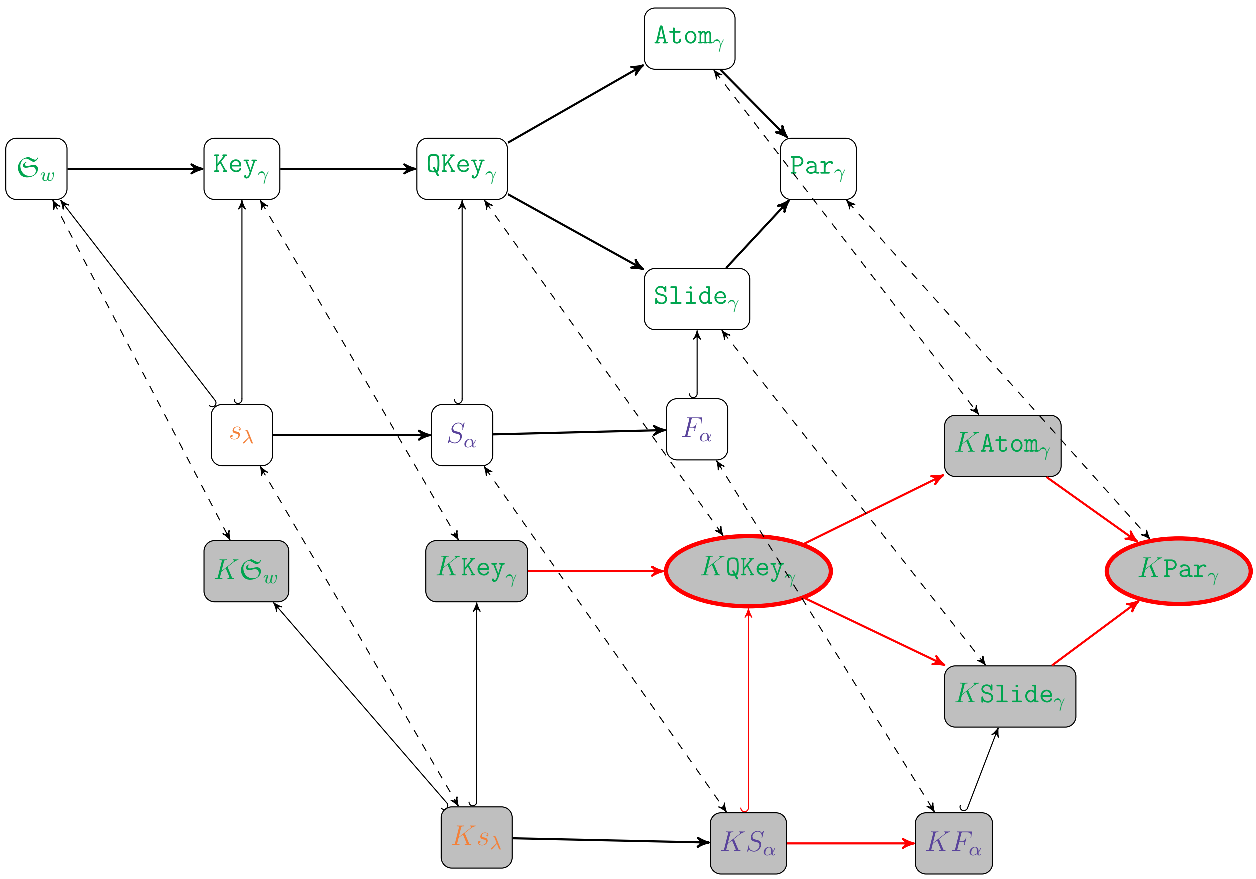


Figure 4.2: The nine homogeneous families of polynomials of Figure 4.1 are shown as white nodes, while their respective K-deformations are shown as grey nodes. Those families that are introduced in this chapter are outlined in red. The dotted arrows denote, in one direction, taking an inhomogeneous deformation and, in the other direction, specializing to $\beta = 0$. As in Figure 4.1, families depicted in orange are bases of Sym, those in purple are bases of QSym, and those in green are bases of Pol_n. The thinner hooked arrows pointing up denote that the basis at the tail lifts to the basis at the head. The thicker arrows pointing to the right denote that the basis at the tail combinatorially expands in the basis at the head. Some of these latter arrows are red; these correspond to the expansions established here for the first time.

□

For example, for $\gamma = (0,6,6,2)$,

$$\sum_{\delta \in \mathsf{WC}} M_\gamma^\delta = 16\beta^3 + 66\beta^2 + 80\beta + 31.$$

Again substituting in $\beta = -1$ gives us 1. This has also been checked for all γ with at most 3 zeros and $|\gamma| \leq 7$.

4.2 Kaons

In this section, we prove that the kaons have the properties of the basis ★ in Section 4.1.2, namely those listed in Theorem 4.19.

Proposition 4.24. *The fundamental particles are the $\beta = 0$ specialization of kaons, i.e. the kaons are an inhomogeneous deformation of the fundamental particles.*

Proof. Since any mesonic slide of γ is a mesonic glide of γ, this is clear from the definitions of the two families of polynomials. □

4.2.1 Kaons and glide polynomials

Proposition 4.25.

$$K\mathtt{Slide}_\gamma = \sum_{\substack{\delta \in \mathsf{Expand}(\gamma^+) \\ \delta \geq_D \gamma}} K\mathtt{Par}_\delta.$$

In particular, every glide polynomial combinatorially expands in the kaons.

Proof. Let γ be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. Suppose g is a glide of γ. Then there are $0 = i_0 < i_1 < \cdots < i_\ell$ satisfying conditions (G1), (G2), and (G3) of Definition 4.14. We obtain g from γ via a 2-step process. First, apply (m1) repeatedly to move each nonzero entry of γ from position n_j to position i_j. Call the resulting weak composition δ. Note that δ satisfies $\delta \geq_D \gamma$ and $\delta^+ = \gamma^+$. Second, apply some sequence of (m1), (m2) and (m3) to obtain the weak komposition g from δ. In this second step, note that we never apply (m1) at positions $i_j - 1$ and i_j for any j.

Hence every glide g of γ is a mesonic glide of a δ with $\delta \geq_D \gamma$ and $\delta^+ = \gamma^+$. Conversely, every mesonic glide of such a weak composition δ is clearly a glide of γ.

Thus to complete the proof, we only need to show that there is at most one δ with $\delta^+ = \gamma^+$ such that b is a mesonic glide of δ. Let $\alpha = \delta^+ = (\alpha_1, \ldots, \alpha_\ell)$. By the definition of mesonic glide, we know that

$$\alpha_j = \delta_{n_j} = b_{n_{j-1}+1} + \ldots + b_{n_j} - \mathsf{ex}(b_{n_{j-1}+1}, \ldots, b_{n_j}),$$

the leftmost nonzero entry among $b_{n_{j-1}+1}, \ldots, b_{n_j}$ is black, and $b_{n_j} \neq 0$. We claim there is only one sequence $n_1, \ldots, n_\ell$ that satisfies these conditions. Suppose to the contrary that there are two $m_1, \ldots, m_\ell$ and $n_1, \ldots, n_\ell$ and let i be the lowest index such that $m_i \neq n_i$. Without loss of generality, assume $m_i > n_i$. Then,

$$\begin{aligned}
\alpha_i &= b_{m_{i-1}+1} + \ldots + b_{m_i} - \mathsf{ex}(b_{m_{i-1}+1}, \ldots, b_{m_i}) \\
&= b_{m_{i-1}+1} + \ldots + b_{n_i} + b_{n_i+1} + \ldots + b_{m_i} - \mathsf{ex}(b_{m_{i-1}+1}, \ldots, b_{n_i}) - \mathsf{ex}(b_{n_i+1}, \ldots, b_{m_i}) \\
&= \alpha_i + b_{n_i+1} + \ldots + b_{m_i} - \mathsf{ex}(b_{n_i+1}, \ldots, b_{m_i}),
\end{aligned}$$

and so we have

$$0 = b_{n_i+1} + \ldots + b_{m_i} - \mathsf{ex}(b_{n_i+1}, \ldots, b_{m_i}).$$

This is only possible if each of $b_{n_i+1}, \ldots, b_{m_i}$ is either 0 or a red 1. Since $b_{m_i} \neq 0$, there is at least one red 1 in this set of entries. However, the first nonzero entry of $b_{n_i+1}, \ldots, b_{n_{i+1}}$ is required to be black, a contradiction. □

In light of Proposition 4.24, setting $\beta = 0$ in Proposition 4.25 recovers [Sea17, Proposition 4.4] on the expansion of fundamental slides into fundamental particles. Note that although our K-theoretic deformations of those polynomials are significantly larger, the matrix of basis change is exactly the same as for fundamental slides into fundamental particles.

Theorem 4.26. *The set*

$$\{\beta^k K\mathtt{Par}_\gamma : k \in \mathbb{Z}_{\geq 0} \textit{ and } \gamma \textit{ is a weak composition of length } n\}$$

is an additive basis of the free $\mathbb{Z}$-module $\mathbb{Z}[x_1, x_2, \ldots, x_n; \beta]$.

Proof. By Proposition 4.25, every glide polynomial can be written as a positive sum of kaons, and indeed the transition matrix is unitriangular with respect to the lexicographic total order on weak compositions. Furthermore, the expansion of a glide polynomial into kaons is finite as there are finitely many compositions with $\gamma^+ = \delta^+$ and $\delta \geq_D \gamma$. Hence, the transition matrix is invertible over $\mathbb{Z}$, and the theorem follows from the fact that glide polynomials are an additive basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n; \beta]$ [PS17, Theorem 2.6].

We can also see that the expansion of any polynomial into kaons is finite by a similar argument to the proof of Theorem 3.10. If δ is a mesonic glide of γ, then $\max(\delta) \leq \max(\gamma)$. Thus, for any monomial $\mathbf{x}^\gamma$, there are finitely many terms (determined solely by γ) that can be introduced in the process of expanding $\mathbf{x}^\gamma$ in the kaons. $\square$

The kaon basis does not have positive structure coefficients. Nonetheless, we conjecture the following:

Conjecture 4.27. *For any weak compositions γ and δ, the product*

$$K\mathtt{Par}_\gamma \cdot K\mathtt{Slide}_\delta$$

expands combinatorially in the kaon basis.

For example,

$$K\mathtt{Par}_{(2,0,1)} K\mathtt{Slide}_{(1,0,2)} = K\mathtt{Par}_{(3,0,3)} + \beta K\mathtt{Par}_{(3,1,3)} + \beta K\mathtt{Par}_{(3,2,2)} + \beta^2 K\mathtt{Par}_{(3,2,3)} + \beta^2 K\mathtt{Par}_{(3,3,2)}.$$

This has been checked for all γ, δ with at most 3 zeros and $|\gamma|, |\delta| \leq 5$.

4.2.2 Kaons and Lascoux atoms

Recall from Definition 3.1 that $\mathsf{SetSkyFill}(\gamma, \mathbf{b})$ is the set of all semistandard set-valued skyline fillings of shape γ and basement $\mathbf{b}$ and that a set-valued skyline filling is semistandard if

(S1) entries do not repeat in a column,

(S2) rows are weakly decreasing where sets $A \geq B$ if $\min A \geq \max B$,

(S3) every triple of anchor entries is an inversion triple, and

(S4) free entries are with the smallest possible anchor entry such that (S2) is not broken.

Let $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. Then for every i appearing in T, let i^+ be the smallest non-basement entry greater than i appearing in T. If k is the greatest entry in T, we set $k^+ = k + 1$.

Definition 4.28. *Let γ be a weak composition and $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. We say T is meson-highest if, for every i appearing in T, either*

- *the leftmost i is in the leftmost column and is an anchor, or*
- *there is a i^+ weakly right of the leftmost i and in a different box.*

Let $\mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)$ be the set of all meson-highest $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. Two examples of meson-highest fillings are below. The filling on the left is meson-highest as the leftmost 1, 3, 4 are anchor in the left-most column and the leftmost 2 (in column 1) has a 3 in column 2. The filling on the right is not meson-highest since there is a free 2 in column 1 and no 3 weakly to the right.

1	**1**		
2			
3	**3**2	**2**	**2**1
4	**4**	**4**31	

1	**1**		
2			
3	**3**2	**2**	**2**1
4	**4**	**4**1	

Theorem 4.29.
$$K\mathtt{Atom}_\gamma = \sum_{T \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|T|-|\gamma|} K\mathtt{Par}_{c(T)}.$$
In particular, the Lascoux atoms expand combinatorially in the kaons.

Setting $\beta = 0$ in Theorem 4.29 recovers [Sea17, Theorem 4.11].

To prove Theorem 4.29, we must first develop properties of a **destandardization** map, denoted dst, on $\mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. Fix $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. Consider the least integer i with the property that the leftmost i in T is

- not an anchor in the leftmost column and
- has no i^+ weakly to its right in a different box;

replace every i in T with an $i+1$. If this results in two instances of $i+1$ in a single box, delete one. Repeat this replacement process until no further replacements can be made: the final result is $\mathsf{dst}(T)$. In fact, the order in which we perform replacements does not affect the resulting destandardization. Nonetheless, it is convenient to fix the explicit replacement order chosen here. This algorithm necessarily terminates, as we only perform replacement on entries i that are strictly less than the maximum entry k of T; this is because k is guaranteed to appear as an anchor in the leftmost column of T.

Example 4.30. On the filling above that is not meson-highest, the following is the result of the destandardization map:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & \mathbf{1} \\ \hline 2 \\ \hline 3 & \mathbf{3}2 & \mathbf{2} & \mathbf{2}1 \\ \hline 4 & \mathbf{4} & \mathbf{4}1 \\ \hline \end{array} \qquad \overset{\mathsf{dst}}{\Longrightarrow} \qquad \begin{array}{|c|c|c|c|} \hline 1 & \mathbf{1} \\ \hline 2 \\ \hline 3 & \mathbf{3} & \mathbf{3} & \mathbf{3}1 \\ \hline 4 & \mathbf{4} & \mathbf{4}1 \\ \hline \end{array} = S$$

Lemma 4.31. *If* $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$*, then*

$$\mathsf{dst}(T) \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i).$$

Moreover,

$$\mathsf{dst}(T) = T \textit{ if and only if } T \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i) \subseteq \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i).$$

Proof. Fix the weak composition γ. By definition, if $T \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)$, then $\mathsf{dst}(T) = T$ as dst changes exactly the entries that don't exist when T is meson-highest. Moreover, if $T \notin \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)$, then by definition $\mathsf{dst}(T) \neq T$. Hence, the second claim of the lemma is clear.

It remains to establish the first claim of the lemma, so fix $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$. It is enough to show that $\mathsf{dst}(T) \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$, for then by definition we must have $\mathsf{dst}(T) \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)$, as the destandardization algorithm does not terminate until the extra conditions defining $\mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)$ are satisfied. Since destandardization is defined as a sequence of replacements, it is enough by induction to show that any single such replacement produces an element of $\mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$.

Suppose we replace the letters i in T and the result is T'. Then, by assumption, the leftmost $i \in T$

- is not an anchor in the leftmost column of T, and
- does not have an i^+ weakly to its right in T and in a different box.

We want to show that T' satisfies the conditions (S.1)–(S.4).

(S1): If there is no column of T containing both i and $i+1$, then it is clear that T' satisfies (S1). Hence, suppose column c of T contains both i and $i+1$. Then, $i^+ = i+1$. Since T

has then no $i+1$ weakly to the right of the leftmost i, c must be the column of the leftmost i. Moreover, i and $i+1$ must appear in the same box b of column c in T. Thus, replacing the i's results in two instances of $i+1$ in b, one of which we then delete. Thus, T' has no repeated entries in any column.

(S2): If row i of T contains an entry i, then by (S2) for T, row i has an anchor i in the first column. Thus, in this case, the leftmost i is an anchor i in the first column.

Therefore, every entry i in T is in some row j with $j > i$. Moreover, for each such $j > i$, we have by (S2) for T that all labels strictly left of the leftmost i in row j are strictly greater than i. Hence, replacing every i in T with with $i+1$ preserves the rows being weakly decreasing, and does not increase an entry beyond the basement value at the start of the row.

(S3): The four triples we will discuss in this case are pictured below, from left to right:

$$\begin{array}{cc} \boxed{c} & \boxed{i+1} \\ & \vdots \\ & \boxed{i} \end{array} \qquad \begin{array}{cc} \boxed{i} & \boxed{a} \\ & \vdots \\ & \boxed{i+1} \end{array} \qquad \begin{array}{cc} \boxed{i} & \\ \vdots & \\ \boxed{c} & \boxed{i+1} \end{array} \qquad \begin{array}{cc} \boxed{i+1} & \\ \vdots & \\ \boxed{i} & \boxed{a} \end{array} .$$

To see that no type A coinversion triples appear in T', suppose first that T has a type A inversion triple with $c \geq a > b$. This could become a coinversion triple in T' only if $b = i$ and $a = i+1 = i^+$. However, in this case, T has i and i^+ in distinct boxes of the same column, contradicting our assumptions on the number i.

Now, suppose instead that T has a type A inversion triple with $b > c \geq a$. This could become become a coinversion triple in T' only if $b = i+1 = i^+$ and $c = i$. However, in this case, T has $i+1$ appearing strictly right of i, again contradicting our assumptions on the number i.

To see that no type B coinversion triples appear in T', suppose first that T has a type B inversion triple with $c \geq a > b$. This could become a coinversion triple in T' only if $b = i$ and $a = i+1 = i^+$. However, then T would have an i^+ strictly right of an i, contradicting our assumptions on i.

Finally, suppose T has a type B inversion triple with$b > c \geq a$. This could become a coinversion triple in T' only if $b = i+1 = i^+$ and $c = i$. However, then T would have an i and an $i+1$ in distinct boxes of the same column, again contradicting our assumptions on i.

(S4): If a free entry i of T becomes a free entry of $i+1$ of T', and is not deleted, then its anchor entry j is larger than $i+1$ in both T and T'. In particular, since j was the smallest anchor entry in this column accept a free entry i in T (by (S4)), j is still the smallest anchor entry accepting a free entry $i+1$ in T'.

If an anchor entry i of T becomes the anchor entry $i+1$ of T', then since any other anchor entry in this column is either greater than $i+1$ or smaller than i, any free entries in the cell of this anchor entry are still with the smallest possible anchor entry. Any other free entries in a column where an anchor entry i becomes an anchor entry $i+1$ are also still with the smallest possible anchor entry, again since other anchor entries in this column are either greater than $i+1$ or smaller than i. $\square$

Lemma 4.32. *Let* $S \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)$. *Then*

$$K\mathtt{Par}_{c(S)} = \sum_{T \in \mathsf{dst}^{-1}(S)} \beta^{|T|-|S|}\mathbf{x}^T.$$

Proof. We must establish a weight-preserving bijection between mesonic glides of $c(S)$ and fillings $T \in \mathsf{dst}^{-1}(S)$.

Fix $T \in \mathsf{dst}^{-1}(S)$ with $c(T) = (c_1, \ldots, c_\ell)$. Define the **colored weight** $\mathsf{kwt}(T)$ of T to be the weak komposition obtained from $c(T)$ by coloring c_{i+1} red if an $i+1$ is deleted after replacing every i with $i+1$ during a step of destandardization. For example, for the filling in Example 4.30, $c(S) = (3,0,3,2)$ and $\mathsf{kwt}(T) = (3,3,\textcolor{red}{1},2)$ since a 3 is deleted when the 2's are changed to 3's. We observe that $(3,3,\textcolor{red}{1},2)$ is a mesonic glide of $(3,0,3,2)$.

We claim that in general $\mathsf{kwt}(T)$ is a mesonic glide of $c(S)$. Consider the process of destandardization of T to produce S. Each time we replace every i in T' by $i+1$ to produce T'', we change the ith and $(i+1)$st entries of the colored weight by

$$(q, \textcolor{red}{r}) \mapsto (0, q+r-1),$$

if a duplicate $i+1$ is deleted, or

$$(q, r) \mapsto (0, q+r),$$

if not. Note that when we replace the i's with $i+1$'s, we can remove at most one $i+1$. If there were two boxes containing two $i+1$'s after replacing the i's, then T had an $i+1$ weakly to the right of an i and in a different box and so we would not have replaced the i's. Since these are the inverses of the local move (m3) in the first case and either (m1) or (m2) in the second case, it follows then that $\mathsf{kwt}(T)$ is a glide of the weak composition $c(S)$.

Let $c(S)$ have nonzero entries in positions $n_1 < \cdots < n_\ell$. A local change to the colored weight

$$(q, 0) \mapsto (0, q)$$

in positions $n_j - 1$ and n_j for some j would correspond to a step of destandardization replacing every instance of $n_j - 1$ with n_j in a T' that contains no label n_j. Since S is meson-highest, the entries n_j of S satisfy the meson-highest condition. Since T' destandardizes to S and all entries $n_j - 1$ change to n_j, the entries $n_j - 1$ of T' therefore also satisfy the meson-highest condition. This contradicts the application of such a destandardization step to T'. Hence, $\mathsf{kwt}(T)$ is a mesonic glide of $c(S)$.

For the other direction, let b be a mesonic glide of the weak composition $c(S)$. We construct (the unique) $T \in \mathsf{dst}^{-1}(S)$ such that $\mathsf{kwt}(T) = b$ as follows. Suppose $c(S)$ has nonzero entries in positions $n_1 < \cdots < n_\ell$.

For $i = 1, \ldots, \ell$ carry out the following process. Construct the string str_i as follows. For each nonzero $b_{n_{j-1}+1}, \ldots, b_{n_j}$ if b_i is black, record b_i black i's and if b_i is red, record one red i followed by $b_i - 1$ black i's. Now we will replace the instances of n_1 in S with the entries of str_1 as follows.

In the rightmost box of S containing an n_1, replace that n_1 with the first entry of str_1. If the next entry of str_1 is red, place it in the same box the previous entry was placed in, otherwise, place it in the next box of S to the left that contains an n_1, replacing that n_1. Continue in this manner until all entries of str_1 have been placed in S. This procedure is well-defined since by (S1) no more than one entry n_1 appears in any column of S; and since the number $c(S)_{n_1}$ of n_1's in S is exactly the length of the string str_1 minus the number of red entries (by (G1)), each black entry goes in a different box of S, and each red entry goes in a box along with a black entry. Repeat this process with $n_2, \ldots, n_\ell$ in order.

For example, consider the following meson-highest filling S below. We have $c(S) = (0, 3, 0, 2)$ and $(2, 1, 1, 2)$ is a mesonic glide of $c(S)$. Thus, $\mathsf{str}_1 = 112$ and $\mathsf{str}_2 = 344$ and we make the following replacements in S

$$S = \begin{array}{|c|c|c|c|} \hline 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 \\ \hline 4 & 4 & 4 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{|c|c|c|c|} \hline 1 \\ \hline 2 & 2 & 1 & 1 \\ \hline 3 \\ \hline 4 & 4 & 4 \\ \hline \end{array}. \quad \Rightarrow \quad \begin{array}{|c|c|c|c|} \hline 1 \\ \hline 2 & 2 & 1 & 1 \\ \hline 3 \\ \hline 4 & 4 & 43 \\ \hline \end{array} = T.$$

If the resulting filling T is a valid semistandard skyline filling, it is unique as there was no choice at any step of the process. Furthermore, by construction, the resulting filling T has colored weight b and destandardizes to S. Thus, we only need to show that $T \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)$.

To see this, notice that the entries of str_j (which replace the entries n_j in S) are all strictly larger than n_{j-1} and weakly smaller than n_j. This fact implies that all the inequalities between entries of boxes in S are preserved, and thus all of (S1), (S2), (S3) and (S4) are preserved. Since $b_{n_j} \neq 0$, the leftmost n_j in S is still a n_j. This guarantees that the anchor entry of the leftmost column remains equal to its row index, and thus (S3) is satisfied including the basement boxes. □

Proof of Theorem 4.29. We have

$$\begin{aligned}\sum_{S \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|S|-|\gamma|} K\mathtt{Par}_{c(S)} &= \sum_{S \in \mathsf{MesonSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|S|-|\gamma|} \sum_{T \in \mathsf{dst}^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^T \\ &= \sum_{U \in \mathsf{SetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|U|-|\gamma|} \mathbf{x}^U \\ &= K\mathtt{Atom}_\gamma,\end{aligned}$$

where the first inequality is Lemma 4.32, the second is Lemma 4.31, and the third is Definition 3.2. □

4.3 QuasiLascoux polynomials

In this section, we describe a combinatorial model for the quasiLascoux polynomial, giving the monomial expansion directly. We also prove that the quasiLascoux polynomials satisfy the properties of ■, i.e. the properties listed in Theorem 4.21.

Definition 4.33. *A quasistandard set-valued skyline filling is a filling of a skyline diagram such that*

- *(S1) entries do not repeat in a column,*
- *(S2) rows are weakly decreasing including the basement,*
- *(S3') every triple of anchor entries is an inversion triple, excluding triples with any entry in the basement,*

(S4) free entries are with the smallest possible anchor entry that does not violate (S2),

(S5) and the anchor entries of the leftmost column (strictly) increase from top to bottom.

Let$\mathsf{QSetSkyFill}(\gamma, \mathbf{b})$ to be the collection of quasistandard set-valued skyline fillings of shape γ and basement $\mathbf{b}$.

Proposition 4.34.

$$K\mathtt{QKey}_\gamma = \sum_{S \in \mathsf{QSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{\mathsf{ex}(S)} \mathbf{x}^S.$$

Proof. There is a weight-preserving bijection

$$\mathsf{QSetSkyFill}(\gamma, \mathbf{b}_i) \quad \longleftrightarrow \bigcup_{\substack{\delta \in \mathsf{Expand}(\gamma^+) \\ \delta \geq_D \gamma}} \mathsf{SetSkyFill}(\delta, \mathbf{b}_i)$$

where the image of $T \in \mathsf{QSetSkyFill}(\gamma, \mathbf{b})$ is obtained by moving each row of T upwards so that each entry in the first column is equal to its row index. This is well-defined since moving rows does not affect the status of any triple, as long at the rows retain their relative order. The triple condition applying to entries in the basement merely forces the anchor entries of the first column be equal to their basement value. The statement then follows from Definitions 3.2 and 4.20. □

For example, the set $\mathsf{QSetSkyFill}((1, 0, 2), \mathbf{b}_i)$ consists of the ten diagrams in Figure 4.3. Therefore, we have

$$K\mathtt{QKey}_{(1,0,2)} = \mathbf{x}^{(1,0,2)} + \mathbf{x}^{(1,1,1)} + \mathbf{x}^{(1,2,0)} + \beta\mathbf{x}^{(1,1,2)} + \beta\mathbf{x}^{(2,0,2)} + \beta\mathbf{x}^{(1,2,1)} +$$

$$\beta\mathbf{x}^{(2,1,1)} + \beta\mathbf{x}^{(2,2,0)} + \beta^2\mathbf{x}^{(2,1,2)} + \beta^2\mathbf{x}^{(2,2,1)}.$$

Theorem 4.35. *The set*

$$\{\beta^k K\mathtt{QKey}_\gamma : k \in \mathbb{Z}_{\geq 0} \textit{ and } \gamma \textit{ is a weak composition of length } n\}$$

is an additive basis of the free $\mathbb{Z}$-module $\mathbb{Z}[x_1, x_2, \ldots, x_n; \beta]$.

Proof. Since the proof of Lemma 3.9 depends only on (S1) and (S2) which hold for the

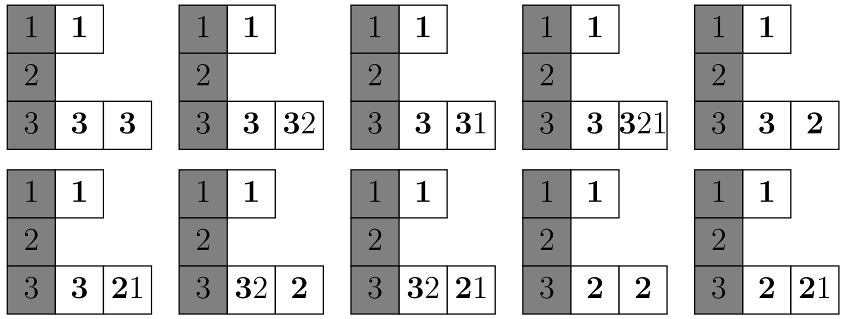


Figure 4.3: The ten elements of $\mathsf{QSetSkyFill}((1,0,2),\mathbf{b}_i)$.

quasistandard fillings, we have

$$K\mathtt{QKey}_\gamma = \mathbf{x}^\gamma + \sum_{\substack{\delta\in\mathsf{S}(\gamma)\\ \delta\geqslant\gamma}} b_{\gamma,\delta}\beta^{|\delta|-|\gamma|}\mathbf{x}^\delta.$$

Then replacing $K\mathtt{Atom}_\gamma$ with $K\mathtt{QKey}_\gamma$ in the proof of Proposition 3.10 completes the proof that the quasiLascoux polynomials form a basis. □

The quasiLascoux polynomials simultaneously generalize the quasikey polynomials of [AS16] and the quasisymmetric Grothendieck polynomials defined in Chapter 3. Clearly

$$K\mathtt{QKey}_\gamma(\mathbf{x};0) = \mathtt{QKey}_\gamma(\mathbf{x}), \tag{4.1}$$

and if all nonzero entries of γ occur in an interval with γ_k the last nonzero entry of γ, then

$$K\mathtt{QKey}_\gamma = KS_{\gamma^+}(x_1,\ldots,x_k). \tag{4.2}$$

Moreover, we have

Proposition 4.36. *Let γ be a weak composition. Then the stable limit of the quasiLascoux polynomial $K\mathtt{QKey}_\gamma$ is the quasisymmetric Grothendieck function $KS_{\gamma^+}(x_1,x_2,\ldots)$.*

Proof. Let $m>0$ and consider the polynomial $K\mathtt{QKey}_{0^m\times\gamma}(x_1,\ldots,x_m)$. For any weak composition δ, the Lascoux atom $K\mathtt{Atom}_\delta$ is divisible by x_{b_i} whenever $b_i>0$, so if $K\mathtt{Atom}_\delta$ appears in the Lascoux atom expansion of $K\mathtt{QKey}_{0^m\times\gamma}$, it is annihilated on restriction to m variables. Thus by definition

$$K\texttt{QKey}_{0^m\times\gamma}(x_1,\ldots,x_m) = \sum_{\substack{\delta^+=\gamma^+\\ \ell(\delta)\le m}} K\texttt{Atom}_\delta = KS_{\gamma^+}(x_1,\ldots,x_m).$$

The statement then follows by letting $m\to\infty$. $\square$

4.3.1 Glide expansion

Definition 4.37. *For $S \in \mathsf{QSetSkyFill}(\gamma,\mathbf{b}_i)$, we say S is quasiYamanouchi if, for every integer i appearing in T, either*

- *the leftmost i is anchor in box $(i,1)$, or*
- *there is a $i+1$ in weakly right of the leftmost i and in a different box.*

Let $\mathsf{YamQSetSkyFill}(\gamma,\mathbf{b})$ be the set of all quasiYamanouchi $S\in\mathsf{QSetSkyFill}(\gamma,\mathbf{b})$. For example, the first and third fillings of Figure 4.3 are the only quasi-Yamanouchi fillings of $\mathsf{QSetSkyFill}((1,0,2),\mathbf{b}_i)$. Observe that the quasiYamanouchi condition is exactly the meson-highest condition of Definition 4.28, with i^+ replaced by $i+1$. The change to the first requirement being in box $(i,1)$ is implied as an anchor i in the leftmost column of a filling of $\mathsf{SetSkyFill}(\gamma,\mathbf{b}_i)$ is forced to be in row i.

Theorem 4.38.

$$K\texttt{QKey}_\gamma = \sum_{S\in\mathsf{YamQSetSkyFill}(\gamma,\mathbf{b}_i)} \beta^{|S|-|\gamma|}K\texttt{Slide}_{c(S)}.$$

In particular, every quasiLascoux polynomial $K\texttt{QKey}_\gamma$ is a positive sum of glide polynomials.

To prove Theorem 4.38, we introduce a destandardization map dst_Q on $\mathsf{QSetSkyFill}(\gamma,\mathbf{b})$. Fix $T\in\mathsf{QSetSkyFill}(\gamma,\mathbf{b})$. Consider the least integer i appearing in T with the property that

- the leftmost i in T is not an anchor in box $(i,1)$, and
- it has no $i+1$ weakly to its right in a different box;

replace every i in T with an $i+1$. If this results in two instances of $i+1$ in a single box, delete one. Repeat this replacement process until no further replacements can be made; the final result in the destandardization $\mathsf{dst}_Q(T)$. The destandardization map dst_Q is exactly the destandardization map dst of Section 4.2 with i^+ replaced by $i+1$.

Continuing our example, the first, second, fifth, seventh and ninth fillings of Figure 4.3 destandardize to the first filling; the remaining fillings destandardize to the third filling.

Lemma 4.39. *If* $T \in \mathsf{QSetSkyFill}(\gamma, \mathbf{b}_i)$ *then*

$$\mathsf{dst}_Q(T) \in \mathsf{YamQSetSkyFill}(\gamma, \mathbf{b}_i).$$

Moreover,

$$\mathsf{dst}_Q(T) = T \textit{ if and only if } T \in \mathsf{YamQSetSkyFill}(\gamma, \mathbf{b}_i) \subset \mathsf{QSetSkyFill}(\gamma, \mathbf{b}_i).$$

Proof. Identical to the proof of Lemma 4.31, with i^+ replaced by $i+1$. $\square$

Lemma 4.40. *Let* $S \in \mathsf{YamQSetSkyFill}(\gamma, \mathbf{b}_i)$*. Then*

$$K\mathtt{Slide}_{c(S)} = \sum_{T \in \mathsf{dst}_Q^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^T.$$

Proof. Nearly identical to the proof of Lemma 4.32. The only difference is that the anchor entries of the leftmost column no longer have to be equal to their row index, which corresponds to the fact that the glide is not necessarily mesonic, and therefore it is possible $b_{n_j} = 0$. $\square$

Proof of Theorem 4.38. We have

$$\begin{aligned}\sum_{S \in \mathsf{YamQSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|S|-|\gamma|} K\mathtt{Slide}_{c(S)} &= \sum_{S \in \mathsf{YamQSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|S|-|\gamma|} \sum_{T \in \mathsf{dst}_Q^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^T \\ &= \sum_{U \in \mathsf{QSetSkyFill}(\gamma, \mathbf{b}_i)} \beta^{|U|-|\gamma|} \mathbf{x}^{c(U)} \\ &= K\mathtt{QKey}_{\gamma},\end{aligned}$$

where the first equality is Lemma 4.40, the second is Lemma 4.39, and the third is Proposition 4.34. $\square$

Setting $\beta = 0$ in the statement of Theorem 4.38 yields a positive formula for the fundamental slide expansion of a quasikey polynomial in terms of quasiYamanouchi quasistandard skyline fillings. A formula in terms of these objects was alluded to in [Sea17] but not stated explicitly.

Corollary 4.41. *The quasisymmetric Grothendieck functions expand combinatorially in the multi-fundamental quasisymmetric functions.*

Proof. By Theorem 4.38, any quasiLascoux (and thus quasisymmetric Grothendieck) polynomial expands positively in the glide basis. The statement then follows from Proposition 4.36 and the fact [PS17] that the stable limits of glide polynomials are the multi-fundamental quasisymmetric functions. □

4.4 Lascoux Polynomials

Recall from Theorem 1.26,

$$\mathtt{Key}_\gamma = \sum_{\substack{\delta \in \mathsf{PermutWC}(\lambda(\gamma)) \\ \delta \ll \gamma}} \mathtt{Atom}_\gamma.$$

We have seen several different ways to define the Lascoux polynomials:

- The original operator definition of A. Lascoux (see Definition 1.34),
- Using the ghostly Kohnert diagrams of C. Ross and A. Yong [RY15], and
- Using set-valued skyline diagrams (see Definition 3.3).

Conjecturally, these definitions are all equivalent, however, there are no known proofs of the equivalence of any of these definitions. While Theorem 3.17 shows

$$\widehat{K\mathtt{Key}}_\gamma = \sum_{\substack{\delta \in \mathsf{PermutWC}(\lambda(\gamma)) \\ \delta \ll \gamma}} \widehat{K\mathtt{Atom}}_\delta$$

for the Lascoux polynomials defined by divided difference operators, this decomposition has not been shown for the other definitions of the Lascoux polynomials or Lascoux atoms.

However, for this chapter, we take this decomposition using combinatorial Lascoux atoms as the definition of the Lascoux polynomials. Conjecture 3.18 combined with Theorem 3.17 would imply this definition is equivalent to the others.

Definition 4.42. *The combinatorial Lascoux polynomial $K\mathtt{Key}_\gamma$ is*

$$K\mathtt{Key}_\gamma = \sum_{\substack{\delta \in \mathsf{PermutWC}(\lambda(\gamma)) \\ \delta \ll \gamma}} K\mathtt{Atom}_\delta.$$

Recall from Definition 4.8 that given a weak composition γ, a left swap is the exchange of two parts $\gamma_i \leq \gamma_j$ where $i < j$ and $\mathsf{lswap}(\gamma)$ is the set of weak compositions δ that can be obtained by a (possibly empty) sequence of left swaps starting with γ.

Lemma 4.43.

$$K\mathtt{Key}_\gamma = \sum_{\delta \in \mathsf{lswap}(\gamma)} K\mathtt{Atom}_\delta.$$

Proof. By the definition of combinatorial Lascoux polynomials, this amounts to showing that $\mathsf{lswap}(\gamma) = \{\delta | \delta \ll \gamma\}$. This is proved in [Sea17, Lemma 3.1]. $\square$

Recall that $\mathsf{Qlswap}(\gamma)$ is the set of all $\delta \in \mathsf{lswap}(\gamma)$ such that for all $\epsilon \in \mathsf{lswap}(\gamma)$ with $\epsilon^+ = \delta^+$, we have $\epsilon \geq_D \delta$.

Theorem 4.44.

$$K\mathtt{Key}_\gamma = \sum_{\delta \in \mathsf{Qlswap}(\gamma)} K\mathtt{QKey}_\delta.$$

In particular, the (combinatorial) Lascoux polynomials combinatorially expand in the quasi-Lascoux polynomials.

Proof. If $\delta \in \mathsf{Qlswap}(\gamma)$ and $\epsilon \geq_D \delta$ with $\epsilon^+ = \delta^+$, then clearly $\epsilon \in \mathsf{lswap}(\gamma)$. By definition of lswap and Qlswap, every $\epsilon \in \mathsf{lswap}(\gamma)$ is either in $\mathsf{Qlswap}(\gamma)$ or dominates some $\delta \in \mathsf{Qlswap}(\gamma)$ with $\epsilon^+ = \delta^+$. This establishes the second equality in the following:

$$K\mathtt{Key}_\gamma = \sum_{\epsilon \in \mathsf{lswap}(\gamma)} K\mathtt{Atom}_\epsilon = \sum_{\delta \in \mathsf{Qlswap}(\gamma)} \sum_{\substack{\epsilon \geq_D \delta \\ \epsilon^+ = \delta^+}} K\mathtt{Atom}_\epsilon = \sum_{\delta \in \mathsf{Qlswap}(\gamma)} K\mathtt{QKey}_\delta,$$

where the first equality is by Lemma 4.43 and the third equality is by the definition of quasiLascoux polynomials. $\square$

While the products of Lascoux polynomials do not expand combinatorially in the Lascoux basis, we conjecture that they do expand combinatorially in the Lascoux atoms.

Conjecture 4.45. *The product $K\mathtt{Key}_\gamma \cdot K\mathtt{Key}_\delta$ is a positive sum of Lascoux atoms.*

For example,

$$K\mathtt{Key}_{(0,2)} K\mathtt{Key}_{(0,1)} = K\mathtt{Atom}_{(0,3)} + K\mathtt{Atom}_{(1,2)} + 2\beta K\mathtt{Atom}_{(1,3)} + K\mathtt{Atom}_{(2,1)} + \beta K\mathtt{Atom}_{(2,2)} +$$

$$\beta^2 K\mathtt{Atom}_{(2,3)} + K\mathtt{Atom}_{(3,0)} + 2\beta K\mathtt{Atom}_{(3,1)} + \beta^2 K\mathtt{Atom}_{(3,2)}.$$

This has been checked for all γ, δ with at most 3 zeros such that $|\gamma|, |\delta| \leq 5$. Specializing Conjecture 4.45 at $\beta = 0$ recovers a conjecture of V. Reiner–M. Shimozono on products of key polynomials (see [Pun16] for discussion and partial results).

CHAPTER 5

Conjectures about Schubert Calculus

5.1 Introduction

We take this opportunity to include some conjectures about (equivariant) Schubert calculus. Let us begin with a brief history of the subject, culminating with the connection to Schur polynomials, which we alluded to in Chapter 1. Our reference is S. Kleiman's [Kle76].

Schubert calculus dates back to the nineteenth century when H. Schubert, among others, was trying to answer questions in enumerative geometry. A well-known introductory question of this type is the "four lines problem" that asks given four random lines in 3-space, how many lines intersect these 4 lines? The calculations of Schubert would conclude there are 2 in the following manner: first we deform our four arbitrary lines so that we have two pairs of intersecting lines. Then there is one line in between the two points of intersection and for the other line, each pair of intersecting lines spans a plane and the two planes intersect in a second line. Since deforming the lines was a continuous function, the number of lines must be a continuous function as well. Thus it is a constant function and so there are always two such lines.

It should be noted that the above problem should really be formulated in $\mathbb{CP}^3$, and the notion of "random" is not defined. Indeed, Hilbert's fifteenth problem was to put a rigorous foundation for Schubert's problems and methods, such as the "principle of conservation of number" utilized above. The modern formulation and solution to Schubert's four line problem involves the Grassmannian and its cohomology ring; this is part of intersection theory [Ful98]. Our reference for the cohomology ring of the Grassmannian and its basis in Schubert cycles is [Man01].

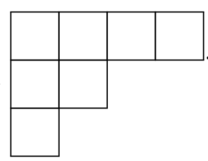
Let X be the Grassmannian of k-planes in $\mathbb{C}^n$. The group $\mathsf{G} = \mathsf{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n$ and this action passes to an action on X. The Borel subgroup B of G is the group of invertible, upper triangular matrices, while the maximal torus T of G is the group of invertible diagonal matrices.

We can represent a point in the Grassmannian as a $k \times n$ matrix of rank k where the k-plane is the row space of the matrix. However, the choice of matrix is not unique as performing row operations on the matrix does not change the k-plane spanned by the rows. Thus, we can consider points in the Grassmannian to be full rank $k \times n$ matrices in reduced row-echelon form. We can partition the Grassmannian into **Schubert cells** by classifying matrices on the positions of the ones:

$$\begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since our matrix must be full rank, we know there must be k pivot entries. The left-most possible positions for these entries are columns $1, \ldots, k$. Furthermore for all i, we know that the pivot in column $i+1$ must be further to the right than the pivot in row i. Thus, if we remove i entries from row i and count the remaining number of entries to the left of the pivot of row i from bottom-to-top, we obtain a partition λ. Furthermore, $\lambda \subset (n-k) \times k$. We can then index the Schubert cells by a partition, denoted X_λ°:

$$\begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Leftrightarrow$$



The closure of X_λ° is the **Schubert variety** X_λ. Any matrix in X_λ will have pivots to the right of those for X_λ°. Thus,

$$X_\lambda = \bigcup_{\lambda \subseteq \mu} X_\mu.$$

The **Schubert class** σ_λ is the Poincaré dual of the Schubert variety X_λ and since the Schubert cells provide a cell decomposition X, the Schubert classes are a $\mathbb{Z}$-linear basis for the cohomology ring $H^*(X)$. Products of Schubert classes in $H^*(X)$ correspond to taking intersections of the Schubert varieties. To fully understand the product structure of $H^*(X)$, it suffices to understand the product

$$\sigma_\lambda \smile \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\sigma_\nu.$$

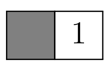
Remarkably, the coefficients $c^\nu_{\lambda,\mu}$ are the exact Littlewood-Richardson coefficients as in the product of Schur functions (see Section 1.2.2):

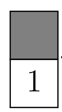
$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

Returning to the four lines problem, let X be the Grassmannian of 2-planes in $\mathbb{C}^4$. Since our four lines are in projective 3-space, they can be considered as points of X. If we fix a point in X, the Schubert variety $X_{(1)}$ is the collection of all points of X that intersect our fixed point in at least a line. Thus to find the number of lines that meet 4 random lines, we merely need to calculate $\sigma^4_{(1)}$. Then using the Littlewood-Richardson rule and the fact that the Schubert cells of X are indexed by partitions that fit inside a 2×2 box, we have the following calculations:

$$\sigma^2_{(1)} = \sigma_{(1,1)} + \sigma_{(2)}$$

with the witnessing fillings





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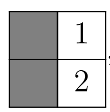
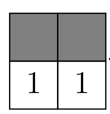
Then

$\sigma^2_{(1,1)} = \sigma_{(2,2)}$ due to 

,

and

$\sigma^2_{(2)} = \sigma_{(2,2)}$ due to 

.

Finally, $\sigma_{(1,1)}\sigma_{(2)} = 0$ and thus

$$\sigma^4_{(1)} = 2\sigma_{(2,2)}.$$

Therefore, there are 2 lines that meet any 4 random lines, as calculated before.

We now generalize this discussion to the equivariant cohomology ring of X; see [TY13] and the references therein. Since each X_λ is stable with respect to the T action, they also admit classes $\sigma^{\mathsf{T}}_\lambda$ in the T-equivariant cohomology ring of X. The equivariant Schubert classes form a basis of $H_{\mathsf{T}}(X, \mathbb{Q})$ as a module over $H_{\mathsf{T}}(\mathrm{pt}) \cong \mathbb{Z}(\beta_1, ..., \beta_{n-1})$. Thus the equivariant structure constants are defined

$$\sigma_\lambda^{\mathsf{T}} \cdot \sigma_\mu^{\mathsf{T}} = \sum_\nu \mathsf{C}_{\lambda,\mu}^{\nu}(\beta_1, \ldots, \beta_{n-1})\sigma_\nu^{\mathsf{T}}.$$

In [Gra01, Corollary 4.1], W. Graham proved that $\mathsf{C}_{\lambda,\mu}^{\nu}$ is a polynomial with positive coefficients in the β_i. The first [Gra01]-positive combinatorial rule for the Grassmannian was given by A. Knutson and T. Tao [KT03, Theorem 1] using **puzzles**. Tableau rules were given by V. Kreiman [Kre10] and A. Molev [Mol09], however H. Thomas and A. Yong [TY13, Theorem 1.2] give a rule in terms of **edge-labeled tableaux** and ***jeu de taquin***, a tableaux operation.

While the equivariant cohomology of the Grassmannian is well understood, there are other spaces of interest where this problem is still open, most particularly the miniscule flag varieties; see [TY09a] and [BL11, Chapter 9]. More generally, let G be a complex, connected, semisimple Lie group with Weyl group W and fix a Borel subgroup $\mathsf{B} \subseteq \mathsf{G}$. Let B_- denote the opposite subgroup and $\mathsf{T} = \mathsf{B} \cap \mathsf{B}_-$ the maximal torus. A subgroup P is parabolic when $\mathsf{B} \subseteq \mathsf{P} \subseteq \mathsf{G}$ and associated to P we have $\mathsf{W}_\mathsf{P} \subseteq \mathsf{W}$. Then $X = \mathsf{G}/\mathsf{P}$ is a generalized flag variety.

The **Bruhat decomposition** of X is the decomposition of X into B_--orbits as follows. Each coset of W_P has a unique minimal length representative w. For $wW_P \in W/W_P$, the **Schubert cell** is defined $X_w^\circ = B_- wP/P$ and the **Schubert variety** X_w is the closure of X_w°. Finally, the **Schubert class** σ_w is the Poincaré dual of the Schubert variety X_w and the set of Schubert classes $\{\sigma_w\}$ where w is a minimal coset representative forms a basis of the cohomology ring $H^*(\mathsf{G}/\mathsf{P}, \mathbb{Q})$.

The Schubert structure problem in the cohomology ring $H^*(\mathsf{G}/\mathsf{P}, \mathbb{Q})$ is to give a combinatorial rule for the structure coefficients $\mathsf{C}_{u,v}^{w}$ in

$$\sigma_u \cdot \sigma_v = \sum_w \mathsf{C}_{u,v}^{w} \sigma_w.$$

Miniscule flag varieties correspond to certain maximal parabolic subgroups, and a root-system uniform combinatorial rule for the Schubert structure problem was given for the these flag varieties by H. Thomas and A. Yong [TY09a].

Since each X_w is stable with respect to the T-action, they admit classes σ_w^{T} in the T-

equivariant cohomology ring of G/P, denoted $H_{\mathsf{T}}(\mathsf{G}/\mathsf{P};\mathbb{Q})$. It is a fact [KT03] that

$$\mathsf{C}_{u,v}^{w}(\beta_1,\ldots,\beta_r) = \begin{cases} 0 & \ell(u)+\ell(v)<\ell(w) \\ c_{u,v}^{w} & \ell(u)+\ell(v)=\ell(w) \end{cases}.$$

Moreover, W. Graham's positivity result that $\mathsf{C}_{u,v}^{w}$ as an element of $\mathbb{Z}[\beta_1,...,\beta_r]$ is a polynomial with nonnegative integer coefficients applies in this more general setting [Gra01, Corollary 4.1].

The cominiscule flag varieties are (see [TY09a, Table 1])

- Type A_n, the Grassmannian $Gr(k,n)$,
- Type B_n, the odd orthogonal Grassmannian $OG(n,2n+1)$,
- Type C_n, the Lagrangian Grassmannian $LG(n,2n)$,
- Type D_n, the orthogonal Grassmannian $OG(n,2n)$,
- Type E_6, the Cayley plane $\mathbb{OP}^2$, and
- Type E_7, the Freudenthal variety, $G_\omega(\mathbb{O}^3,\mathbb{O}^6)$.

Giving a root-system uniform and [Gra01]-positive combinatorial rule for $\mathsf{C}_{u,v}^{w}$ when G/P is miniscule is an open problem. As far as we are aware, there are no rules known for any of the nontrivial minuscule cases outside of the Grassmannian. Indeed the main interesting step towards a root-system uniform rule is to give a [Gra01]-positive combinatorial rule for $C_{u,v}^{w}$ when G/P is a maximal orthogonal Grassmannian (type B) or Lagrangian Grassmannian (type C). In these two cases, the Schubert classes are indexed by shifted partitions as described in Section 2.3.

While a precise conjectural rule has eluded us, there are some computational conjectures that shed light on some constraints any rule must adhere to. First, there is a well-known relationship between the ordinary structure constants of odd orthogonal Grassmannian $OG(n,2n+1)$ (Type B) and Lagrangian Grassmannian $LG(n,2n)$ (Type C). Let $c_{\lambda,\mu}^{\nu}$ be the structure coefficient of the odd orthogonal Grassmannian and $d_{\lambda,\mu}^{\nu}$ the structure coefficient of the Lagrangian Grassmannian. Then,

$$c_{\lambda,\mu}^{\nu} = 2^{\ell(\nu)-\ell(\lambda)-\ell(\mu)} d_{\lambda,\mu}^{\nu}.$$

λ	μ	ν	$C^{\nu}_{\lambda,\mu}$	$D^{\nu}_{\lambda,\mu}$
(1)	(1)	(1)	β_1	β_1
(1)	(1)	(2)	1	2
(1)	(2)	(2)	$\beta_1+\beta_2$	$\beta_1+2\beta_2$
(1)	(2)	$(2,1)$	1	1
(1)	$(2,1)$	$(2,1)$	$2\beta_1+\beta_2$	$2\beta_1+2\beta_2$
(2)	(2)	(2)	$\beta_1\beta_2+\beta_2^2$	$\beta_1\beta_2+2\beta_2^2$
(2)	(2)	$(2,1)$	$2\beta_1+2\beta_2$	$\beta_1+2\beta_2$
(2)	$(2,1)$	$(2,1)$	$2\beta_1^2+3\beta_1\beta_2+\beta_2^2$	$\beta_1^2+3\beta_1\beta_2+2\beta_2^2$
$(2,1)$	$(2,1)$	$(2,1)$	$2\beta_1^3+3\beta_1^2\beta_2+\beta_1\beta_2^2$	$\beta_1^3+3\beta_1^2\beta_2+2\beta_1\beta_2^2$

Table 5.1: The nonzero calculations of $C^{\nu}_{\lambda,\mu}$ and $D^{\nu}_{\lambda,\mu}$ for $\lambda,\mu,\nu\subseteq(2,1)$.

This can be derived from the fact that $Q_\lambda = 2^{\ell(\lambda)}P_\lambda$ (see Equation (2.8)) where P_λ is the Schur P-function, representing classes of the orthogonal Grassmannian and Q_λ is the Schur Q-function, representing classes of the Lagrangian Grassmannian. We generalize this relationship to the equivariant structure constants.

Conjecture 5.1. *Let $C^{\nu}_{\lambda,\mu}$ be the equivariant structure constant of the odd orthogonal Grassmannian and $D^{\nu}_{\lambda,\mu}$ the equivariant structure coefficient of the Lagrangian Grassmannian. Then $C^{\nu}_{\lambda,\mu}$ and $D^{\nu}_{\lambda,\mu}$ are polynomials in the simple roots with the same support and for each β^α that appears, there exists a (possibly negative) integer k_α such that*

$$[\beta^\alpha]D^{\nu}_{\lambda,\mu} = 2^{k_\alpha}[\beta^\alpha]C^{\nu}_{\lambda,\mu}.$$

This has been checked exhaustively for all $\lambda,\mu,\nu\subseteq(5,4,3,2,1)$. The nonzero calculations for $\lambda,\mu,\nu\subseteq(2,1)$ are listed in Table 5.1, given $C^{\nu}_{\lambda,\mu}=C^{\nu}_{\mu,\lambda}$ and $D^{\nu}_{\lambda,\mu}=D^{\nu}_{\mu,\lambda}$.

5.2 The Horn Inequalities

We first briefly recall the history of the Horn inequalities and their connection to Schubert calculus. Our references for this material are [Ful99, ARY13] and the references within.

The question of the Horn inequalities dates back to the nineteeth century and it asks when A and B are Hermitian matrices, what are the possible eigenvalues for

$$A+B=C?$$

Hermitian matrices have real eigenvalues and we consider these eigenvalues in weakly decreasing order. Suppose A has eigenvalues $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_2$, B has eigenvalues $\mu = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, and C has eigenvalues $\nu = \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$. We note here that these values are arbitrarily real values, not necessarily nonnegative integers. The only obvious relation between λ, μ, and ν concerns the trace of A, B, and C:

$$\sum_i \lambda_i + \sum_j \mu_j = \sum_k \nu_k.$$

We will call (λ, μ, ν) a **Hermitian triple** if there exists A, B, C with the specified eigenvalues such that $C = A + B$.

In 1912, H. Weyl gave a list of necessary inequalities on λ, μ, and ν for $A + B = C$. For the next fifty years, various mathematians proved certain inequalities are necessary but a complete list was not provided until 1962. A. Horn defined a system of inequalities of the form

$$\nu_{k_1} + \ldots \nu_{k_r} \leq \lambda_{i_1} + \ldots \lambda_{i_r} + \mu_{j_1} + \ldots + \mu_{j_r}$$

where the triples are in a finite set $T_{r,n}$. He computed $T_{r,n}$ for $n \leq 8$ and gave a highly recursive procedure for defining it in general. He proved that these inequalities were necessary and conjectured that they were also sufficient, i.e. that (λ, μ, ν) satisfied the inequalities defined with $T_{r,n}$ and the trace equality if and only if (λ, μ, ν) was a Hermitian triple.

In 1982, G. J. Heckman showed if the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu} \neq 0$ then (λ, μ, ν) is a Hermitian triple. Then in 1998, A. Kylachko

- defined a different set of inequalities on (λ, μ, ν),
- proved that (λ, μ, ν) is a Hermitian triple if and only if they satisfy his inequalities (thus giving the first complete solution to the eigenvalue problem).
- Furthermore, he proved if $c^{\nu}_{\lambda,\mu} \neq 0$ then (λ, μ, ν) satisfies his inequalities, and
- if (λ, μ, ν) satisfies his inequalities then there exists an integer N such that $c^{N\nu}_{N\lambda,N\mu} \neq 0$ where $N\lambda = N\lambda_1 \geq N\lambda_2 \geq \ldots N\lambda_n$.

A consequence of this result with some previous work reduces the problem of the Horn inequalities down to the saturation conjecture

$$c^{\nu}_{\lambda,\mu} \neq 0 \Longleftrightarrow c^{N\nu}_{N\lambda,N\mu} \text{ for all } N.$$

Finally, in 1999, A. Knutson and T. Tao proved the saturation conjecture, thus completing the proof that

$$c^{\nu}_{\lambda,\mu} \neq 0 \iff (\lambda,\mu,\nu) \text{ satisfy Horn's inequalities } \iff (\lambda,\mu,\nu) \text{ is a Hermitian triple.}$$

In 2000, S. Friedland generalized the eigenvalue question by asking for which eigenvalues (λ, μ, ν) do we have Hermitian matrices

$$A + B \geq C$$

where $M \geq N$ if $M - N$ has nonnegative eigenvalues. He provided a solution consisting of the inequalities of A. Kylachko, a trace *inequality*, and some additional inequalities. These additional inequalities were proven unnecessary by W. Fulton [Ful00]. In [ARY13], D. Anderson, E. Richmond, and A. Yong prove that the same inequalities that govern $A + B \geq C$ also govern the vanishing of the equivariant Schubert structure constants.

In [PS08], K. Purbhoo and F. Sottile extended the Horn inequalities to the Schubert calculus of minuscule G/P's. We wish to extend this work to equivariant Schubert calculus of the maximal orthogonal Grassmannian. The inequalities for the maximal orthogonal Grassmannian are given more explicitly combinatorially in [PS06, Theorem 10].

Fix n and r and let α be an ordinary partition contained in the $r \times (n-r)$ rectangle. Define

$$I_n(\alpha) = \{n - r + 1 - \alpha_1, n - r + 2 - \alpha_2, \ldots, n - \alpha_r\}.$$

For example, for $n = 6, r = 3, \alpha = (3, 2, 2)$,

$$I_6(\alpha) = \{6 - 3 + 1 - 3, 6 - 3 + 2 - 2, 6 - 3 + 3 - 2\} = \{1, 3, 4\}.$$

Let Δ_n denote the n-staircase $(n, n-1, \ldots, 1)$ and then let λ be a strict partition contained in Δ_n. We will write λ in the upper-right corner of Δ_n. For example, for $\lambda = (5, 3, 2)$ we depict $\lambda \subseteq \Delta_6$ as the white boxes in Figure 5.1.

Then, we number the corners of Δ_n from 1 to n from top to bottom. For $0 < r < n$ and $\alpha \subseteq r \times (n-r)$, define $[\lambda]_\alpha$ as the number of boxes of λ that remain after crossing the row to the right and column below the corners numbered by $I_n(\alpha)$. For an example, see Figure 5.2.

For $\lambda \subseteq \Delta_n$, λ^c is strict partition with the remaining boxes of Δ_n when λ is written in the upper right corner as in Figure 5.1. For $\alpha \subseteq r \times (n-r)$, α^c is the partition of remaining boxes

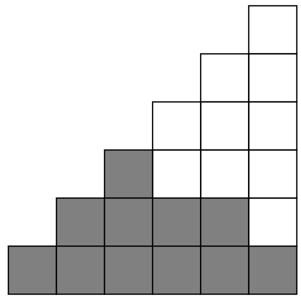


Figure 5.1: The strict partition $\alpha = (5, 3, 2)$ contained in the staircase Δ_6.

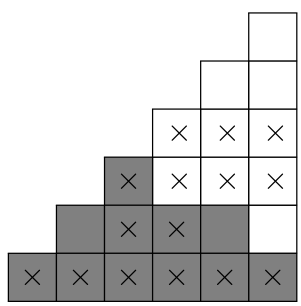


Figure 5.2: Let $\alpha = (3, 2, 2)$ and recall $I_6(\alpha) = \{1, 3, 4\}$. Then the boxes with $\times$ are those removed by $I_6(\alpha)$ and for $\lambda = (5, 3, 2)$, $[\lambda]_\alpha = 4$.

of $r \times (n-r)$ when α is written in the upper right corner. A triple of ordinary partitions α, β, γ is **feasible** if the Littlewood-Richardson coefficient $c_{\beta,\gamma}^{\alpha^c} \neq 0$ and a triple of strict partitions λ, μ, ν is **feasible** if $a_{\mu,\nu}^{\lambda^c} \neq 0$ where $a_{\mu,\nu}^{\lambda^c}$ is the structure constant of the orthogonal Grassmannian.

Theorem 5.2 (Purbhoo-Sottile, Theorem 10 [PS06]). *The triple λ, μ, ν contained in the n-staircase is a feasible triple if and only if the*

- *homogeneity equality $|\lambda| + |\mu| + |\nu| = \binom{n+1}{2}$ holds and*
- *for all $0 < r < n$ and all feasible $\alpha, \beta, \gamma \subseteq r \times (n-r)$, $[\lambda]_\alpha + [\mu]_\beta + [\nu]_\gamma \leq \binom{n+1-r}{2}$.*

A triple of strict partitions λ, μ, ν is **equivariant feasible** if $\mathsf{C}_{\mu,\nu}^{\lambda^c} \neq 0$ where $\mathsf{C}_{\mu,\nu}^{\lambda^c}$ is the equivariant structure constant of the orthogonal Grassmannian.

Conjecture 5.3. *For the maximal Orthogonal Grassmannian, λ, μ, ν contained in the n-staircase (and not a smaller staircase) is an equivariant feasible triple if and only if for $k = |\lambda| + |\mu| + |\nu| - \binom{n+1}{2}$,*

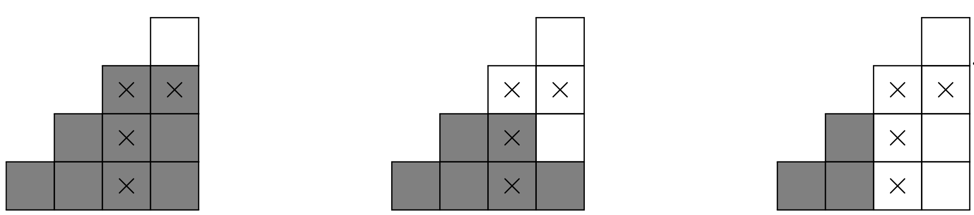
- *the homogeneity inequality holds, i.e. $k \geq 0$, and*
- *for all $0 < r < n$ and all feasible $\alpha, \beta, \gamma \subseteq r \times (n-r)$, $[\lambda]_\alpha + [\mu]_\beta + [\nu]_\gamma - k \leq \binom{n+1-r}{2}$.*

We have checked this for all $\lambda, \mu, \nu \subseteq (5,4,3,2,1)$.

For example,

$$\mathsf{C}_{(3,1),(4,3)}^{(4,3,2)} = 8\beta_1^2 + 16\beta_1\beta_2 + 8\beta_2^2 + 16\beta_1\beta_3 + 16\beta_2\beta_3 + 8\beta_3^2 + 8\beta_1\beta_4 + 8\beta_2\beta_4 + 8\beta_3\beta_4 + 2\beta_4^2 \neq 0.$$

Thus $(1), (3,1), (4,3)$ is an equivariant feasible triple with $k = 1 + 4 + 7 - 10 = 2$. Clearly, $k \geq 1$ and the smallest staircase containing $(1), (3,1), (4,3)$ is Δ_4. Thus the second condition states we need to consider $0 < r < 4$ and thus let $r = 1$. For $r = 1$, there is only one feasible triple $(1), (1), (1)$. Then we have $I_4 = \{3\}$ and $[(1)]_{(1)} = 1$, $[(3,1)]_{(1)} = 2$ and $[(4,3)]_{(1)} = 3$ as



Then as desired $4 = 1 + 2 + 3 - 2 \leq \binom{4+1-1}{2} = 6$. We repeat this for the six feasible triples for $r = 2$ and the single feasible triple for $r = 3$.

While the proof of the saturation conjecture was essential to the story of the Type A Horn inequalities, saturation does not hold in the maximal orthogonal Grassmannian, even nonequivariantly. The Littlewood-Richardson coefficient $c_{(2,1),(2,1)}^{(3,2,1)} = 0$ but $c_{(4,2),(4,2)}^{(6,4,2)} = 8$.

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