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METRIC GEOMETRY OF THE GRUSHIN PLANE AND GENERALIZATIONS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

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Abstract

Given $\alpha > 0$, the α -Grushin plane is \mathbb{R}^2 equipped with the sub-Riemannian metric generated by the vector fields $X = \partial_1$ and $Y = |x_1|^\alpha \partial_2$. It is a standard example in sub-Riemannian geometry, as a space which is Riemannian except on a small singular set—here the vertical axis, where the vector field Y vanishes.

The main purpose of this thesis is to study various problems related to the metric geometry of the α -Grushin plane and a generalization of it, termed *conformal Grushin spaces*. One such problem is the embeddability of these spaces in some Euclidean space under a bi-Lipschitz or quasisymmetric mapping. Building on work of Seo [Seo11] and Wu [Wu15a], we prove a sharp embedding theorem for the α -Grushin plane and a general embedding theorem for conformal Grushin spaces under appropriate hypotheses. We also study quasiconformal homeomorphisms of the α -Grushin plane.

In the final section, we solve a separate problem regarding quasiconformal mappings in metric spaces. The main result states that if a metric space homeomorphic to \mathbb{R}^2 can be quasiconformally parametrized by a domain in \mathbb{R}^2 , then one can find a mapping which improves the dilatation to within a universal constant. A non-sharp theorem of this type was recently proved by Rajala in [Raj17]; our theorem gives the sharp bounds for this problem.

Acknowledgments

I would like to deeply thank the many people who have made this journey possible, first and foremost my advisor Jeremy Tyson for his time and encouragement over the past four years. I am also greatly indebted to Jang-Mei Wu, Denka Kutzarova and Kai Rajala for their generosity with their time and expertise. I thank my fellow students Anton Lukyanenko, Vyron Vellis, Colleen Ackermann, Chris Gartland, and Derek Jung for always being willing to discuss mathematics, collaborate on problems, and enjoy time together. Finally, I am grateful for the consistent love, support, and encouragement of my family, beginning long before my interests turned to mathematics.

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Chapter 1

Introduction

This thesis is based on the following four papers:

- [Rom16] Matthew Romney, *Conformal Grushin spaces*. Conform. Geom. Dyn., 20: 97–115, 2016.
- [RV17] Matthew Romney and Vyron Vellis, *Bi-Lipschitz embedding of the generalized Grushin plane in Euclidean spaces*, Math. Res. Lett., 24(4): 1179–1205, 2017.
- [GJR17] Chris Gartland, Derek Jung, and Matthew Romney, *Quasiconformal mappings on the Grushin plane*, Math. Z., 297(3-4): 915–928, 2017.
- [Romar] Matthew Romney, *Quasiconformal parametrization of metric surfaces with small dilatation*, Indiana Univ. Math. J. (to appear).

The first three papers study the metric geometry of a particular metric space, the *Grushin plane*, as well as a generalization that I termed *conformal Grushin spaces*. The final paper deals with quasiconformal mappings in an abstract metric setting and has no direct relationship to the other three.

Here is a brief summary of each. The papers [Rom16] and [RV17] both concern the *bi-Lipschitz embedding problem* for the α -Grushin plane and its conformal Grushin space generalization. It is shown that, under suitable mild hypotheses, such spaces can be embedded in Euclidean space under a bi-Lipschitz mapping. In the case of the α -Grushin plane and certain other specific cases, an embedding with sharp target dimension can be constructed. These papers generalize earlier results by Seo [Seo11] and Wu [Wu15a].

The paper [GJR17] discusses various questions concerning the definitions and properties of quasiconformal mappings from the Grushin plane to itself, or to \mathbb{R}^2 . This continues the work of Ackermann [Ack15].

Finally, the paper [Romar] answers an open problem of Rajala from [Raj17]. The result states that, if a metric space homeomorphic to \mathbb{R}^2 admits some quasiconformal parametrization by a domain in \mathbb{R}^2 , then there always exists a parametrization with improved (that is, smaller) dilatation. The result in [Romar] gives the sharp bounds for this problem.

My hope for this thesis is to go beyond simply repeating the material of my papers. Hence a large portion will be devoted to giving motivation and background not present in the various papers, at a level accessible

to a beginning graduate student entering the field of analysis on metric spaces. Consistent with this goal, I have omitted proofs of the longer theorems; the interested reader can of course find the whole story in the respective paper. On the other hand, I have included many of the shorter proofs, to at least convey the flavor of the technical arguments.

One particular goal is to give, as far as possible, an exhaustive review of the literature related to the Grushin plane. My work included, researchers at the University of Illinois have published seven recent papers specifically on the Grushin plane and its generalizations [Ack15], [Seo11], [Wu15a], [Wu15b] and two papers partially dedicated to it [DHLT14], [Tys06]. There are other research groups with interest in the Grushin plane, coming from different mathematical backgrounds, and we will encounter their work throughout this thesis.

Chapter 2

Overview of the Grushin plane

Stated concisely, the *Grushin plane* is the set \mathbb{R}^2 consisting of real tuples (x_1, x_2) , equipped with the sub-Riemannian metric d generated by the vector fields $X = \partial_1$ and $Y = x_1\partial_2$. We call d the *Grushin metric*. When the Grushin metric is understood, we write \mathbb{G} in place of \mathbb{R}^2 .

2.1 Explanation of the definition

The Grushin plane is one example of a much more general notion of *sub-Riemannian manifold*. We will not develop the general framework of sub-Riemannian geometry here but simply illustrate it for the case of the Grushin plane. See the book of Montgomery [Mon02] for a friendly introduction aimed at a broader mathematical audience, including applications to the sciences. The surveys of Bellaïche [Bel96] and Gromov [Gro96] are also recommended.

First, as a sub-Riemannian manifold, the Grushin plane is an example of a *length metric space*, meaning that the distance between two points is defined as the infimum of the set of lengths of paths connecting them. Informally, we can consider the metric as measuring the shortest possible travel time between two points in the plane. The vector fields $X = \partial_1$ and $Y = x_1\partial_2$ specify how fast one can travel in a given direction at a given point. For example, the length of the horizontal path from $(0, b)$ to (a, b) is exactly a , while the length of the vertical path from $(a, 0)$ to (a, b) is b/a (here $a \neq 0$).

More precisely, let $\gamma : [t_0, t_1] \rightarrow \mathbb{R}^2$ be an absolutely continuous curve, defined on an interval $[t_0, t_1] \subset \mathbb{R}$. Write γ in coordinates as $\gamma(t) = (x(t), y(t))$. Then the *length* of γ , denoted by $\ell(\gamma)$, is

$$\ell(\gamma) = \int_a^b \sqrt{x'(t)^2 + \frac{1}{x(t)^2} y'(t)^2} dt. \tag{2.1}$$

The absolute continuity condition on γ is the minimal requirement for this integral to be defined. We recall one characterization of this property: a curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ is absolutely continuous if and only if it is rectifiable (of bounded variation) and satisfies Lusin's condition (N): if $E \subset [a, b]$ has measure zero, then so

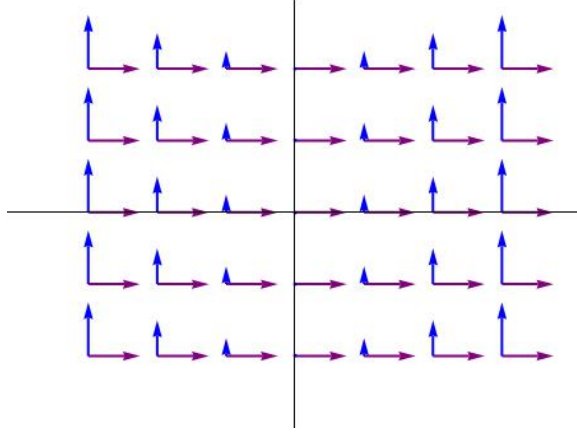


Figure 2.1: The vector fields $X = \partial_1$, $Y = x_1\partial_2$

does the image of E under γ . Equation (2.1) may be written more concisely as

$$\ell(\gamma) = \int_{\gamma} \sqrt{dx^2 + \frac{1}{x^2}dy^2}.$$

It should be clear, with a little thought, that (2.1) is the line element consistent with the orthonormal frame $X = \partial_1$, $Y = x_1\partial_2$. As simple examples, consider again the case of the horizontal path from $(0, b)$ to (a, b) , and the case of the vertical path from $(a, 0)$ to (a, b) . For a derivation of (2.1), as formalized in control theory, see Bellaïche [Bel96, p. 4-6].

Let \mathcal{Z} be the set on which Y vanishes, that is, $\mathcal{Z} = \{x : x_1 = 0\}$. We will call \mathcal{Z} the *singular line* of the Grushin plane. Its complement will be called the *Riemannian set*.

2.2 The α -Grushin plane

We will immediately make the following generalization to the α -Grushin plane. Take as before $X = \partial_1$, but consider instead $Y = |x_1|^\alpha \partial_2$ as the second vector field, for some $\alpha > 0$. By the same line of reasoning, we arrive at the integral formula

$$\ell(\gamma) = \int_a^b \sqrt{x'(t)^2 + \frac{1}{|x(t)|^{2\alpha}} y'(t)^2} dt. \quad (2.2)$$

Notice that, if we take $\alpha = 0$, then we just recover the Euclidean metric.

The reason for introducing the α -Grushin plane separately is due to a technicality in the definition of sub-Riemannian manifold. This technicality is unimportant in the case of the Grushin plane, but it will still be useful to give an explanation. In the standard framework of sub-Riemannian geometry, one considers a system of smooth vector fields X_1, X_2, \dots, X_m on a smooth n -manifold M . Typically, one requires that

the iterated Lie brackets $[X_i, X_j]$, $[[X_i, X_j], X_k]$, \dots , where $[X, Y] = XY - YX$, generate the full tangent space at each point of M . The importance of this is the so-called *Chow's condition* [Cho39]; it can be shown that when this is satisfied then any two points in M can be connected by a rectifiable path. The minimum number of terms in the Lie bracket to achieve this at every point is called the *step* of the sub-Riemannian manifold.

If α is an integer, then Y is a smooth vector field (for odd integers α , we must replace Y with $-Y$ on the left half-plane). A short calculation then shows that $\alpha + 1$ is the step of the α -Grushin plane and we have a *bona fide* sub-Riemannian manifold. On the other hand, if α is not an integer, then Chow's condition is not satisfied for \mathbb{G}_α . However, the rectifiable connectedness of the α -Grushin plane is trivial to establish directly: for two points (x_1, y_1) and (x_2, y_2) , pick any value $t > \max\{0, x_1, x_2\}$ and the polygonal path from (x_1, y_1) to (t, y_1) to (t, y_2) to (x_2, y_2) . Hence it is most natural to take α to be any positive real number.

2.3 Historical summary

The Grushin plane is named after the Russian mathematician Victor Vasilievich Grushin (also spelled as Grušin), who studied a certain class of hypoelliptic differential operators in a 1970 paper [Gru70]. The prototype of Grushin's class of operators is that of the form $\partial_1^r + |x_1|^{2\alpha} \partial_2^r$ for some $r, \alpha \in \mathbb{N}$.

The Grushin plane, as the metric space introduced above, was first studied by Franchi and Lanconelli [FL83] in 1983. Somewhat similarly to Grushin's paper, this was studied as part of a broader class of examples, of which the Grushin plane is the most basic example.

Both Grushin and Franchi-Lanconelli were motivated by connections to partial differential equations, and more specifically the notion of hypoelliptic differential operator. Let $\Omega \subset \mathbb{R}^n$ be a domain. A differential operator D is *hypoelliptic* if, for any domain $V \subset \Omega$ and any distribution u on V such that $Du \in C^\infty(V)$, it holds that $u \in C^\infty(V)$.

In 1967, Hörmander gave an essentially complete characterization of second-order hypoelliptic linear differential operators with smooth coefficients [Hör67]. The main result of Hörmander says that if D can be written (locally) in the form $D = X_1^2 + \dots + X_m^2 + X_0 + g$ for some homogeneous first-order operators X_0, \dots, X_m with smooth coefficients and some smooth function g , and if the iterated commutators $[X_i, X_j]$, $[[X_i, X_j], X_k]$, \dots span the whole tangent space, then D is hypoelliptic. The bracket-generating condition is often called Hörmander's condition in PDE; notice that this is none other than Chow's condition. Already, this gives an indication of the underlying connection between PDE and geometry. Even before the result in [Hör67], Hörmander's work on hypoelliptic operators played a role in him receiving the Fields medal in 1962.

On a more general level, there arose interest in studying metrics associated to such a differential operator. An influential paper in this time period is due to Nagel, Stein, and Wainger [NSW85]. This is one of several origin stories for the area of math now known as sub-Riemannian geometry.

Carathéodory first proved the corank-one case of Chow’s theorem [Car09]. The topic of his paper was thermodynamics, a subject pioneered in the early 19th century by the French physicist Nicolas Carnot. This is the origin of the term *Carnot-Carathéodory metric*, a term often used in the literature for sub-Riemannian metric.

2.4 The Heisenberg group

When discussing standard examples in sub-Riemannian geometry, the Grushin plane is usually overshadowed by another object: the *Heisenberg group*. This is the simplest example of a sub-Riemannian manifold which is fully sub-Riemannian, in the sense that at every point the horizontal subspace has dimension strictly smaller than that of the full tangent space. On a related note, it is the simplest nontrivial example of a class of objects called *Carnot groups*.

The Heisenberg group is \mathbb{R}^3 equipped with the sub-Riemannian metric generated by the vector fields $X = \partial_1 - \frac{x_2}{2}\partial_3$ and $Y = \partial_2 + \frac{x_1}{2}\partial_3$. In this case, it is less obvious that any two points can be connected by a path of finite length. Nevertheless, we observe that $[X, Y] = \partial_3$, so Chow’s theorem does guarantee the existence of finite-length paths between points. Denote the resulting space by \mathbb{H}_1 (as implied by the notation, there is a Heisenberg group \mathbb{H}_n for each odd dimension $2n + 1$, though we omit the definition here).

The Heisenberg group is a very rich mathematical object, appearing naturally in a number of mathematical contexts, and has been studied extensively. As the name indicates, it has a natural Lie group structure. The group operation is

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + (ab' - ba')/2).$$

It can be checked that the Heisenberg vector fields $X = \partial_1 - \frac{x_2}{2}\partial_3$ and $Y = \partial_2 + \frac{x_1}{2}\partial_3$ are left-invariant with respect to this group operation; in particular, the resulting metric is also left-invariant.

The mathematical origin of the Heisenberg group comes from quantum mechanics. The German physicist Werner Heisenberg studied the Lie algebra associated with \mathbb{H}_n , which arises in a quantum description of the position and momentum of a particle. A separate mathematical context where the Heisenberg group appears naturally is in complex analysis of several variables, where the boundary of the unit ball in \mathbb{C}^n can be identified with \mathbb{H}_n . The book of Stein [Ste93] contains a nice introduction to the topic in its final chapters;

see also the book of Capogna, Danielli, Pauls, and Tyson [CDPT07].

There is a close connection between the two objects: the Grushin plane appears as a quotient of the Heisenberg group. This could be considered an occurrence of the Grushin plane in “mathematical nature”. This fact is already observed in work of Rothschild and Stein, where they prove a general result along these lines [RS76]. A paper of Arcozzi and Baldi [AB08] contains a detailed discussion of this connection between the Heisenberg group and the Grushin plane.

The precise connection is as follows. For any choice of horizontal tangent vector v at the origin, one obtains a one-parameter subgroup in \mathbb{H}_1 via the exponentiation map, denoted \mathcal{S}_v . This gives rise to the right-quotient space $\mathcal{S}_v \backslash \mathbb{H}_1$. A metric on $\mathcal{S}_v \backslash \mathbb{H}_1$ is induced by infimizing the sub-Riemannian Heisenberg metric over all points in the respective cosets. If $v = \partial_3$, then $\mathcal{S}_v \backslash \mathbb{H}_1$ is isometric to \mathbb{R}^2 ; otherwise, $\mathcal{S}_v \backslash \mathbb{H}_1$ is isometric to \mathbb{G} . See [AB08, Thm. 1] for a proof.

2.5 Higher dimensions

Let us return now to the Grushin setting. As already alluded to in Section 2.3, the defining procedure outlined in Section 2.1 is not restricted to two dimensions. The original paper of Franchi and Lanconelli considers \mathbb{R}^n equipped with the sub-Riemannian metric generated by the vector fields X_j , $j = 1, \dots, n$, of the form $X_j = \lambda_j(x)\partial_j$, where $\lambda_1 = 1$, $\lambda_j(x) = \lambda_j(x_1, \dots, x_{j-1})$ satisfy certain smoothness and growth properties. The exact statement is somewhat lengthy and will be omitted here. But, following [Wu15b], one can have in mind the case $\lambda_j(x) = |x_{j-1}|^{\alpha_{j-1}}\lambda_{j-1}(x)$, where $\alpha_i \in [0, \infty)$.

There is a smaller body of literature for the higher-dimensional setting, as results are more difficult to prove. However, one paper of mention is a Liouville theorem in this setting due to Morbidelli [Mor09]. A second paper on higher-dimensional Grushin spaces is due to Wu [Wu15b], investigating the quasiconformal geometry of these spaces—the same types of problems as those outlined in Section 4 below. There are numerous cases that are considered separately, even in the three-dimensional case; some of these were left unresolved.

Within this thesis, the papers [RV17] and [GJR17] deal exclusively with the Grushin plane. On the other hand, the main purpose of the paper [Rom16] is to introduce another way of generalizing the Grushin plane to higher dimensions. These are the *conformal Grushin spaces* and they will be defined in Section 5.1. The results in [Rom16] apply to all dimensions without any additional difficulty to the proofs.

2.6 Almost Riemannian manifolds

In this section, we give a short overview of another line of research connected to the Grushin plane. A research group centered around Agrachev, Boscain and Sigalotti recently introduced the notion of an *almost Riemannian manifold*, for which the Grushin plane is the prototype. Besides the mathematical appeal of enlarging the scope of portions of Riemannian geometry, their motivation stems from physical applications to certain control systems. Papers on this subject include [ABC⁺10], [ABS08], [BCG13], [BCGS13], [BL13], [BS08].

Their work deals almost exclusively with the two-dimensional case. Concisely, an almost Riemannian 2-manifold is a sub-Riemannian manifold M with the additional property that for any open set $U \subset M$, the (possibly rank-varying) horizontal distribution is generated by two vector fields, but not one. As above, the *singular set* \mathcal{Z} is the set on which the two vector fields are linearly dependent. The set \mathcal{Z} must be small, in a precise sense described in Proposition 2.6.1 below.

The starting point for their investigations is a proposition on the structure of a *generic* almost Riemannian 2-manifold. That is, an almost Riemannian manifold corresponds to an element in $\text{Vec}(M) \times \text{Vec}(M)$, where $\text{Vec}(M)$ is the space of smooth vector fields on M , endowed with the C^2 -Whitney topology. (Informally, two elements in $\text{Vec}(M)$ are close relative to this topology if there is a point at which their second order Taylor expansions are close.) A property is *generic* if it holds for a dense open set in $\text{Vec}(M) \times \text{Vec}(M)$.

Proposition 2.6.1. For a generic almost Riemannian 2-manifold, generated by the vector fields X and Y and with singular set \mathcal{Z} , the following hold.

- (i) The set \mathcal{Z} is an embedded smooth manifold of dimension 1.
- (ii) $\{X, Y, [X, Y]\}$ spans the tangent space except on a set of isolated points in \mathcal{Z} .
- (iii) $\{X, Y, [X, Y], [[X, Y], X], [[X, Y], Y]\}$ spans the tangent space at every point in \mathcal{Z} .

This proposition is an application of the transversality theorem, a standard result in differential topology which states, roughly, that two generic submanifolds of a manifold intersect transversally. The proposition allows the introduction of local normal forms for the vector fields X and Y , which are used as a basic tool in the papers cited above.

The set of points for which $\{X, Y, [X, Y]\}$ does not span the tangent space are termed *tangency points*. By property (ii) in Proposition 2.6.1, these points are isolated. The basic example of a tangency point is provided by the vector fields $X = \partial_1$ and $Y = (x_2 - x_1^2)\partial_2$. The corresponding almost Riemannian manifold has a tangency point at the origin.

In this setting, they prove a Gauss-Bonnet-type formula, relating the total curvature of a compact

oriented almost-Riemannian manifold to the Euler characteristic [ABS08], [ABC⁺10]. They also study the heat equation and Schrödinger equation on almost-Riemannian manifolds [BL13]. Their results imply that heat cannot flow across the singular line in the Grushin plane, nor can a quantum particle cross the singular line.

2.7 Other recent research directions

An important paper of Monti and Morbidelli [MM04] solves the isoperimetric problem for the α -Grushin plane. This problem asks for the sets $E \subset \mathbb{G}_\alpha$ with Lebesgue measure $|E| = 1$ which minimizes an appropriate notion of perimeter. This is shown to be the set

$$E_\alpha = \left\{ x \in \mathbb{R}^2 : |x_2| < \int_{\arcsin |x_1|}^{\pi/2} \sin^{\alpha+1}(t) dt, |x_1| < 1 \right\},$$

together with its dilations (see Section 3.1) and vertical translations.

This result is particularly noteworthy for its connection with the Heisenberg group. Consider E_1 as a subset of \mathbb{R}^3 , where the x_2 -axis in \mathbb{G}_α is identified with the vertical axis of \mathbb{R}^2 . The set formed by rotating E_1 about the vertical axis is conjectured to be the solution to the isoperimetric problem in the Heisenberg group. This is a long-standing open problem which has inspired a full-length book [CDPT07].

A careful study of geodesics in the Grushin plane has been carried out in a series of papers by Calin, Chang, Li and coauthors [CC05], [CCGK05], [CL12]. The primary motivation of these studies lies in understanding heat flow on the Grushin plane. Geodesics in the Grushin plane will be discussed in Section 3.4 below.

On a more general note, geometric analysis in the setting of the Grushin plane and other Grushin spaces has become a fairly popular research direction in recent years. There are now too many papers to discuss individually, but topics include the p -Laplace equation and other PDE and stochastic accessibility. See [Bie02], [Bie05], [Bie07], [Bie09], [Bie11], [BG06], [BV11], [Cal14], [CUT14a], [CUT14b], [FV09], [Liu17], [TU17].

2.8 Notation

At various points, we use the notation \simeq to denote comparability. More precisely, for $a, b > 0$ (depending on certain parameters), the statement $a \simeq b$ means there exists a constant C (independent of said parameters) such that $C^{-1} \leq a/b \leq C$.

Chapter 3

Basic geometry of the Grushin plane

The section is intended to provide a better intuition for the geometry of the Grushin plane in preparation for the deeper work that follows. The basic source for most of this material is Bellaïche [Bel96].

3.1 The dilation property

The starting point for a study of the geometry of the Grushin plane is the following *dilation property*. This states that, for all $\lambda \in \mathbb{R}$,

$$|\lambda|d((x_1, y_1), (x_2, y_2)) = d((\lambda x_1, \lambda|\lambda|^\alpha y_1), (\lambda x_2, \lambda|\lambda|^\alpha y_2)). \quad (3.1)$$

This can be proved by a change of variables in (2.1).

The dilation property tells us something about the shape of balls. Consider the square S_0 with vertices at $(\pm 1, \pm 1)$. It is simple to check that for all $x \in S_0$, $1 \leq d((0, 0), x) \leq 3$. By the dilation property, for all $\lambda > 0$, the square S_λ with vertices at $(\pm\lambda, \pm\lambda|\lambda|^\alpha)$ has the property that, for all $x \in S_\lambda$, $\lambda \leq d((0, 0), x) \leq 3\lambda$.

For $x \in \mathbb{G}$ and $r > 0$, write $B(x, r; d_\alpha)$ for the ball at x of radius r in the α -Grushin metric. We have

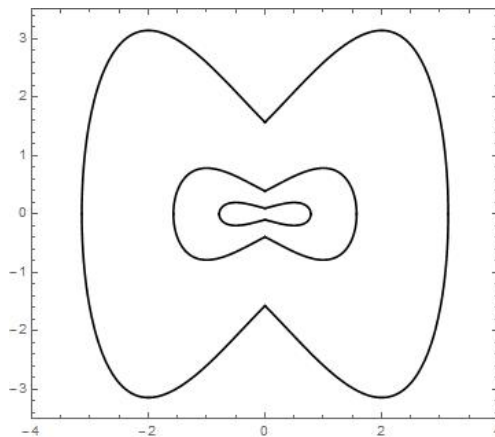


Figure 3.1: Grushin balls of varying radii centered at the origin

shown that

$$[-r/3, r/3] \times [-(r/3)^{1+\alpha}, (r/3)^{1+\alpha}] \subset B((0,0), r; d_\alpha) \subset [-r, r] \times [-r^{1+\alpha}, r^{1+\alpha}].$$

This fact is often known as the *ball-box theorem* and gives us a rough sense of the shape of balls centered on the singular line in the Grushin metric. Outside the singular line, small balls are roughly Euclidean in shape, since the metric there is Riemannian. In Section 3.4, we will give a more precise description of the shape of Grushin balls.

3.2 The singular line

One can verify the following fact. For a metric space (X, d) and some $\beta \in (0, 1)$, the function $d^\beta : X \times X \rightarrow [0, \infty)$ defined by $d^\beta(x, y) = d(x, y)^\beta$ is also a metric on X . The metric d^β is called the β -*snowflake* of d .

The terminology arises from the important case of $X = \mathbb{R}$, equipped with the Euclidean distance. Let $C \subset \mathbb{R}^2$ be the standard (unbounded) von Koch snowflake, and $f : \mathbb{R} \rightarrow C$ the standard parametrization of C . It can be shown that the map $f : (\mathbb{R}, |\cdot|^\beta) \rightarrow (C, |\cdot|)$ is bi-Lipschitz, where $\beta = \log 3 / \log 4$. In fact, for any $\beta \in (0, 1)$, there is a similar bi-Lipschitz embedding of $(\mathbb{R}, |\cdot|^\beta)$ onto a von Koch snowflake-type curve in \mathbb{R}^N , where $N = \lfloor 1/\beta \rfloor + 1$. The dimension N cannot be lowered. This well-known classical result is due to Assouad [Ass83].

With this terminology in hand, we make a simple but important observation from (3.1) that the Grushin metric, restricted to the singular line $\mathcal{Z} = \{(x, y) : x = 0\}$, takes the form

$$d_\alpha((0, y_1), (0, y_2)) = C|y_2 - y_1|^{1/(1+\alpha)},$$

for some constant $C = C(\alpha)$. That is, the Grushin metric on \mathcal{Z} is the $(1 + \alpha)$ -snowflake of the Euclidean metric on \mathcal{Z} , up to rescaling.

3.3 The Grushin quasidistance

In this section, we present an estimate of the Grushin metric that gives a good qualitative feel for its behavior. We define the *Grushin quasidistance* q_α by

$$q_\alpha(x, y) = |x_1 - y_1| + \min \left\{ |x_2 - y_2|^{1/(1+\alpha)}, \frac{|x_2 - y_2|}{|x_1|^\alpha} \right\}. \quad (3.2)$$

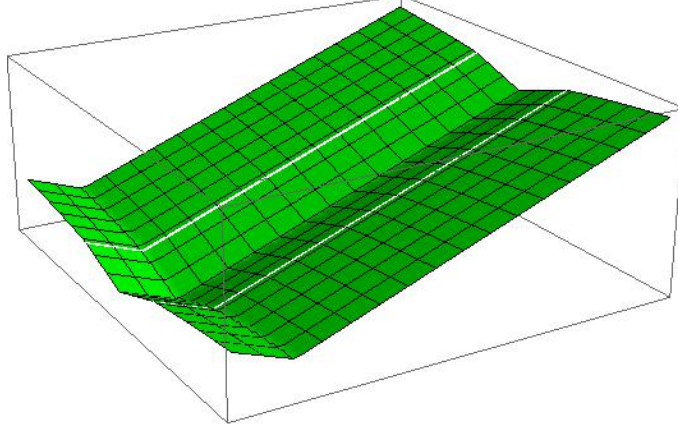


Figure 3.2: The Grushin quasidistance $q_1(x, \cdot)$ for fixed x

In the next section, we consider explicit formulas for geodesics in the Grushin plane; however, these can be unwieldy for applications.

As just indicated, the quasidistance q_α is bi-Lipschitz equivalent to the Grushin metric d_α . The proof is based on cases and is somewhat tedious, but it is instructive to work out the details.

Assume that $|x_1| \neq 0$; the case that $x_1 = 0$ is similar but simpler, so we ignore it. By dilating, translating, and reflecting we can assume that $x_1 = 1$, $x_2 = 0$, and $y_2 > 0$. We divide into 5 cases, based on which term of $q_\alpha(x, y)$ dominates and based on the minimum in the second term. For each case, we must prove that $d_\alpha(x, y) \simeq q_\alpha(x, y)$. In the following, let $C_\alpha = d_\alpha((0, 0), (0, 1))$.

Case 1. $y_2 \leq |y_1 - 1|$ and $y_2 \leq 1$. Considering the path from $(1, 0)$ to $(1, y_2)$ to (y_1, y_2) gives $d_\alpha(x, y) \leq |y_1 - 1| + y_2 = q_\alpha(x, y)$. On the other hand, $q_\alpha(x, y) \leq 2|1 - y_1| \leq 2d_\alpha(x, y)$.

Case 2. $y_2 \geq |y_1 - 1|$ and $y_2 \leq 1$. As in Case 1, $d_\alpha(x, y) \leq |y_1 - 1| + y_2 = q_\alpha(x, y)$. Next, observe that $d_\alpha(x, y) \leq 2$, and so the geodesic $\gamma(t) = (z_1(t), z_2(t))$ from x to y must satisfy $|z_1(t)| \leq 3$ for all t . This implies that $q_\alpha(x, y) \leq 2y_2 \leq 2 \cdot 3^\alpha d_\alpha(x, y)$.

Case 3. $|y_1 - 1| \geq \frac{1}{4}C_\alpha y_2^{1/(1+\alpha)}$ and $y_2 \geq 1$. Consider the straight-line path from $(1, 0)$ to $(0, 0)$, followed by a geodesic to $(0, y_2)$, followed by the straight-line paths to $(1, y_2)$ and then to (y_1, y_2) . This yields

$$\begin{aligned} d_\alpha(x, y) &\leq 1 + C_\alpha y_2^{1/(1+\alpha)} + 1 + |y_1 - 1| \\ &\leq (2 + C_\alpha)(y_2^{1/(1+\alpha)} + |y_1 - 1|) = (2 + C_\alpha)q_\alpha(x, y). \end{aligned}$$

For the other direction, we have

$$\begin{aligned} q_\alpha(x, y) &= y_2^{1/(1+\alpha)} + |y_1 - 1| \\ &\leq (1 + 4/C_\alpha)|y_1 - 1| \leq (1 + 4/C_\alpha)d_\alpha(x, y). \end{aligned}$$

Case 4. $|y_1 - 1| \leq \frac{1}{4}C_\alpha y_2^{1/(1+\alpha)}$ and $y_2 \geq (8/C_\alpha)^{1+\alpha}$. The proof that $d_\alpha(x, y) \lesssim q_\alpha(x, y)$ is identical to Case 3.

Next, let γ be an arbitrary curve from x to y , and let $\tilde{\gamma}$ be the curve consisting of the straight-line path from $(0, 0)$ to x , followed by γ , followed by the straight line paths to $(1, y_2)$ and then to $(0, y_2)$. Then $C_\alpha y_2^{1/(1+\alpha)} \leq \ell(\tilde{\gamma})$. This implies that

$$\begin{aligned} \ell(\gamma) &\geq C_\alpha y_2^{1/(1+\alpha)} - |y_1 - 1| - 2 \\ &\geq \frac{1}{2}C_\alpha y_2^{1/(1+\alpha)} + |y_1 - 1| - 2 \\ &\geq \frac{1}{4}C_\alpha y_2^{1/(1+\alpha)} + |y_1 - 1| \geq \min\{c_\alpha/4, 1\}q_\alpha(x, y) \end{aligned}$$

Infimizing over all paths γ from x to y shows that $d_\alpha(x, y) \gtrsim q_\alpha(x, y)$.

Case 5. $|y_1 - 1| \leq \frac{1}{4}C_\alpha y_2^{1/(1+\alpha)}$ and $1 \leq y_2 \leq (8/C_\alpha)^{1+\alpha}$. This is immediate from the compactness of the set in consideration.

The quasidistance plays a useful role in the papers [Mey11], [Wu15a], [Ack15], and [RV17]. In the paper [Rom16], analogous ideas are present in the geometric lemmas; see particularly Lemma 5.2.3 below.

3.4 Grushin geodesics

From our discussion up to this point, we have a rough but nevertheless accurate understanding of many features of Grushin geometry. Here, we refine this by considering geodesics in the Grushin plane. To fix terminology, we use the term *geodesic* for local length-minimizers, which are solutions to the Hamilton-Jacobi equations where the metric is Riemannian. The contrasts with *length-minimizing curves*, which naturally are global length minimizers. Note that the completeness of the Grushin plane guarantees the existence of a length minimizing curve between any two points in \mathbb{G}_α , by the Hopf-Rinow theorem.

Let

$$H(x, \xi) = \frac{1}{2} (\xi_1^2 + |x_1|^{2\alpha} \xi_2^2)$$

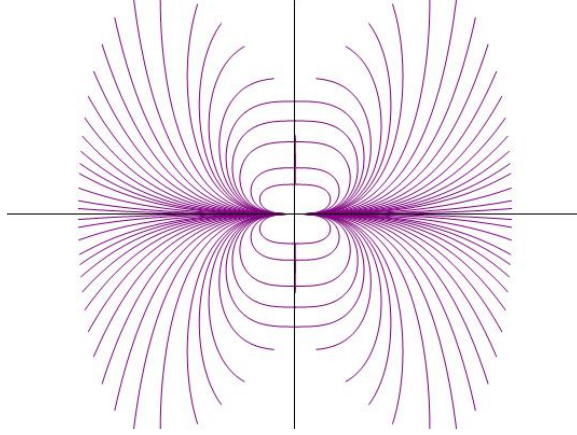


Figure 3.3: Grushin geodesics of fixed length emanating from the origin

be the so-called Hamiltonian function. Then a geodesic $\gamma(t) = (x_1(t), x_2(t))$ is the projection onto the spatial coordinates of a solution to the Hamilton-Jacobi system

$$\begin{aligned} x_1' &= \partial_{\xi_1} H(x, \xi) = \xi_1 \\ x_2' &= \partial_{\xi_2} H(x, \xi) = |x_1|^{2\alpha} \xi_2 \\ \xi_1' &= \partial_{x_1} H(x, \xi) = -\alpha x_1^{2\alpha-1} \xi_2^2 \\ \xi_2' &= \partial_{x_2} H(x, \xi) = 0 \end{aligned}$$

for some initial data. This can be found, for instance, in [MM04, Sect. 4].

In the standard case $\alpha = 1$, there is a relatively simple formula for solutions of this system. For this case, the family of geodesics through the origin can be parametrized as

$$\gamma_b(t) = \left(\frac{1}{b} \sin(bt), \frac{1}{b} \left(\frac{t}{2} - \frac{\sin(2bt)}{4b} \right) \right), \quad (3.3)$$

with the parameter $b \in \mathbb{R}$ determining the direction of γ_b . A similar formula for arbitrary α is available in terms of hypergeometric functions. Reversing the roles of t and b , equation (3.3) also gives a parametrization for the sphere of a fixed radius centered at the origin.

As noted earlier, the papers [CC05], [CCGK05], [CL12] study Grushin geodesics in more depth.

3.5 Measure and curvature

A natural choice of measure on the Grushin plane is the Hausdorff 2-measure with respect to the sub-Riemannian metric, which we denote by \mathcal{H}_α^2 . We recall this fundamental definition. Let $r \geq 0$. For a set

$A \subset X$ and all $\delta > 0$, let

$$\mathcal{H}_\delta^r(A) = \inf \left\{ \frac{\omega_n}{2^n} \sum_{j=1}^{\infty} \text{diam}(A_j)^r : A_j \text{ closed, } A \subset \bigcup_{j=1}^{\infty} A_j \right\}.$$

Here, ω_n denotes the Lebesgue n -measure of the Euclidean n -ball. Then the r -Hausdorff measure $\mathcal{H}^r(A)$ is defined by $\mathcal{H}^r(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^r(A)$, which may be infinite.

Notice that we have defined the Hausdorff measure with a normalization constant. This ensures that the Hausdorff n -measure on \mathbb{R}^n coincides with Lebesgue measure. For the 2-dimensional case, which is our main consideration, this constant is $\pi/4$. This constant is not essential, but it usually is convenient. The choice of normalization constant plays a role in the statement of Theorem 7.2.3 below, since this is a sharp result.

In the case of the Grushin plane, the Hausdorff measure outside the singular set is given by integrating the volume element $|x_1|^{-\alpha}$. That is, for a Borel set $A \subset \mathbb{R}^2 \setminus \mathcal{Z}$,

$$\mathcal{H}_\alpha^r(A) = \int_A \frac{dx_1 dx_2}{|x_1|^\alpha}.$$

If $\alpha \geq 1$, then \mathcal{H}_α^2 is not locally finite at any point on the singular line. It is true that certain small subsets of the singular line can have Grushin Hausdorff dimension 2, but this observation is not important to any of the results that follow. One reason for this is a lemma (Proposition 3.2 in [GJR17]) stating that any rectifiable curve in \mathbb{G}_α^2 ($\alpha \geq 1$) must intersect the singular line \mathcal{Z} in a set whose Grushin Hausdorff 1-measure is zero.

We conclude this section with a word on curvature, which adds a new perspective to our picture of the Grushin plane. Curvature, in its various guises, is one of the main geometric properties studied in Riemannian geometry. In two dimensions, which is the case for the Grushin plane, the story remains relatively simple. Here, we need only consider the sectional curvature at a point. This is given by Brioschi's formula: for a Riemannian metric $ds^2 = E dx_1^2 + G dx_2^2$, the sectional curvature K is

$$K = \frac{-1}{2\sqrt{EG}} \left(\left(\frac{G_{x_1}}{\sqrt{EG}} \right)_{x_1} + \left(\frac{E_{x_2}}{\sqrt{EG}} \right)_{x_2} \right).$$

Applying this to the Grushin line element gives $K = -\alpha(\alpha + 1)|x|^{-2}$ (see Theorem 8.5 in [DHLT14]). From this, we see that the Grushin plane has negative curvature everywhere on its Riemannian part, with curvature that explodes to negative infinity near the singular line. This might suggest some parallels with hyperbolic geometry; nevertheless, the presence of a singular set changes the situation drastically. For example, there are two length-minimizing curves between any two points on the singular line $\mathcal{Z} \subset \mathbb{G}_\alpha$, and many more geodesics—a phenomenon that cannot happen in hyperbolic geometry.

Chapter 4

Some concepts from metric geometry

With the Grushin plane now fully introduced, we now turn our attention to some concepts from metric space geometry. These concepts apply to general metric spaces; our eventual goal is to investigate them in the context of Grushin-type spaces.

4.1 Quasisymmetric mappings

The notion of quasisymmetry is one of two basic notions of equivalence for metric spaces. The other, bi-Lipschitz equivalence, will be discussed in Section 4.3.

A homeomorphism $f : (X, d_X) \rightarrow (W, d_W)$ between metric spaces is a *quasisymmetry* (or, is *quasisymmetric*) if there exists a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that

$$\frac{d_W(f(x), f(y))}{d_W(f(x), f(z))} \leq \eta(t) \tag{4.1}$$

for all distinct points $x, y, z \in X$ satisfying $d_X(x, y) \leq t d_X(x, z)$. The function η is called the *control function*. In words, the quasisymmetry condition states that the relative distance between triples of points is not distorted too greatly. On the other hand, a quasisymmetry can highly distort *absolute* distance.

A key observation is that this definition applies to any metric space without additional assumptions. It is a standard exercise to show that an inverse of a quasisymmetry is also quasisymmetric, as is the composition of two quasisymmetries. Also, the class of quasisymmetries from X to W has nice compactness properties, under mild hypotheses on W . All these facts make quasisymmetries a natural class of geometry-preserving mappings to study.

Historically, quasisymmetric mappings arose as a generalization of quasiconformal mapping (see Section 6.1) to the metric space setting. The defining condition (4.1) first appears in work of Ahlfors and Beurling [BA56] in the context of boundary behavior of quasiconformal maps in the plane. The condition was isolated and studied by Tukia and Väisälä in [TV80]. Their paper is recommended as a clear introduction to the topic, including the properties mentioned in the previous paragraph; see also Chapters 10-11 in the book by

Heinonen [Hei01].

Later, in Section 6.1, we give a short overview of the basic theory of quasiconformal mappings. It is a fundamental fact that the class of quasisymmetric homeomorphisms of \mathbb{R}^n coincides with the class of quasiconformal homeomorphisms of \mathbb{R}^n . (For domains in \mathbb{R}^n , the same is true if one replaces *quasisymmetric* with *locally quasisymmetric*.)

A metric property which is quasisymmetrically invariant is that of being doubling. A metric space is *doubling* if there exists a constant $C \geq 1$ such that all metric balls $B(x, r)$ in X can be covered by C balls of radius $r/2$. The doubling property is a basic niceness property which is necessary for many results, such as the bi-Lipschitz embeddability in Euclidean space, as discussed below in Section 4.3. A standard exercise is to show that the quasisymmetric image of a doubling metric space is also doubling.

It is a natural mathematical problem to study when two metric spaces are quasisymmetrically equivalent. Such spaces necessarily have similar geometric features. A particular important case is when one space is some Euclidean space \mathbb{R}^n (the *quasisymmetric uniformization problem*). There is much literature on this and related problems, far beyond the scope of this thesis. We only remark on a few important results. This sort of problem was first studied in depth by Semmes in [Sem96a]. Laakso provided an important example of a highly non-Euclidean metric on \mathbb{R}^2 which nevertheless is quasisymmetrically equivalent to the Euclidean metric in [Laa02]. We discuss Laakso's example in more detail in Section 4.4. Finally, Bonk and Kleiner solved the quasisymmetric uniformization problem for the two dimensional case, under the assumption of Euclidean mass bounds, in [BK02]. Here is a statement of their result.

Theorem 4.1.1 (Bonk–Kleiner). Let (X, d) be a metric space homeomorphic to \mathbb{S}^2 which is Ahlfors 2-regular. Then there exists a quasisymmetric mapping from X onto \mathbb{S}^2 if and only if f is linearly locally connected.

A metric space X is *linearly locally connected* if there exists $\lambda > 0$ such that any two points in a metric ball $B(x, r)$ can be connected by a continuum in $B(x, \lambda r)$, and any two points in $X \setminus B(x, r)$ can be connected by a continuum in $B(x, r/\lambda)$. A metric measure space (X, d, μ) is *Ahlfors p -regular* (for some $p > 0$) if there exists a constant $C > 0$ such that $r^p/C \leq \mu(B(x, r)) \leq Cr^p$ for every metric ball $B(x, r) \subset X$.

We return again to this theme in Section 7 of this thesis, which discusses some of my work in this line of research.

4.2 Quasisymmetric parametrization of the Grushin plane

In light of the preceding discussion, it is natural to ask whether the Grushin plane can be quasisymmetrically parametrized by \mathbb{R}^2 . It turns out (in hindsight, not too surprisingly) that the answer is yes. This was first explicitly shown in 2011 by W. Meyerson in [Mey11]. In this paper, Meyerson shows that the mapping $\varphi_\alpha : \mathbb{G}_\alpha \rightarrow \mathbb{R}^2$ defined by

$$\varphi_\alpha(x_1, x_2) = ((1 + \alpha)^{-1}x_1|x_1|^\alpha, x_2)$$

is quasisymmetric. The proof is done by cases using the Grushin quasidistance (3.2). Meyerson appears to have arrived at this mapping on his own. However, it can be found in other papers, since φ_α is a conformal map between Riemannian manifolds outside the singular line. To my knowledge, it is first used by Beckner in [Bec01]. It is also used by Monti and Morbidelli in [MM04].

In Theorem 5.2.1 below, we prove a general result on quasisymmetric parametrization of conformal Grushin spaces, under mild hypotheses. This includes Meyerson's result as a special case.

As one simple application, the existence of a quasisymmetric parametrization is one way to verify that the Grushin plane is doubling. It is fairly simple to verify this directly from the dilation property; still, it's interesting to note that the α -Grushin Hausdorff 2-measure are doubling measures. (A measure μ on X is doubling if there exists a $C \geq 1$ such that $\mu(B(x, r)) \leq C\mu(B(x, r/2))$ for all metric balls $B(x, r)$.) Any metric space with a doubling measure must be doubling. A complete doubling metric space supports a doubling measure (for the Grushin plane, the pushforward of Lebesgue measure under φ_α^{-1}), but it may not be a natural measure such as the Hausdorff measure.

4.3 Bi-Lipschitz mappings

A topological embedding f of metric space (X, d_X) into the metric space (W, d_W) is *bi-Lipschitz* if there exists $L \geq 1$ such that

$$L^{-1}d_X(x, y) \leq d_W(f(x), f(y)) \leq Ld_X(x, y)$$

for all $x, y \in X$. In words, a bi-Lipschitz map quasi-preserved distance (note, however, that the term *quasi-isometry* refers to a slightly different concept). It is easy to see that any bi-Lipschitz map is also quasisymmetric (with $\eta(t) = L^2t$), so this forms a more restrictive class of mapping.

Another natural, broad problem in metric geometry is the bi-Lipschitz embedding problem. This asks under what conditions a given metric space may be embedded in some given model space (very often, some Euclidean space \mathbb{R}^n) under a bi-Lipschitz mapping. On a philosophical level, such an embedding gives one

a concrete description of an abstract object, as subsets of a well-understood model space.

A famous theorem in differential geometry is the Nash embedding theorem, which states that any Riemannian manifold can be isometrically embedded in some Euclidean space of fairly small dimension. However, the term *isometric* in this context has a different meaning, referring only to the infinitesimal behavior of the mapping. Such a Nash embedding is typically not a bi-Lipschitz embedding, in this global sense we are using.

The classical result on the embedding problem is *Assouad's theorem* [Ass83]. This states that, for any doubling metric space (X, d) and $\beta \in (0, 1)$, the snowflaked space (X, d^β) embeds quantitatively into some Euclidean space. As already mentioned in Section 3.2, the snowflaked line (\mathbb{R}, d^β) embeds in $\mathbb{R}^{\lfloor 1/\beta \rfloor + 1}$, where this target dimension is sharp.

In a somewhat similar spirit to the quasimetric uniformization problem described in Section 4.1, the bi-Lipschitz embedding problem is a broad and difficult one that has led to a large body of literature. It is once again Semmes who gets credit for several lengthy and detailed papers pioneering the topic in the 1990's [Sem93], [Sem96b], [Sem99]. However, more recent research has seen a large cast of contributors with various mathematical motivations. Two significant early papers come from Bonk, U. Lang, and Plaut [BL03], [LP01]. A research program focused on metric spaces supporting a Poincaré inequality has been led by Cheeger and Kleiner [CK13], [CKN11], [Che99]. One class of examples in this framework are metric graphs which support such a Poincaré inequality (often termed *Laakso graphs* after a separate paper of Laakso [Laa00]). Other work has come from the direction of theoretical computer science, based on connections to the Sparsest Cut problem. Naor and J. Lee are two of the main contributors [ALN05], [KLMN05], [LM10], [Nao10], [LN14]. In many of these works, the focus is on embeddings into various Banach spaces (for example, Laakso graphs embed into L^1 , while the Heisenberg group does not). Finally, we mention an interesting result on quantitative embedding of Riemannian manifolds into Euclidean spaces by Eriksson-Bique [EB15]. His main theorem states that any bounded subset of a complete Riemannian n -manifold of bounded sectional curvature K can be L -bi-Lipschitz embedded into some Euclidean space \mathbb{R}^N , where L and N depend only on n , K , and the diameter of the set.

4.4 Two examples

In addition to the papers listed above, we discuss briefly two important examples that help place our investigations of the Grushin plane in a broader mathematical context.

The first of these is a paper of David and Toro [DT99]. To motivate it, recall from Section 3.2 the result of Assouad that $(\mathbb{R}, |\cdot|^\beta)$, $\beta \in (0, 1)$, is bi-Lipschitz equivalent to a von Koch-type curve in $\mathbb{R}^{\lfloor 1/\beta \rfloor + 1}$. The

target dimension is sharp. A natural question is whether a similar sharp snowflake embedding is possible if \mathbb{R} is replaced by \mathbb{R}^n . It was included as a notable open problem at the time in a survey article of Heinonen and Semmes [HS97].

This has proved to be a very difficult problem, eventually solved by David and Toro for certain ranges of β , depending on the dimension n . More precisely, for β sufficiently close to 1, the snowflaked space (\mathbb{R}^n, d^β) can be embedded in \mathbb{R}^{n+1} under a bi-Lipschitz mapping. Moreover, its image is a quasilane (the image of an n -dimensional linear subspace under a quasisymmetric self-map of \mathbb{R}^n). While the result does not address the case of arbitrary $\beta \in (0, 1)$, their proof is already of such difficulty that it seems unlikely to be improved upon.

The second example is the paper of Laakso [Laa02], briefly mentioned in Section 4.1. We discuss this example because it provides an interesting but accessible application of the bi-Lipschitz embedding problem and connects it to the quasisymmetric uniformization problem discussed in Section 4.1. It also provides a point of contrast with the conformal Grushin spaces studied in Section 5, whose definition is similar.

As was the case with the work of David–Toro, Laakso’s paper originated as a solution to an open problem in the survey article [HS97]. It gives a negative answer to a conjectured resolution to the quasiconformal Jacobian problem for \mathbb{R}^2 . This requires some explanation. A locally integrable, nonnegative function ω on \mathbb{R}^n is an A^∞ -weight if there exists $\delta, \epsilon > 0$ such that for all n -cubes $Q \subset \mathbb{R}^n$ and Borel subsets $E \subset Q$, $|E| \geq \delta|Q|$ implies that $\int_E \omega \, dm \geq \epsilon \int_Q \omega \, dm$. This is a standard notion in harmonic analysis, specifically the theory of *Muckenhoupt weights*. A related notion was introduced by David and Semmes [DS90]. For its statement, for $x, y \in \mathbb{R}^n$, let B_{xy} denote the ball of diameter $|x - y|$ containing both x and y . A *strong A^∞ -weight* is a locally integrable, nonnegative function ω on \mathbb{R}^n such that the measure induced by $\omega \, dm$ is doubling and the quasidistance

$$D_\omega(x, y) = \left(\int_{B_{xy}} \omega \, dm \right)^{1/n}$$

is bi-Lipschitz equivalent to a metric. A bit more concretely, the function D_ω induces the metric $d_\omega(x, y) = \inf\{\sum_{j=1}^n D_\omega(x_{j-1}, x_j) : x_j \in \mathbb{R}^n (0 \leq j \leq n), x_0 = x, x_n = y\}$. The strong A^∞ weight asks for the induced metric to be comparable to the original function D_ω . As the name implies, a strong A^∞ -weight is also an A^∞ -weight.

A landmark paper of Gehring from 1973 shows, as a consequence of the main theorem, that the Jacobian of a quasiconformal (equivalently, quasisymmetric) mapping of the plane is a strong A^∞ -weight [Geh73]. The question is whether the converse holds: is every A^∞ -weight comparable to the Jacobian of a quasiconformal mapping? If true, this would provide a complete characterization of weights which arise as Jacobians of a quasiconformal mapping. However, Laakso answered this problem in the negative by finding an A^∞ -weight

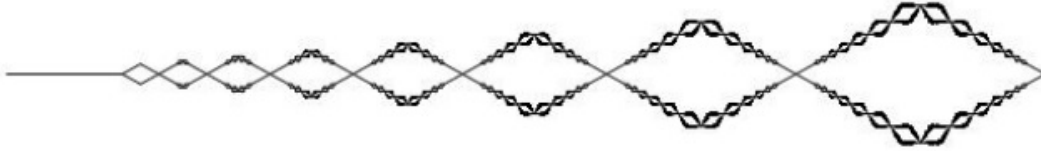


Figure 4.1: The set \mathcal{Z} used by Laakso

in the plane which is not comparable to any quasiconformal Jacobian. Other examples for higher dimensions had previously been found by Semmes in [Sem96b].

The method of proof is based on the observation that if ω is the Jacobian of a quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, then (\mathbb{R}^2, D_ω) is bi-Lipschitz equivalent to \mathbb{R}^2 . This is the connection with bi-Lipschitz embeddings: in his paper, Laakso constructs a strong A^∞ -weight which does not bi-Lipschitz embed in any Euclidean space. In fact, such a weight cannot be the Jacobian of any quasisymmetric mapping of \mathbb{R}^2 onto an Ahlfors 2-regular subset of \mathbb{R}^n .

For his construction, Laakso considers the fractal set \mathcal{Z} illustrated in Figure 4.1, with what amounts to the metric induced by the weight $\omega = d(\cdot, \mathcal{Z})^s$. For a suitable choice of s , this gives a metric on \mathbb{R}^2 , notwithstanding the fact that ω vanishes on the set \mathcal{Z} . Laakso shows that this metric space is not bi-Lipschitz equivalent to any “rounded ball space” (for instance, a uniformly convex Banach space), thus ruling out a bi-Lipschitz embedding in \mathbb{R}^n .

On the other hand, it is a short task to show that it is quasisymmetrically equivalent to \mathbb{R}^2 . One moral is that there is no general connection between the existence of a quasisymmetric parametrization by Euclidean space of one dimension and the existence of a bi-Lipschitz embedding in Euclidean space of another dimension.

4.5 Bi-Lipschitz embedding of Grushin plane

The bi-Lipschitz embedding problem is most interesting for non-Riemannian spaces. Aside from snowflake spaces and metric graphs, the most basic example of such a space is the Heisenberg group. Semmes showed that the Heisenberg group does not embed in any Euclidean space under a bi-Lipschitz mapping, not even locally [Sem96b]. A more general framework for approaching this sort of problem was developed by Cheeger in [Che99]. In this paper, Cheeger proves a differentiability theorem for Lipschitz functions on metric

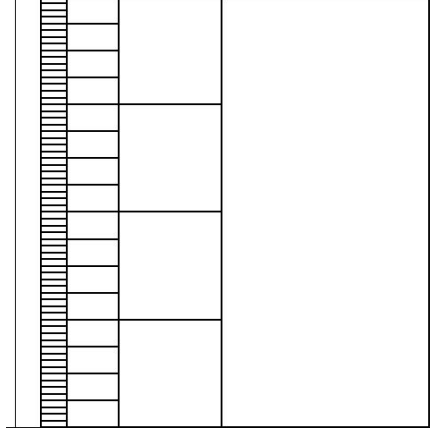


Figure 4.2: A Christ-Whitney decomposition of the Grushin plane relative to the singular line

spaces supporting a Poincaré inequality. As the Heisenberg group and \mathbb{R}^n have non-isomorphic differential structures, which are defined at almost every point, this rules out any bi-Lipschitz embedding.

Cheeger’s theorem does not apply to the Grushin plane: the differential structures studied in his paper are defined only almost everywhere, while the singular part of the Grushin plane has Lebesgue measure zero. Hence it was unknown whether a bi-Lipschitz embedding of the Grushin plane existed. The problem was taken up by Seo, at the time a student of Tyson. She was able to give an affirmative answer as part of her thesis work, published as [Seo11]. Seo framed her embedding method in terms of a general metric space, and we include it as the following theorem.

Theorem 4.5.1 (Seo). A metric space (X, d) admits a bi-Lipschitz embedding into some Euclidean space if and only if there exist $M_1, M_2 \in \mathbb{N}$, $L_1, L_2 \geq 1$ such that the following hold:

- (1) (X, d) is doubling.
- (2) There is a closed subset $Z \neq \emptyset$ of X which admits an L_1 -bi-Lipschitz embedding into \mathbb{R}^{M_1} .
- (3) There is a Christ-Whitney decomposition of $X \setminus Z$ such that each cube admits an L_2 -bi-Lipschitz embedding into \mathbb{R}^{M_2} .

The bi-Lipschitz constant and target dimension of this embedding depend only on M_1, M_2, L_1, L_2 , the doubling constant of X , and the data of the Christ-Whitney decomposition.

A *Whitney decomposition* of a domain $\Omega \subsetneq \mathbb{R}^n$ is a decomposition into essentially disjoint n -cubes Q such that $\text{diam } Q \simeq d_E(Q, \partial\Omega)$. A *Christ-Whitney decomposition* is a similar decomposition for metric spaces, based on work of Christ [Chr90]. Because it is fairly lengthy and technical, we omit a precise statement here.

Seo’s proof is, in part, an adaptation of the proof of Assouad’s embedding theorem [Ass83]. It uses a coloring argument to embed the infinitely-many Christ-Whitney cubes into finitely-many dimensions, using

condition (2) to get sufficient separation between the cubes for the map to satisfy a co-Lipschitz inequality.

Seo’s paper does not attempt to find the optimal target dimension. A few years later, this was shown to be dimension 3 by Wu for the standard Grushin plane ($\alpha = 1$) [Wu15a]. This was shown by a direct construction, based on earlier work of Bonk and Heinonen [BH04]. Wu’s result is analogous to the David–Toro paper [DT99] in proving sharp, low-dimensional embedding of a snowflaked or partially-snowflaked space. Indeed, Heinonen, in his Math Review of [DT99], asks whether a similar embedding is possible for “intermediate-type fractal surfaces”. While the phrasing is somewhat imprecise, the Grushin plane is a metric space which naturally fits this description. In both the David–Toro and Wu constructions, the embedded image is shown to be a quasiplane in the target space.

Finally, let us describe one potential application of such a bi-Lipschitz embedding of the Grushin plane. A substantial and ongoing area of research within the field of analysis on metric spaces is the study of Sobolev spaces of mappings with metric space targets. The notion of Sobolev space requires a linear structure on the target, and hence one considers a metric space as a subset of some larger Banach space. If the metric space W is separable, this can always be done via an isometric embedding in ℓ_∞ (Frechét’s theorem; see [Hei01, Exer. 12.6]). This is a very large space, and so one can expect stronger theorems by replacing ℓ_∞ with a smaller space such as some Euclidean space. Prior to the publication of [Wu15a], an initial investigation into Sobolev mappings with the Grushin plane as a target had been carried out in [DHLT14], using ℓ_∞ as the underlying Banach space. It is not known whether the conclusions of their theorems might be improved by using a finite-dimensional embedding.

4.6 Sharp bi-Lipschitz embedding of Grushin plane for arbitrary

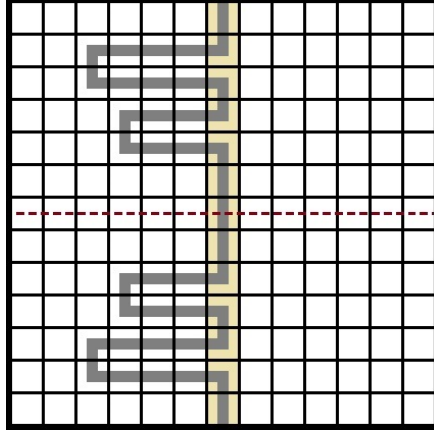
$$\alpha \geq 0$$

In joint work with Vellis, we extended Wu’s result to the case of arbitrary $\alpha \geq 0$. Here is a precise statement.

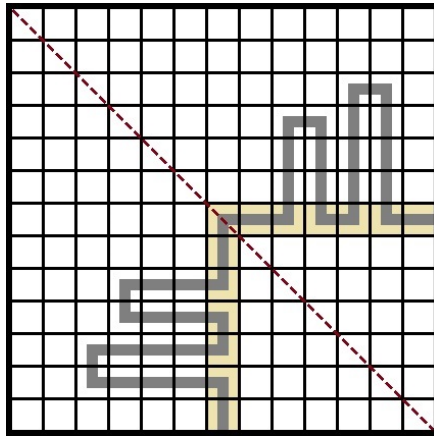
Theorem 4.6.1. For all integers $N \geq 0$ and $n \geq 1$, there exists $L > 1$ depending only on N, n such that for any $\alpha \in [N, N + \frac{n-1}{n}]$, there exists an L -bi-Lipschitz homeomorphism of \mathbb{G}_α onto a 2-dimensional quasiplane \mathcal{P}_α in \mathbb{R}^{N+2} .

Our proof is an adaption of the proof of Wu, the main difficulty being how to navigate certain higher-dimensional issues. Much of the proof is more technical than interesting, but we give an overview of some of the main ideas involved.

The proof itself is a detailed construction of an appropriate quasiconformal self-map $F_\alpha : \mathbb{R}^{N+2} \rightarrow \mathbb{R}^{N+2}$. One then shows that the composition $F_\alpha \circ \varphi_\alpha : \mathbb{G}_\alpha \rightarrow \mathbb{R}^{N+2}$ is actually bi-Lipschitz, using the quasidistance



(a) A schematic representation of an I -tube in two dimensional case



(b) A schematic representation of an L -tube in two dimensional case

Figure 4.3: Building blocks of the construction

estimates in (3.2). As before, $\varphi_\alpha : \mathbb{G}_\alpha \rightarrow \mathbb{R}^2$ is the canonical quasisymmetric parametrization of the α -Grushin plane, where we now consider the image \mathbb{R}^2 as a subset of \mathbb{R}^{N+2} in the usual way.

The construction of the map F_α proceeds in stages. We give a very rough overview. One defines an I -tube and an L -tube by beginning with an n -cube and removing a tubular neighborhood of a path J_I or J_L which runs from bottom to top (for the I -tube), or from bottom to one of the other faces (for the L -tube). See Figure 4.3 for a schematic representation in the two-dimensional case. The length of this path is carefully chosen, as is explained below.

The paths J_I and J_L themselves are constructed by concatenating some combination of straight line segments and ninety-degree turns, as illustrated in Figure 4.3. This allows us to iterate the construction: replace each straight-line segment with a scaled copy of the I -tube, and each ninety-degree turn with a scaled copy of the L -tube. This is repeated for all scales.

Let τ denote an n -cube with a certain tubular neighborhood of a straight-line path from bottom to top

removed. One can show there exists a bi-Lipschitz map from the I -tube onto τ which is well-behaved on the boundary (to allow for iteration at different scales). Likewise there exists a bi-Lipschitz map from the L -tube onto τ which is also well-behaved on the boundary. This is achieved by first defining a map on the boundary, and then justifying an extension to a bi-Lipschitz mapping of the interior.

In dimension three, which is the case of the original paper of Wu, the existence of such an extension follows from standard metric Schoenflies-type theorems. In arbitrary dimensions, this existence is not obvious. This step presented the largest difficulty in proving Theorem 4.6.1. We were eventually able to leverage an existing extension theorem of Väisälä for piecewise linear manifolds, which we quote here.

Theorem 4.6.2 ([Väi86, Corollary 5.20]). Let $n \geq 2$ and $\Sigma \subset \mathbb{R}^n$ be a compact piecewise linear manifold of dimension n or $n - 1$ with or without boundary. Then, there exist $L', L > 1$ depending on Σ , such that every L -bi-Lipschitz embedding $f : \Sigma \rightarrow \mathbb{R}^n$ extends to an L' -bi-Lipschitz map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

This does not apply directly to the case under consideration; instead, we must use a factoring argument. That is, we consider an I -tube or L -tube as arising from a piecewise-linear bi-Lipschitz isotopy from a straightened tube; this allows us to factor the boundary mapping into bi-Lipschitz maps of small distortion, to which Väisälä's theorem applies. We then obtain the desired bi-Lipschitz extension as the composition of mappings with smaller distortion.

Having obtained bi-Lipschitz mappings from the I -tube and L -tube onto τ , one can iterate this map at all scales. This yields, in the limit, a quasiconformal mapping of the n -cube onto itself. This is extended via backwards iteration to give a quasiconformal self-mapping of \mathbb{R}^{N+2} .

A second difficulty not present in Wu's paper is in producing a construction schema that works for any real value of α , and which still attains the (at that point, conjectured) sharp target dimension of $\lfloor \alpha \rfloor + 2$. The approach in our paper was to construct a model for integer values of α , and then alternate between them. This proved the result for rational α . The result for arbitrary α then follows from a standard Arzela–Ascoli-type limiting argument.

Chapter 5

Generalization to conformal Grushin spaces

This section is an exposition of the paper [Rom16]. Its main purpose is to introduce and study a natural generalization of the Grushin plane, termed *conformal Grushin spaces*.

5.1 The conformal definition

We will state the main definition first, and then discuss it. We use the notation d_E for Euclidean distance (between either points or sets), and ds_E for the Euclidean line element.

Definition 5.1.1. Let $n \in \mathbb{N}$, let $Z \subset \mathbb{R}^n$ be a nonempty closed set, and let $\beta \in [0, 1)$. The (Z, β) -Grushin space is the space \mathbb{R}^n equipped with the metric determined by the line element

$$ds = \frac{ds_E}{d_E(\cdot, Z)^\beta}.$$

More explicitly, the distance between two points $z_1, z_2 \in \mathbb{R}^n$ is given by $\inf \int_\gamma ds$, the infimum taken over all absolutely continuous paths from z_1 to z_2 . If $n = 2$ we call this space the (Z, β) -Grushin plane.

The definition as given has no requirements on Z beyond being closed and nonempty. Thus, it allows the possibility of infinite distance between points, in which case the (Z, β) -Grushin space is not a true metric space. For instance, this happens if Z has nonempty interior. A niceness assumption will be inevitable for proving results; this takes the form of a Hölder condition on the metric.

I decided to call these *conformal Grushin spaces*. This is because the metric is defined via a weight on \mathbb{R}^n which is a conformal deformation whenever it is finite. The terminology is not entirely satisfying, since the metric is certainly not conformally equivalent to \mathbb{R}^n at points where the weight is infinite. But, a term like *generalized Grushin space* would be more misleading, since that would most logically refer to the larger class of sub-Riemannian Grushin spaces studied by Franchi and Lanconelli.

If one takes $n = 2$ and Z to be the vertical axis, then one obtains the original Grushin plane, up to rescaling of the metric. This is seen by taking the original Grushin line element and pushing it forward under the quasisymmetry φ_α from Section 4.2. Observe the relationship $\beta = 1/(1 + \alpha)$.

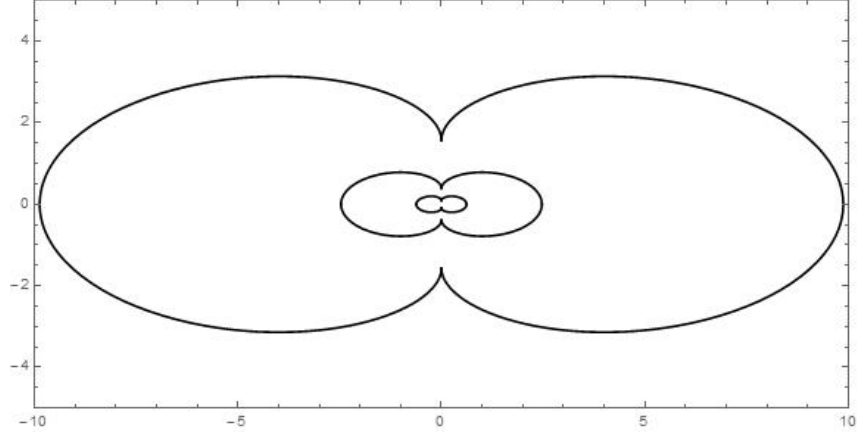


Figure 5.1: Balls centered at the origin in the conformal Grushin plane

One may wonder about the originality of Definition 5.1.1. It is reminiscent of a standard concept from complex analysis: the quasihyperbolic metric, first studied by Gehring and Palka [GP76]. There is a large literature on this topic; see a survey article by Koskela for a brief introduction [Kos98]. The definition of the quasihyperbolic metric is similar to Definition 5.1.1, except that one restricts to a proper domain $\Omega \subset \mathbb{R}^n$ and takes $Z = \partial\Omega$ and $\beta = 1$. Hence another name for the conformal Grushin metric could be *sub-quasihyperbolic metric*. There is a fundamental difference, though, in the behavior of the quasihyperbolic metric: the boundary $\partial\Omega$ lies at infinite distance from any point in Ω . This is typically not the case for the set Z in Definition 5.1.1.

Essentially the same metric as Definition 5.1.1 turns up in work of Gehring and Martio [GM85], though considered only on a proper domain $\Omega \subset \mathbb{R}^2$. They were motivated by connections with Lipschitz mappings. See also work of Langmeyer [Lan98].

Tyson proposed Definition 5.1.1 as a potential starting point for my thesis research in 2014. Our motivating question was the following: under what generality can one obtain geometric parametrization or embedding results, in particular the bi-Lipschitz embedding result of Seo in [Seo11]? These questions will be answered in Section 5.2 and Section 5.3.

First, here is a simple concrete example.

Example 5.1.2. Take $Z = \{0\} \subset \mathbb{R}^2$. Representing points in the (Z, β) -Grushin plane in polar coordinates by (r, θ) , the map

$$(x, y, z) = \left(r^{1-\beta} \cos \theta, r^{1-\beta} \sin \theta, r^{1-\beta} \sqrt{(1-\beta)^{-2} - 1} \right)$$

is a path-isometry between the (Z, β) -Grushin plane and a cone S in \mathbb{R}^3 with angular defect (total curvature) $2\pi\beta$. In this case the intrinsic metric on S is bi-Lipschitz equivalent to the Euclidean metric. Composing the above map with the projection map into the (x, y) -plane gives a bi-Lipschitz map between (\mathbb{R}^2, d_Z) and

(\mathbb{R}^2, d_E) .

Next, we give a precise statement of the Hölder condition mentioned above.

Definition 5.1.3. The (Z, β) -Grushin space satisfies the *Hölder condition* if there exists $H > 0$ such that $d_Z(x, y) \leq H d_E(x, y)^{1-\beta}$ for all $x, y \in \mathbb{R}^n$. That is, d_Z is $(1-\beta)$ -Hölder continuous as a function on (\mathbb{R}^n, d_E) .

This definition is our working assumption, sufficient to obtain the results that follow. Fortunately, it turns out that this Hölder condition is fairly mild. This is the content of the next proposition. We recall a standard definition needed for its statement. A domain $\Omega \subset \mathbb{R}^n$ is *uniform* (or, more explicitly, *C-uniform*) if there exists $C > 0$ such that for any two points $x, y \in \Omega$, there exists an arc γ from x to y such that $\ell_E(\gamma) \leq C d_E(x, y)$ and $d(z, \partial\Omega) \geq C^{-1} \min\{\ell_E(\gamma_{x,z}), \ell_E(\gamma_{z,y})\}$ for all $z \in \text{Im } \gamma$. Here $\gamma_{z,w}$ refers to the subarc of γ from z to w .

Proposition 5.1.4. Let $X \subset \mathbb{R}^n$ be a nonempty closed set such that $\Omega = \mathbb{R}^n \setminus X$ is the union of finitely many C -uniform domains and $\bar{\Omega} = \mathbb{R}^n$. Then for all $\beta \in [0, 1)$ and any nonempty closed subset $Z \subset X$, the (Z, β) -Grushin space satisfies the Hölder condition. The Hölder constant depends only on C , β , and the number of components of Ω , denoted by N .

The proposition follows almost immediately from the definition of uniform domain; the proof is omitted. The non-trivial content of Proposition 5.1.4 lies in the fact that most domains one typically encounters are uniform: for example, Z can be taken to be many Cantor-type and other fractal sets. The standard reference on uniform domains is [Väi88].

5.2 Quasisymmetric parametrization

The Hölder condition is sufficient in order to obtain quasisymmetric parametrizability.

Theorem 5.2.1. Assume the (Z, β) -Grushin space satisfies the Hölder condition with constant H . The identity map $\iota : (\mathbb{R}^n, d_E) \rightarrow (\mathbb{R}^n, d_Z)$ is quasisymmetric. The control function η depends only on β and H .

The proof is elementary and relatively clean, so we include it here. It requires two geometric lemmas.

Lemma 5.2.2. For any $x \in \mathbb{R}^n$,

$$d_Z(x, Z) = \frac{d_E(x, Z)^{1-\beta}}{1-\beta}.$$

Proof. Let $x \in \mathbb{R}^n$. By integrating over a straight-line path that realizes the Euclidean distance from x to Z , it is clear that $d_Z(x, Z) \leq (1-\beta)^{-1} d_E(x, Z)^{1-\beta}$. For the reverse inequality, let γ be an arbitrary path

from Z to x . Then

$$\ell_Z(\gamma) \geq \int_0^{\ell_E(\gamma)} \frac{dt}{t^\beta} \geq \int_0^{d_E(x,Z)} \frac{dt}{t^\beta} = \frac{d_E(x,Z)^{1-\beta}}{1-\beta}.$$

□

Lemma 5.2.3. Fix $c \geq 0$. For any $x, y \in \mathbb{R}^n$ satisfying $d_E(x, Z) \leq cd_E(x, y)$,

$$d_Z(x, y) \geq \frac{1}{1-\beta} ((1+c)^{1-\beta} - c^{1-\beta}) d_E(x, y)^{1-\beta}.$$

Proof. Consider an arbitrary path γ from x to y , parametrized by Euclidean arc length. Then

$$\begin{aligned} \ell_Z(\gamma) &\geq \int_0^{d_E(x,y)} \frac{dt}{(t + d_E(x, Z))^\beta} \\ &= \frac{1}{1-\beta} ((d_E(x, y) + d_E(x, Z))^{1-\beta} - d_E(x, Z)^{1-\beta}). \end{aligned}$$

Taking the infimum over all paths yields the same inequality with $d_Z(x, y)$ in place of $\ell_Z(\gamma)$. If $c = 0$, then $d_E(x, Z) = 0$ as well and we are done.

Otherwise, write the above inequality in the form

$$d_Z(x, y) \geq \frac{1}{1-\beta} \left(\left(1 + \frac{d_E(x, Z)}{d_E(x, y)}\right)^{1-\beta} - \left(\frac{d_E(x, Z)}{d_E(x, y)}\right)^{1-\beta} \right) d_E(x, y)^{1-\beta}.$$

Let $h(t) = ((1+t)^{1-\beta} - t^{1-\beta})$, defined on $(0, c]$. We compute

$$h'(t) = (1-\beta)((1+t)^{-\beta} - t^{-\beta}) < 0.$$

This shows that $h(t)$ is decreasing, and in particular that

$$h\left(\frac{d_E(x, Z)}{d_E(x, y)}\right) \geq h(c).$$

This yields the desired inequality. □

We now prove Theorem 5.2.1.

Proof. We need to show that, for all $t \geq 0$, there exists a value $\eta(t)$ such that, for all distinct points $x, y, z \in \mathbb{R}^n$ satisfying $d_E(x, y) \leq td_E(x, z)$, $d_Z(x, y) \leq \eta(t)d_Z(x, z)$, with $\lim_{t \rightarrow 0} \eta(t) = 0$. Let $t \geq 0$ and suppose that $x, y, z \in \mathbb{R}^n$ are three such points. We consider two cases.

First, assume that $d_E(x, Z) \leq 2td_E(x, z)$. By Lemma 5.2.3 there exists $c(t, \beta) > 0$ such that $d_Z(x, z) \geq c(t, \beta)d_E(x, z)^{1-\beta}$. Now by the Hölder condition there exists H such that $d_Z(x, y) \leq Hd_E(x, y)^{1-\beta}$. It follows that

$$\frac{d_Z(x, y)}{d_Z(x, z)} \leq Hc(t, \beta) \left(\frac{d_E(x, y)}{d_E(x, z)} \right)^{1-\beta} \leq Hc(t, \beta)t^{1-\beta}.$$

Next, assume that $d_E(x, Z) > 2td_E(x, z)$. First consider the straight-line path γ' from x to y . Notice that for any point $w \in \text{Im } \gamma'$, $d_E(w, Z) \geq d_E(x, Z)/2$. From this we obtain

$$d_Z(x, y) \leq \int_{\gamma'} \frac{ds_E}{d_E(\gamma'(\cdot), Z)^\beta} \leq \int_{\gamma'} \frac{ds_E}{(d_E(x, Z)/2)^\beta} = 2^\beta \frac{d_E(x, y)}{d_E(x, Z)^\beta}.$$

Now let γ be any path from x to z . Then

$$\begin{aligned} \ell_Z(\gamma) &= \int_0^{\ell_E(\gamma)} \frac{dt}{d_E(\cdot, Z)^\beta} \geq \frac{d_E(x, z)}{(d_E(x, Z) + d_E(x, z))^\beta} \\ &\geq \left(1 + \frac{1}{2t}\right)^{-\beta} \frac{d_E(x, z)}{d_E(x, Z)^\beta}. \end{aligned}$$

Since γ is arbitrary the same inequality holds with $d_Z(x, z)$ in place of $\ell_Z(\gamma)$. We obtain

$$\frac{d_Z(x, y)}{d_Z(x, z)} \leq 2^\beta \left(1 + \frac{1}{2t}\right)^\beta \left(\frac{d_E(x, y)}{d_E(x, z)}\right) \leq 2^\beta \left(1 + \frac{1}{2t}\right)^\beta t.$$

Take $\eta(t)$ to be the larger of the two upper bounds. It is easy to see that $\lim_{t \rightarrow 0} \eta(t) = 0$. \square

5.3 Bi-Lipschitz embedding of conformal Grushin spaces

The main result of [Rom16] is the following.

Theorem 5.3.1. Let $n \in \mathbb{N}$, let $Z \subset \mathbb{R}^n$ be nonempty and closed, and let $\beta \in [0, 1)$. If the (Z, β) -Grushin space satisfies the Hölder condition (Definition 5.1.3), then it is bi-Lipschitz embeddable in some Euclidean space. The bi-Lipschitz constant and target dimension depend only on β , n , and the Hölder constant H .

We give a short discussion of the proof. It is proved by verifying Seo's embeddability criterion Theorem 4.5.1. The doubling property holds by Theorem 5.2.1. The singular set Z , equipped with the Grushin metric, also embeds in some Euclidean space; this follows from Assouad's theorem, since the Grushin metric on Z is comparable to a snowflaked metric on \mathbb{R}^n . The final, and non-trivial, step is to verify the uniform embeddability of Christ-Whitney cubes.

This step is done by estimating the metric in an individual Christ-Whitney cube; one obtains an estimate

similar to the second case in the Grushin quasidistance (3.2). From there, a suitable dilation gives a uniform embedding for that cube. The details become somewhat technical, as one must work with the many pieces of data appearing in the definition of Christ-Whitney cube.

5.4 Another sharp target dimension result

We end this section with an additional result on sharp embedding dimensions for a particular class of conformal Grushin spaces; this broadens the scope somewhat of the main result in [RV17]. In its statement, we use the term ϵ -snowflake line in \mathbb{R}^2 for the image of $(\mathbb{R}, |\cdot|^\epsilon)$ under a bi-Lipschitz map into \mathbb{R}^2 (relative to the Euclidean metric). Notice that necessarily $\epsilon \in (1/2, 1]$. If $\epsilon = 1$, such a set is often referred to as a *chord-arc line*.

Theorem 5.4.1. Let $Z \subset \mathbb{R}^2$ be an ϵ -snowflake line for some $\epsilon \in (1/2, 1]$, let $\tilde{\beta} = 1 - \epsilon$, and let $\tilde{\alpha} = (\tilde{\beta} + \beta - \tilde{\beta}\beta)/(1 - \tilde{\beta} - \beta + \tilde{\beta}\beta)$. Then for all $\beta \in [0, 1)$, the (Z, β) -Grushin plane is bi-Lipschitz embeddable in $\mathbb{R}^{[\tilde{\alpha}] + 2}$. The target dimension is sharp.

Observe that $\tilde{\alpha}$ grows to infinity as either $\beta \rightarrow 0$ or $\tilde{\beta} \rightarrow 0$. The proof of Theorem 5.4.1 relies on the following result from [RV17] (Corollary 1.3).

Theorem 5.4.2. If $\alpha \in (0, 1)$, then any bi-Lipschitz embedding of the singular line of \mathbb{G}_α^2 into \mathbb{R}^2 extends to a bi-Lipschitz homeomorphism of \mathbb{G}_α^2 onto \mathbb{R}^2 .

Proof. Suppose that $\alpha \in (0, 1)$ and g is a bi-Lipschitz embedding of the singular line \mathcal{Z} into \mathbb{R}^2 . We show that g extends to a bi-Lipschitz embedding of \mathbb{G}_α^2 onto \mathbb{R}^2 .

Let $f: \mathbb{G}_\alpha^2 \rightarrow \mathbb{R}^2$ be the bi-Lipschitz mapping of Theorem 4.6.1. Then, $g(\mathcal{Z})$ and $f(\mathcal{Z})$ are quasilinear in \mathbb{R}^2 and $g \circ f^{-1}$ is a bi-Lipschitz homeomorphism between these quasilinear sets. Consider an η -quasisymmetric mapping $h: \mathbb{R} \rightarrow f(\mathbb{G})$. By the Beurling-Ahlfors quasiconformal extension [BA56], there exists a K -quasiconformal extension $H: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of h , with K depending only on η that satisfies

$$\text{diam } H(I) \simeq |DH(x)| \text{diam } I$$

for every arc $I \subset \mathbb{R} \times \{0\}$ and every point $x \in \mathbb{R}^2$ for which $\text{dist}(x, I) \simeq |I|$. (See Section 6.1 below for an explanation of quasiconformal mappings). Here the ratio constants depend only on η . Similarly, there exists a quasiconformal mapping $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ extending $g \circ f^{-1} \circ h$ satisfying the same property.

We claim that $F = G \circ H^{-1} \circ f$ is bi-Lipschitz extension of g . Indeed, for any point $x \in \mathbb{R}^2$ we have $|DH(x)|/|DG(x)| \simeq \text{diam } H(I)/\text{diam } G(I)$ for some suitable $I \subset \mathbb{R} \times \{0\}$. Since $g \circ f^{-1}$ is bi-Lipschitz, the

last ratio is comparable to 1. Therefore, $|DH(x)| \simeq |DG(x)|$ and $G \circ H^{-1}$ is bi-Lipschitz. We see that F is bi-Lipschitz, being the composition of bi-Lipschitz mappings. \square

We continue now with the proof of Theorem 5.4.1.

Proof. Let $\tilde{\beta} = 1 - \epsilon$ and let $\tilde{Z} = \{(0, v) : v \in \mathbb{R}\}$. There exists a global bi-Lipschitz map $\Gamma : (\mathbb{R}^2, d_{\tilde{Z}, \tilde{\beta}}) \rightarrow (\mathbb{R}^2, d_E)$ such that $\Gamma(\tilde{Z}) = Z$; this is a restatement of Theorem 5.4.1 in terms of our new notation for Grushin-type surfaces. We claim that Γ is also a global bi-Lipschitz map between the $(\tilde{Z}, \tilde{\beta} + \beta - \tilde{\beta}\beta)$ -Grushin plane and the (Z, β) -Grushin plane. To check this, note that $d_E(\Gamma(\cdot), \tilde{Z}) \simeq d_{\tilde{Z}, \tilde{\beta}}(\cdot, \tilde{Z}) \simeq d_E(\cdot, \tilde{Z})^{1-\tilde{\beta}}$ by Lemma 5.2.2. For any path γ in \mathbb{R}^2 , we have

$$\int_{\gamma} \frac{ds_E}{d_E(\cdot, \tilde{Z})^{\tilde{\beta}}} \simeq \int_{\Gamma \circ \gamma} ds_E.$$

It follows from this that

$$\int_{\gamma} \frac{ds_E}{d_E(\cdot, \tilde{Z})^{(1-\tilde{\beta})\beta + \tilde{\beta}}} \simeq \int_{\Gamma \circ \gamma} \frac{ds_E}{d_E(\cdot, Z)^{\beta}}$$

for any path γ in \mathbb{R}^2 . This shows that Γ is bi-Lipschitz as a mapping from the $(\tilde{Z}, \tilde{\beta} + \beta - \tilde{\beta}\beta)$ -Grushin plane to the (Z, β) -Grushin plane. The $(\tilde{Z}, \tilde{\beta} + \beta - \tilde{\beta}\beta)$ -Grushin plane is isometric (up to rescaling) to the usual Grushin plane $\mathbb{G}_{\tilde{\alpha}}^2$, where $\tilde{\alpha} = (\tilde{\beta} + \beta - \tilde{\beta}\beta)/(1 - \tilde{\beta} - \beta + \tilde{\beta}\beta)$. This space is embeddable in $\mathbb{R}^{[\tilde{\alpha}] + 2}$ by Theorem 4.6.1. \square

Chapter 6

Quasiconformal mappings on the Grushin plane

The theory of quasiconformal mappings in metric spaces has been an active area of research since seminal papers of Korányi and Reimann [KR85], Margulis and Mostow [MM95], and Heinonen and Koskela [HK95], [HK98]. The standard setting in much of the literature is Ahlfors regular metric spaces, often with an additional assumption of supporting a Poincaré inequality. The Grushin plane is a simple setting in which the existing theory does not apply, and so it is natural to investigate quasiconformal mappings on the Grushin plane.

This endeavor was initiated by Ackermann, another recent student of Tyson. In [Ack15], she studies a version of the Beltrami equation for the Grushin plane (see the commentary following Definition 6.1.3) and shows that it is satisfied by a class of mappings considered earlier by Payne in [Pay06]. There are some other natural questions on the topic of quasiconformal mappings on the Grushin plane. Does the infinitesimal metric definition of quasiconformal map (Definition 6.1.1) imply the global quasisymmetry condition? Is the inverse of a quasiconformal mapping still quasiconformal? Can one characterize the class of conformal homeomorphisms of the Grushin plane?

I explored these questions in collaboration with Gartland and Jung, two other graduate students at Illinois. We also considered a deeper question of whether there is a meaningful geometric definition of quasiconformality (Definition 6.1.2). This is complicated by certain technical and philosophical issues, but in the end we were able to give what I consider a satisfactory answer to the problem.

Before proceeding further, we offer some background to the area of quasiconformal mappings.

6.1 Basic theory of quasiconformal mappings

Complex analysis, at a basic level, is the study of *holomorphic* or *analytic* functions of a complex variable. Such a function has the property of being *conformal*, or angle-preserving. Beginning in the 1920s, this was generalized to the notion of quasiconformal mapping (the term *map* or *mapping* meaning an injective function). The field of quasiconformal mappings, developed first in the plane, later in Euclidean spaces \mathbb{R}^n of arbitrary dimension, and finally in various metric space settings, is one outgrowth of complex analysis

developed over the past century.

At the foundations lie the questions of definitions and regularity of a quasiconformal map. There are three main definitions, commonly referred to as the *metric*, *analytic*, and *geometric* definitions. The metric definition is the weakest; the other two definitions either contain or imply strong properties of a quasiconformal mapping. We give these definitions for the metric space setting.

Definition 6.1.1. A homeomorphism $f : (X, d_X) \rightarrow (W, d_W)$ of two metric spaces is *metrically quasiconformal* if there exists $H \geq 1$ such that, for all $x \in X$,

$$\limsup_{r \rightarrow 0} \frac{\sup\{d_W(f(x), f(y)) : y \in X, d_X(x, y) \leq r\}}{\inf\{d_W(f(x), f(y)) : y \in X, d_X(x, y) \geq r\}} \leq H.$$

Next is the geometric definition, which applies to any two metric measure spaces (X, d_X, μ) and (W, d_W, ν) . For a fixed $p \geq 1$, define the *p-modulus* of a family Γ of curves in X as

$$\text{Mod}_p \Gamma := \inf \int_X \rho^p d\mu,$$

where the infimum is taken over all Borel measurable functions $\rho : X \rightarrow [0, \infty]$ with the property that $\int_\gamma \rho ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. Such a function ρ is called *admissible*, and the value p is called the *exponent*.

Definition 6.1.2. A homeomorphism $f : (X, d_X, \mu) \rightarrow (W, d_W, \nu)$ between metric measure spaces is *geometrically quasiconformal with exponent p* if there exists a constant K such that

$$K^{-1} \text{Mod}_p \Gamma \leq \text{Mod}_p f\Gamma \leq K \text{Mod}_p \Gamma \tag{6.1}$$

for all curve families Γ .

In the case of $X = W = \mathbb{R}^n$, one takes the exponent $p = n$. One often encounters a variation on Definition 6.1.2, which we explain now as it will be relevant in the next section. For disjoint continua $E, F \subset X$, we let $\Gamma(E, F)$ denote the family of curves with one endpoint in E and another endpoint in F . In this variation, one requires that (6.1) hold only for curve families $\Gamma(E, F)$, for all disjoint continua $E, F \subset X$. It is customary to refer to the complement of $E \cup F$ as a *ring* or *ring domain*. We will refer to such a mapping f as *geometrically quasiconformal for ring domains*. In the Euclidean case, and even rather broad metric space settings (see [Wil12]), these definitions coincide. This is nontrivial, and probably difficult to prove directly. Instead, one shows that this weaker variant of the geometric definition implies the analytic definition (Definition 6.1.3 below), and then one shows that the analytic definition, in turn, implies the stronger definition.

We conclude now with the analytic definition. We could state this definition for an arbitrary metric measure space, similar to Definition 6.1.2. However, this would require some notion of metric Sobolev space, which would take some time to develop; we refer the interested reader to [HKST01] or [HKST15]. Instead, we give the definition for the Euclidean case.

Definition 6.1.3. A homeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (where $n \geq 2$) is *analytically quasiconformal* if f is in the Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, and there exists $K \geq 1$ such that

$$|Df(x)|^n \leq K J_f(x) \tag{6.2}$$

for a.e. $x \in \mathbb{R}^n$. Here J_f is the Jacobian of f , and $|Df|$ is the operator norm of Df .

If $n = 2$, after identifying \mathbb{R}^2 with \mathbb{C} , the condition (6.2) is equivalent to satisfying the Beltrami equation: there exists a measurable function $\mu : \mathbb{C} \rightarrow \mathbb{C}$, $\|\mu\|_\infty \leq (K - 1)/(K + 1) < 1$, such that $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$ a.e. Also, the requirement that $f \in W_{\text{loc}}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ is equivalent to the so-called ACL^n property (see for instance Rickman [Ric93, Theorem 1.2]). The map f is *absolutely continuous on lines* (ACL) if for each closed n -interval $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$, f is absolutely continuous on n -a.e. line segment in Q parallel to a coordinate axis (that is, the family of those line segments where f fails to be absolutely continuous has n -modulus zero). Then f is said to be ACL^n if additionally the partial derivatives (which must exist almost everywhere) are locally n -integrable.

In the classical setting of homeomorphisms of \mathbb{R}^n , all three definitions are equivalent (taking $p = n$ in the geometric definition). They are also equivalent to the notion of quasisymmetry, defined in Section 4.1. This equivalence is quantitative in the sense that worst-case data corresponding to one definition can be computed from the data of the other. Still the main reference for this material, even with its typewriter typesetting, is the book of Väisälä [Väi71].

6.2 Equivalence of definitions for the Grushin case

The main theorem in [GJR17] is the following. For a homeomorphism $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$, we call $Y \cup f^{-1}(Y)$ the *singular set* of f and its complement the *Riemannian set* of f .

Theorem 6.2.1. Let $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ be a homeomorphism. The following are equivalent.

- (a) f is quasisymmetric.
- (b) f is metrically quasiconformal.

- (c) f is metrically quasiconformal on its Riemannian set.
- (d) f is geometrically quasiconformal for ring domains, for the Grushin Hausdorff 2-measure \mathcal{H}_α^2 and exponent $p = 2$.
- (e) f is uniformly locally geometrically quasiconformal on its Riemannian set. That is, there exists a $K \geq 1$ such that, for any point x in the Riemannian set of f , there exists a neighborhood U_x such that $f|_{U_x} : U_x \rightarrow \mathbb{G}_\alpha^2$ is geometrically quasiconformal with constant K .

Theorem 6.2.1 is quantitative in the sense that worst-case data for one definition can be computed from the data of any other definition. The quantitative relationship is written down explicitly as Remark 3.8 in [GJR17].

The proof strategy for Theorem 6.2.1 is to exploit the corresponding fact for the Euclidean plane. This is achieved via the standard quasisymmetry $\varphi_\alpha : \mathbb{G}_\alpha^2 \rightarrow \mathbb{R}^2$. Here is an example which works out one of the implications in Theorem 6.2.1.

Theorem 6.2.2. Let $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ be a homeomorphism which is metrically quasiconformal on its Riemannian set. Then f is quasisymmetric.

Proof. The result essentially follows from a negligibility theorem of Väisälä [Väi71] for Euclidean quasiconformal mappings. Let $\tilde{f} = \varphi_\alpha \circ f \circ \varphi_\alpha^{-1}$, and let \tilde{Z} denote the vertical axis as a subset of \mathbb{R}^2 . Notice that $\mathbb{R}^2 \setminus \tilde{f}^{-1}(\tilde{Z})$ has two components, which we denote by E_1 and E_2 . Now, \tilde{f} is metrically quasiconformal on each component of $E_i \setminus \tilde{Z}$. We apply the theorem of Väisälä [Väi71, Theorem 35.1], which states that any set of σ -finite 1-dimensional Lebesgue measure is negligible, to conclude that \tilde{f} is metrically quasiconformal on both E_1 and E_2 . It follows then that \tilde{f}^{-1} is metrically quasiconformal on $\mathbb{R}^2 \setminus \tilde{Z}$. By the same theorem of Väisälä, \tilde{f}^{-1} is metrically quasiconformal on all of \mathbb{R}^2 . This suffices to show that \tilde{f} , and hence f itself, is quasisymmetric. \square

Corollary 6.2.3. The inverse of a metrically quasiconformal mapping $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ is metrically quasiconformal.

Proof. For such an f , the preceding theorem says that f is quasisymmetric. Thus f^{-1} is also quasisymmetric, hence quasiconformal in the metric sense. \square

The most difficult piece of Theorem 6.2.1 is the following.

Theorem 6.2.4. A quasisymmetric map $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ is geometrically quasiconformal for ring domains.

The proof of this is fairly lengthy and involves a number of lemmas. Here is a sketch of the main ideas. It is easy to see that Theorem 6.2.4 follows from showing that the canonical quasisymmetry $\varphi_\alpha : \mathbb{G}_\alpha^2 \rightarrow \mathbb{R}^2$ is modulus-preserving for ring domains. The main obstacle is that there are many unrectifiable curves in \mathbb{G}_α^2 whose images under φ_α are rectifiable in \mathbb{R}^2 . Hence it is possible, in principle, for φ_α to increase the modulus of curve families.

Fix subsets $E, F \subset \mathbb{G}_\alpha^2$. The task becomes a matter of finding a full-modulus subset of the curve family $\Gamma(\varphi_\alpha E, \varphi_\alpha F)$ that are well-behaved under φ_α^{-1} : they remain rectifiable and the usual change-of-variables formula holds. This task is accomplished, first, by using the Riemann mapping theorem for doubly connected domains to map $\mathbb{R}^2 \setminus (E \cup F)$ conformally onto an annulus. The family of radial curves has full modulus, and hence so does the corresponding subfamily of $\Gamma(\varphi_\alpha E, \varphi_\alpha F)$. The second ingredient is using an approximation procedure for such curves near the singular line, to obtain curves whose images under φ_α^{-1} are also rectifiable. See [GJR17, Sec. 3] for details.

6.3 A family of curves

Observe that the statement of Theorem 6.2.4 refers only to ring domains. This is a necessity: we can show by example that a quasisymmetry of the Grushin plane need not quasi-preserve the modulus of an arbitrary curve family.

Theorem 6.3.1. For $\alpha \geq 1$, there exists a family Γ of nonrectifiable curves in \mathbb{G}_α^2 with $\text{Mod } \Gamma > 0$.

Proof. We continue to use (x, y) for coordinates in \mathbb{G}_α^2 and (u, v) for coordinates in \mathbb{R}^2 , and to use \sim to indicate objects in \mathbb{R}^2 .

Consider the family $\tilde{\Gamma}$ of curves $\tilde{\gamma}_a(t) = (t^{1+\alpha}/(1+\alpha), -t^\alpha/\log t + a) : (0, 1/2] \rightarrow (\mathbb{R}^2, d_E)$, where $0 \leq a \leq 1$. We will show that this family has positive modulus. This has the same modulus as the family Γ consisting of the curves $\gamma_a = \varphi_\alpha^{-1} \circ \tilde{\gamma}_a$.

Let D be the domain in \mathbb{R}^2 which is foliated by the curves in $\tilde{\Gamma}$. In coordinates

$$\begin{cases} u &= \frac{t^{1+\alpha}}{1+\alpha} \\ v &= \frac{-t^\alpha}{\log t} + a \end{cases} \quad \begin{cases} t &= ((1+\alpha)u)^{1/(1+\alpha)} \\ a &= v + \frac{((1+\alpha)u)^{\alpha/(1+\alpha)}}{\log((1+\alpha)u)^{1/(1+\alpha)}} \end{cases}.$$

Write $\tilde{\gamma}(a, t)$ in place of $\tilde{\gamma}_a(t)$, observing that $|J_{\tilde{\gamma}}(a, t)| = t^\alpha$ and $|J_{\tilde{\gamma}^{-1}}(u, v)| = ((1+\alpha)u)^{-\alpha/(1+\alpha)}$.

Let $\tilde{\rho}$ be an admissible function for $\tilde{\Gamma}$, which we may assume to be supported on D . This means that

$\int_{\tilde{\gamma}_a} \tilde{\rho} ds \geq 1$ for all $a \in [0, 1]$. This can be written as

$$\begin{aligned} 1 &\leq \int_{\tilde{\gamma}_a} \tilde{\rho} ds_E = \int_0^{1/2} (\tilde{\rho} \circ \tilde{\gamma}_a) \sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2} dt \\ &= \int_0^{1/2} (\tilde{\rho} \circ \tilde{\gamma}_a) \sqrt{t^{2\alpha} + \left(\frac{t^{-1+\alpha}}{\log^2 t} - \frac{\alpha t^{-1+\alpha}}{\log t}\right)^2} dt. \end{aligned}$$

Then

$$\begin{aligned} 1 &\leq \int_0^1 \int_0^{1/2} (\tilde{\rho} \circ \tilde{\gamma}_a) \sqrt{t^{2\alpha} + \left(\frac{t^{-1+\alpha}}{\log^2 t} - \frac{\alpha t^{-1+\alpha}}{\log t}\right)^2} dt da \\ &= \int_D \tilde{\rho} \sqrt{\left((1+\alpha)u\right)^{2\alpha/(1+\alpha)} + \left(\frac{t(u)^{-1+\alpha}}{\log^2 t(u)} - \frac{\alpha t(u)^{-1+\alpha}}{\log t(u)}\right)^2} \frac{1}{\left((1+\alpha)u\right)^{\alpha/(1+\alpha)}} dm. \end{aligned}$$

Applying Hölder's inequality gives

$$1 \leq \left(\int_D \tilde{\rho}^2 dm \right)^{1/2} \left(\int_D 1 + \frac{1}{\left((1+\alpha)u\right)^{2\alpha/(1+\alpha)}} \left(\frac{t(u)^{-1+\alpha}}{\log^2 t(u)} - \frac{\alpha t(u)^{-1+\alpha}}{\log t(u)}\right)^2 dm \right)^{1/2}.$$

To evaluate the rightmost integral, we pull back to the (a, t) -plane to get

$$\begin{aligned} \int_D \left(1 + \frac{1}{t^{2\alpha}} \left(\frac{t^{-1+\alpha}}{\log^2 t} - \frac{\alpha t^{-1+\alpha}}{\log t}\right)^2 \right) t^\alpha dm &= \int_0^1 \int_0^{1/2} t^\alpha + \frac{1}{t^\alpha} \left(\frac{t^{-1+\alpha}}{\log^2 t} - \frac{\alpha t^{-1+\alpha}}{\log t}\right)^2 dt da \\ &= \int_0^{1/2} t^\alpha + \frac{1}{t^\alpha} \left(\frac{t^{-1+\alpha}}{\log^2 t} - \frac{\alpha t^{-1+\alpha}}{\log t}\right)^2 dt. \end{aligned}$$

The convergence of this integral follows from that of $\int_0^{1/2} -t^{-2+\alpha}(\log t)^{-2} dt$, for $\alpha \geq 1$. This gives a positive lower bound for $\int_D \tilde{\rho}^2 dm$ in the case that $\alpha \geq 1$.

However, we show that the curves $\gamma_a(t) = (t, \frac{-t^\alpha}{\log t} + a)$ are not rectifiable for $\alpha > 0$. We compute

$$\begin{aligned} \int_{\gamma_a} ds_\alpha &= \int_0^{1/2} \sqrt{1 + \frac{1}{t^{2\alpha}} \left(\frac{t^{-1+\alpha}}{\log^2 t} - \frac{\alpha t^{-1+\alpha}}{\log t}\right)^2} dt \\ &\geq \int_0^{1/2} \frac{|1 - \alpha \log t|}{t \log^2 t} dt = \infty, \end{aligned}$$

where divergence follows from the divergence of $\int_0^{1/2} (-t \log t)^{-1} dt$, for $\alpha > 0$. □

Based on the same example we also obtain the following collection of results.

Corollary 6.3.2. Let $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ ($\alpha \geq 1$) be a quasimetric homeomorphism.

- (1) The modulus of a curve family may fail to be quasi-preserved by f . Of necessity, such a curve family does not correspond to a ring domain.
- (2) (failure of absolute continuity on lines (ACL) property) The map f may fail to be absolutely continuous on almost every closed, compact curve.
- (3) (failure of Lusin condition N) The map f may map a set of zero measure (here, Hausdorff 2-measure) onto a set of positive measure.

Proof. Let Γ' be the family of curves in \mathbb{G}_α^2 obtained by adding the left endpoint 0 into the domain of each curve in Γ in the example of Theorem 6.3.1. Then Γ' consists only of non-locally rectifiable curves. The definition of modulus implies that any curve family with no non-locally rectifiable curves has modulus zero. However, the family $\varphi_\alpha \Gamma'$ has the same modulus as the family $\tilde{\Gamma}$ in the proof of Theorem 6.3.1, which is positive. Hence it is not true that the map φ_α quasi-preserved the modulus of all curve families not associated to some disjoint continua E, F .

Define $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\tilde{f}(u, v) = (u + 1, v)$, and $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ by $f = \varphi_\alpha^{-1} \circ \tilde{f} \circ \varphi_\alpha$. Since $\varphi_\alpha \Gamma'$ and $f\Gamma'$ have the same modulus, we obtain claim (1) of the theorem.

To verify claim (2), consider again the curve family Γ' and the function f defined in the previous paragraph. The family $f\Gamma'$ has positive modulus and contains only rectifiable curves, but every curve in Γ' is nonrectifiable. This shows that f^{-1} is not absolutely continuous on almost every closed rectifiable curve.

To verify claim (3) observe that $f(\mathcal{Z})$ is a locally rectifiable line in \mathbb{G}_α^2 , whose image under f^{-1} , namely \mathcal{Z} itself, has positive 2-measure whenever $\alpha \geq 1$. □

In summary, we have shown that various properties regarding curve families and absolute continuity, all of which are well-known in the Euclidean case, do not hold for the Grushin case. In particular, we have seen a difference between the property of quasi-preserving the modulus of all curve families associated to a ring domain, and the property of quasi-preserving the modulus of an arbitrary curve family.

6.4 Conformal mappings of the Grushin plane

An auxiliary theorem in [GJR17] is a characterization of conformal mappings of the Grushin plane, which we define in the metric sense (Definition 6.1.1, with $H = 1$).

Theorem 6.4.1. Let $f : \mathbb{G}_\alpha^2 \rightarrow \mathbb{G}_\alpha^2$ ($\alpha > 0$) be an orientation-preserving homeomorphism which is metrically conformal. Then there exists $\lambda > 0, a \in \mathbb{R}$ such that $f(x, y) = (\pm\lambda x, \pm\lambda^{1+\alpha}y + a)$.

Proof. Define $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\tilde{f} = \varphi_\alpha \circ f \circ \varphi_\alpha^{-1}$. Note that \tilde{f} is a conformal mapping on the image of the Riemannian set of f . By the argument of Proposition 6.2.2, we conclude that \tilde{f} is in fact conformal on all of \mathbb{R}^2 . It follows that

$$\tilde{f}(u, v) = (a_1 + \lambda_0((\cos \theta)u - (\sin \theta)v), a_2 + \lambda_0((\sin \theta)u + (\cos \theta)v))$$

for some $a_1, a_2 \in \mathbb{R}, \lambda_0 > 0, \theta \in [0, 2\pi)$. The theorem will follow by showing $\theta \in \{0, \pi\}$ and $a_1 = 0$. This in turn will follow from showing that f maps the singular line $\mathcal{Z} = \{0\} \times \mathbb{R}$ onto itself.

An explicit formula for f obtained by conjugating \tilde{f} by φ_α is fairly complicated. However, we can work with \tilde{f} directly by considering instead the conformal definition of the Grushin plane, as developed in Section 5.1, rather than the standard definition. In the conformal Grushin plane, all balls centered on the singular line are self-similar (relative to the Euclidean metric) by the corresponding version of the dilation property (3.1) and have (Euclidean) dilatation strictly larger than 1. Recall Figure 5.1 above. Since \tilde{f} is a similarity map, it preserves the Euclidean dilatation of metric balls; this implies that \tilde{f} cannot map a point on the singular line to a point off the singular line, as small balls in the conformal Grushin metric which are off the singular line have roughly Euclidean shape. We conclude that \tilde{f} and hence f maps the singular line onto itself, and the result follows. \square

A Liouville-type theorem on the rigidity of conformal mappings between domains in higher-dimensional Grushin spaces was proved by Morbidelli [Mor09]. Morbidelli does not address the two-dimensional case explicitly, instead referring the reader to a paper of Payne [Pay06]. Theorems 4.4 and 4.5 of [Pay06] describe a sequence of one-parameter families of conformal homeomorphisms of \mathbb{G}_α or subdomains, including the conformal mappings in Theorem 6.4.1 above. However, Payne does not address the converse question of whether these one-parameter families generate all conformal mappings.

Chapter 7

Quasiconformal parametrization of metric surfaces

This section is dedicated to a result on quasiconformal parametrizations for abstract metric surfaces with locally finite Hausdorff measure. This is a very general setting, though observe that it does not include the Grushin plane—the Grushin Hausdorff 2-measure is not locally finite.

7.1 Rajala’s uniformization theorem

The motivation comes from the quasimetric uniformization problem, already discussed in Section 4.1. In place of quasimetric mappings, one may relax the problem to the class of quasiconformal mappings, in the sense of Definition 6.1.2. This is the geometric definition; throughout this section, the term *quasiconformal* will refer specifically to this definition. A result for quasiconformal mappings analogous to the Bonk–Kleiner theorem was obtained by Kai Rajala in [Raj17]. It is based on the following definition.

For its statement, we use the following terminology. A *quadrilateral* is a topological closed disk with four designated boundary edges, denoted in cyclic order by ζ_1 , ζ_2 , ζ_3 , and ζ_4 . Also, for a set $\Omega \subset X$ and disjoint continua $E, F \subset \Omega$, let $\Gamma(E, F; \Omega)$ denote the family of curves in Ω intersecting both E and F .

Definition 7.1.1. The metric space (X, d) is *reciprocal* if there exists $\kappa \geq 1$ such that for all quadrilaterals Q in X ,

$$\text{Mod } \Gamma(\zeta_1, \zeta_3; Q) \text{Mod } \Gamma(\zeta_2, \zeta_4; Q) \leq \kappa \tag{7.1}$$

and

$$\text{Mod } \Gamma(\zeta_1, \zeta_3; Q) \text{Mod } \Gamma(\zeta_2, \zeta_4; Q) \geq 1/\kappa, \tag{7.2}$$

and for all $x \in X$ and $R > 0$ such that $X \setminus B(x, R) \neq \emptyset$,

$$\lim_{r \rightarrow 0} \text{Mod}(\Gamma(B(x, r), X \setminus B(x, R); \overline{B(x, R)})) = 0. \tag{7.3}$$

The Euclidean plane \mathbb{R}^2 is reciprocal, in fact with $\kappa = 1$. This follows from standard computations of modulus: for $R = [0, a] \times [0, b] \subset \mathbb{R}^2$, with $\zeta_1 = [0, a] \times \{0\}$, etc., it holds that $\text{Mod } \Gamma(\zeta_1, \zeta_3; R) = a/b$ and

$\text{Mod } \Gamma(\zeta_2, \zeta_4; R) = b/a$. By the Riemann mapping theorem, any quadrilateral Q is conformally equivalent to such a rectangle R , where the ratio a/b does not depend on the choice of R . Another calculation shows that $\text{Mod } \Gamma(B(x, r), \mathbb{R}^2 \setminus B(x, R); \overline{B(x, R)}) = 2\pi \log(R/r)^{-1}$, which verifies condition (7.3). Since \mathbb{R}^2 is reciprocal, it is immediate that any metric space (X, d) quasiconformally equivalent to \mathbb{R}^2 must also be reciprocal.

In the main result of [Raj17], Rajala shows that Definition 7.1.1 is sufficient to construct “by hand” a quasiconformal mapping onto a domain in \mathbb{R}^2 . While the proof is lengthy and difficult, Rajala’s uniformization theorem is simple to state.

Theorem 7.1.2. Let (X, d) be a metric space homeomorphic to \mathbb{R}^2 or \mathbb{S}^2 with locally finite Hausdorff 2-measure. There exists a domain Ω in \mathbb{R}^2 or \mathbb{S}^2 and a quasiconformal mapping $f : (X, d) \rightarrow \Omega$ if and only if X is reciprocal.

The original Bonk–Kleiner theorem (Theorem 4.1.1) can then be deduced as a corollary to Theorem 7.1.2. This follows from two facts, also proved in [Raj17]. First, any space satisfying the upper mass bound $\mathcal{H}^2(B(x, r)) \leq Cr^2$ is reciprocal. Second, under that same assumption and the local linear connectivity property, a quasiconformal mapping is also quasisymmetric. Hence Rajala’s result is indeed a generalization of the earlier work of Bonk–Kleiner.

7.2 Minimizing dilatation

Let $f : X \rightarrow W$ be a quasiconformal map between the metric spaces X and W . We use the following notation. The smallest value K_O such that $\text{Mod } \Gamma \leq K_O \text{Mod } f(\Gamma)$ for all curve families Γ in X is called the *outer dilatation* of f . Similarly, the smallest value K_I such that $\text{Mod } f(\Gamma) \leq K_I \text{Mod } \Gamma$ for all curve families Γ in X is the *inner dilatation*.

The same paper of Rajala also examines a related question: if such a quasiconformal parametrization exists, can one find a quasiconformal mapping which improves the dilatation constants K_O and K_I to within some universal constants? If so, what is the best result of this type? Rajala obtains the following theorem [Raj17, Thm. 1.5]:

Theorem 7.2.1. (Rajala) Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and (X, d) a metric space of locally finite Hausdorff 2-measure. There exists a quasiconformal homeomorphism $f : X \rightarrow \Omega$ if and only if there exists a 2-quasiconformal homeomorphism $f : X \rightarrow \Omega$.

This result is proved using two classical results. First is the measurable Riemann mapping theorem, the fundamental existence result for quasiconformal mappings in the plane. This is usually formulated using complex notation, where a quasiconformal mapping is a homeomorphic solution of the Beltrami equation.

Here is a precise statement. Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $\mu : \Omega \rightarrow \mathbb{C}$ a measurable function with $\|\mu\|_\infty < 1$. Then there exists a quasiconformal mapping $g : \Omega \rightarrow \Omega$ such that $\partial_{\bar{z}}g(z) = \mu(z)\partial_zg(z)$ for a.e. $z \in \Omega$. The function μ is called the *complex dilatation*. Its relation to the dilatation in Definitions 6.1.2 and 6.1.3 is this: $\|\mu\|_\infty \leq k < 1$ if and only if g is K -quasiconformal for $K = (1+k)/(1-k)$.

The second result invoked in proving Theorem 7.2.1 is John's theorem on convex bodies. This theorem asserts, in part, that any convex body A in \mathbb{R}^n contains a unique ellipsoid E of maximal volume satisfying $E \subset A \subset \sqrt{n}E$, where the constant \sqrt{n} is the best possible. The constant 2 in Theorem 7.2.1 is derived from the constant $\sqrt{2}$ in John's theorem for dimension two.

To give some context to Theorem 7.2.1, we mention an important example.

Example 7.2.2. This is Example 2.2 in [Raj17]. Take $X = \mathbb{R}^2$, equipped with the ℓ_∞ -metric. The identity map $\iota : X \rightarrow \mathbb{R}^2$ satisfies

$$\frac{2}{\pi} \text{Mod } \Gamma \leq \text{Mod } f\Gamma \leq \frac{4}{\pi} \text{Mod } \Gamma.$$

Moreover, any quasiconformal mapping $f : X \rightarrow \mathbb{R}^2$ must have $K_O \geq \pi/2$ and $K_I \geq 4/\pi$.

A similar fact holds with the ℓ_1 -metric in place of the ℓ_∞ -metric. Both of these represent extremal norms on \mathbb{R}^2 , so (in light of the proof of Theorem 7.2.1, see Section 7.4) it is natural to conjecture that the modulus bounds in Example 7.2.2 are also extremal.

In the paper [Romar], I was able to prove the following improvement to Theorem 7.2.1. The given bounds had been conjectured by Rajala in [Raj17].

Theorem 7.2.3. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and (X, d) a metric space of locally finite Hausdorff 2-measure. There exists a quasiconformal homeomorphism $f : X \rightarrow \Omega$ if and only if there exists a quasiconformal homeomorphism $f : X \rightarrow \Omega$ satisfying

$$\frac{2}{\pi} \text{Mod } \Gamma \leq \text{Mod } f\Gamma \leq \frac{4}{\pi} \text{Mod } \Gamma. \tag{7.4}$$

Rajala's techniques, together with standard volume ratio estimates (see for instance [Bal97, Thm. 6.2]), guarantee the existence of a quasiconformal map $f_O : X \rightarrow \Omega$ with outer dilatation $K_O \leq \pi/2$, and a quasiconformal map $f_I : X \rightarrow \Omega$ with inner dilatation $K_I \leq 4/\pi$. The improvement in Theorem 7.2.3 is in finding a map which satisfies both modulus inequalities simultaneously. As already indicated in Example 7.2.2, inequality (7.4) cannot be improved.

The simple connectedness assumption is essential to Theorem 7.2.3. For example, any K -quasiconformal mapping f between the annular regions $\{x \in \mathbb{R}^2 : 1 < |x| < a\}$ and $\{x \in \mathbb{R}^2 : 1 < |x| < b\}$, $b \geq a$, must satisfy $K \geq \log b / \log a$. In particular, for any $K' \geq 1$ there exist annuli $A_1, A_2 \subset \mathbb{R}^2$ such that any

K -quasiconformal map $f : A_1 \rightarrow A_2$ must satisfy $K \geq K'$. A similar fact holds for wedge domains in \mathbb{R}^n , $n \geq 3$, which is one indication that a result like Theorem 7.2.3 is only possible in dimension two. See Väisälä [Väi71, Sec. 39-40] for a discussion of quasiconformal mappings between annular and wedge domains.

My contribution was in stating and proving a volume ratio inequality adapted to the problem at hand. I consulted a couple experts on the subject of convex bodies, and it appears to be an original result despite its elementary statement.

7.3 Volume ratio lemma for planar convex bodies

First, we fix some notation and recall some definitions. In the following, $|E|$ will denote the area of the set $E \subset \mathbb{R}^2$. We also let $L(E) = \sup\{|z| : z \in E\}$ denote the outer radius of E and $\ell(E) = \inf\{|z| : z \notin E\}$ denote the inner radius of E . A *convex body* is a compact convex set A in \mathbb{R}^2 with nonempty interior; it is *symmetric* if $z \in A$ implies $-z \in A$. There is a natural correspondence between the set of norms on \mathbb{R}^2 and the set of symmetric convex bodies in \mathbb{R}^2 . Namely, the unit ball for a norm on \mathbb{R}^2 is a symmetric convex body, while for any symmetric convex body A the function $p(x) := \inf\{t > 0 : x/t \in A\}$ defines a norm on \mathbb{R}^2 . Terms such as *ellipse* and *polygon* should be understood as including the interior of the respective objects.

For a convex body $A \subset \mathbb{R}^2$ and a linear transformation $T \in GL(2, \mathbb{R})$, let $r(A, T) = L(TA)/\ell(TA)$. Set $\rho(A) = \inf r(A, T)$, the infimum taken over all $T \in GL(2, \mathbb{R})$. Notice that $\rho(A) \leq \sqrt{2}$ by John's theorem. Expressed in different terms, $\rho(A)$ is the (multiplicative) Banach-Mazur distance between A and the closed Euclidean unit ball in \mathbb{R}^2 .

It is easy to verify that there is a matrix $T \in GL(2, \mathbb{R})$ such that $r(A, T)$ attains $\rho(A)$. Consider the family $\mathcal{T} = \{T \in GL(2, \mathbb{R}) : 1/\sqrt{2} \leq \ell(TA) \leq L(TA) \leq 1\}$. By John's theorem, restricting to $T \in \mathcal{T}$ does not affect the infimal value of $r(A, T)$. For such a map

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we must have $a^2 + c^2 \leq \ell(A)^{-2}$ and $b^2 + d^2 \leq \ell(A)^{-2}$. This is seen by looking at the action of T on the test points $\ell(A)e_1$ and $\ell(A)e_2$. We also have $|\det A| = |ad - bc| \geq L(A)^{-2}/2$. Hence the set \mathcal{T} is compact as a subset of $GL(2, \mathbb{R})$ and it follows that a nonzero map T minimizing $r(A, T)$ must exist.

Lemma 7.3.1. Let $A \subset \mathbb{R}^2$ be a symmetric convex body and $T \in GL(2, \mathbb{R})$ a linear map such that $r(A, T) = \rho(A)$. Then the image of A satisfies $2L(TA)^2 \leq |TA| \leq 4\ell(TA)^2$.

Proof. Let $\tilde{A} = TA$. Without loss of generality we can assume that the outer radius satisfies $L(\tilde{A}) = 1$. Let $\ell = \ell(\tilde{A})$. Then from John's theorem it follows that $2^{-1/2} \leq \ell \leq 1$. Where convenient, we will use complex notation for points in \mathbb{R}^2 . For $\theta \in [0, 2\pi)$, let z_θ denote the unique point in $\partial\tilde{A} \cap \{e^{i\theta}r : r > 0\}$. By rotating if necessary, we will assume that $|z_0| = 1$.

We first need a fact about the existence of contact points with the circles $S(0, \ell)$ and $S(0, 1)$. Specifically, we claim that there exist values $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 < \pi$ such that $|z_{\theta_0}| = |z_{\theta_2}| = 1$ and $|z_{\theta_1}| = |z_{\theta_3}| = \ell$. Suppose this does not hold. Then there exist $0 < \theta_1 \leq \theta_3 < \pi$ such that $|z_{\theta_1}| = |z_{\theta_3}| = \ell$, $|z_\theta| < 1$ whenever $\theta_1 < \theta < \theta_3$, and $|z_\theta| > \ell$ whenever $0 < \theta < \theta_1$ or $\theta_3 < \theta < \pi$. Observe that if θ, θ' are such that $|z_\theta| = 1$ and $|z_{\theta'}| = \ell$, then $|\theta - \theta'| \geq \cos^{-1}(\ell)$. In particular, $\theta_1 \geq \cos^{-1}(\ell)$ and $\theta_3 \leq \pi - \cos^{-1}(\ell)$.

Consider now a small linear stretch in the direction $(\theta_1 + \theta_3)/2$. Expressed in a suitable orthonormal basis $\{v_1, v_2\}$ for \mathbb{R}^2 , this linear stretch takes the form $T_\lambda : (x, y) \mapsto (\lambda x, y)$ for some sufficiently small parameter $\lambda > 1$.

Let $\theta = (\theta_3 - \theta_1)/2$. For sufficiently small $\epsilon > 0$, consider the function

$$R_\epsilon(\lambda) = \frac{\lambda^2 \cos^2(\theta + \cos^{-1}(\ell) - \epsilon) + \sin^2(\theta + \cos^{-1}(\ell) - \epsilon)}{\lambda^2 \ell^2 \cos^2(\theta + \epsilon) + \ell^2 \sin^2(\theta + \epsilon)}.$$

We obtain the functions $R_\epsilon(\lambda)$ by considering the (Euclidean) norm of the image of the points $e^{i(\theta_1 - \cos^{-1}(\ell) + \epsilon)}$ (the numerator) and $\ell^2 e^{i(\theta_1 - \epsilon)}$ (the denominator), as expressed relative to the basis $\{v_1, v_2\}$. Then $R_\epsilon(\lambda)$ is an upper bound for $r(A, T_\lambda T)$ for sufficiently small λ and satisfies $R_\epsilon(1) = r(A, T)$. In particular, $\frac{d}{d\lambda} r(A, T_\lambda T) \leq R'_0(\lambda)$. We compute

$$R'_0(\lambda) = \frac{-2\lambda\sqrt{1 - \ell^2} \sin(2\theta + \cos^{-1}(\ell))}{\ell^2 (\sin^2 \theta + \lambda^2 \cos^2 \theta_1)^2}.$$

Since $2\theta_1 + \cos^{-1}(\ell) < \pi$, we see that $R'(1) < 0$. This contradicts the minimality of $r(A, T)$. The existence of the desired values $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 < \pi$ now follows.

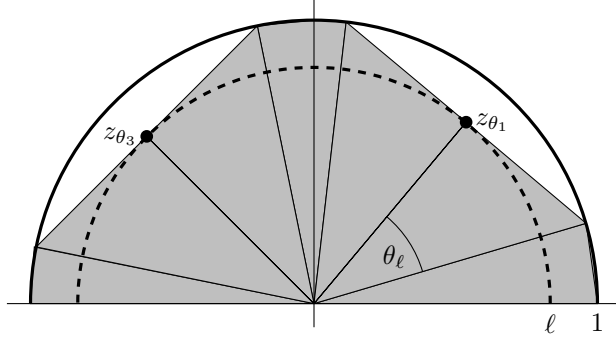
We now estimate $|\tilde{A}|$ from above. Write $\theta_\ell = \cos^{-1}(\ell)$. By covering \tilde{A} with the triangles $[0, e^{i(\theta_1 - \theta_\ell)}, e^{i(\theta_1 + \theta_\ell)}]$, $[0, 1, e^{i(\theta_3 - \theta_\ell)}, e^{i(\theta_3 + \theta_\ell)}]$ and the set

$$\{re^{i\theta} : 0 \leq r \leq 1, \theta \in [0, \theta_1 - \theta_\ell] \cup [\theta_1 + \theta_\ell, \theta_2 - \theta_\ell] \cup [\theta_2 + \theta_\ell, \pi]\},$$

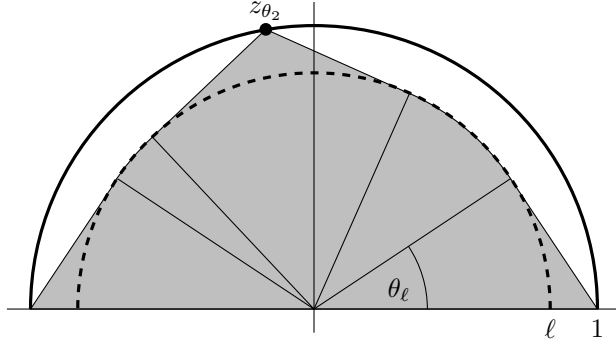
along with their reflections about the origin, we obtain

$$|\tilde{A}| \leq M(\ell) := \pi - 4 \cos^{-1}(\ell) + 4\ell\sqrt{1 - \ell^2}.$$

See Figure 7.1a. Observe that $M(2^{-1/2}) = 2 = 4(2^{-1/2})^2$, so the right inequality $|\tilde{A}| \leq 4\ell^2$ holds for



(a) The convex body \tilde{A} is contained in the shaded region and its reflection about the origin.



(b) The convex body \tilde{A} contains the shaded region and its reflection about the origin.

Figure 7.1: Estimating the area of \tilde{A} .

$\ell = 2^{-1/2}$. Next, compute $M'(\ell) = 8\sqrt{1-\ell^2}$. Since this satisfies $M'(\ell) \leq 8\ell$ when $2^{-1/2} \leq \ell \leq 1$, we obtain $|\tilde{A}| \leq 4\ell^2$ holds for all $\ell \in [2^{-1/2}, 1]$.

We can estimate $|\tilde{A}|$ from below using the polygons $[0, 1, \ell e^{i\theta_\ell}]$, $[0, -1, \ell e^{i(\pi-\theta_\ell)}]$, $[0, \ell e^{i\theta_2-\theta_\ell}, e^{i\theta_2}, \ell e^{i\theta_2+\theta_\ell}]$ and the set

$$\{re^{i\theta} : 0 \leq r \leq \ell, \theta \in [\theta_\ell, \theta_2 - \theta_\ell] \cup [\theta_2 + \theta_\ell, \pi - \theta_\ell]\}.$$

See Figure 7.1b. This gives

$$|\tilde{A}| \geq m(\ell) := (\pi - 4 \cos^{-1}(\ell))\ell^2 + 4\ell\sqrt{1-\ell^2}.$$

Now $m(2^{-1/2}) = 2$, so the left inequality $2L^2 \leq |\tilde{A}|$ holds when $\ell = 2^{-1/2}$. Since $m'(\ell) = 2\pi\ell + 4\sqrt{1-\ell^2} - 8\ell \cos^{-1}(\ell) \geq 0$, we obtain $2 \leq |\tilde{A}|$ for all $\ell \in [2^{-1/2}, 1]$. This completes the proof. \square

7.4 Outline of main result

A self-contained proof of Theorem 7.2.3 would be lengthy. We follow the same presentation as that given in [Romar], which depends on auxiliary facts from the original paper [Raj17]. The proof of Theorem 7.2.3 is very close to that of Rajala's original result, Theorem 7.2.1; the significant difference is that we apply Lemma 7.3.1 in place of John's theorem.

As mentioned above, the main tool is the measurable Riemann mapping theorem. The idea is that by post-composing the map $f : X \rightarrow \Omega \subset \mathbb{R}^2$ with a suitable planar quasiconformal mapping, we can cancel out most of the dilatation in the original mapping.

The other tool employed is a metric differentiability theorem due to Kirchheim [Kir94]. Let Z be a metric space, which by Fréchet's theorem we may assume to be a subset of some larger Banach space, with norm $\|\cdot\|$. For a Lipschitz function $g : \Omega \subset \mathbb{R}^2 \rightarrow Z$, define the *metric differential* $MD(g, x)$ at the point $x \in \Omega$ by

$$MD(g, x)(u) = \lim_{r \rightarrow 0} \frac{1}{r} \|g(x + ru) - g(x)\|,$$

where $u \in \mathbb{R}^2$, if this limit exists. It is shown in [Kir94, Thm. 2] that $MD(g, x)$ is a seminorm on \mathbb{R}^2 for a.e. $x \in \Omega$.

While a quasiconformal mapping need not be Lipschitz, it can nevertheless be decomposed into Lipschitz pieces on a subset of full measure. As explained in [Raj17, Lem. 14.1, 14.2], for every quasiconformal map $h : \Omega \rightarrow X$ there exist disjoint measurable sets Ω_j ($j = 1, 2, \dots$) covering Ω up to a set of measure zero such that $h|_{\Omega_j}$ is j -Lipschitz. The map $h|_{\Omega_j}$ can be extended to a Lipschitz map $h_j : \mathbb{R}^2 \rightarrow \ell_\infty(X)$. Then for all $j \in \mathbb{N}$ and a.e. $x \in \Omega$, $MD(h_j, x)$ is a non-zero norm on \mathbb{R}^2 .

We proceed now with the proof sketch. Recall that we are assuming the existence of a quasiconformal homeomorphism $h = f^{-1} : \Omega \rightarrow (X, d)$. For a.e. $x \in \Omega$, we obtain a non-zero norm G_x on \mathbb{R}^2 from the metric derivative of the function h_j described above, where j is such that $x \in \Omega_j$. For each such norm G_x , the set $C_x = \{y \in \mathbb{R}^2 : G_x(y) \leq 1\}$ is a symmetric convex body in \mathbb{R}^2 .

Let T_x be an invertible linear mapping for which $L(T_x C_x)/\ell(T_x C_x) = \rho(C_x)$. Let $E_x = T_x^{-1}(B(0, \ell(T_x C_x)))$; this gives an ellipse field on Ω defined for a.e. $x \in \Omega$. It can be shown that the ellipse E_x does not depend on our choice of T_x . Setting $E_x = B(x, 1)$ for the remaining points in Ω gives an ellipse field defined on all Ω . The associated complex dilatation is measurable and has a uniform bound less than 1.

Applying the measurable Riemann mapping theorem gives a quasiconformal mapping $\nu : \Omega \rightarrow \Omega$ such that

$$D\nu(x)(E_x) = B(0, r_x)$$

for a.e. $x \in \Omega$ and some $r_x > 0$. Let $C'_x = D\nu(x)(C_x)$, observing that $D\nu(x)$ differs from T_x by a scaling factor and orthogonal transformation.

Define $H = h \circ \nu^{-1} : \Omega \rightarrow X$. As above we obtain Lipschitz pieces $H_j = H|_{\Omega'_j}$ for disjoint sets $\Omega'_j \subset \Omega$. Then there exists $R_{x'} > 0$ such that $C'_x = \{y \in \mathbb{R}^2 : MD(H_j, x')(y) \leq R_{x'}\}$, for a.e. $x' \in \Omega$, where j is such that $x' \in \Omega'_j$ and $x = \nu^{-1}(x')$. Hence for a.e. $x' \in \Omega$, the metric derivative satisfies $|MD(H_j, x')| = R_{x'}/r_x$, and the Jacobian J_H is given by $J_H(x') = \pi R_{x'}^2 / |C'_x|$. By Lemma 7.3.1 we see that

$$\frac{|MD(H_j, x')|^2}{J_H(x')} = \frac{|C'_x|}{\pi r_x^2} \leq \frac{4}{\pi}.$$

Following [Raj17], this suffices to show the inequality $\text{Mod } \Gamma \leq 4\pi^{-1} \text{Mod } H\Gamma$ for all curve families Γ in Ω .

Similarly we have

$$\ell(MD(H_j, x')) := \inf_{|z|=1} |MD(H_j, x')z| = \frac{R_{x'}}{L(C'_x)},$$

valid for a.e. $x' \in \Omega$. This gives by Lemma 7.3.1

$$\frac{J_H(x')}{\ell(MD(H_j, x'))^2} = \frac{\pi L(C'_x)^2}{|C'_x|} \leq \frac{\pi}{2},$$

which suffices to show that $\text{Mod } H\Gamma \leq (\pi/2) \text{Mod } \Gamma$ for all curve families Γ in Ω .

Chapter 8

Open problems

We conclude with some potential research questions and directions. First are some questions that deal with the Grushin plane.

1. Harmonize the notion of conformal Grushin space with the work on almost-Riemannian manifolds carried out by Agrachev and co-authors. This would include extending, if possible, the embedding results of Theorems 5.2.1 and 5.3.1 to this setting.
2. Study quasiregular mappings on the Grushin plane. A quasiregular mapping is essentially a quasiconformal mapping without the requirement of injectivity. There is a well-developed theory of quasiregular mappings in both the Euclidean setting and various metric space settings; a starting point is the book of Rickman [Ric93]. One point to notice is that negligibility theorem used in the proof of Theorem 6.2.2 is not true in general for quasiregular mappings, hence a deeper investigation is needed.
3. Extend the results on quasiconformal mappings in Chapter 6 to higher-dimensional Grushin spaces or to conformal Grushin spaces. This is challenging due to the failure of the Riemann mapping theorem in higher dimensions.

Next are some problems related to quasiconformal mappings in metric spaces, inspired by the paper of Rajala [Raj17] and the result in Chapter 7. For all of these, let (X, d) be a metric space with locally finite Hausdorff 2-measure homeomorphic to \mathbb{R}^2 .

4. If X satisfies the reciprocity condition (7.3), does there exist a homeomorphism $h : \mathbb{R}^2 \rightarrow (X, d)$ such that $\text{Mod } \Gamma \leq K \text{Mod } h\Gamma$ for all curve families Γ and some fixed constant $K \geq 1$? This would be a one-sided version of Theorem 7.1.2. We could also ask a similar question without assuming (7.3), but such a parametrization would not be a homeomorphism in general. Consider, for example, a conformal weight on \mathbb{R}^2 which vanishes on a line segment; even though the line segment collapses to a point, the resulting metric space is still homeomorphic to \mathbb{R}^2 .

5. Suppose there exists a homeomorphism $f : (X, d) \rightarrow \mathbb{R}^2$ which quasi-preserves modulus of quadrilaterals: there exists $K \geq 1$ such that for all quadrilaterals $Q \subset X$,

$$K^{-1} \text{Mod } f\Gamma(\zeta_1, \zeta_3; Q) \leq \text{Mod } \Gamma(\zeta_1, \zeta_3; Q) \leq K \text{Mod } f\Gamma(\zeta_1, \zeta_3; Q).$$

Then is f K -quasiconformal? That is, must the same inequality hold for an arbitrary curve family?

6. Is it true that (X, d) satisfies the upper reciprocity bound for quadrilaterals (7.1) if and only if it satisfies the analogous upper reciprocity bound on ring domains? If true, then the reciprocity condition could be framed equivalently in terms of rings.
7. In modulus stable under convergence of sets in this setting? For instance, if $\text{Mod } \Gamma(\overline{B(x, r)}, X \setminus B(x, R), \overline{B(x, R)}) \geq K$ for all $r \in (0, R)$, does it follow that $\text{Mod } \Gamma(\{x\}, X \setminus B(x, R), \overline{B(x, R)}) \geq K$?

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