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IMPROVEMENTS IN ADAPTIVE IIR FILTERING THEORY AND APPLICATION

Majid Nayeri

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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It is also shown how different filter structures of an adaptive filter leads to a change in the characteristics of the corresponding error surface, and consequently, to a change in the corresponding convergence rate and minimum mean square error. A general theory, based on an analysis of stationary points, is presented which shows that whenever a direct form IIR filter with a unimodal MSE surface is transformed into an alternate realization, the MSE surface associated with the new structure may have additional stationary points, which are either new equivalent minima, or saddle points. The general theory is specialized for the parallel and cascade forms.

IMPROVEMENTS IN ADAPTIVE IIR FILTERING THEORY AND APPLICATION

BY

MAJID NAYERI

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Majid Nayeri, Ph.D.
Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign, 1988

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CHAPTER 1

INTRODUCTION

The term *filter* is often used to describe a device in the form of a piece of physical hardware or computer software that is applied to a set of noisy data in order to extract information about a prescribed quantity of interest. This quantity can vary from the heart beat in an electrocardiogram (ECG) to transmitted signals in communication channels. The predesigned filters used in the communication channels have been effective in a wide variety of applications. Most of these filters are, of course, optimal in a *fixed* environment. For instance, a matched filter used in radar applications is optimal in achieving the minimum probability of error in deciding which signal in a set of *a priori known* signals was transmitted in an *a priori known* fixed channel. But the change in the environment degrades the performance of some of these filters to an unacceptable degree. If the characteristics of the channel change, for instance, the matched filter has to be redesigned in order to offset the effect of this channel variation. This may not be possible from a practical point of view. This shows the lack of robustness of such filters under varying conditions. However, filters which are self-designing to adjust (or to adapt) to these variations have a great potential in replacing the fixed filters. The broad range of applications of adaptive filters has led to a global research in many disciplines. As a result, many algorithms have been proposed and studied for adaptive filtering [1]-[26], adaptive echo cancellation [27]-[32], adaptive channel equalization [33]-[37], adaptive control [38]-[40], and system identification [41]-[47], some of which are known to have desirable convergence properties if certain *a priori* information can be assumed from a knowledge of system characteristics and signal properties. These algorithms and their analysis can be found in detail in some recent books [48]-[53].

1.1. Adaptive Filtering

The design of optimal filters requires *a priori* information about the statistics of the data to be processed. When this information is not known completely, it is feasible to use an *adaptive filter* which relies on a recursive algorithm for its operation. Figure 1.1 shows the general adaptive filtering configuration in which the coefficients of the adaptive filter are adjusted to minimize the mean-square value of the error signal e(n). It is assumed that the available information signals, x(n) and y(n), are correlated; therefore, the adaptive filter tries to remove that component of y(n) which is correlated to x(n). The complete removal of this component depends on the nature of the correlation. Two subclasses of adaptive filters can be distinguished analogous to conventional digital filters: adaptive finite impulse response (FIR) and adaptive infinite impulse response (IIR) filters.

The FIR adaptive filters have been extensively considered for specific applications such as adaptive noise cancellation [4], echo cancellation [27]-[32], and channel equalization [33]-[37]. The signal $\hat{y}(n)$ is generated by

$$\begin{aligned}\hat{y}(n) &= w_0(n)\,x(n) + w_1(n)\,x(n-1) + \cdots + w_{N-1}(n)\,x(n-N+1) \\ &= W^T(n)\,X(n)\end{aligned} \qquad (1.1.1)$$

where

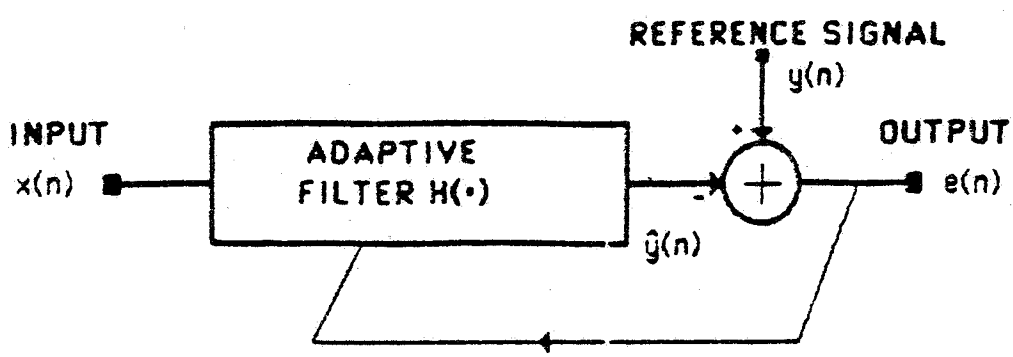


Fig. 1.1. General form of adaptive filters.

$$X(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \cdot \\ \cdot \\ \cdot \\ x(n-N+1) \end{bmatrix} \quad , \quad W(n) = \begin{bmatrix} w_0(n) \\ w_1(n) \\ \cdot \\ \cdot \\ \cdot \\ w_{N-1}(n) \end{bmatrix}$$

which is an N-weighted sum of the present and past inputs. The wide applications of FIR adaptive filters are due to the simplicity of the FIR structure which makes the analysis of the adopted algorithms simpler. It is well known that for FIR adaptive filters in a stationary environment, the mean square function is given by

$$E\, e^2(n) = W^T R W - 2\, W^T P + E\, y^2(n) \tag{1.1.2}$$

with

$$R \triangleq E\left[X(n)\, X(n)^T\right] \quad , \quad P \triangleq E\left[y(n) X(n)\right]$$

which represents a quadratic error surface[1] that has a single global minimum point [4]. The unique minimum is the well-known Wiener solution given by

$$W^* = R^{-1} P \ . \tag{1.1.3}$$

Note that the linear prediction of x(n+1) based on the observation vector X(n) is a special case of the adaptive filtering problem in Fig. 1.1. Thus, the adaptive filters can function as *linear predictors*, as well. Some well-known algorithms within the discipline of adaptive systems are now introduced.

LMS Algorithm:

The least mean square (LMS) algorithm [1] is a naive, yet attractive, algorithm for its performance and its low computational complexity. It uses a simple estimate of the MSE gradient to update the filter's weights in the direction of the negative gradient (i.e., downhill) through

$$W(n+1) = W(n) + 2\mu e(n) X(n) \ . \tag{1.1.4}$$

1 Error surface is an (N+1) dimensional surface which is a plot of $E\, e^2(n)$ as a function of N adaptive coefficients.

It has been shown [1] that if $0 < \mu < \frac{1}{\lambda_{max}}$, where λ_{max} is the largest eigenvalue of R, then (1.1.4) will be stable and W(n) converges to W^* in the mean. However, if a vanishing adaptive gain is used, the convergence will be guaranteed *almost surely* at the expense of taking away the tracking capability of the algorithm.

RLS Algorithm:

The recursive least square (RLS) algorithm, which is formulated through the equation error method [20], enjoys the unimodality of the error surface in the FIR case. This is because the algorithm is essentially reduced to a Newton type algorithm (or known as sequential regression (SER) algorithm) given by

$$\begin{aligned} W(n+1) &= W(n) + 2\,M(n)X(n)[\,y(n) - W^T(n)X(n)] \\ M(n) &= M(n-1) - \frac{M(n-1)X(n)X^T(n)M(n-1)}{1 + X^T M(n-1)X(n)} \end{aligned} \tag{1.1.5}$$

where M(n) is an estimate of R^{-1}. For this reason, RLS is normally faster than the LMS algorithm. The difference between the two becomes much more evident as the input signal x(n) becomes more colored. In other words, as the eigenvalue spread of the input autocorrelation matrix, measured by the condition number, gets larger, the LMS algorithm convergence rate decreases. Meanwhile, the RLS algorithm normalizes this spread by estimating the inverse of the input autocorrelation matrix within the algorithm.

FDLMS Algorithm:

The drastic degradation of LMS in the colored input case was resolved by the introduction of the frequency domain LMS (FDLMS) algorithm [8],[9], which has the effect of whitening the input to the adaptive FIR filter. This is achieved by passing the input through a bank of band-pass filters whose outputs are used as the inputs to the taps. It was concluded that this bank of band-pass filters does an FFT operation on the input which whitens it to some degree. The update equation is given by

$$W(n+1) = W(n) + 2\mu e(n)\Lambda^{-2}Z^*(n) \tag{1.1.6}$$

where

$$\Lambda^2 = \text{diag}\left[\sigma_0^2\ \sigma_2^2 \ldots \sigma_{N-1}^2\right] \quad , \quad \sigma_i^2 = E\left[|z_i(n)|^2\right]$$

$$Z(n) = T\,X(n) \quad , \quad T = [t_{kl}]$$

$$t_{kl} = e^{j\frac{2\pi}{N}kl} \quad , \ k\,,l = 0\,,1\,,\cdots,N-1\ .$$

These and other algorithms have made the adaptive FIR filtering literature quite rich. On the contrary, the literature for adaptive IIR filtering is scarce.

The advantages of IIR filters over FIR filters are well known, e.g., for the same performance, IIR filters require much fewer computations than FIR filters, and IIR filters can usually match physical systems well whereas FIR filters often give only a rough approximation of them. This is also true in the field of adaptive filtering. However, unlike adaptive FIR filtering, the error surfaces for adaptive IIR filters may not be unimodal, and the poles may move outside the unit circle during the adaptation. These considerations make the adaptive IIR filtering much more difficult. In order to discuss several issues in IIR adaptive filtering, let us assume that the reference signal y(n) is generated by the input x(n) through some unknown system. Then, the adaptive filtering problem becomes a system identification problem, as shown in Fig. 1.2. It is further assumed that

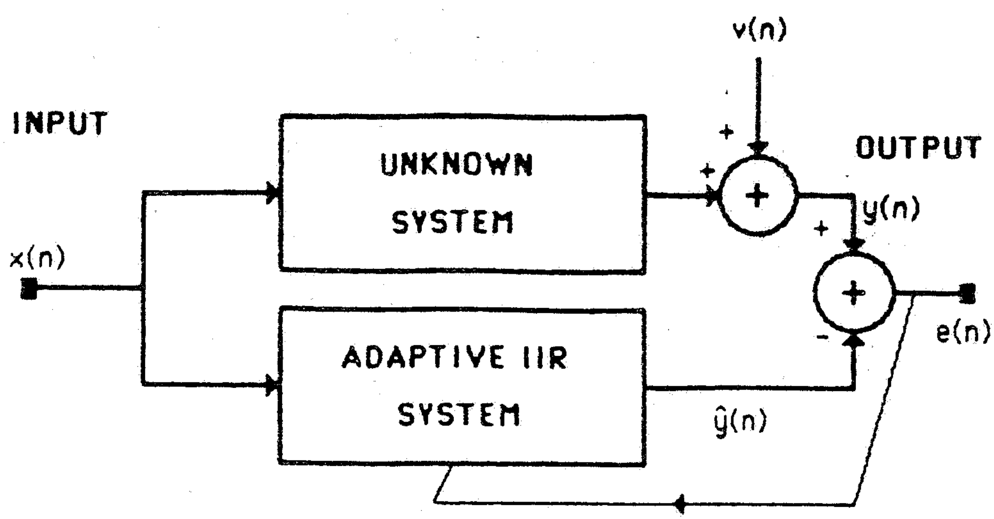


Fig. 1.2. System identification configuration.

$$y(n) = \frac{D(q^{-1})}{C(q^{-1})} x(n) + v(n) \tag{1.1.7}$$

where

$$D(q^{-1}) = d_0 + d_1 q^{-1} + \ldots + d_{n_d} q^{-n_d},$$

$$C(q^{-1}) = 1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c}$$

are coprime polynomials of the unit delay operator q^{-1} and $v(n)$ is additive noise. The transfer function of the unknown system is $\frac{D(z^{-1})}{C(z^{-1})}$ whose poles are assumed to lie strictly inside the unit circle. The additive noise $v(n)$ is a zero mean stochastic process that is independent of $x(n)$. Let the adaptive system be an IIR filter whose input-output relation is governed by

$$\hat{y}(n) = \frac{B(q^{-1},n)}{A(q^{-1},n)} x(n), \tag{1.1.8}$$

where

$$B(q^{-1},n) = b_0(n) + b_1(n) q^{-1} + \ldots + b_{n_b}(n) q^{-n_b},$$

$$A(q^{-1},n) = 1 + a_1(n) q^{-1} + \ldots + a_{n_a}(n) q^{-n_a}$$

Figure 1.3 shows the system identification mode of adaptive IIR filtering with the assumption that (1.1.7) and (1.1.8) hold.

In 1981 Stearns [12] conjectured that if the input excitation $x(n)$ is a white process and if the adaptive filter has a "sufficient order" with respect to the unknown system, i.e., $n_a \geqslant n_c$ and $n_b \geqslant n_d$, then the error surface will have no local minima. Based on this conjecture, the error surfaces of the IIR adaptive filters can be classified [22] as:

1) sufficient order with white noise excitation;

2) sufficient order with colored (noise) excitation;

3) insufficient order with white noise excitation;

4) insufficient order with colored (noise) excitation.

It is understood that to ensure possible parameter convergence, the input should always be *persistently exciting* which necessitates the following definition by [51]:

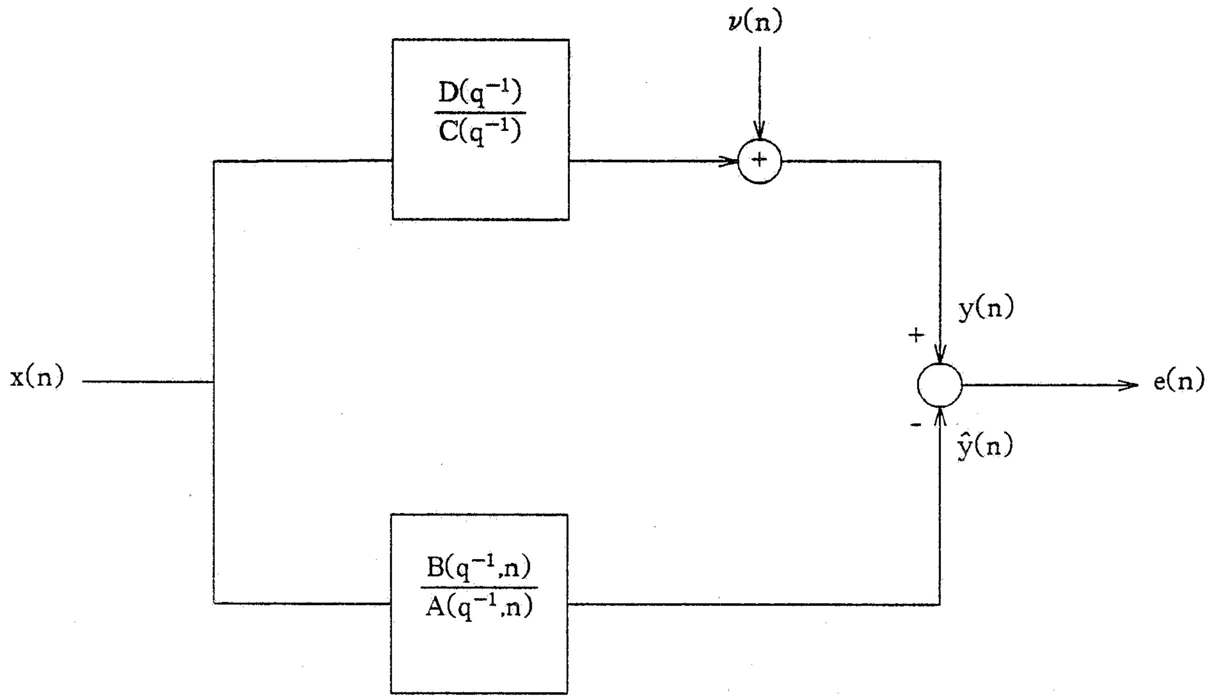


Fig. 1.3. System identification with linear time invariant model.

Definition:

A process x(n) is persistently exciting of the order k, in a stationary environment if

a) its kxk autocorrelation matrix R is positive definite,

b) it is ergodic or mixing.

The stationary input x(n) is ergodic when its asymptotic time average is equal to its ensemble average. But the mixing condition is a weaker condition than the ergodicity requirement. It merely guarantees that future input carries information that is independent of the present and past inputs. This is essential for the convergenceof the adaptive algorithm.

1.1.1. RLS vs. gradient algorithm for IIR filters

The autoregressive-moving average (ARMA) model set up under the equation error approach is an IIR filter formulated in an FIR setting. To clarify, the output of this adaptive filter $\hat{y}_e(n)$ is considered to be generated by

$$\hat{y}_e(n) = \left[1 - A(q^{-1},n)\right] y(n) + B(q^{-1},n)\, x(n) \tag{1.1.9}$$

Hence, $\hat{y}_e(n)$ is not recursive. This formulation has resulted in the development of many algorithms, such as RLS, whose convergence analyses are straightforward. In the sufficient order case, the unique attracting solution of many such algorithms corresponds to the global minimum point of the error surface. If we define

$$e_e(n) \triangleq y(n) - \hat{y}_e(n) = A(q^{-1},n)\, y(n) - B(q^{-1},n)\, x(n) \tag{1.1.10}$$

then, in the steady state, Fig. 1.3 can be represented alternatively as shown in Fig. 1.4.

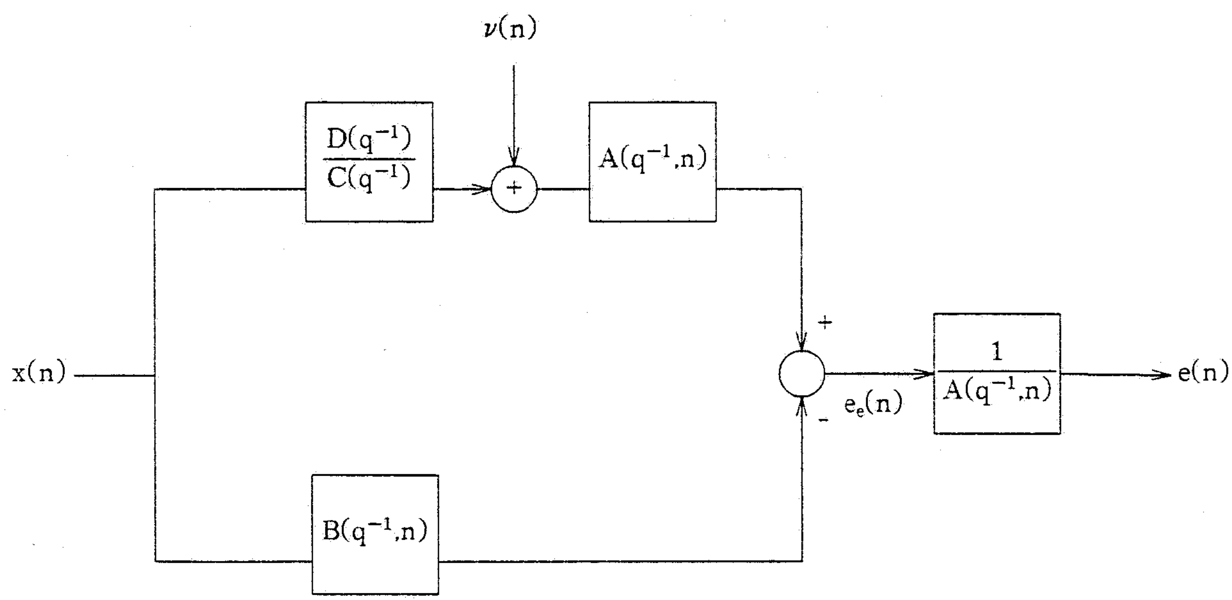


Fig. 1.4. Alternative representation of Fig. 1.3.

The equation error approach minimizes the function $\left[\frac{1}{K}\sum_{0}^{K-1} e_e^2(n)\right]$ (or $E\,e_e^2(n)$ by ergodicity assumptions) while the output error approach minimizes $E\,e^2(n)$. But minimization of $E\,e_e^2(n)$ through the RLS algorithm (1.1.5) by replacing $W(n)$ and $X(n)$ with $\theta = [a_1(n) \cdots a_{n_a}(n)\, b_0(n) \cdots b_{n_b}(n)]^T$ and $\phi = [y(n-1) \cdots y(n-n_a)\, x(n) \cdots x(n-n_b)]^T$, respectively, produces biased estimates in this case [46]. To cure this, the algorithm proposed by Fan and Jenkins [22] suggests that the operation $\frac{1}{A(q^{-1},n)}$ is done on the input first and then the equation error approach is used for optimization. This alternatively results in the Steiglitz-McBride algorithm [45] which has been analyzed for uniqueness of the solution and for convergence properties.

However, the algorithms based on a gradient search for updating the coefficients of the IIR adaptive filter are complex to analyze directly. This is due to the nonlinear nature of the algorithms. For instance, Stearns' algorithm, which is the simplest gradient algorithm with constant step size, is given by

$$\begin{aligned} a_i(n+1) &= a_i(n) - 2\mu e(n)\frac{1}{A(q^{-1}\,,\,n)}\, y(n-i) \\ b_j(n+1) &= b_j(n) + 2\mu e(n)\frac{1}{A(q^{-1}\,,\,n)}\, x(n-j) \end{aligned} \tag{1.1.11}$$

But the error surface can determine the possible behavior of such algorithms. For instance, unimodality means the uniqueness of the estimate and if certain regularity conditions are satisfied the algorithm converges to a neighborhood of the unique solution.

The ordinary differential equation (ODE) method [48],[55] is also a nice tool to understand the transient behavior of many algorithms and their convergence to the solution. It has been proven [48] that if the adaptive algorithm uses a vanishing gain, the convergence to the solution is in an *almost everywhere* sense while constant adaptive gain guarantees the convergence only to a neighborhood of the solution in the *probability* sense (weak convergence). This method was used in [22] to show the weak convergence of Fan-Jenkins algorithm to the unique solution. This method can be applied to the gradient techniques, as well [56].

For insufficient order adaptive filters, gradient search algorithms may be more suitable. The reason is that the algorithms formulated under the equation error approach may very well have solutions, although unique, which do not coincide with the global minimum of the corresponding error surface [20]. Therefore, the study of error surfaces in this case is essential to find the MMSE solution attained by gradient methods.

The alternative realizations of conventional (fixed) filters is a classical subject in almost every signal processing textbook [57],[58]. The advantages of the parallel and the cascaded structures over the direct form are well known. For instance, the low sensitivity of these structures to roundoff and quantization error noises makes them more robust than their direct form counterparts. These realizations have also been considered for adaptive filtering applications [10],[11],[24]-[26] and some promising behavior has been observed with regard to convergence and pole monitoring capabilities.

1.2. Overview of the Presentation

The goals of this thesis are primarily twofold: (1) to address the question regarding the uniqueness of the estimates obtained by the gradient algorithms in the field of adaptive IIR filtering, and (2) to introduce and analyze the alternative structures of IIR filters. They are achieved in two, essentially self-contained, Chapters (2 and 3), respectively.

In Chapter 2, the shapes of the four classes of error surfaces introduced above are investigated. In particular, Stearns' conjecture which concludes the unimodality of class 1 error surfaces is restated and a counter example is developed through the notion of *degenerated* solutions introduced by Soderstrom[41]. In addition, a case of overparameterization is sited which shows the creation of local minima on the performance surface. This adverse effect can have serious practical implications. Also, some sufficient conditions regarding multimodality of error surfaces of insufficient order adaptive filters are introduced. The derived conditions are then applied to some already existing examples in the literature [12],[15]. The analysis in this chapter results in a unification of the existing knowledge in the system identification with adaptive filtering.

To overcome certain shortcomings of the direct form IIR adaptive filters, such as complicated pole monitoring and slow rate of convergence, alternative structures of IIR adaptive filters will be introduced in Chapter 3. Numerical examples are used to point out some desirable behavior of the parallel form. Some analyses are then carried out to relate the characteristics of the error surfaces in the direct form with those of equivalent realizations. Several examples are introduced which set the groundwork to generalize important characteristics of the error surfaces in the parallel and cascade forms. The analyses given have a great potential to be applied to the bandpass adaptive IIR filtering algorithm, introduced by Shynk *et al.*[10],[11] which has shown promising performance in various applications.

CHAPTER 2

CHARACTERISTICS OF ERROR SURFACES FOR ADAPTIVE IIR FILTERS

2.1. Introduction

In 1981 Stearns [12] conjectured that if the input excitation $x(n)$ is a white process and if the adaptive filter is of sufficient order with respect to a linear unknown system, then the error surface will have no local minima. In 1982 Soderstrom [43] proved Stearns' conjecture indirectly for the special case in which the degree of the numerator in the adaptive filter plus 1 is at least as great as the number of poles of the unknown linear system.

In this chapter it is shown by example that when this additional constraint imposed by Soderstrom is not satisfied, Stearns' conjecture can be false. To show the multimodality of the error surface associated with this example, we use the notion of *degenerated points* introduced by Soderstrom [41]. It is concluded that Soderstrom's constraint is sufficient but not necessary. In addition, another sufficient order, but overparameterized, filter is considered which shows the sensitivity issue surrounding exact knowledge of the unknown system's order. Lack of this knowledge will lead to underparameterization or overparameterization, which are shown to be undesirable. We present a necessary and sufficient condition which is less stringent than Soderstrom's. The definition of degenerated points is extended when $x(n)$ is white noise and some sufficient conditions are given for multimodality of the error surfaces of "insufficient order" adaptive filters. Finally, some insight regarding the pole locations of the minimizing solutions with respect to the unknown system parameters is presented.

2.2. Degenerated Points and Saddle Points

Let us consider the model representation in Fig. 1.3 where the adaptive IIR filter in (1.1.8) is considered frozen. In other words, it does not depend on the time variable n. Then, the mean square error (MSE) is given by

$$E[e^2(n)] = E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})}\right)x(n)\right]^2 + E[v^2], \tag{2.2.1}$$

which is a function of the parameter vector $\boldsymbol{\theta}(\underline{a},\underline{b})$ where

$$\underline{a} = [a_1 \cdots a_{n_a}],$$

$$\underline{b} = [b_0\ b_1 \cdots b_{n_b}].$$

The stationary points of the MSE functional are the solutions of

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})}\right)x(n) \cdot \frac{B(q^{-1})}{A^2(q^{-1})}\, x(n-i)\right] = 0\ , \quad 1 \leqslant i \leqslant n_a, \tag{2.2.2a}$$

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})}\right)x(n) \cdot \frac{1}{A(q^{-1})} x(n-j)\right] = 0\ , \quad 0 \leqslant j \leqslant n_b, \tag{2.2.2b}$$

which are obtained by setting to zero the gradient of $E[e^2(n)]$ with respect to $\underline{a}$ and $\underline{b}$. Let us assume that $\boldsymbol{\theta}^*$ is a stationary point and correspondingly $A^*(q^{-1})$ and $B^*(q^{-1})$ satisfy (2.2.2a) and (2.2.2b). Then, if (2.2.2b) is multiplied by b_j^* and summed over j (or alternatively, (2.2.2a) is multiplied by a_i^* and summed over i), we get

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B^*(q^{-1})}{A^*(q^{-1})}\right)x(n) \cdot \frac{B^*(q^{-1})}{A^*(q^{-1})}x(n)\right] = 0. \tag{2.2.3}$$

The MSE (2.2.1) at this stationary point could be expanded to

$$E[e^2(n)] = E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B^*(q^{-1})}{A^*(q^{-1})}\right)x(n) \cdot \left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B^*(q^{-1})}{A^*(q^{-1})}\right)x(n)\right] + E[v^2]. \tag{2.2.4}$$

Using Equation (2.2.3), the expression in (2.2.4) is reduced to

$$E[e^2(n)] = E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B^*(q^{-1})}{A^*(q^{-1})}\right)x(n) \cdot \frac{D(q^{-1})}{C(q^{-1})}x(n)\right] + E[v^2]$$

$$= E\left[\frac{D(q^{-1})}{C(q^{-1})}x(n)\right]^2 - E\left[\frac{B^*(q^{-1})}{A^*(q^{-1})}x(n) \cdot \frac{D(q^{-1})}{C(q^{-1})}x(n)\right] + E[v^2]. \tag{2.2.5}$$

Now, if $\frac{D(q^{-1})}{C(q^{-1})}$ in the second term after equality is replaced by $\left(\frac{D(q^{-1})}{C(q^{-1})} - \frac{B^*(q^{-1})}{A^*(q^{-1})}\right) + \frac{B^*(q^{-1})}{A^*(q^{-1})}$, then using (2.2.3) Equation (2.2.5) is reduced to

$$E[e^2(n)] = E\left[\frac{D(q^{-1})}{C(q^{-1})}x(n)\right]^2 - E\left[\frac{B^*(q^{-1})}{A^*(q^{-1})}x(n)\right]^2 + E[v^2]. \tag{2.2.6}$$

Note the close similarity of (2.2.6) with the minimum norm obtained by the minimum norm estimator in linear spaces. As a matter of fact, if an adaptive FIR is used, i.e., when $A(q^{-1}) = 1$, there is only one stationary point, which corresponds to the Wiener solution of the underlying linear vector space.

Soderstrom [41] introduced the notion of *degenerated points* for adaptive filters with $b_0 \equiv 0$. Since we intend to use the more general filter structure (1.1.8), a slightly more general definition of degenerated points is introduced when x(n) is a white process. Based on this definition, it will be shown that a degenerated stable stationary point is a saddle point. Existence of such a saddle point which was to be equivalent to the existence of multiple minima is the key to establishing a set of sufficient conditions which result in the multimodality of error surfaces when x(n) is white. (x(n) is assumed white throughout unless stated otherwise.)

Definition:

The degenerated points are the points in the parameter space such that $B(q^{-1}) \equiv 0$ if $n_b < n_a$, and $B(q^{-1}) \equiv L(q^{-1})A(q^{-1})$ if $n_b \geq n_a$ where $L(q^{-1}) = l_0 + l_1 q^{-1} + \cdots + l_{n_b - n_a} q^{-(n_b - n_a)}$, which includes $B(q^{-1}) \equiv 0$.

Theorem 2.1:

A stable degenerated solution of (2.2.2) is a saddle point.

Proof: Based on the definition of degenerated points we consider two cases:

Case 1: $n_b < n_a$

Since $B(q^{-1}) \equiv 0$, (2.2.2) is reduced to

$$E\left[\frac{D(q^{-1})}{C(q^{-1})}x(n) \cdot \frac{1}{A(q^{-1})}x(n-j)\right] = 0 \;,\; 0 \leqslant j \leqslant n_b, \tag{2.2.7}$$

It has been proven in [41] that the solutions of (2.2.7) are saddle points. See Appendix A for the details.

Case 2: $n_b \geqslant n_a$

The degenerated points are of the form $B(q^{-1}) = L(q^{-1})A(q^{-1})$ and (2.2.2) is reduced to

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - L(q^{-1})\right)x(n) \cdot \frac{1}{A(q^{-1})}x(n-j)\right] = 0 \;,\; 0 \leqslant j \leqslant n_b \,. \tag{2.2.8}$$

Now it can be shown that stable solutions of (2.2.8) are saddle points. If the polynomial $B(q^{-1})$ is represented as

$$B(q^{-1}) = L(q^{-1}) \cdot A(q^{-1}) + \beta(q^{-1})$$

where

$$\beta(q^{-1}) = \sum_{i=n_b-n_a+1}^{n_b} \beta_i q^{-i},$$

then minimizing (2.2.1) is equivalent to minimizing

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})} - L(q^{-1}) - \frac{\beta(q^{-1})}{A(q^{-1})}\right)x(n)\right]^2 \tag{2.2.9}$$

with respect to the parameters a_i, l_j and β_k. This is so because there is a one-to-one correspondence between $(\underline{a}, \underline{b})$ and $(\underline{a}, \underline{l}, \underline{\beta})$ points. If the gradient of (2.2.9) with respect to l_j parameter is set to zero, we will have

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})}-L(q^{-1})-\frac{\beta(q^{-1})}{A(q^{-1})}\right)x(n)\cdot x(n-j)\right]=0\ ,\quad 0\leqslant j\leqslant n_b-n_a. \tag{2.2.10}$$

Note that

$$E\left[\frac{\beta(q^{-1})}{A(q^{-1})}x(n)\cdot x(n-j)\right]=0\ ,\quad 0\leqslant j\leqslant n_b-n_a, \tag{2.2.11}$$

with x(n) being white. This clearly shows that the optimal $L(q^{-1})$ polynomial, denoted by $L^*(q^{-1})$, is the same for all the stationary points of (2.2.8) and satisfies

$$E\left[\left(\frac{D(q^{-1})}{C(q^{-1})}-L^*(q^{-1})\right)x(n)\cdot x(n-j)\right]=0\ ,\quad 0\leqslant j\leqslant n_b-n_a. \tag{2.2.12}$$

If $L^*(q^{-1})$ is substituted in (2.2.9), $\left[\frac{D(q^{-1})}{C(q^{-1})}-L^*(q^{-1})\right]$ is then fixed. The degenerated point solutions corresponding to $\beta(q^{-1})\equiv 0$ are the solutions of

$$E\left[\frac{1}{A^*(q^{-1})}x(n-j)\cdot\left(\frac{D(q^{-1})}{C(q^{-1})}-L^*(q^{-1})\right)x(n)\right]=0\ ,\quad n_b-n_a+1\leqslant j\leqslant n_b, \tag{2.2.13}$$

which corresponds to a saddle point by [41]. □

This theorem simply states that the degenerated points are not extreme points. However, the significance of these points is expressed by the next theorem which predicts the multimodality of the error surface if such points exist. In other words, non-uniqueness of the estimates could be concluded.

Theorem 2.2:

If $n_a > n_b = 0$ or $n_a = n_b = 1$, the existence of a stable degenerated solution implies the multimodality of the error surface of (2.2.1).

Proof: Since a stable degenerated solution is a saddle point by Theorem 2.1, the proof is immediate by [41]. □

It should be pointed out that the analysis of degenerated solutions is independent of the parameter $n^* \triangleq \min(n_a-n_c, n_b-n_d)$. This means that Theorem 2.1 and Theorem 2.2 are true for both the sufficient order case, i.e., when $n^* \geqslant 0$, and insufficient order case, $n^* < 0$. We now apply Theorem 2.2 to the sufficient order case to construct a counterexample to Stearns' conjecture.

2.2.1. Stearns' conjecture and its counterexample

Stearns [12] examined the behavior of the error surfaces of first-order and second-order IIR filters driven by white noise. In particular, he realized when first- and second-order filters were of sufficient order, i.e., ones for which $n^* = \min(n_a-n_c, n_b-n_d) \geqslant 0$, the error surfaces did not attain any local minimum. Based on this observation he introduced the following conjecture:

Consider the system (1.1.7) in which $x(n)$ is white. Then, with $n^* \geqslant 0$, the solution of (2.2.2) is given by

$$A(q^{-1}) = C(q^{-1})H(q^{-1}),$$
$$B(q^{-1}) = D(q^{-1})H(q^{-1}),$$

where $H(q^{-1}) = 1 + h_1 q^{-1} + \ldots + h_{n^*} q^{-n^*}$ is an arbitrary polynomial restricted only to have all zeros inside the unit circle. □

The indirect proof of this conjecture by Soderstrom [43] confirmed its validity when

$$(n_b+1) - n_c \geqslant 0, \qquad (2.2.14)$$

i.e., when the number of free parameters in the numerator of the filter is at least as great as the number of poles of the unknown linear system. An alternative proof has been given in Appendix B. Note that this condition is always satisfied for the first-order case. Hence, no local minimum exists for first-order IIR adaptive filters of sufficient order. However, for the second-order case, violation of this condition seems to result in no contradiction to Stearns' observation. In fact, the only situations in which the condition can be violated are when $n_b=n_d=0$ and $n_a=n_c=2$. Then, the second-order filter $\frac{b_0}{1+a_1z^{-1}+a_2z^{-2}}$ is of sufficient order to identify the unknown linear system $\frac{d_0}{1+c_1z^{-1}+c_2z^{-2}}$ which does not satisfy (2.2.14). Despite the fact that Soderstrom's proof does not cover this case, it was shown by Stearns in [12] only by examples that this second-order case has a unimodal error surface. This behavior can be proven by Theorem 2.2 as follows. To find a degenerated solution we let $b_0 = 0$ in (2.2.2). Equation (2.2.2a) is trivially satisfied and (2.2.2b) is reduced to

$$E\left[\frac{d_0}{1+c_1z^{-1}+c_2z^{-2}}x(n) \cdot \frac{1}{1+a_1z^{-1}+a_2z^{-2}}x(n)\right] = 0. \tag{2.2.15}$$

Since $d_0 \neq 0$, (2.2.15) can be written equivalently as

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{1+a_1z+a_2z^2} \cdot \frac{1}{1+c_1z^{-1}+c_2z^{-2}} \Phi_{xx} \frac{dz}{z} = 0, \tag{2.2.16}$$

where $\Phi_{xx}=1$ for white x(n). Using the residue theorem, (2.2.16) is reduced to

$$1 - a_2c_2 = 0 . \tag{2.2.17}$$

Since $|c_2|<1$, (2.2.17) implies that $|a_2|>1$ which corresponds to an unstable solution. Therefore, there exists no degenerated point within the stability region such that (2.2.2) holds.

However, if we consider a third-order filter, the conclusion is different (see [59],[60]). Let the unknown system be given by the transfer function $\frac{d_0}{1+c_1z^{-1}+c_2z^{-2}+c_3z^{-3}}$ which is being identified by a filter of the form $\frac{b_0}{1+a_1z^{-1}+a_2z^{-2}+a_3z^{-3}}$. According to Theorem 2.2, if (2.2.2) has a stable solution, multiple minima exist on the MSE surface. When $b_0 = 0$, Equation (2.2.2b) can be equivalently written as

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{1+a_1z+a_2z^2+a_3z^3} \cdot \frac{1}{1+c_1z^{-1}+c_2z^{-2}+c_3z^{-3}} \frac{dz}{z} = 0, \tag{2.2.18}$$

which can be evaluated by the residue theorem. This leads to the condition that

$$c_3^2a_3^2-(a_1c_1c_3-c_1c_2+2c_3)a_3-a_2c_2+a_1a_2c_3+1 = 0. \qquad (2.2.19)$$

This is a three-dimensional surface in the $a_1-a_2-a_3$ plane, and if this surface has a nonempty intersection with the stability region shown in Fig. 2.1 (see [61] for details), then multiple minima exist. Let us suppose that $c_1 = -2.4$, $c_2 = 1.91$ and $c_3 = -0.504$, which correspond to the unknown system having poles at 0.7, 0.8 and 0.9. A stable solution is easily found to be

$$a_1=a_2=0, \quad a_3=0.2854$$

which indicates the existence of the saddle point (0,0,0.2854, 0) on the error surface. Figure 2.2 (courtesy of Prof. Fan [62]) shows 3-D contours of the normalized reduced error surface [12] as a function of a_1, a_2 and a_3. Two separate bodies are evident which indicate the presence of two minima. The global minimum is at $(a_1,a_2,a_3) = (-2.4,1.91,-0.504)$ with zero normalized MSE, and

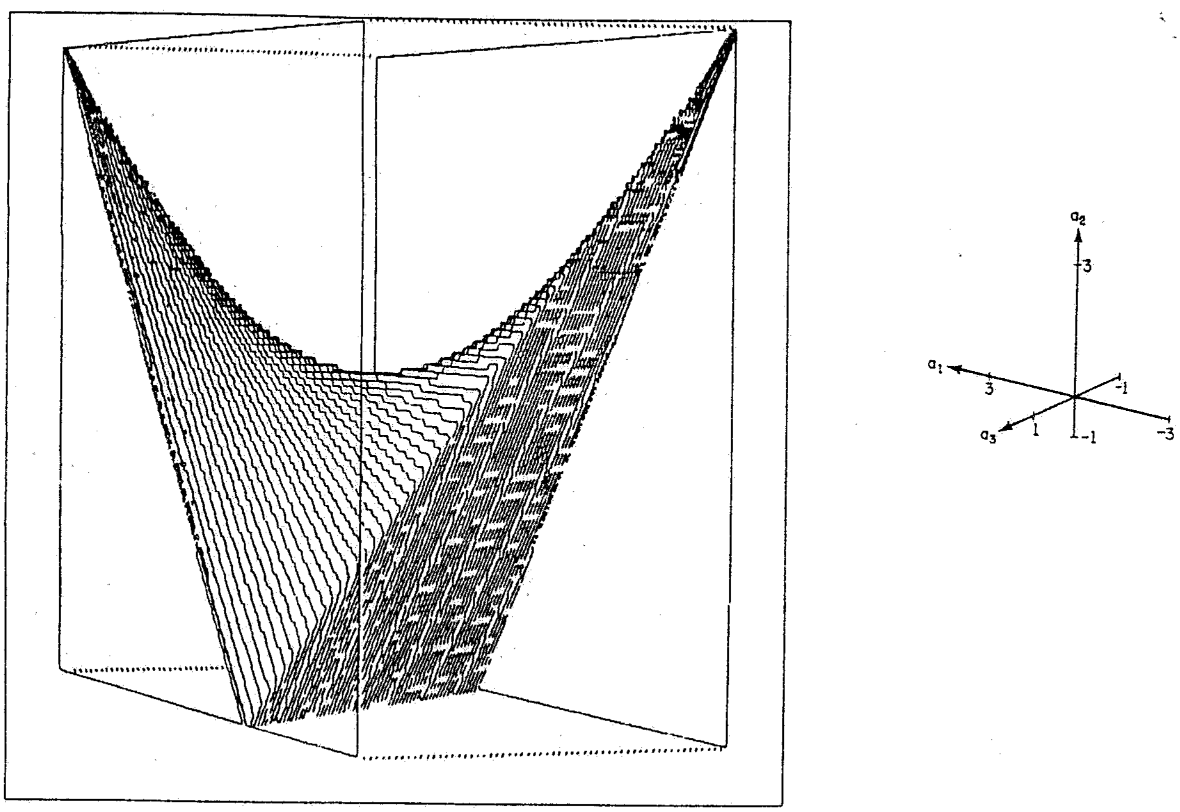
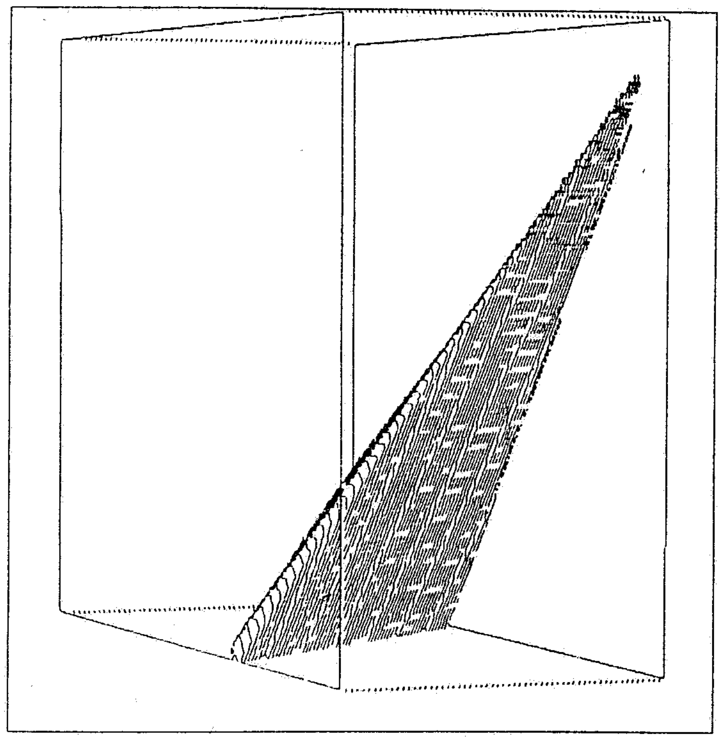
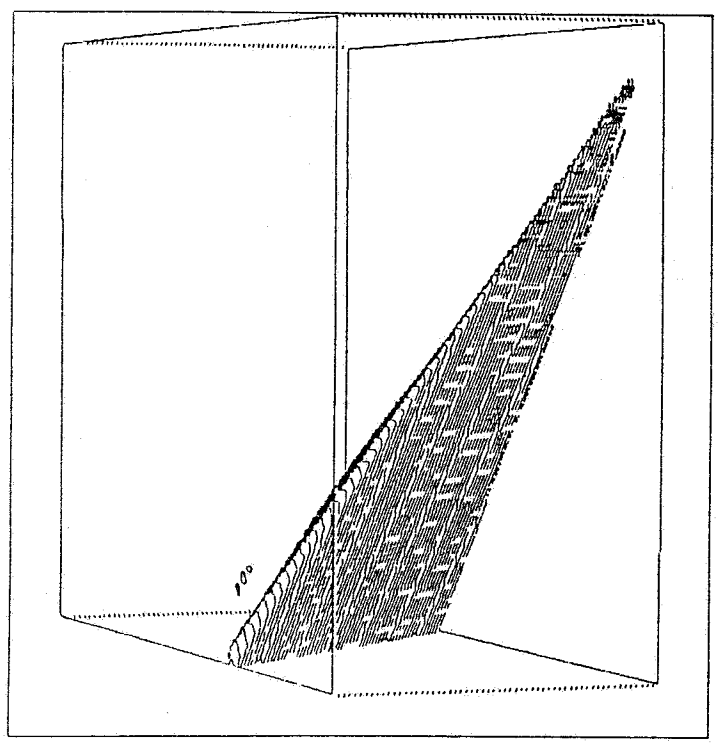


Fig. 2.1. The stability region of a third-order polynomial viewed with the indicated axis directions.

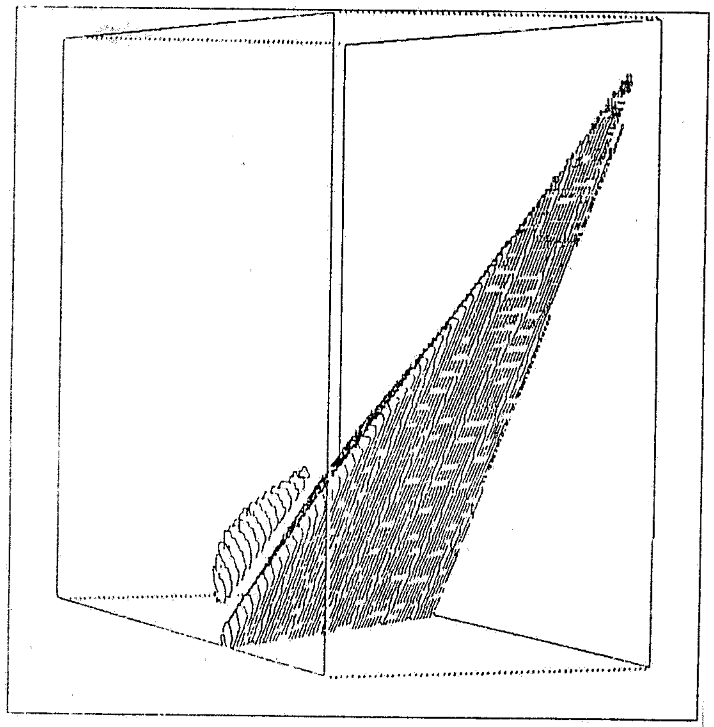


(a) NMSE = 0.99948

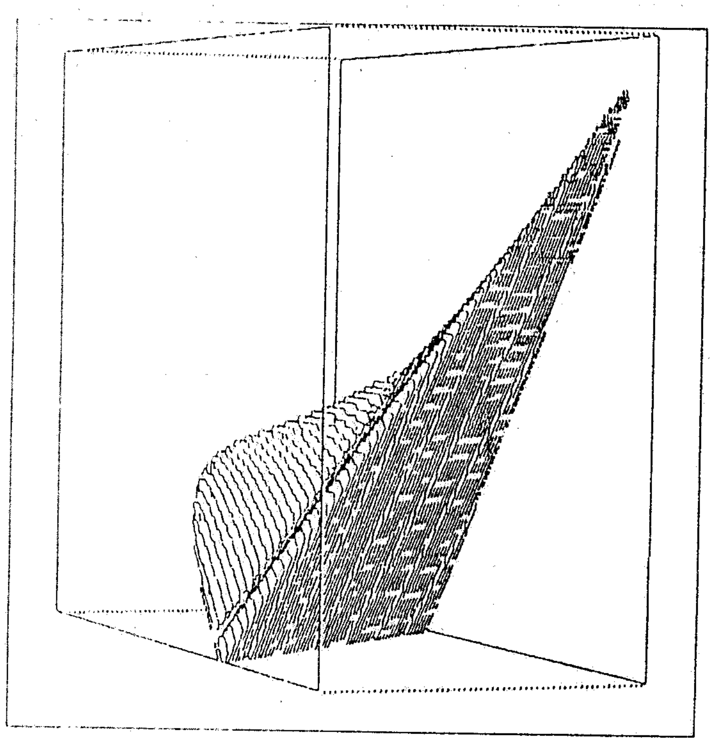


(b) NMSE = 0.99950

Fig. 2.2. The 3-D error contours of the counterexample with normalized MSE values shown.



(c) NMSE = 0.99965



(d) NMSE = 0.99990

Fig. 2.2. Continued.

the local minimum point is approximately at $(-0.6563, -0.4727, 0.8167)$ with normalized MSE of approximately 0.999488. Note that the whole 3-D surface is contained in the stability region. The surface of saddle points (2.2.19) intersects the stability region and separates the two bodies in Fig. 2.2. □

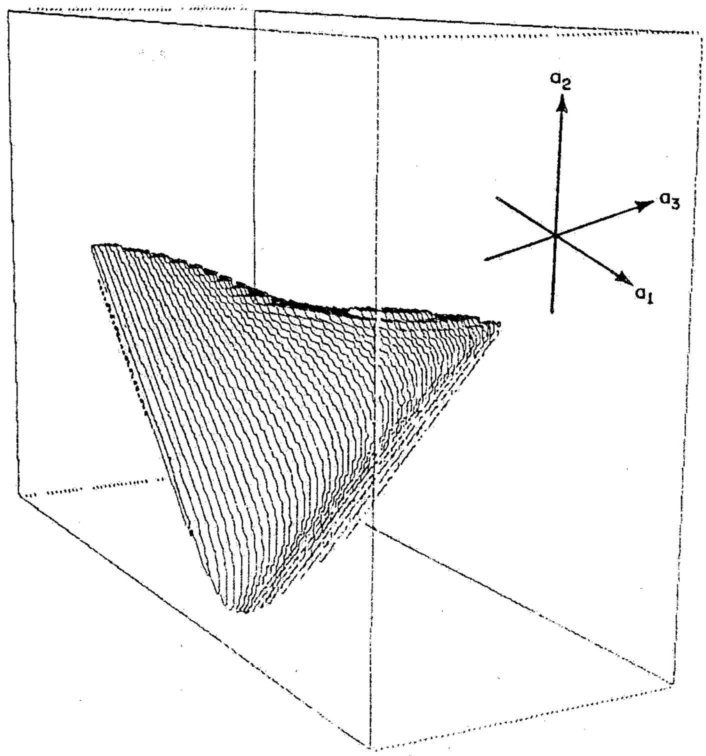
A deeper local minimum is expected as the order of the unknown system increases and the left-hand side of (2.2.14) grows more negative.

2.2.2. Overparameterization

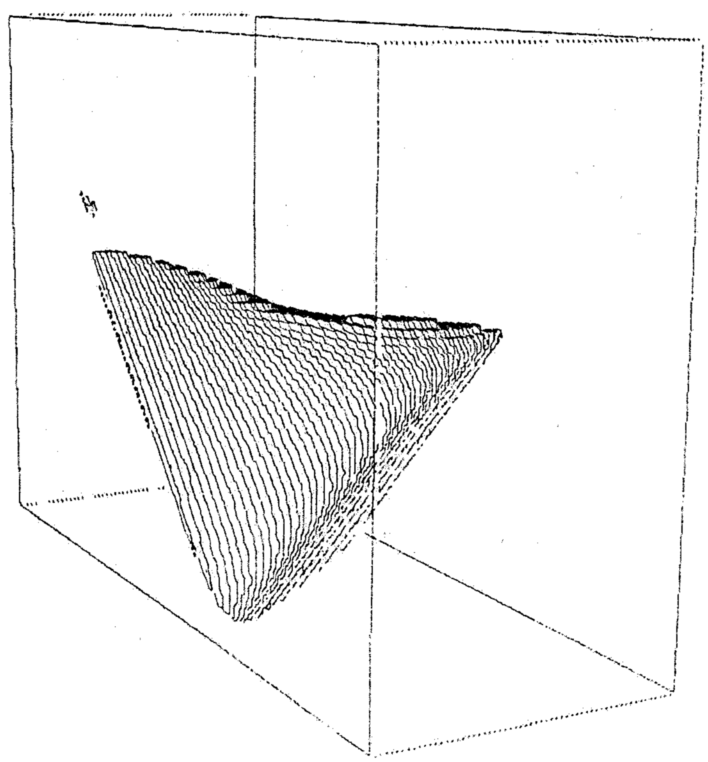
When there is uncertainty in the order of the unknown system, which is usually the case, the tendency is to overparameterize the adaptive filter. As Stearns' conjecture suggests, the overparameterization should not cause any problem for practical purposes as long as Soderstrom's condition is satisfied as well. Otherwise, local minima may appear on the error surface. To see this, consider the counterexample which was just introduced. In the general form, let $c_3 = 0$ in which case a second-order unknown system is being identified with a third-order adaptive filter. The analysis of the degenerated points given previously is still valid and all that needs to be done is to set c_3 equal to zero. This reduces (2.2.19) to

$$c_1 c_2 a_3 - a_2 c_2 + 1 = 0 \,, \tag{2.2.20}$$

which is a hyperplane parallel to the a_1 axis for a given c_1 and c_2. Depending on how this hyperplane cuts the 3-D stability region shown in Fig. 2.1, we could have two, one or no local minima on the error surface. For example, if $c_1 = -0.2$ and $c_2 = 0.82$, then Fig. 2.3 (courtesy of Prof. Fan [62]) shows that the hyperplane partitions the stability region into three parts. In each part, there is a minimum, two of which are local minima.

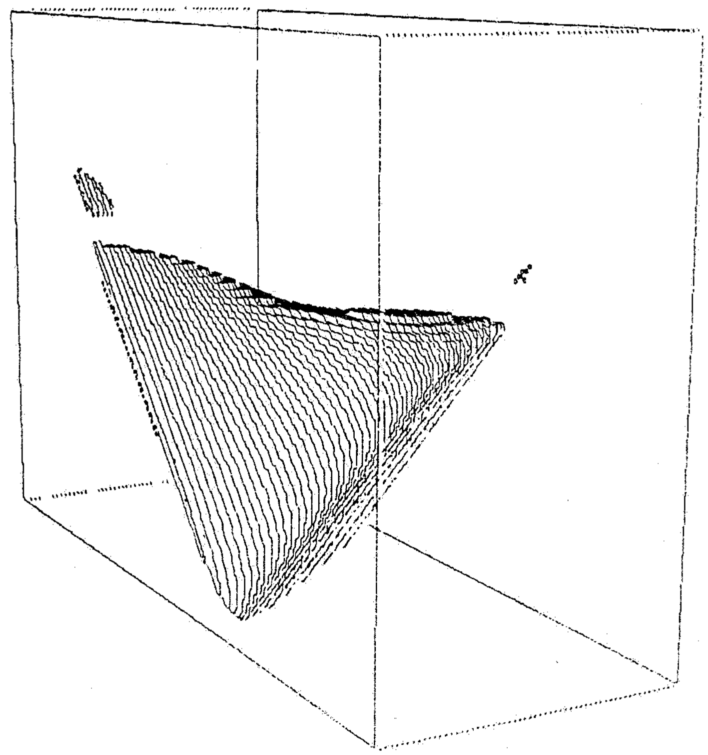


(a) NMSE = 0.9950

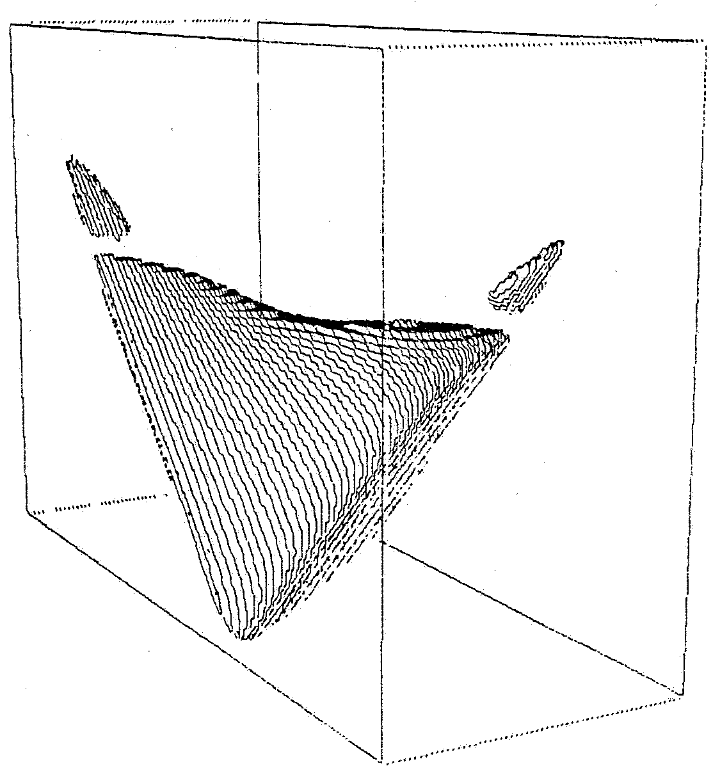


(b) NMSE = 0.9955

Fig. 2.3. The 3-D error contours of the overparameterization example with normalized MSE values shown.

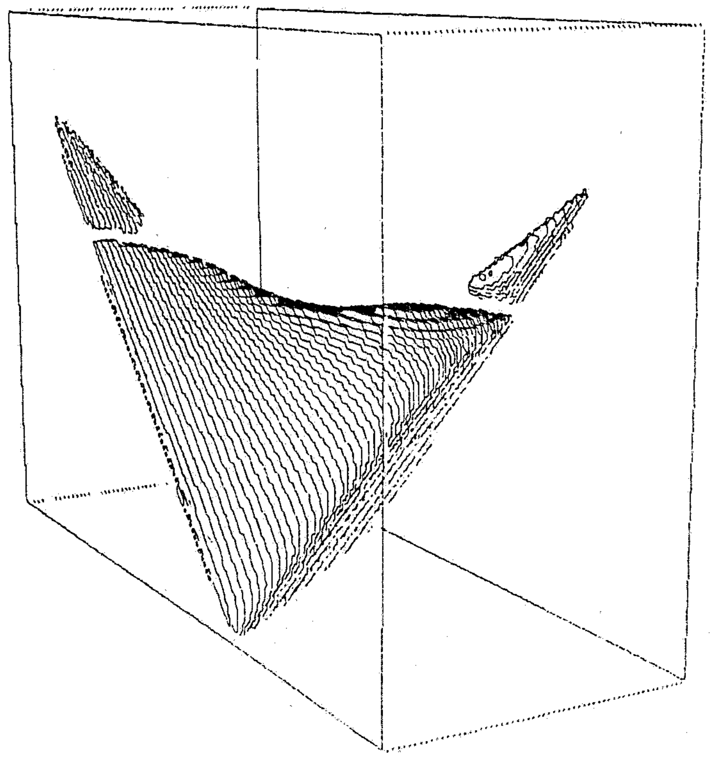


(c) NMSE = 0.9965

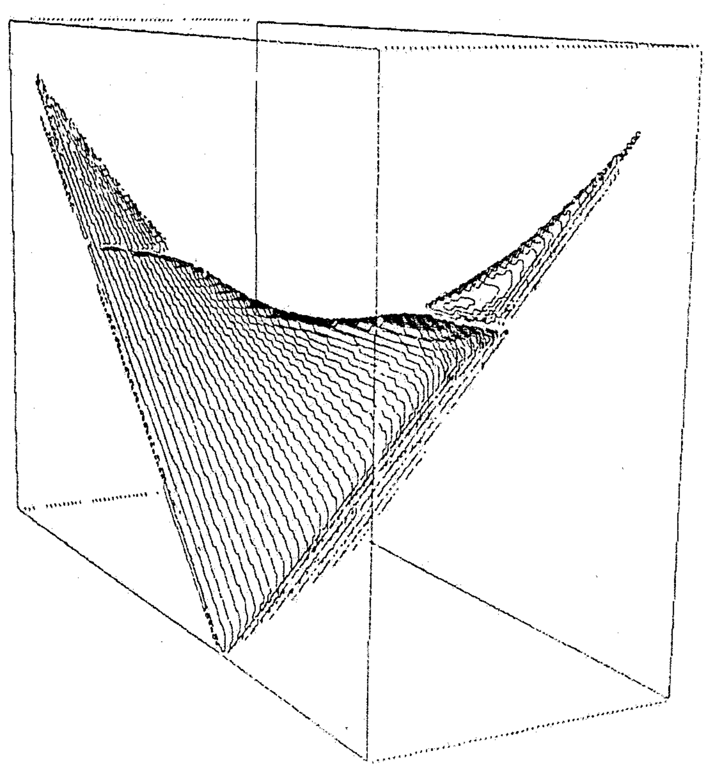


(d) NMSE = 0.9980

Fig. 2.3. Continued.



(e) NMSE = 0.9990



(f) NMSE = 0.9999

Fig. 2.3. Continued.

2.3. Insufficient Order Filters

To simplify the task of analyzing error surfaces, Theorem 2.2 can also be applied to insufficient order filters where $n^* < 0$. It provides the necessary tool to establish conditions for existence of multiple minima even for insufficient order filters. We now give two examples which are analyzed for comparison purposes using both the conventional method and Theorem 2.2.

Example 2.1:

Suppose that the unknown system in Fig. 1.3 is given by $d_0 + d_1 z^{-1} + d_2 z^{-2}$ and an adaptive IIR filter of the form $\frac{b_0 + b_1 q^{-1}}{1 - aq^{-1}}$ is used to minimize the mean square error. Equation (2.2.2) yields

$$E \begin{bmatrix} \frac{b_0 + b_1 q^{-1}}{(1 - aq^{-1})^2} x(n-1) \\ \frac{1}{1 - aq^{-1}} x(n) \\ \frac{1}{1 - aq^{-1}} x(n-1) \end{bmatrix} \left[(d_0 + d_1 q^{-1} + d_2 q^{-2} - \frac{b_0 + b_1 q^{-1}}{1 - aq^{-1}}) x(n) \right] = 0, \qquad \text{(E2.1.1)}$$

which can also be written as

$$\begin{bmatrix} 0 & b_0 & b_1 \\ 1 & -a & 0 \\ 0 & 1 & -a \end{bmatrix} \mathbf{R} \ \mathbf{h} = 0, \qquad \text{(E2.1.2)}$$

where

$$\mathbf{R} = E \begin{bmatrix} \frac{1}{(1 - aq^{-1})^2} x(n) \\ \frac{1}{(1 - aq^{-1})^2} x(n-1) \\ \frac{1}{(1 - aq^{-1})^2} x(n-2) \end{bmatrix} \left[\frac{1}{1 - aq^{-1}} x(n) \quad \frac{1}{1 - aq^{-1}} x(n-1) \quad \frac{1}{1 - aq^{-1}} x(n-2) \quad \frac{1}{1 - aq^{-1}} x(n-3) \right],$$

and

$$\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} d_0-b_0 \\ d_1-ad_0-b_1 \\ d_2-ad_1 \\ -ad_2 \end{bmatrix}.$$

First assume that $ab_0 + b_1 \neq 0$ so that in (E2.1.2) $\mathbf{R}\,\mathbf{h} = \mathbf{0}$. This simply implies that $\mathbf{h}$ is in the null space of $\mathbf{R}$. Here $x(n)$ is a white process with unit variance. Then, it is found that

$$\mathbf{R} = \frac{1}{(1-a^2)^2} \begin{bmatrix} 1 & a(2-a^2) & a^2(3-2a^2) & a^3(4-3a^2) \\ a & 1 & a(2-a^2) & a^2(3-2a^2) \\ a^2 & a & 1 & a(2-a^2) \end{bmatrix},$$

and

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = -h_3 \begin{bmatrix} 1 & a(2-a^2) & a^2(3-2a^2) \\ a & 1 & a(2-a^2) \\ a^2 & a & 1 \end{bmatrix}^{-1} \begin{bmatrix} a^3(4-3a^2) \\ a^2(3-2a^2) \\ a(2-a^2) \end{bmatrix} = h_3 \begin{bmatrix} 0 \\ a^2 \\ -2a \end{bmatrix}.$$

Therefore

$$\mathbf{h} = \gamma \begin{bmatrix} 0 \\ a^2 \\ -2a \\ 1 \end{bmatrix}, \qquad \text{(E2.1.3)}$$

for some γ. But $\mathbf{h}$ is given in (E2.1.2) and can be expressed as

$$\mathbf{h} = d_0 \begin{bmatrix} 1 \\ -a \\ 0 \\ 0 \end{bmatrix} + d_1 \begin{bmatrix} 0 \\ 1 \\ -a \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -a \end{bmatrix} - \begin{bmatrix} b_0 \\ b_1 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(E2.1.4)}$$

Consequently, equating (E2.1.3) to (E2.1.4) results in the following equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & -a^2 \\ 0 & -a & 1 & 2a \\ 0 & 0 & -a & -1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \gamma \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(E2.1.5)}$$

Using Cramer's rule it is found that

$$d_0 = b_0 \qquad d_1 = \frac{(ab_0+b_1)}{(1-a^2)^2}(1-2a^2)$$

$$d_2 = \frac{(ab_0+b_1)}{(1-a^2)^2}a \qquad \gamma = -\frac{(ab_0+b_1)}{(1-a^2)^2}a^2.$$

Obviously, $\frac{d_2}{d_1} = \frac{a}{1-2a^2}$ where $\frac{a}{1-2a^2}$ is the monotonically increasing function shown in Fig. 2.4. When $|\frac{d_2}{d_1}| < 1$ there is a unique stable solution for a in the (-1,1) interval which represents the global minima. Otherwise, if $|\frac{d_2}{d_1}| > 1$, there are two solutions for a which correspond to two minima.

Now consider the case where $ab_0 + b_1 = 0$, which corresponds to the degenerated point solution (Theorem 2.2), $B(q^{-1}) = L(q^{-1})A(q^{-1})$. In this case (E2.1.2) yields

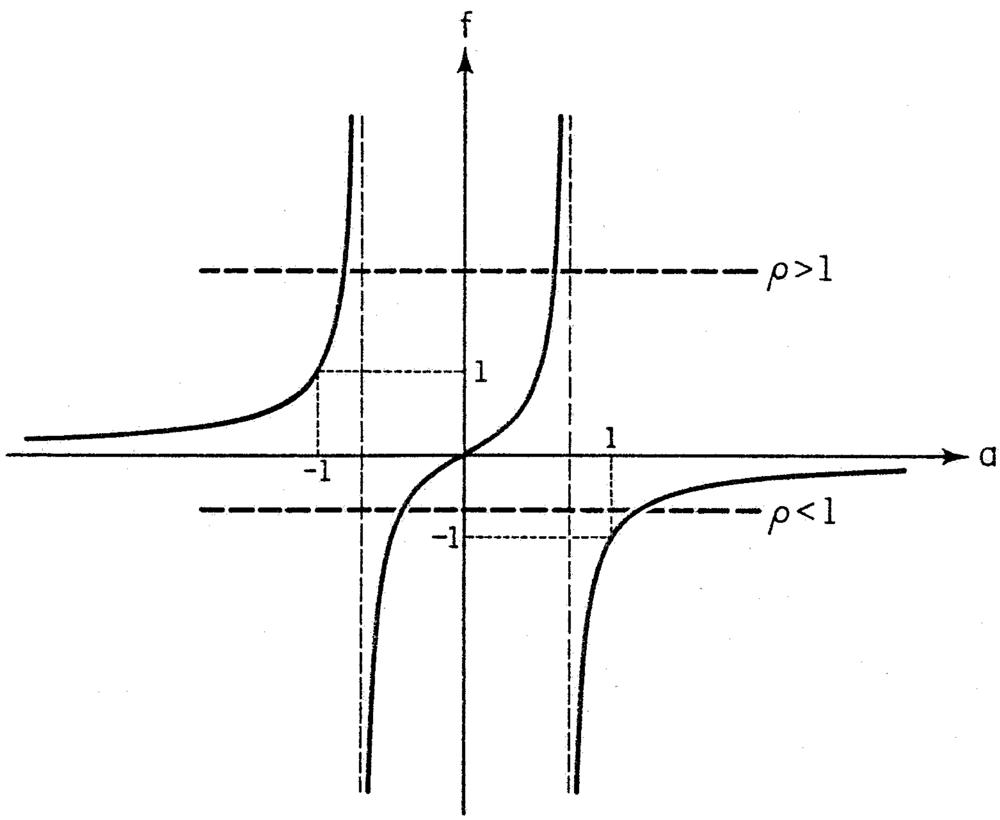


Fig. 2.4. Existence of stable solutions in terms of the model parameter in Example 2.1 where $\rho \triangleq \frac{d_2}{d_1}$.

$$d_1 + ad_2 = 0, \qquad b_0 = d_0+ad_1+a^2d_2. \tag{E2.1.6}$$

Since a should be stable, there is a solution to the first equation in (E2.1.6) if there is a local minimum. The condition for this to happen is $|\frac{d_2}{d_1}| > 1$ which is precisely what we just concluded from the tedious calculations above. Then, the second equation implies that $b_0 = d_0$. Notice the pole-zero cancellation which occurs for this case. □

Example 2.2:

Suppose that the unknown system in Fig. 1.3 is given by $d_0 + d_1z^{-1}$ and an adaptive IIR filter of the form $\frac{b}{1-a_1q^{-1}+a_2q^{-2}}$ is used to minimize the mean square error. Setting $\nabla_\theta E\, e^2 = 0$, we get

$$E\begin{bmatrix} \frac{b}{(1-a_1q^{-1}+a_2q^{-2})^2}\, x(n-1) \\ \frac{b}{(1-a_1q^{-1}+a_2q^{-2})^2}\, x(n-2) \\ \frac{1}{1-a_1q^{-1}+a_2q^{-2}}\, x(n) \end{bmatrix}\left[(d_0+d_1q^{-1} - \frac{b}{1-a_1q^{-1}+a_2q^{-2}})\, x(n)\right] = 0, \tag{E2.2.1}$$

which is equivalently written as

$$\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 1 & -a_1 & a_2 \end{bmatrix} \mathbf{R}\ \mathbf{h} = 0, \tag{E2.2.2}$$

where

$$\mathbf{R} = E\begin{bmatrix} \frac{1}{(1-a_1q^{-1}+a_2q^{-2})^2}\, x(n) \\ \frac{1}{(1-a_1q^{-1}+a_2q^{-2})^2}\, x(n-1) \\ \frac{1}{(1-a_1q^{-1}+a_2q^{-2})^2}\, x(n-2) \end{bmatrix}\left[\frac{1}{1-a_1q^{-1}+a_2q^{-2}}x(n) \quad \cdots \quad \frac{1}{1-a_1q^{-1}+a_2q^{-2}}x(n-3)\right],$$

and

$$\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} d_0 - b \\ d_1 - a d_0 \\ a_2 d_0 - a_1 d_1 \\ a_2 d_1 \end{bmatrix}.$$

Let us first assume that $b \neq 0$ so that in (E2.2.2) $\mathbf{R}\,\mathbf{h} = \mathbf{0}$. This simply implies that $\mathbf{h}$ should be in the null space of $\mathbf{R}$. If we represent $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_{-1} & r_0 & r_1 & r_2 \\ r_{-2} & r_{-1} & r_0 & r_1 \end{bmatrix}, \tag{E2.2.3}$$

then it is found after numerous polynomial manipulations that

$$r_{-2} = \frac{a_1^2 - a_2(1 + a_2)^2}{\Delta} \qquad r_{-1} = \frac{a_1(1 - a_2^2)}{\Delta}$$

$$r_0 = \frac{(1 + a_2)^2 - a_1^2 a_2}{\Delta} \qquad r_1 = \frac{2a_1(1 + a_2) - a^3}{\Delta}$$

$$r_2 = \frac{(a_2^3 - 2a_2)(1 + a_2)^2 + a_1^2(-2a_2^3 + 4a_2 + 3) + a_1^4(a_2 - 2)}{\Delta}$$

$$r_3 = \frac{a_1(2a_2^5 + 5a_2^4 - 9a_2^2 - 6a_2) + a_1^3(-4a_2^3 - 2a_2^2 + 8a_2 + 4) + a_1^5(2a_2 - 3)}{\Delta},$$

where

$$\Delta = \left[(1 - a_2)(1 + a_1 + a_2)(1 - a_1 + a_2)\right]^2. \tag{E2.2.4}$$

Note that Δ is the square of the product of the stability triangle sides associated with the second-order adaptive filter. Now, since $\mathbf{R}\,\mathbf{h} = \mathbf{0}$, it is obvious that $\mathbf{h}$ is in the null space of $\mathbf{R}$. Then

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = -h_3 \begin{bmatrix} r_0 & r_1 & r_2 \\ r_{-1} & r_0 & r_1 \\ r_{-2} & r_{-1} & r_0 \end{bmatrix}^{-1} \begin{bmatrix} r_3 \\ r_2 \\ r_1 \end{bmatrix} = h_3 \begin{bmatrix} a_1 a_2 (a_2 - 2) \\ a_1^2 - a_2(a_2^2 - 2) \\ -2a_1 \end{bmatrix}, \tag{E2.2.5}$$

and

$$\mathbf{h} = \gamma \begin{bmatrix} a_1 a_2 (a_2 - 2) \\ a_1^2 - a_2(a_2^2 - 2) \\ -2a_1 \\ 1 \end{bmatrix}, \tag{E2.2.6}$$

for some γ. Another way to derive the same result for **h** is to write $\mathbf{R}\,\mathbf{h} = \mathbf{0}$ as

$$\begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_{-1} & r_0 & r_1 & r_2 \\ r_{-2} & r_{-1} & r_0 & r_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma \end{bmatrix}, \tag{E2.2.7}$$

and then apply Cramer's rule to solve for h_i in terms of γ:

$$h_0 = -\frac{\begin{vmatrix} r_1 & r_2 & r_3 \\ r_0 & r_1 & r_2 \\ r_{-1} & r_0 & r_1 \end{vmatrix}}{\Lambda}\gamma \qquad h_1 = \frac{\begin{vmatrix} r_0 & r_2 & r_3 \\ r_{-1} & r_1 & r_2 \\ r_{-2} & r_0 & r_1 \end{vmatrix}}{\Lambda}\gamma \qquad h_2 = -\frac{\begin{vmatrix} r_0 & r_1 & r_3 \\ r_{-1} & r_0 & r_2 \\ r_{-2} & r_{-1} & r_1 \end{vmatrix}}{\Lambda}\gamma \qquad h_3 = \gamma,$$

where

$$\Lambda = \begin{vmatrix} r_0 & r_1 & r_2 \\ r_{-1} & r_0 & r_1 \\ r_{-2} & r_{-1} & r_0 \end{vmatrix}. \tag{E2.2.8}$$

Then, taking the inverse of the matrix in (E2.2.5) will not be necessary. It was found that

$$\Lambda = \left[(1 - a_2)^2 \cdot \Delta\right]^{-1}. \tag{E2.2.9}$$

But **h** is given in (E2.2.2) and therefore could be expressed as

$$\mathbf{h} = d_0 \begin{bmatrix} 1 \\ -a_1 \\ a_2 \\ 0 \end{bmatrix} + d_1 \begin{bmatrix} 0 \\ 1 \\ -a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{E2.2.10}$$

Consequently, equating (E2.2.6) to (E2.2.10) results in the following equation

$$\begin{bmatrix} 1 & 0 & -a_1 a_2 (a_2 - 2) \\ -a_1 & 1 & -a_1^2 + a_2(a_2^2 - 2) \\ a_2 & -a_1 & 2a_1 \\ 0 & a_2 & -1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \gamma \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{E2.2.11}$$

In order to have a consistent system of equations in (E2.2.11), we first found that there is a special relationship between a_1 and a_2 given by

$$a_1^2 = a_2 (1 + a_2)^2. \tag{E2.2.12}$$

Then, it is straightforward to solve (E2.2.11) to obtain

$$d_0 = \frac{b\,(1-2a_2)}{(1-a_2)^3\,(1+a_2)} \qquad d_1 = \frac{b\,a_2}{a_1\,(1-a_2)^3\,(1+a_2)} \qquad \gamma = a_2\,d_1. \tag{E2.2.13}$$

Using (E2.2.12) and (E2.2.13), it is found that

$$\left[\frac{d_1}{d_0}\right]^2 = \frac{a_2}{[\,(1+a_2)(1-2a_2)\,]^2}.$$

From (E2.2.12) we conclude that the admissible a_2 is positive and $a_2 < 1$ by the stability condition. Figure 2.5 shows the plot of the function $\frac{a_2}{[\,(1+a_2)(1-2a_2)\,]^2}$. Then, the intersection of the line $\left[\frac{d_1}{d_0}\right]^2$ with this plot on (0,1) determines how many solutions there are for a_2. Obviously, if $\left|\frac{d_1}{d_0}\right| < \frac{1}{2}$ there is a unique solution. Otherwise, there are two solutions which

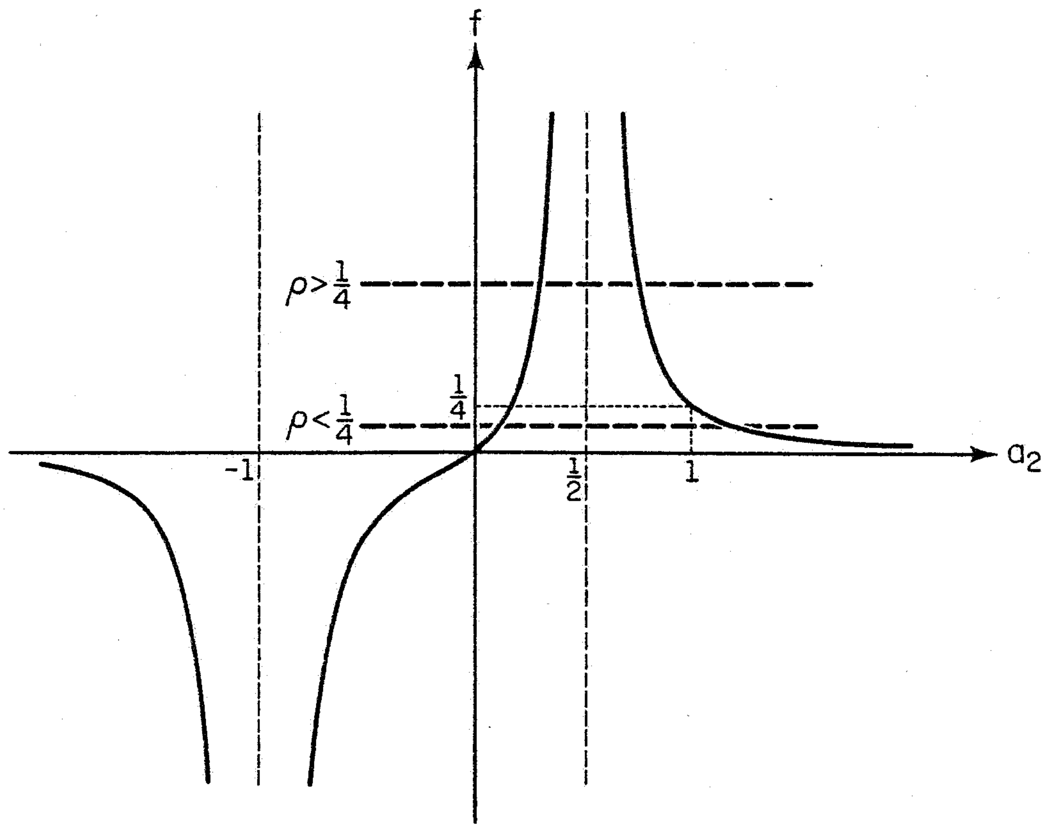


Fig. 2.5. Existence of stable solutions in terms of the model parameter in Example 2.2 where $\rho \triangleq \frac{d_1}{d_0}$.

correspond to two minima. Note that this result confirms the observation in [12].

Now, if we consider the case where b = 0 in (E2.2.2) corresponding to the degenerated point solution, then it is a matter of simple computation to find that $a_1 = -\frac{d_0}{d_1}$. Since a_1 is in the interval (-2,2) by stability requirements, we conclude that an acceptable solution for a_1 exists iff $|\frac{d_1}{d_0}| > \frac{1}{2}$, i.e., when there are multiple minima. □

In order to establish a systematic method to find multiple minima on MSE surfaces, we can exploit Theorem 2.2 even further in obtaining conditions, with respect to the unknown system characteristics, which guarantee the existence of multiple minima. These conditions will then be used to demonstrate that Example 2.1, for the insufficient order case, can be studied in a simpler way.

Theorem 2.3:

Consider the system (1.1.7) in which x(n) is white and $n^* < 0$.

1. When $n_b = n_a - 1$, stable degernerated solutions exist if $D(z^{-1})$ has at least (n_b+1) zeros outside the unit circle. For the case in which $n_a = 1$ and $n_b = 0$, this means that the error surface has multiple minima.

2. When $n_b \geq n_a$, let $L^*(z^{-1})$ satisfy (2.2.12). Then, stable degenerated solutions exist if $[D(z^{-1}) - L^*(z^{-1})C(z^{-1})]$ has at least (n_b+1) zeros outside the unit circle. For the case in which $n_a = n_b = 1$, this means that the error surface has multiple minima.

Proof:

Equation (2.2.8) at the solutions $A^*(z^{-1})$ and $L^*(z^{-1})$ could be rewritten as

$$0=\frac{1}{2\pi i}\oint_{|z|=1}\frac{1}{A^*(z^{-1})}\left[\frac{D(z)}{C(z)}-L^*(z)\right]z^{-j}\,\frac{dz}{z}\ ,\ 0\leqslant j\leqslant n_b$$

$$=\frac{1}{2\pi i}\oint_{|z|=1}\frac{1}{z^{n_a}A^*(z^{-1})}\ \frac{[D(z)-L^*(z)C(z)]}{C(z)}\ \frac{z^{\max(0,p)}}{z^{\max(0,-p)}}\ z^{n_b-j}\,dz,\ 0\leqslant j\leqslant n_b \tag{2.3.1}$$

where $p=n_a-n_b-1$.

Note that (2.2.7) is embedded in (2.3.1). Now, we use the result of Astrom and Soderstrom [44] which is of central importance in analyzing Eq. (2.3.1).

Lemma 2.1:[44]

Let $f(z)=\frac{g(z)}{h(z)}$, where g(z) is analytic inside and on the unit circle and h(z) is a polynomial of degree n with all zeros strictly inside the unit circle. Assume that

$$\frac{1}{2\pi i}\oint_{|z|=1} f(z)z^k\,dz=0\ ,\quad 0\leqslant k\leqslant (n-1)\,.$$

Then *all* zeros of h(z) are also zeros of g(z), i.e., f(z) is analytic inside the unit circle. □

From Equation (2.3.1), it is observed that

$$\text{number of poles} = n_a+\max(0,n_b-n_a+1)$$
$$= \max(n_a,n_b+1)$$
$$\text{number of equations} = (n_b+1)$$

Since the number of poles should be less than or equal to the number of equations if we wish to apply Lemma 2.1, then we require

$$\boxed{(n_b+1)-n_a\geqslant 0}$$

We now distinguish two cases:

1. $\mathbf{n_b = n_a - 1}$

   As discussed before, $B^*(q^{-1})\equiv 0$ or equivalently $L^*(q^{-1})\equiv 0$ for this case. Thus $\left[\frac{1}{z^{n_a}A^*(z^{-1})}\cdot\frac{D(z)}{C(z)}\right]$, in Equation (2.3.1), should be analytic inside the unit circle. This can occur only when D(z) has at least n_a zeros inside the unit circle so that the cancelled poles are stable ones. Thus we conclude that if $D(z^{-1})$ has at least n_a zeros *outside* the unit circle, stable degenerated solutions exist. Note that this excludes $n^* \geqslant 0$, in which case, $D(z^{-1})$ can not have n_a zeros because $n_a > n_b \geqslant n_d$. Exclusion of sufficient order filters is further justified by Soderstrom's condition being automatically satisfied in this case since $(n_b+1) = n_a \geqslant n_c$. Existence of a stable degenerated solution in the special case in which $n_a = 1$ and $n_b = 0$ implies multimodality of the error surface according to Theorem 2.2.

2. $\mathbf{n_b \geqslant n_a}$

Note that $L^*(q^{-1})$ is not necessarily zero and its solution is found by (2.2.12). But (2.3.1) is reduced to

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{z^{(n_b-n_a+1)}\, z^{n_a} A^*(z^{-1})} \cdot \frac{[D(z)-L^*(z)C(z)]}{C(z)} z^{n_b-j} dz = 0 \ , \ 0 \leqslant j \leqslant n_b \, . \qquad (2.3.2)$$

We then conclude that $[D(z)-L^*(z)C(z)]$ must have an (n_b-n_a+1)th order zero at z=0 and that the sufficient condition for the existence of a stable $A^*(z^{-1})$ is that among the remaining zeros of $D(z)-L^*(z)C(z)$ at least n_a of them are inside the unit circle. Note that since $\deg(D(z)-L^*(z)C(z)) = n_b-n_a-n^*$ and since $n^*<0$, the degree requirement is automatically satisfied. Application of Theorem 2.2 implies multimodality of the error surface when $n_a = n_b = 1$. □

Example 2.3:

Let us use the adaptive filter $\frac{b}{1-az^{-1}}$ to find the MMSE estimate of the stable unknown system $\frac{d_0+d_1z^{-1}}{1+c_1z^{-1}+c_2z^{-2}}$. Since $n_b = 0$ and $n_a = 1$, by Theorem 2.3, the function

$$\left[\frac{1}{z-a^*} \cdot \frac{d_0+d_1z}{1+c_1z+c_2z^2} \right] \qquad (E2.3.1)$$

has to be analytic inside the unit circle. Let $d_1 \neq 0$. If $D(z^{-1}) = d_0 + d_1 z^{-1}$ has its zero *outside* the unit circle, i.e., $|\frac{d_0}{d_1}| < 1$, then $a^* = -\frac{d_0}{d_1}$ is a stable degenerated solution. This guarantees the existence of a local minimum. For instance, if the unknown system is given by $\frac{0.05-0.4z^{-1}}{1-1.131z^{-1}+0.25z^{-2}}$, Fig. 2.6 shows the normalized error surface which already appeared in [15]. On the other hand, if $d_1=0$, then (E2.3.1) is analytic only if $|a^*| > 1$ and no stable degenerated point exists. The unimodal characteristic of the error surface when the unknown system is given by $\frac{0.05}{1-1.75z^{-1}+0.81z^{-2}}$ was shown in [15]. □

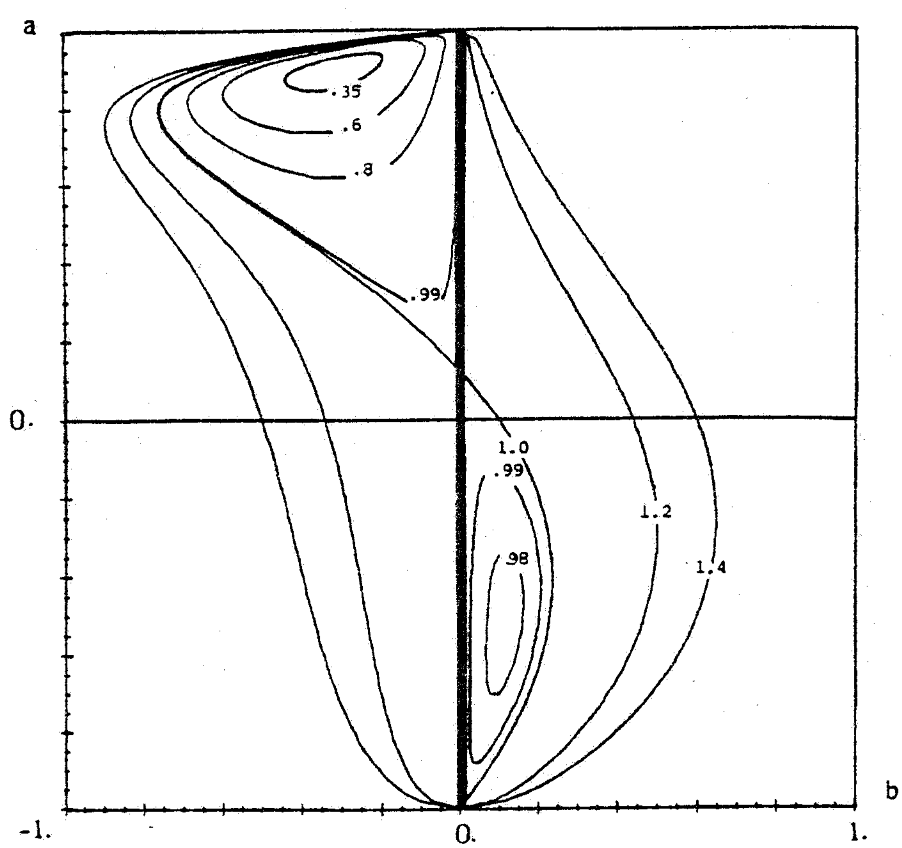


Fig. 2.6. Normalized error contours of Example 2.3. All the degenerated points are shown by the heavy line.

Example 2.4:

Consider Example 2.1 in the last section. From the identity $B^*(z^{-1})=L^*(z^{-1})A^*(z^{-1})$, we find that $L^*(z^{-1})=b_0^*$ and $a^*b_0^*+b_1^*=0$. Since $D(z)-L^*(z)C(z)=(d_0-b_0^*)+d_1z+d_2z^2$ has a first-order zero at $z=0$, it is concluded that $b_0^*=d_0$. (Consistent with Eq. (2.2.12).) Also, $(d_1z+d_2z^2)$ has a nonzero root inside the unit circle if $|\frac{d_1}{d_2}|<1$ which implies that $\frac{d_1z+d_2z^2}{z-a^*}$ is analytic inside the unit circle with $a^*=-\frac{d_1}{d_2}$ and $|a^*|<1$. Thus, multiple minima exist if $|\frac{d_1}{d_2}|<1$. This is consistent with the development in Example 2.1. It is seen that if $|\frac{d_1}{d_2}|>1$, then no stable degenerated solution exists and Example 2.1 shows the unimodal characteristic of the error surface. □

We now introduce the following conjecture based on the examples that we have studied. This conjecture provides a necessary condition for multimodality (and a necessary and sufficient condition for unimodality) of error surfaces of adaptive filters with white noise input.

Conjecture:

If no stable degenerated solution of (2.2.2) exists with white noise input x(n), the error surfaces of the adaptive filters are unimodal.

2.4. Colored Inputs

The formulation in [41] is also valid for the colored input x(n) as long as it is persistently exciting of sufficient order. Thus, Theorem 2.2 in which $n_a > n_b = 0$ holds for this class of colored inputs. A sufficient order multimodal example was given in [41]. An insufficient order example can be easily constructed by generalizing the example in [41], as given in Example 2.5.

Example 2.5: [22]

Let the unknown system in Fig. 1.3 be $\frac{1}{(1-\alpha z^{-1})^3}$, the adaptive filter be $\frac{b}{1+a_1q^{-1}+a_2q^{-2}}$, and $x(n)$ be generated by a white noise process $\nu(n)$ through $x(n) = (1-\alpha q^{-1})^2(1+\alpha q^{-1})^2\nu(n)$. Then, it can be shown that the degenerated stationary points for $b=0$ are given by

$$\alpha^2 a_1^2 - 2\alpha a_1 + \alpha^3 a_1 a_2 - 3\alpha^2 a_2 + 4(1-\alpha^4)^2 - 3 = 0 \qquad \text{(E2.5.1)}$$

for $|\alpha| < 1$. This is a hyperbola which intersects the stability region for $\alpha = 0.6$, thus giving a multimodal error surface (see Fig. 2.7). Note that for some other values of α, (e.g., $\alpha = 0.1$) the hyperbola does not intersect the stability region. This gives another example of a unimodal surface for insufficient order cases. □

Although, for the general case in which $n_a \leqslant n_b$, a stable degenerated solution of the form $B(q^{-1})=L(q^{-1})A(q^{-1})$ can be shown to be a saddle point, multimodality of the error surface does not follow. Therefore, we have to find all the stationary points and to examine each for minimality. Under some conditions, it is possible to give some expressions which facilitate this task. We now explore this possibility.

The stationary points of the MSE function satisfy (2.2.2a) and (2.2.2b) which alternatively can be written as

$$S(B,A)\,E\begin{bmatrix} \frac{1}{A(q^{-1})A(q^{-1})}\,x(n) \\ \vdots \\ \frac{1}{A(q^{-1})A(q^{-1})}\,x(n-m) \end{bmatrix}\left[\frac{1}{C(q^{-1})A(q^{-1})}\,x(n)\cdots\frac{1}{C(q^{-1})A(q^{-1})}\,x(n-r)\right]\begin{bmatrix} h_0 \\ \vdots \\ h_r \end{bmatrix} = 0\,, \qquad \text{(2.4.1)}$$

where

$$m = n_a + n_b\ , \quad r = \max(n_a + n_d\, , n_c + n_b)\,,$$

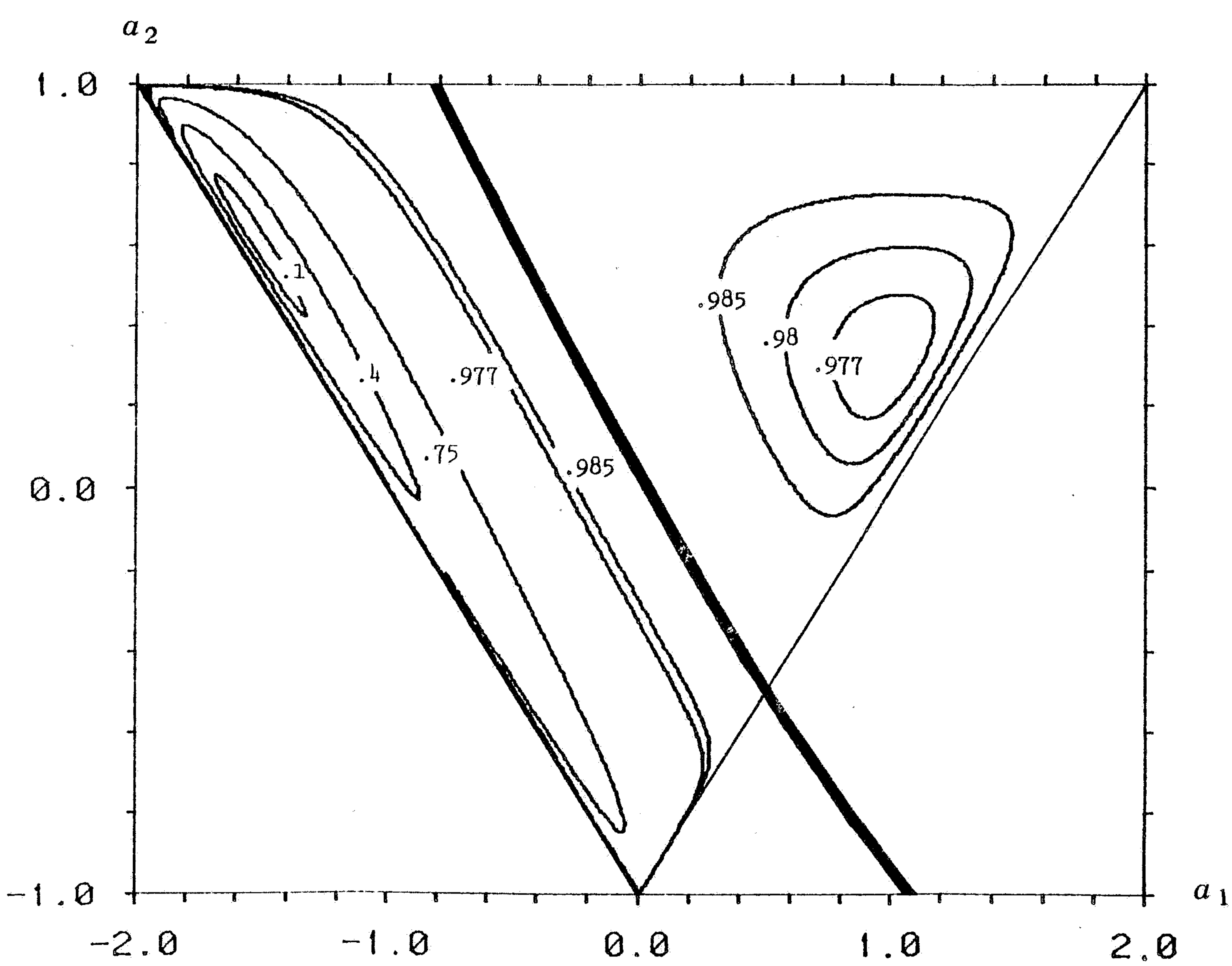


Fig. 2.7. Normalized reduced error contours of Example 2.5 for insufficient order filter with colored noise input (α=0.6). All the degenerated point solutions are shown by the heavy line.

$$\sum_{i=0}^{r} h_i q^{-i} \triangleq D(q^{-1})A(q^{-1}) - B(q^{-1})C(q^{-1}) .$$

and

$$S(B,A) = \begin{bmatrix} 0 & b_0 & b_1 & \cdots & b_{n_b} & & & \\ & 0 & b_0 & b_1 & \cdots & b_{n_b} & & \\ & & & \ddots & & & & \\ & & & 0 & b_0 & b_1 & \cdots & b_{n_b} \\ 1 & a_1 & a_2 & \cdots & a_{n_a} & & & \\ & 1 & a_1 & a_2 & \cdots & a_{n_a} & & \\ & & & \ddots & & & & \\ & & & 1 & a_1 & a_2 & \cdots & a_{n_a} \end{bmatrix}$$

is an $(m + 1) | (m + 1)$ Sylvester matrix. This formulation was used to find the stationary points in Examples 2.1 and 2.2. Note that although this formulation is valid in all cases, we consider the insufficient order filters, i.e., cases where $n^* < 0$. Let A_0 and B_0 represent a stationary point of the MSE function and also assume that they are coprime. Then $S(B_0, A_0)$ is nonsingular and h is in the null space of R, the cross-correlation matrix in (2.4.1).

Let x(n) be an ARMA process given by

$$G(q^{-1})\, x(n) = F(q^{-1})\, \nu(n),$$

where $\nu(n)$ is white noise and the polynomials

$$F(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_{n_f} z^{-n_f},$$
$$G(z^{-1}) = 1 + g_1 z^{-1} + \ldots + g_{n_g} z^{-n_g},$$

are relatively prime. Assume that $A(z^{-1})$ and $F(z^{-1})$ have no common zero. Then, (2.4.1) is equivalent to

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{A_0^2(z^{-1})} \frac{H(z)}{A_0(z)C(z)} \phi_{xx}(z) z^{-j} \frac{dz}{z} , \quad 0 \leqslant j \leqslant m \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{n_f} F(z^{-1})\, F(z)}{z^{2n_a} A_0^2(z^{-1}) z^{n_g} G(z^{-1})} \cdot \frac{H(z)}{C(z)G(z)} \frac{z^{\max(0,p)}}{z^{\max(0,-p)}} z^{m-j} dz , \quad 0 \leqslant j \leqslant m \end{aligned} \qquad (2.4.2)$$

where

$$H(z) \triangleq A_0(z)D(z) - B_0(z)C(z),$$

and

$$p = n_a + n_g - n_b - n_f - 1.$$

In (2.4.2), we have

$$\text{number of poles} = 2n_a + n_g + \max(0, -p) ,$$

$$\text{number of equations} = m+1 = n_a + n_b + 1 .$$

To use Lemma 2.1, the number of poles must be less than or equal to the number of equations. This leads to the conditions

$$\begin{aligned} n_g &\leqslant (n_b+1) - n_a, \\ n_f &= 0. \end{aligned} \tag{2.4.3}$$

Fulfillment of the conditions in (2.4.3) and application of Lemma 2.1 require that $\Gamma(z) \triangleq \dfrac{H(z)}{z^{2n_a}A_0^2(z^{-1}) \cdot z^{n_g}G(z^{-1})}$ be analytic inside the unit circle. This implies that all the poles are cancelled by the zeros of H(z); hence, $\Gamma(z)$ becomes a polynomial in z. If we denote

$$z^{2n_a+n_g}A_0^2(z^{-1})G(z^{-1}) \triangleq \sum_{i=0}^{2n_a+n_g} \alpha_i z^i,$$

and

$$H(z) \triangleq \sum_{i=0}^{m-n^*} h_i z^i,$$

and

$$\Gamma(z) \triangleq \sum_{i=0}^{n_b-n_a-n_g-n^*} \gamma_i z^i ,$$

then, using the result by Kailath [63] regarding polynimial multiplication in the matrix form, we have

$$h = T(\gamma)\alpha, \tag{2.4.4}$$

where $\alpha^T = [\alpha_0 \cdots \alpha_{2n_a+n_g}]$, $h^T = [h_0 \cdots h^*_{m-n}]$ and $T(\gamma)$ is an $(m-n^*+1) \mid (2n_a+n_g+1)$ lower triangular Toepiitz matrix with the first column being $[\gamma_0 \cdots \gamma_{n_b-n_a-n_g-n^*} 0 \cdots 0]^T$. The vector h given by

(2.4.4) is in the null space of the matrix R. This null space is obviously of dimension $(-n^*)$.

From the identity

$$H(z) \triangleq A_0(z)D(z) - B_0(z)C(z) = \Gamma(z)\cdot[z^{2n_a}A_0^2(z^{-1})]\cdot[z^{n_g}G(z^{-1})], \tag{2.4.5}$$

we should solve $(m-n^*+1)$ equations in $(2n_b-n^*-n_g+1)$ unknowns A_0, B_0 and Γ. Example 2.1 shows the use of h in solving for the unknowns. Also, inspection of examples in [15] reveals the convenience of (2.4.5) in finding the stationary points.

In summary, when $n_b \geqslant n_a$, all the stationary points have to be found and determined to be minimum, maximum or saddle point. However, if (2.4.3) is satisfied then we can take advantage of the equality in (2.4.5) to find the solutions in a more convenient way.

The cases where $A_0(z^{-1})$ and $B_0(z^{-1})$ are not coprime follow similarly by dissolving the common factors in (2.2.2) and then using an equation similar to (2.4.1) to find the stationary points through similar analysis. It would be appropriate to mention that for all the examples of insufficient order case that we have studied, the minimum points are such that no pole zero cancellation occurs.

2.5. Summary

The four classes of error surfaces introduced in Chapter 1 can have local minima. In particular, class 1) which is associated with Stearns' conjecture was shown to be multimodal for a case when Soderstrom's condition (2.2.14) is not satisfied. The same conclusion could be made about class 2) whose examples can be found in [41]. The error surfaces associated with class 3) were investigated and, as Theorem 2.3 suggests, multimodality of some of these surfaces could be conveniently concluded when the unknown system is not minimum phase. Also, a multimodal error surface in class 4) was introduced and analyzed through the notion of degenerated points.

CHAPTER 3

ANALYSIS OF ALTERNATE REALIZATIONS OF ADAPTIVE IIR FILTERS

3.1. Introduction

In recent years there has been growing interest within the communications industry in using adaptive digital filters for noise cancellation, echo cancellation, and channel equalization. This interest is due in part to remarkable advances that have been made in VLSI digital hardware. Computationally powerful single-chip digital signal processors (known generically as DSPs) are now available for these types of applications. DSPs can be programmed for real-time operation and are quite convenient for applications in which space and power are particularly limited.

Noise cancellation, echo cancellation, and channel equalization all require that the adaptive filter adjust to match an unknown transfer function or system impulse response function. Hence, they are all variations of the system identification problem shown in Fig. 1.3. For example, in echo cancellation the filter adjusts to match the transhybrid characteristics. The adjustment of the adaptive filter to the unknown system can be achieved in different ways. The popular equation error and output error methods resulted in the development of many algorithms such as recursive least squares (RLS), least mean square (LMS), and hyperstable adaptive recursive filter (HARF), which have some desirable asymptotic properties. But the fast convergence and low computation requirements led to the modification of many of these algorithms. Hence, fast RLS (FRLS) and simple HARF (SHARF) were introduced. However, these modifications compromised on some very import convergence properties. The global convergence of SHARF disappeared and high sensitivity of FRLS to finite word length implementation surfaced. The trade-off between the convergence properties and computational complexity is viewed differently in each field. Traditionally, devising new algorithms to improve performance while keeping the structure of the filter fixed, mostly in direct form, has been the focus of most researchers.

Virtually all practical echo cancellers that have been reported in the literature use a finite impulse response (FIR) filter as the adaptive element. This is because FIR adaptive filters are well behaved, i.e., they are guaranteed to remain stable in the presence of many diverse channel conditions and they usually converge to a globally optimum condition because the error surface is unimodal [12]. However, from the viewpoint of computational complexity, the FIR structures are quite expensive. For example, when cancelling echoes in a satellite link with an FIR filter of length 128, it is difficult for a DSP to complete all the necessary computations within the limits of the standard 8 KHz sampling rate used for digital voice transmission.

When excessive computational complexity is a major issue, the use of infinite impulse response (IIR) adaptive filters becomes appealing. The poles of the IIR make it possible to obtain well-matched characteristics with a much lower-order structure, and, hence, with many fewer arithmetic operations. For example, a fifth-order IIR filter requiring 9 multiplies and 8 adds per output sample may match the unknown system as well as a 64^{th} order FIR filter that requires 64 multiplies and 63 adds per output sample. Unfortunately, an IIR adaptive structure may not remain well behaved during the adaptive process. Poles may move outside the unit circle, resulting in instability, or the adaptation may converge to a local minimum because the error surface may be multimodal. Therefore, it is important to monitor stability at each iteration and to select an algorithm which promises good convergence.

One of the notable features of IIR adaptive algorithms is their relatively slow convergence rate in comparison with that for FIR filters. Although the poles permit better noise cancellation with lower computational complexity after convergence, they cause sensitivities that inadvertently slow the convergent rate. The FIR class can be summarized as well behaved with a rapid convergence rate, but with high real-time computational requirements and limited abilities to cancel properly after convergence. The IIR class is less well behaved and converges more slowly, although its real-time computational requirements are lower and it has the potential to more effectively match the unknown system after convergence.

There are many published algorithms for adaptive IIRs realized as direct form digital filters. In this form the transfer characteristic is expressed in terms of single numerator and denominator polynomials, and the time domain implementation becomes a single high-order difference equation. The direct form adaptive IIR structure is very common throughout the literature on system identification within the discipline of adaptive control [47]. However, from the signal processing point of view, a decomposed filter structure consisting of parallel or cascaded first- and second-order sections is superior for two reasons: 1) Coefficient sensitivities of the parallel and cascade forms are much lower than that for the direct form [56],[57], and 2) it is simple to monitor stabilities for the parallel and cascade forms by examining the pole locations of the individual second-order sections. Property 1) suggests that the parallel and cascade forms can be more finely adjusted, will have smaller minimum mean-square error, and will be less sensitive to gradient noise and arithmetic quantization effects. Property 2) is extremely important in practice. A major deficiency of the direct form is that stability cannot be easily monitored because it requires that the high order denominator polynomial be factored in real time to observe the pole locations.

3.2. Example: The Parallel Form Realization

Suppose we wish to realize the adaptive filter of Fig. 1.3 as a parallel structure. It will be assumed that the original adaptive filter is characterized by an N^{th} order linear difference equation and that the coefficients of this difference equation are adaptively adjusted until $E\{e^2(n)\}$ is minimized. Figure 3.1 shows the decomposition of the adaptive filter into N/2 (assuming N is even) 2^{nd} order sections, where the i^{th} section is characterized by a second-order difference equation

$$y_i(n) = \sum_{j=0}^{2} b_{ij}\, x(n-j) - \sum_{m=1}^{2} a_{im}\, y_i(n-m). \tag{3.2.1}$$

The instantaneous error is given by

$$e(n) = d(n) - \sum_{i=1}^{N/2} y_i(n). \tag{3.2.2}$$

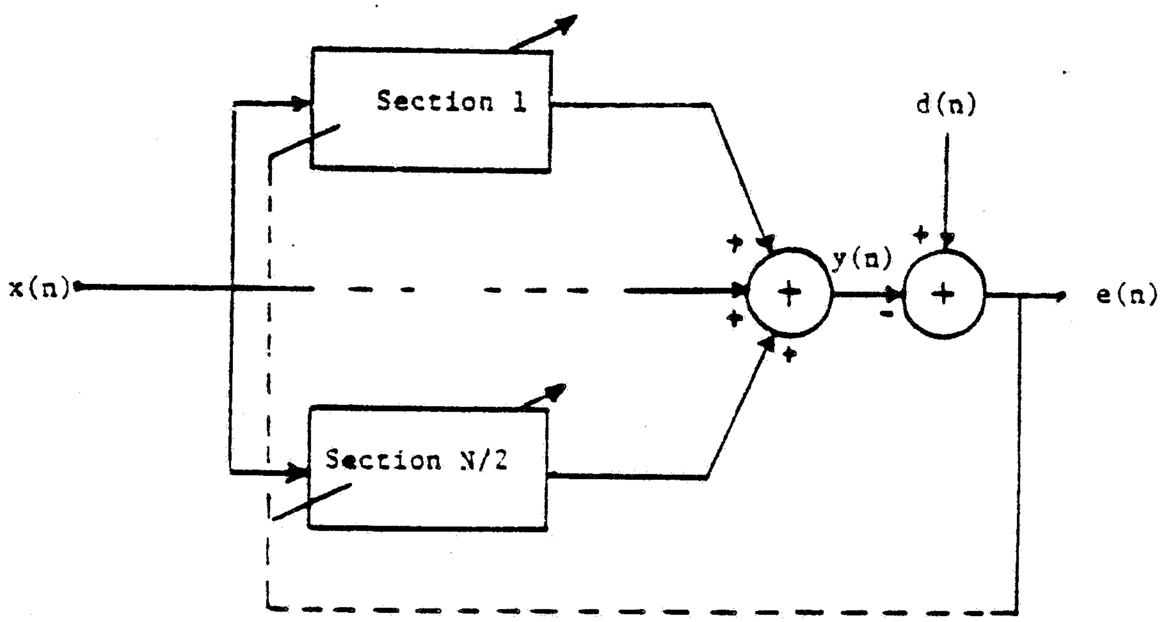


Fig. 3.1. Parallel form adaptive IIR filter.

In this algorithm the mean-square error $E\{e^2(n)\}$ is approximated by the instantaneous squared error $e^2(n)$. In order to use a gradient algorithm, it is necessary to find explicit expressions for

$$\partial e^2(n)/\partial a_{im} = 2\, e(n)[\partial e(n)/\partial a_{im}]$$

and

$$\partial e^2(n)/\partial b_{ij} = 2\, e(n)[\partial e(n)/\partial b_{ij}].$$

The necessary partial derivatives can be calculated from (3.2.2).

$$\frac{\partial e(n)}{\partial a_{im}} = \frac{-\partial y_i(n)}{\partial a_{im}} = y(n-m) + \sum_{r=1}^{2} a_{ir} \frac{\partial y_i(n-r)}{\partial a_{im}} \qquad (3.2.3a)$$

$$\frac{\partial e(n)}{\partial b_{ij}} = \frac{-\partial y_i(n)}{\partial b_{ij}} = -x(n-j) + \sum_{r=1}^{2} a_{ir} \frac{\partial y_i(n-r)}{\partial b_{ij}} , \tag{3.2.3b}$$

where $m = 1, 2$, $j = 0, 1, 2$, and $i = 1, ..., N/2$. Note that (3.2.3) represents two recursive relations for the sensitivities of $e(n)$ with respect to the a_{im}'s and b_{ij}'s, i.e., coefficients of the second-order sections. Using (3.2.3) the coefficient update equations become

$$a_{im}(n+1) = a_{im}(n) - 2\mu e(n)\frac{1}{A_i(q^{-1},n)} y_i(n-m)$$

$$b_{ij}(n+1) = b_{ij}(n) + 2\mu e(n)\frac{1}{A_i(q^{-1},n)} x(n-j) \tag{3.2.4}$$

where

$$m \in [1, 2] , \quad j \in [1, 2, 3] , \quad i \in [1, ..., N/2] ,$$
$$A_i(q^{-1}, n) = \left[1 + a_{i1}(n)q^{-1} + a_{i2}(n)q^{-2}\right] .$$

The variable q^{-1} is the unit time delay operator, and $\frac{1}{A_i(q^{-1},n)}$ is the inverse operator of $A_i(q^{-1},n)$.

Two numerical examples are presented here for the case in which the order of the adaptive filter matches that of the unknown system (sufficient order). In Example 3.1, the poles of the unknown system are not close to the unit circle, and both the direct form and the parallel form converge easily. Example 3.2 has poles that are close to the unit circle; in this case, the direct form fails to converge, whereas the parallel form converges quite readily.

Example 3.1:

This example considers identification of the third-order filter

$$H_p = \frac{2 - 1.7\, z^{-1} + 0.6\, z^{-2}}{1 - 1.7\, z^{-1} + 1.2\, z^{-2} + 0.3\, z^{-3}} \tag{E3.1.1}$$

whose parallel structure consists of first-order and one-second order transfer functions, $\frac{1}{1-0.5z^{-1}}$ and $\frac{1}{1 - 1.2\, z^{-1} + 0.6\, z^{-2}}$, respectively. This filter has poles at $z = 0.5$ and $z = 0.6 \pm j0.49$. The input x(n) is a unit variance white Gaussian pseudo-noise sequence chosen to ensure the richness of the excitation. According to Stearns' conjecture [12] and the development in Chapter 2, in particular (2.2.14), this gradient algorithm should converge to the true minimum MSE by virtue of unimodality of the error surface. Figure 3.2 shows the mean-square error $E\{e^2(n)\}$ starting initially from the origin in the parameter space for both direct and parallel structures. The function $E\{e^2(n)\}$ is estimated by averaging over a window of length 100. The step size μ is 3×10^{-3}, which was chosen after μ was varied over the course of many experiments. The direct form shows a faster convergence initially, although the parallel form starts to show less error after 2500 iterations. After 50k iterations the average square errors are of the order 10^{-2} and 10^{-4} for the direct and parallel structures, respectively. The respective adaptive filters at this point in time are

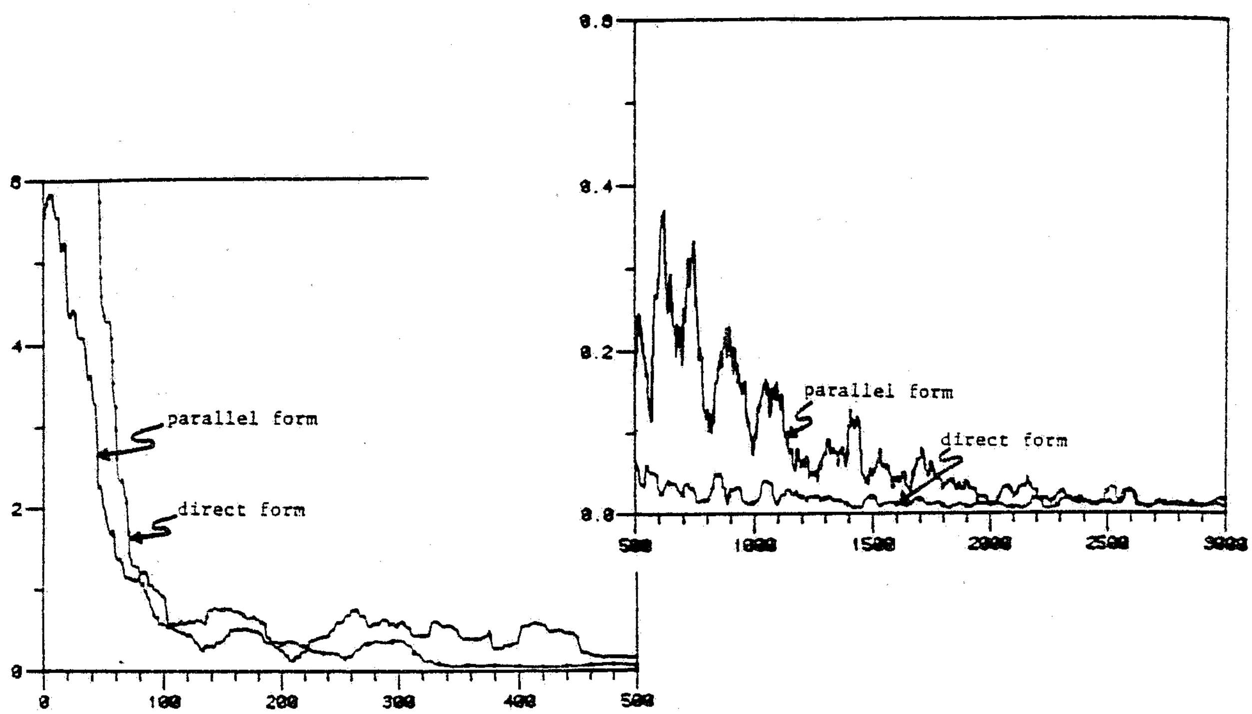


Fig. 3.2. Comparison of MSEs for parallel and direct forms for Example 3.1.

$$H_{direct} = \frac{1.996 + 0.63z^{-1} - 0.07z^{-2} + 0.23z^{-3}}{1 - 0.525z^{-1} - 0.22z^{-2} + 0.307z^{-3}} . \quad \text{(E3.1.2)}$$

and

$$H_{parallel} = \left[\frac{0.947 + 0.04z^{-1} + 0.03z^{-2}}{1 - 1.19\, z^{-1} + 0.60\, z^{-2}} + \frac{1.05 + 0.07z^{-1}}{1 - 0.44\, z^{-1}} \right] . \quad \text{(E3.1.3)}$$

It is likely that reduced gradient noise, which is directly related to the low coefficient sensitivity of the parallel structure, results in a smaller mean square error for the parallel algorithm.

Example 3.2:

Example 3.1 had poles well inside the unit circle. Now consider a model with poles at $z = 0.95$ and $z = 0.8 \pm j\,0.51$,

$$H_p = \left[\frac{1}{1 - 1.6z^{-1} + 0.9z^{-2}} + \frac{1}{1 - 0.95z^{-1}} \right], \quad \text{(E3.2.1)}$$

with a unit variance white Gaussian pseudo noise input, chosen to place the poles closer to the unit circle in the z-plane. Figure 3.3 represents the behavior of the adaptation of the parallel form measured by the mean square error, if the initial adaptive filter is selected to be

$$H_{initial} = \left[\frac{1}{1 - 1.2z^{-1} + 0.6z^{-2}} + \frac{1}{1 - 0.5z^{-1}} \right] \quad \text{(E3.2.2)}$$

with $\mu = 10^{-5}$. The average square error is of the order 10^{-4} after 50k iterations and the adaptive filter

$$H_{parallel} = \left[\frac{1 - 0.0006z^{-1} + 0.0005z^{-2}}{1 - 1.6z^{-1} + 0.90z^{-2}} + \frac{1 + 0.013z^{-1}}{1 - 0.950z^{-1}} \right] \quad \text{(E3.2.3)}$$

closely matches the model. The direct form was also attempted for this case, but convergence was not observed in 50k iterations. It is interesting to note that the maximum μ which stabilized the direct form adaptive (gradient) algorithm was on the order of 10^{-9} which indicates the relatively

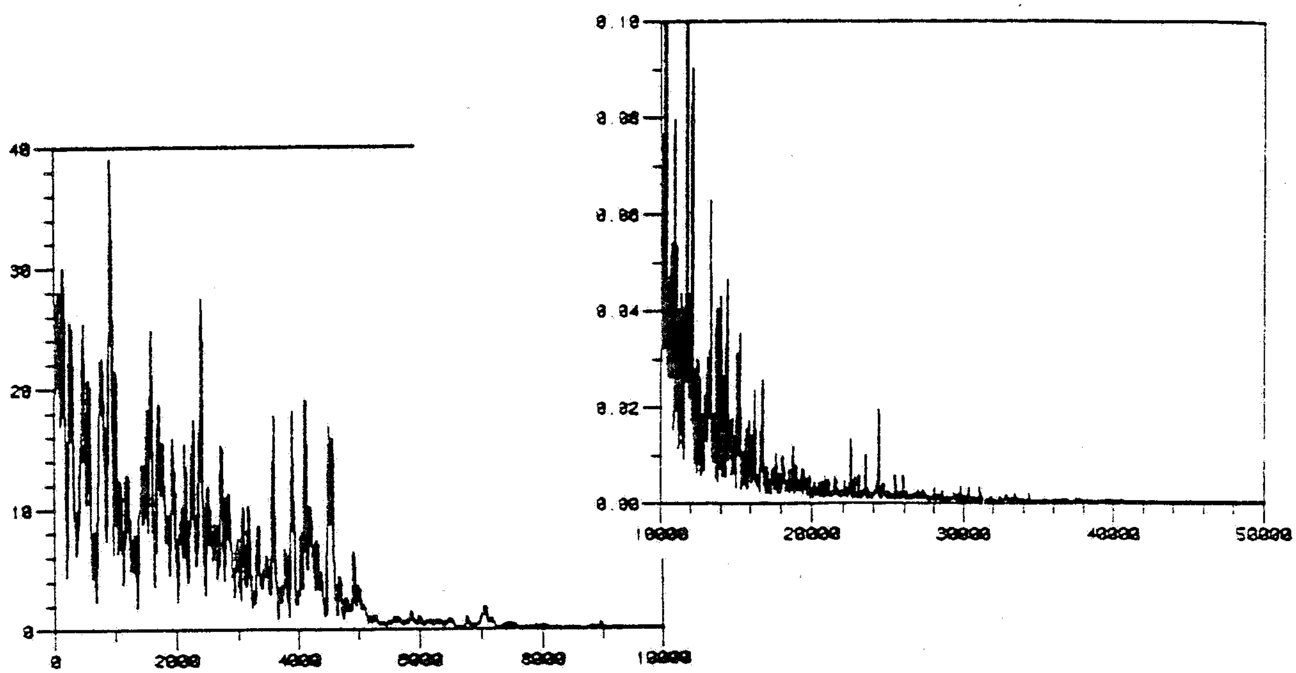


Fig. 3.3. MSE for parallel form for Example 3.2.

high coefficient sensitivity of this form when the poles are close to the unit circle. In contrast, the low sensitivity of the parallel structure has caused a considerable improvement of the simple gradient algorithm toward identification of the poles close to the unit circle. □

Shynk, Gooch and Widrow [10],[11] introduced a parallel structure of first-order sections whose inputs are the outputs of pass-band filters. Near orthogonality of the inputs reduces the dependencies of each Section and the associated Newton algorithm converges quite rapidly. However, complex computations are not desirable for practical purposes. Two important questions are raised regarding the parallel structure considered in section 3.3. 1) Is it possible for any two sections to converge to the same poles? 2) If yes, how do we prevent this from happening? Note that a reduction in the order of the adaptive filter results if 1) is possible. This is very undesirable since it then indicates the failure of the adaptive filter to identify the unknown system. In other words, the LMS is trapped in a local minimum. The complete nature of such solutions is discussed in the next section.

3.3. Equivalent Realizations of Adaptive IIR Filters

Convergence of the algorithm whose update equation is given by (3.2.4) to a filter structure in Fig. 3.1 where any two or more sections have the same poles can create serious problems in any application. This merely shows that a reduction in the overall transfer function has occurred and, hence, the adaptive filter has not been able to identify the unknown system. In this section we characterize this reduced-order structure as a point in the parameter space and analyze its characteristics. It will be concluded that such a point is a saddle point and that this should not cause any problem as long as the initial conditions are chosen suitably. The analysis is of course general enough to accommodate any realization.

Consider the update equation

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \mu \mathbf{F}_k \tag{3.3.1}$$

where $\boldsymbol{\theta}_k$ is the adjustable parameter vector, $\mathbf{F}_k$ is a vector whose direction determines the direction of updating $\boldsymbol{\theta}_k$, and μ is the step size. Each element of $\mathbf{F}_k$ is usually a function of $\boldsymbol{\theta}_k$. If (3.3.1) converges to a point $\boldsymbol{\theta}^*$, then the vector $\mathbf{F}_k$ is identically zero at $\boldsymbol{\theta}^*$. In other words, the stationary point $\boldsymbol{\theta}^*$ could be found, not necessarily uniquely, by solving $\mathbf{F}_k|_{\boldsymbol{\theta}^*} = 0$. Then $\boldsymbol{\theta}^*$ is called a *stable* point of (3.3.1) if any perturbation $\Delta\boldsymbol{\theta}$ introduced in $\boldsymbol{\theta}^*$ results in no difficulty for (3.3.1) to converge back to $\boldsymbol{\theta}^*$. If gradient techniques, in particular the LMS, are adopted, then

$$\mathbf{F}(\boldsymbol{\theta}) \equiv -\nabla_{\boldsymbol{\theta}} E[e^2] \,. \tag{3.3.2}$$

Therefore the equation $\nabla_{\boldsymbol{\theta}} E[e^2] = 0$ determines the stationary points of the gradient methods. Obviously, the minimum points of $E[e^2]$ are the only stable points.

The idea of stationary points is used now to analyze the behavior of the LMS algorithm when other realizations of the adaptive filter are used. Three important questions that must be answered are

1) What happens to the MSE, $E[e^2]$, error surface when other realizations are used? 2) Are new stationary points created as a result of such transformations? 3) Are any new stationary points so created stable or unstable points, i.e., are they minima, maxima, or saddle points? Note that the performance of algorithms that use gradient techniques is highly dependent upon the answers to the above questions. Until these questions are answered, the performance of these algorithms can not be predicted accurately in many situations. In the following discussion the adjustable system parameters of the direct form will be denoted $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_N]$. Next, consider an equivalent realization of the same system in terms of $\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_N]$, another parameter vector.

Definition:

i) Each *realization* can be considered as a continuous mapping $g \in \mathbb{R}^N \times \mathbb{C}^N$, $g: D \rightarrow R$ defined by

$$g(\boldsymbol{\alpha}) \triangleq \boldsymbol{\beta} \tag{3.3.2}$$

where $D \subset S_\alpha$ and $R \subset S_\beta$ and

$$S_x = \left\{ x \mid \text{The adjustable system is stable} \right\}$$

ii) Let $f: D \rightarrow \mathbb{R}$ and $h: R \rightarrow \mathbb{R}$ represent the MSE functional defined on D and R, respectively. By an equivalent realization it is meant that g is a realization such that $\forall\ \boldsymbol{\alpha} \in D$.

$$f(\boldsymbol{\alpha}) = h(g(\boldsymbol{\alpha})) = h(\boldsymbol{\beta}). \tag{3.3.3}$$

The equivalency property is, of course, redundant in signal processing since equivalent realizations of a given system automatically imply this property for any functionals f and h.

The nature of the error surfaces of sufficient order direct from filters with white noise and some classes of colored noise inputs is known (see [12], [43] and Chapter 2). It is important to know how the modality of these error surfaces changes under g.

The differentiability of g plays an important role in this analysis.

If g is differentiable, then by the chain rule it is found that

$$\begin{aligned}\frac{\partial f(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} &= \frac{\partial h(\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}} \text{(by equivalency property)} \\ &= \frac{\partial h(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \cdot \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\alpha}},\end{aligned} \tag{3.3.4}$$

where $\frac{\partial f}{\partial \boldsymbol{\alpha}}$ and $\frac{\partial h}{\partial \boldsymbol{\beta}}$ are N-row vectors and $\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\alpha}}$ is the (N×N) Jacobian matrix. Now if $\boldsymbol{\beta}^*$ is a stationary point of $h(\boldsymbol{\beta})$, then $\boldsymbol{\alpha}^*$ is a stationary point of $f(\boldsymbol{\alpha})$ where $g(\boldsymbol{\alpha}^*) = \boldsymbol{\beta}^*$. That is

$$\frac{\partial h}{\partial \boldsymbol{\beta}}\Big|_{\boldsymbol{\beta}^*} = 0 \Rightarrow \frac{\partial f}{\partial \boldsymbol{\alpha}}\Big|_{\boldsymbol{\alpha}^*} = 0 \tag{3.3.5}$$

where $g(\boldsymbol{\alpha}^*) = \boldsymbol{\beta}^*$. Note that $\boldsymbol{\alpha}^*$ exists since g is an onto mapping. Also since

$$\nabla^2_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}^*} = \left(\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\alpha}}\right)^H_{\boldsymbol{\beta}^*} \cdot \nabla^2_{\boldsymbol{\beta}} h(\boldsymbol{\beta})\Big|_{\boldsymbol{\beta}^*} \cdot \left(\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\alpha}}\right)_{\boldsymbol{\beta}^*} \tag{3.3.6}$$

the nature of the stationary point $\boldsymbol{\alpha}^*$ is that of $\boldsymbol{\beta}^*$'s.

Alternately, let there exist $\boldsymbol{\beta}^+ \in D$ such that $\frac{\partial h}{\partial \boldsymbol{\beta}}\Big|_{\boldsymbol{\beta}^+} = 0$ but $\frac{\partial f}{\partial \boldsymbol{\alpha}}\Big|_{\alpha} \neq 0$ where $\boldsymbol{\beta}^+ = g(\boldsymbol{\alpha}^+)$. This is the situation in which a stationary point is formed at $\boldsymbol{\beta}^+$ under the transformation g while $\boldsymbol{\alpha}^+$ is not a stationary point itself. Clearly (3.3.4) is not applicable; thus, g must be nondifferentiable at $\boldsymbol{\alpha}^+$.

Thus, we have Lemma 3.1.

Lemma 3.1:

The function g is nondifferentiable at any newly formed stationary point on $h(\boldsymbol{\beta})$.

Lemma 3.2 clearly determines the nature of such points.

Lemma 3.2:

All the newly formed stationary points are saddle points.

Proof: If $B(\boldsymbol{\alpha}^+, r)$ is an open ball centered at $\boldsymbol{\alpha}^+$ with the radius $r > 0$, then the image of B under g contains an open ball $B'(\boldsymbol{\beta}^+, r')$, $r' > 0$, since g is continuous. Because $\boldsymbol{\alpha}^+$ is not a minimum (or a maximum) in B, there are directions **d** where $f(\boldsymbol{\alpha}^+ + \mathbf{d}) < f(\boldsymbol{\alpha}^+)$. Since $(\boldsymbol{\alpha}^+ + \mathbf{d}) \in B$ then $g(\boldsymbol{\alpha}^+ + \mathbf{d}) \in B'$ and by equivalency property $h(g(\boldsymbol{\alpha}^+ + \mathbf{d})) < h(\boldsymbol{\beta}^+)$. □

To clarify, the following examples are helpful.

Example 3.3:

Consider the MSE identification problem of a system $H_p(z^{-1}) = \dfrac{2+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}-\frac{1}{3}z^{-2}}$ by the sufficient order filter $H_M(z^{-1}) = \dfrac{2+\alpha_1 z^{-1}}{1+\alpha_1 z^{-1}+\alpha_2 z^{-2}}$. Let the driving input, x(n), be a white process. Then, the error surface of $E[e^2]$, as a function of α_1 and α_2, is unimodal, as depicted in Fig. 3.4. No saddle point is observed on this surface.

Now, consider a parallel realization of $H_M(z^{-1})$ using two first-order sections $H_1(z^{-1}) = \dfrac{1}{1+\beta_1 z^{-1}}$ and $H_2(z^{-1}) = \dfrac{1}{1+\beta_2 z^{-1}}$ for the same identification problem. We can now define the mapping $g \in \mathbf{R}^2 \times \mathbb{C}^2$ as

$$g_1(\alpha_1, \alpha_2) = \frac{\alpha_1+\sqrt{\alpha_1^2 - 4\alpha_2}}{2} = \beta_1$$

$$g_2(\alpha_1, \alpha_2) = \frac{\alpha_1-\sqrt{\alpha_1^2 - 4\alpha_2}}{2} = \beta_2 \qquad \text{(E3.3.1)}$$

Obviously, the mapping g is continuous over the stability region and nondifferentiable only on the parabola $\alpha_2 = \frac{1}{4}\alpha_1^2$.

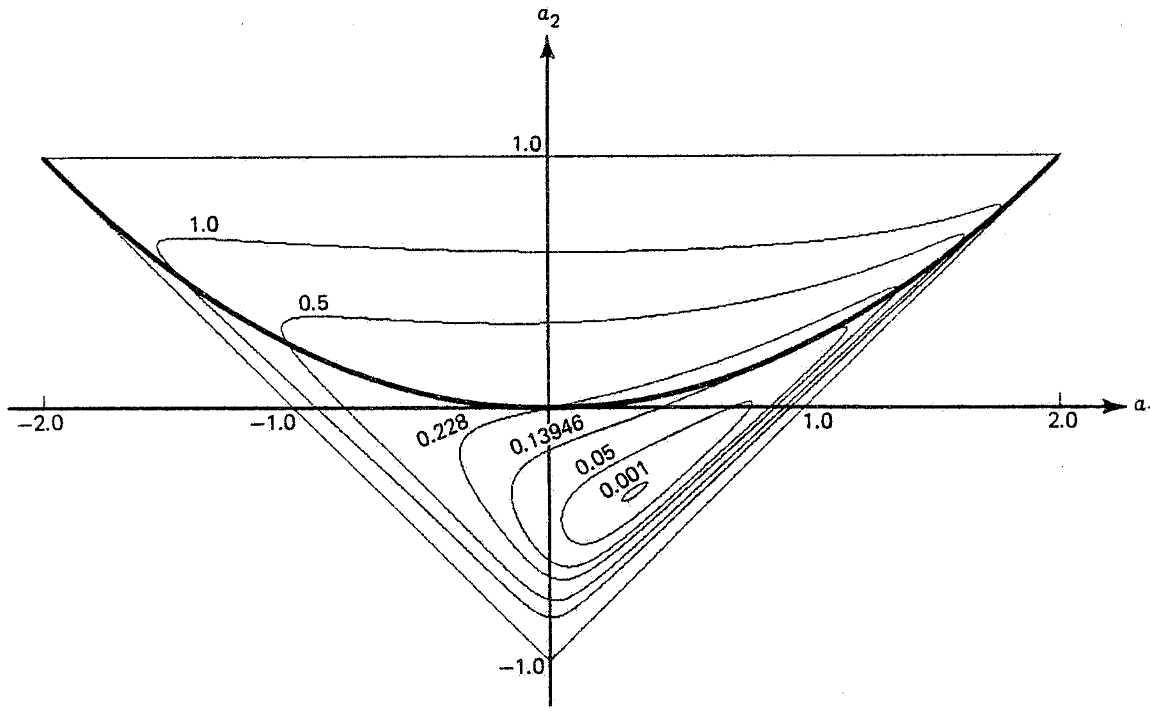


Fig. 3.4. MSE surface for the second-order direct form IIR filter in Example 3.3 (with real poles).

It can now be observed that if $\exists$ a point $\boldsymbol{\alpha}^+ = (\alpha_1^+, \alpha_2^+)$ with the property that $\nabla_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}^+} \neq 0$, and for $\boldsymbol{\beta}^+ \triangleq g(\boldsymbol{\alpha}^+)$ we have $\nabla_{\boldsymbol{\beta}} h(\boldsymbol{\beta})|_{\boldsymbol{\beta}^+} = 0$, then $\boldsymbol{\alpha}^+$ must lie on the parabola $\alpha_2 = \frac{1}{4}\ \alpha_1^2$. To see this, we can write

$$\begin{bmatrix} \frac{\partial h}{\partial \beta_1} & \frac{\partial h}{\partial \beta_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \alpha_1} & \frac{\partial f}{\partial \alpha_2} \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha_1}{\partial \beta_1} & \frac{\partial \alpha_1}{\partial \beta_2} \\ \frac{\partial \alpha_2}{\partial \beta_1} & \frac{\partial \alpha_2}{\partial \beta_2} \end{bmatrix} . \quad \text{(E3.3.2)}$$

Using (E3.3.1), this is then reduced to

$$\nabla_{\boldsymbol{\beta}} h = \nabla_{\boldsymbol{\alpha}} f \begin{bmatrix} 1 & 1 \\ \beta_2 & \beta_1 \end{bmatrix} . \quad \text{(E3.3.3)}$$

Then, for $\boldsymbol{\alpha}^+$ with the above property, it is necessary that $\beta_1^+ = \beta_2^+$ in (13) since the (2×2) matrix will have a nonzero left null space. It is easily seen that $\beta_1^+ = \beta_2^+$ corresponds to the point $(\alpha_1^+, \frac{1}{4}\alpha_1^{+2})$ which is on the aforementioned parabola. It is found that $\frac{df}{d\alpha_1}\Big|_{\alpha^+} = 0$ (i.e., the total derivative) which indicates that the $\boldsymbol{\alpha}^+$ corresponds to the MMSE solution on, and only on, the parabola $\alpha_2 = \frac{1}{4}\alpha_1^2$.

The error surface contours in the $\beta_1 - \beta_2$ plane are shown in Fig. 3.5 for this example where,

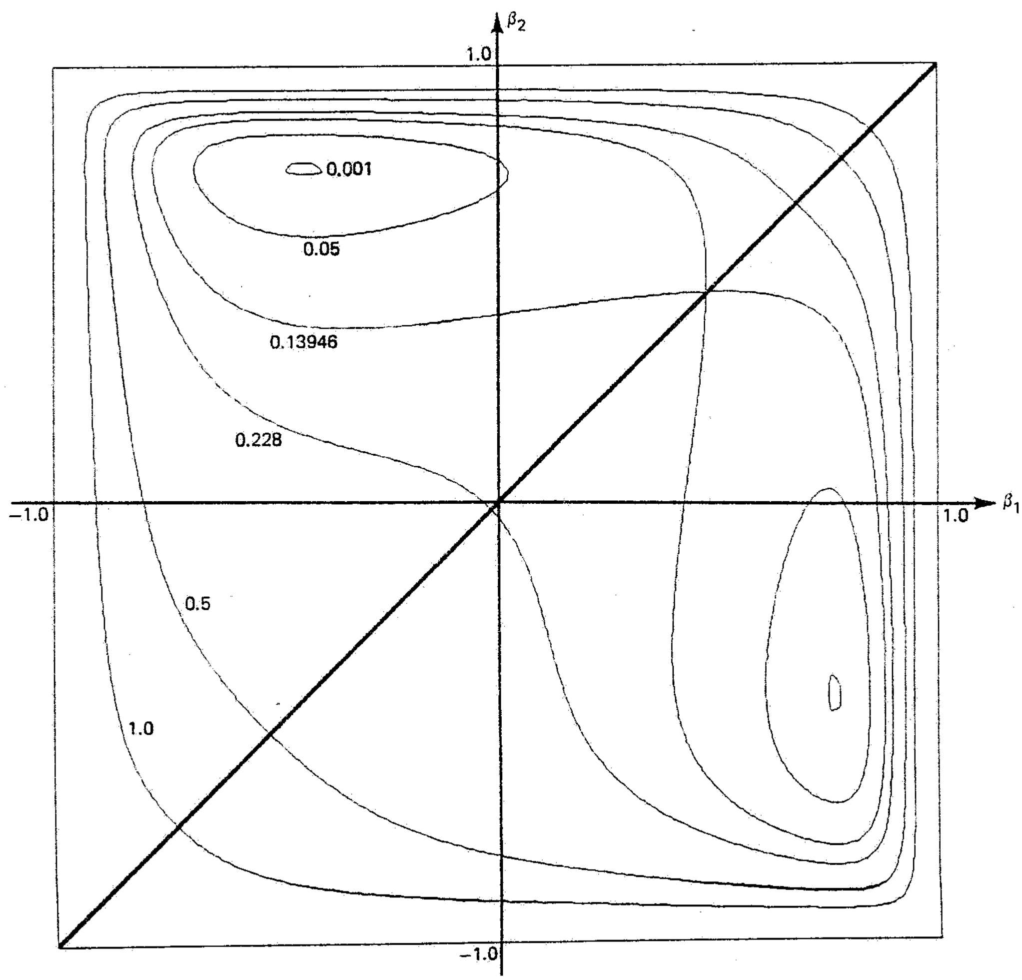


Fig. 3.5. MSE surface for the second-order parallel form IIR filter in Example 3.3 (with real poles).

for the sake of presentation, it has been assumed that β_1 and β_2 take on real values only. This does not violate the identifiability of $H_p(z^{-1})$. It is found that $\boldsymbol{\beta}^+ = (0.362, 0.362)$ and, correspondingly, $\boldsymbol{\alpha}^+ = (0.724, 0.131)$. Figure 3.5 clearly shows that the $\beta_1 = \beta_2$ line is perpendicular to all the contours whose normalized MSE values are greater than 0.13946. It is obvious that $\boldsymbol{\beta}^+$ is a saddle point and, therefore, if an adaptive gradient search algorithm is used to identify $H_p(z^{-1})$ by two first-order filters, there will be no difficulty with convergence as long as the initial conditions for β_1 and β_2 are not selected on the $\beta_1 = \beta_2$ manifold. Existence of noise will work to our advantage in this case since it tends to prevent the solution from remaining on the $\beta_1 = \beta_2$ line. To illustrate this, two experiments were conducted as shown in Fig. 3.6. The LMS algorithm (Eq. (3.1.4) with $a_{12} = a_{22} = 0$) was initialized at (0.95, 0.948) for the two experiments with $\mu = 10^{-3}$ but the white pseudo-noise input sequences were selected to be different. It was observed that the LMS quickly converged to a neighborhood of $\boldsymbol{\beta}^+$ after 20 iterations which is expected since the filter is essentially acting like a first-order filter. But small deviation from the reduced order line eventually caused the break away. The reduced order linear subspace is not invariant simply because of the dependence of the future estimates on the past estimates. Further simulations are shown in Fig. 3.7 in which the initial points were selected at (-0.95, -0.948) and (-0.95, 0.95). The convergence of the parameters to the global minima is clearly shown. □

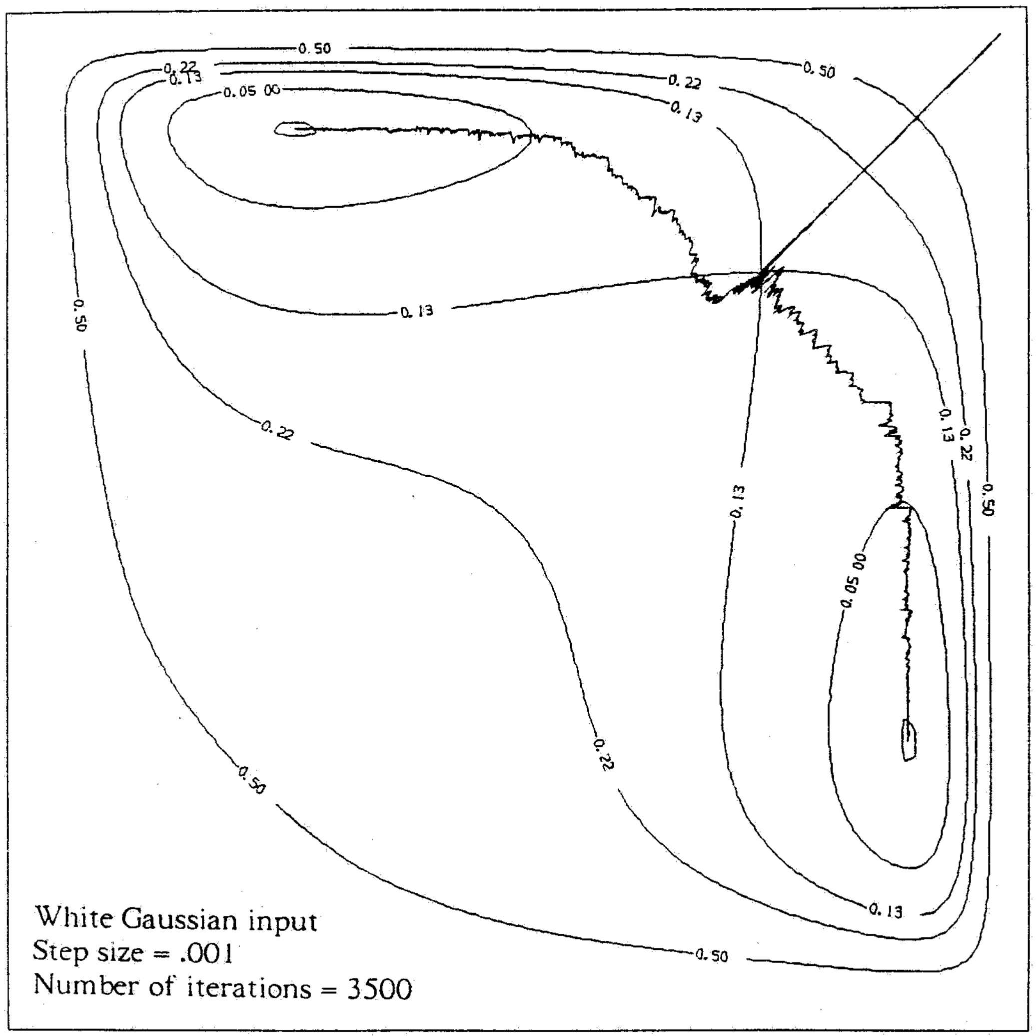


Fig. 3.6. The trajectories of two simulations when the LMS is initialized at the point (0.95,0.948) with indicated specifications.

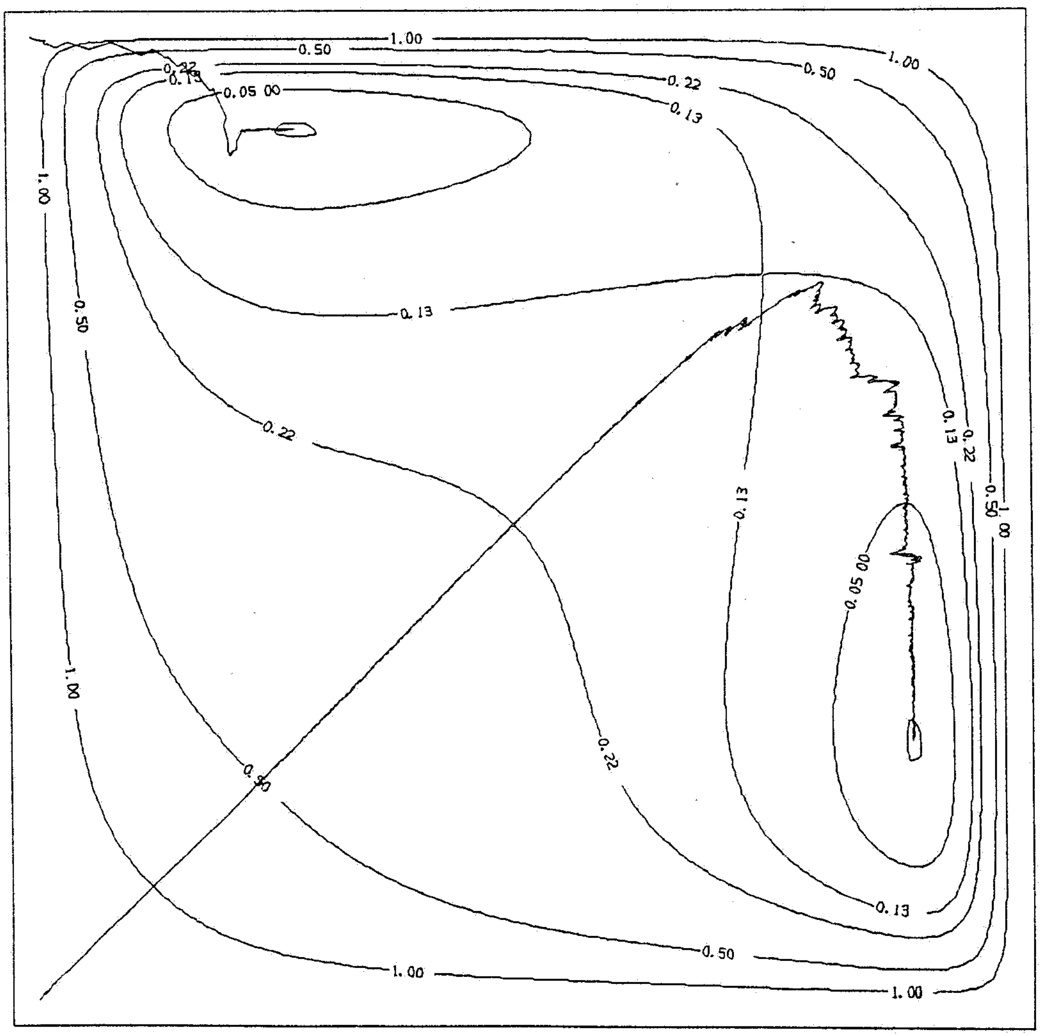


Fig. 3.7. The trajectories of two simulations when the LMS is initialized at the points (−0.95,−0.948) and (−0.95,0.95) with the same specifications in Fig. 3.6..

Example 3.4:

Let $H_M(z^{-1}) = \dfrac{2 + \alpha_3 z^{-1}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-1}}$ be the adaptive filter to identify $H_p(z^{-1})$ as given in Example 3.3. The reduced order MSE surface [12] is shown in Fig. 3.8. Now, let $H_1(z^{-1}) = \dfrac{2}{1 + \beta_1 z^{-1}}$ and $H_2(z^{-1}) = \dfrac{1 + \beta_3 z^{-1}}{1 + \beta_2 z^{-1}}$ denote two cascaded first-order sections used to identify $H_p(z^{-1})$. The reduced order MSE surface when β_1 and β_2 are real is shown in Fig. 3.9. Note the familiar symmetry and the fact that no stationary point has been formed as a result of pole-zero cancellation in $H_2(z^{-1})$. This result is generalized later. □

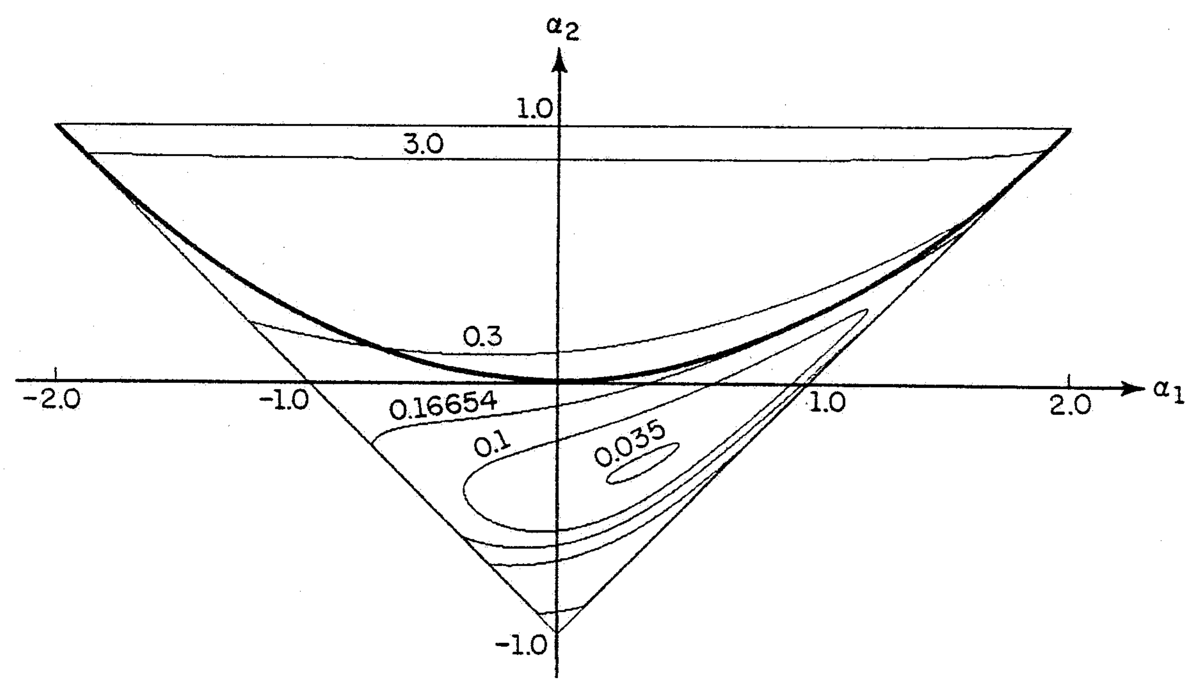


Fig. 3.8. Reduced order MSE surface for the second-order direct form IIR filter in Example 3.4 (with real poles).

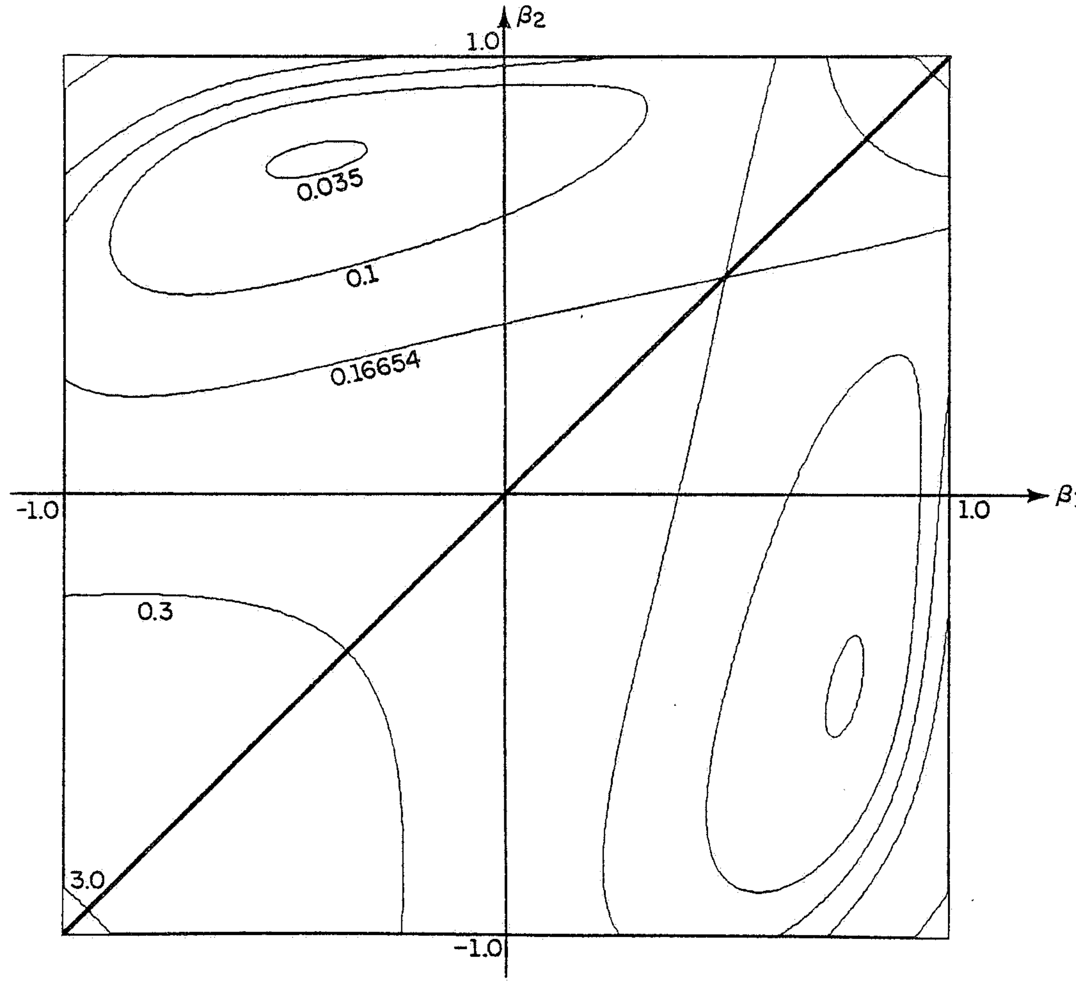


Fig. 3.9. Reduced order MSE surface for the second-order parallel form IIR filter in Example 3.4 (with real poles).

Example 3.5:

If we now consider $H_p(z^{-1}) = \dfrac{2 + z^{-1}}{1 + z^{-1} + \frac{3}{4}z^{-2}}$, then the MSE surface contours are as shown in Fig. 3.10 when $H_M(z^{-1})$ as in Example 3 is used. But the real first-order sections, $H_1(z^{-1})$ and $H_2(z^{-1})$, in parallel, can not identify $H_p(z^{-1})$. The new stationary point $\boldsymbol{\beta}^+$ is a minimum in this case (see Fig. 3.11). However, note that if complex first-order sections are permissible, $\boldsymbol{\beta}^+$ will be a saddle point. In this example, **d** is restricted; therefore, for all feasible **d**, $f(\boldsymbol{\alpha}^+ + \mathbf{d}) > f(\boldsymbol{\alpha}^+)$ which is translated to minimality of $\boldsymbol{\beta}^+$. □

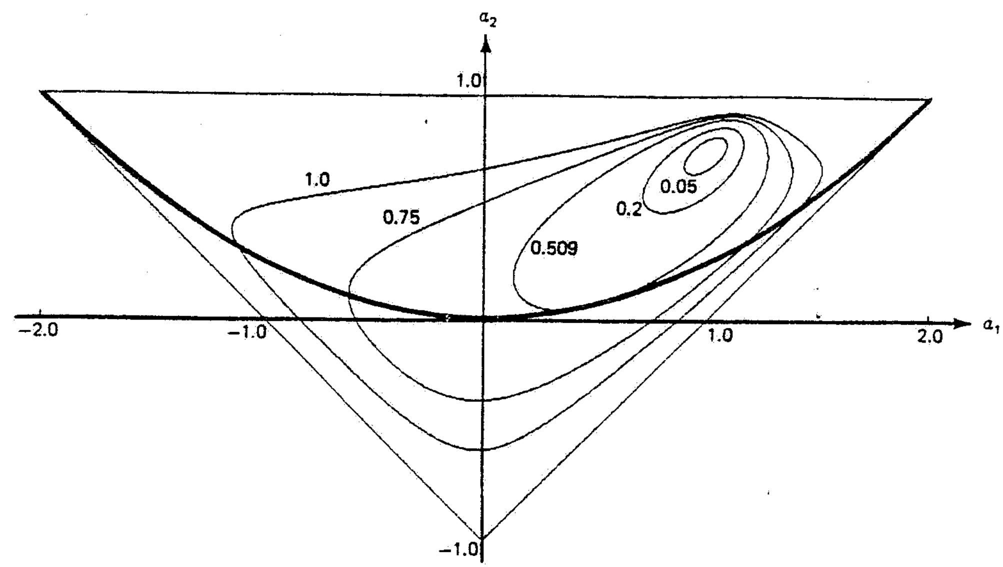


Fig. 3.10. MSE surface for the second-order direct form IIR filter in Example 3.5 (with conjugate poles).

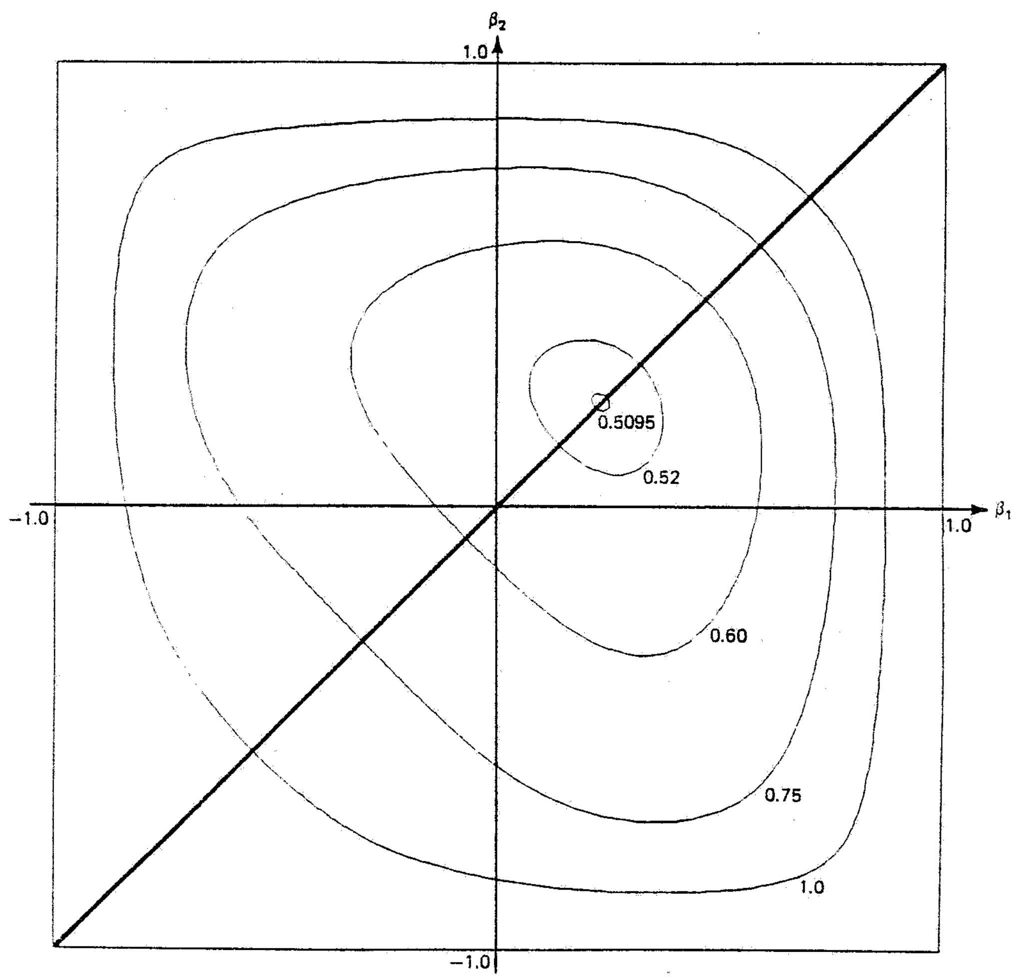


Fig. 3.11. MSE surface for the second-order filter in Example 3.5 when real sections are used.

3.4. Application of the General Theory to the Parallel Form

In this section some general results regarding the parallel form structure are obtained. Consider the identification problem of a system $H_p(z^{-1})$, with *no multiple poles*, given by

$$H_p(z^{-1}) = \frac{c_1 z^{-1} + \dots + c_N z^{-N}}{1 + d_1 z^{-1} + \dots + d_N z^{-N}}. \tag{3.4.1}$$

A sufficient order adaptive filter $H_M(z^{-1})$ of the form

$$H_M(z^{-1}) = \frac{\alpha_1 z^{-1} + \dots + \alpha_N z^{-N}}{1 + \alpha_{N+1} z^{-1} + \dots + \alpha_{2N} z^{-N}} \tag{3.4.2}$$

is able to identify $H_p(z^{-1})$ and the MSE surface is unimodal according to the results in [43] and Chapter 2. Another alternative is to use N banks of first-order filters to do the identification, i.e., we could write

$$H_M(z^{-1}) = \frac{\beta_1 z^{-1}}{1 + \beta_{N+1} z^{-1}} + \frac{\beta_2 z^{-1}}{1 + \beta_{N+2} z^{-1}} + \dots + \frac{\beta_N z^{-1}}{1 + \beta_{2N} z^{-1}} \tag{3.4.3}$$

where there is a direct relationship between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We have the following theorem.

Theorem 3.1:

The newly formed stationary points in (3.4.3) are located on the reduced order subspaces.

Proof: We can write

$$\frac{\partial h}{\partial \boldsymbol{\beta}} = \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\beta}} \tag{3.4.4}$$

where it is found (see Appendix C) that

$$\det\left[\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\beta}}\right] = \left[\prod_{i<j}\left(\beta_{N+i} - \beta_{N+j}\right)\right]^2, \; i, j = 1, \dots, N \; . \tag{3.4.5}$$

Therefore, if there exists $\boldsymbol{\beta}^+ \in \mathbb{C}^N$ such that $\frac{\partial h}{\partial \boldsymbol{\beta}}\big|_{\boldsymbol{\beta}^+} = 0$ but $\frac{\partial f}{\partial \boldsymbol{\alpha}}\big|_{\boldsymbol{\alpha}^+} \neq 0$, where $\boldsymbol{\beta}^+ = g(\boldsymbol{\alpha}^+)$, then

(3.4.4) reduces to

$$0 = \frac{\partial f}{\partial \alpha}\Big|_{\alpha^+} \cdot \frac{\partial \alpha}{\partial \beta}\Big|_{\beta^+} \tag{3.4.6}$$

and $\frac{\partial \alpha}{\partial \beta}\big|_{\beta^+}$ is therefore singular. This is possible if any two or more poles of (3.4.3) are equal, according to Equation (3.4.5). Lemma 3.2 states that $\boldsymbol{\beta}^+$ is a saddle point. □

Every stationary point in the direct form corresponds to (N!) stationary points in the parallel form since there are (N!) permutations of sections to represent (3.4.2).

3.5. Application of the General Theory to the Cascade Form

Now some general results regarding the cascade form can be obtained. Let us assume that the transfer function of the unknown system is given by

$$H_p(z^{-1}) = \frac{c_0 + c_1 z^{-1} + \ldots + c_N z^{-N}}{1 + d_1 z^{-1} + \ldots + d_N z^{-N}} \tag{3.5.1}$$

and the filter

$$H_M(z^{-1}) = \frac{\alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_N z^{-N}}{1 + \alpha_{N+1} z^{-1} + \ldots + \alpha_{2N} z^{-N}} \tag{3.5.2}$$

is used for identification. Let us assume that $c_0 = \alpha_0 = 1$ for simplicity but with no loss of generality of the final conclusion. The alternative form of (3.5.2) is given by

$$H_M(z^{-1}) = \left[\frac{1 + \beta_1 z^{-1}}{1 + \beta_{N+1} z^{-1}}\right] \left[\frac{1 + \beta_2 z^{-1}}{1 + \beta_{N+2} z^{-1}}\right] \cdots \left[\frac{1 + \beta_N z^{-N}}{1 + \beta_{2N} z^{-N}}\right] \tag{3.5.3}$$

which consists of first-order sections in series. We then have the following result.

Theorem 3.2:

The newly formed stationary points in (3.5.3) are on the manifolds corresponding to subspaces where any two or more zeroes, or any two or more poles, are the same and consequently where no pole-zero cancellation takes place in the sections.

Proof: The general form of $\frac{\partial \alpha}{\partial \beta}$ is given by

$$\frac{\partial \alpha}{\partial \beta} = \left[\begin{array}{c|c} \Omega & 0 \\ \hline 0 & \Gamma \end{array} \right] \tag{3.5.4}$$

where Γ is as given in Appendix C and Ω has the same form of Γ with $\beta_1, \beta_2, \ldots, \beta_N$ substituted for $\beta_{N+1}, \beta_{N+2}, \ldots, \beta_{2N}$, respectively. It then easily follows that

$$\det\left(\frac{\partial \alpha}{\partial \beta}\right) = \left[\prod_{i<j}\left(\beta_i - \beta_j\right)\right] \cdot \left[\prod_{i<j}\left(\beta_{N+i} - \beta_{N+j}\right)\right], \qquad i, j = 1, \ldots, N \tag{3.5.5}$$

and the proof immediately follows. □

Theorems 3.1 and 3.2 provide the loci of the new equilibrium points when complex first-order sections are used in parallel or in series to model the unknown system. However, these theorems can be extended easily to second-order sections in which case complex arithmetic is avoided.

3.6. Summary

Some alternative structures are attractive for their convenience and properties. It was shown that changing the underlying direct form parameter space to parallel form parameter space, characterized by a transformation, does not result in formation of any local minimum when no constraint is applied (see Example 3.5 for a constrained case). However, linear manifolds are introduced by this transformation on the MSE surface. It was shown that these manifolds are unstable and pose no threat to the adaptive algorithm as long as the initial conditions are not placed on them. Similar analysis was carried out for cascade form adaptive IIR filters and similar conclusions were obtained.

CHAPTER 4

CONCLUSIONS

The main contributions in Chapter 2 are a clarification of the conditions under which Stearns' conjecture holds and a unification of knowledge that has evolved in the fields of adaptive control and adaptive signal processing. It is shown that although Stearns' conjecture concerning sufficient conditions to guarantee unimodality of error surfaces for adaptive pole-zero filters is valid for first- and second-order filters, it is not true in general without an additional constraint introduced by Soderstrom. Soderstrom's additional condition, which requires that the degree of the numerator of the adaptive filter be greater than the degree of the denominator of the unknown system, was not stated as a binding condition in Stearns' original conjecture. An example of overparameterization showed the creation of local minima on the performance surface. It was also shown that for a class of insufficient order filters, the nonminimum phase characteristic of the unknown system driven by white noise is sufficient for multimodality of the error surface. Finally, a convenient method for finding the stationary points was introduced.

It has been shown in Chapter 3 that the MSE surfaces associated with alternate realizations of adaptive IIR digital filters have different characteristics which result in different adaptive behavior in practice. A general theory, based on an analysis of stationary points, was presented which shows that whenever a direct form IIR filter with a unimodal MSE surface is transformed into an alternate realization, the MSE surface associated with the new structure may have additional stationary points, which are either new equivalent minima (and hence indistinguishable at the filter output), or saddle points, which are unstable solutions in the parameter space. The general theory was specialized for the parallel and cascade forms. For parallel and cascade forms with N first-order sections, the MSE surface contains N! distinct subregions, each containing an equivalent global minimum which corresponds to particular ordering of the second-order sections. In the parallel form, saddle points are created on the boundaries between these subregions of the parameter space;

these boundaries represent reduced order manifolds which are nonattractive, and which will be avoided by a gradient algorithm as long as there is some noise present in the adaptive algorithm. The parallel form should not be used to identify systems with multiple poles unless the adaptive filter structure is modified appropriately. The bottom line here is that an adaptive filter must be defined by a structure that represents the proper partial fraction expansion of the unknown system function.

In particular, it has been shown that for both the parallel and cascade forms, a gradient algorithm will find a global minimum as long as there is some noise present to jitter the solution away from the reduced order manifolds which may contain saddle points. Experimental examples were presented to illustrate that the predicted behavior is indeed observed in practice. Note that this work does not reveal anything directly about the convergence rate of these different forms. However, the lower coefficient sensitivities of the parallel and cascade structures as compared to a direct form suggest that the minimum MSE is likely to be smaller after convergence due to the shallow nature of the MSE surface in the vicinity of the global minimum. At least one example showed that convergence can be achieved with the low sensitive parallel or cascade forms in circumstances in which the unknown system has poles near the unit circle. Further research is needed to fully explain this impact of low sensitivity on adaptive behavior.

APPENDIX A

PROOF OF CASE ONE IN THEOREM 2.1

As in [41], we can express $E[e^2]$ as

$$W(\underline{a}, \underline{b}) \triangleq E[e^2] = P_0 - 2P_1(\underline{a})^T\underline{b} + \underline{b}^T P_2(\underline{a})\underline{b} \tag{A.1}$$

where

$$\begin{aligned} P_0 &= E\left[\frac{D(q^{-1})}{C(q^{-1})}x(n)\right]^2 \\ P_1(\underline{a})_j &= E\left[\frac{D(q-1)}{C(q^{-1})}x(n)\cdot\frac{1}{A(q^{-1})}x(n-j)\right] \qquad 0 \leqslant j \leqslant n_b \\ P_2(\underline{a})_{jk} &= E\left\{\left[\frac{1}{A(q^{-1})}x(n-j)\right]\left[\frac{1}{A(q^{-1})}x(n-k)\right]\right\} \qquad 0 \leqslant j,k \leqslant n_b. \end{aligned} \tag{A.2}$$

Now, since $P_2(\underline{a})$ is always a positive definite matrix for sufficiently rich input $x(n)$, we could optimize $W(\underline{a}, \underline{b})$ with respect to $\underline{b}$ by

$$\underline{b}^0 = P_2(\underline{a})^{-1} P_1(\underline{a}). \tag{A.3}$$

Equation (A.3) set equal to zero yields $P_1(\underline{a}) = 0$ which, of course, is exactly Equation (2.2.7). To show that degenerated points which satisfy (2.2.7) are saddle points, we find two points arbitrarily close to $(\underline{a}^*, 0)$, where $\underline{a}^*$ is a stable solution of (2.2.7), which gives higher and lower costs compared to those for $W(\underline{a}^*, 0) = P_0$. First,

$$W(\underline{a}^*, \delta\underline{b}) = P_0 + \delta\underline{b}^T P_2(\underline{a}^*)\delta\underline{b} > P_0 . \tag{A.4}$$

Second, since (A.3) represents a continuous function of a , then the perturbation $\delta\underline{a}$ induces a perturbation in $\underline{b}^0$. Therefore, we have

$$W(\underline{a}^* + \delta\underline{a}, \underline{b}^0(\underline{a}^* + \delta\underline{a})) = P_0 - P_1(\underline{a}^* + \delta\underline{a})^T P_2^{-1}(\underline{a}^* + \delta\underline{a}) P_1(\underline{a}^* + \delta\underline{a}) < P_0. \tag{A.5}$$

Thus, $(\underline{a}^*, 0)$ is indeed a saddle point. □

APPENDIX B

PROOF OF STEARNS' CONJECTURE WHEN SODERSTROM'S CONDITION IS SATISFIED

We will distinguish between two cases for our analysis:

1. A and B are coprime.

2. A and B have a common factor.

Case 1: When A and B are coprime, the analysis could be divided into two parts:

(a) Let either A be of *true* order n_a (i.e., $a_{n_a} \neq 0$) or B be of *true* order n_b (i.e., $b_{n_b} \neq 0$). Then, simple manipulation of (2.2.2) results in

$$S(B,A)\, E \begin{bmatrix} \frac{1}{A(q^{-1})A(q^{-1})}\, x(n) \\ \vdots \\ \frac{1}{A(q^{-1})A(q^{-1})}\, x(n-m) \end{bmatrix} \left[\frac{1}{C(q^{-1})A(q^{-1})}\, x(n) \cdots \frac{1}{C(q^{-1})A(q^{-1})}\, x(n-r) \right] \begin{bmatrix} \gamma_0 \\ \vdots \\ \gamma_r \end{bmatrix} = 0 , \tag{B.1}$$

where

$$m = n_a + n_b \ , \quad r = \max(n_a + n_d \, , n_c + n_b) \, ,$$

$$\sum_{i=0}^{r} \gamma_i q^{-i} \triangleq D(q^{-1})A(q^{-1}) - B(q^{-1})C(q^{-1}) \, ,$$

and

$$S(B,A) = \begin{bmatrix} 0 & b_0 & b_1 & \cdot & \cdot & \cdot & b_{n_b} & & & & \\ & 0 & b_0 & b_1 & \cdot & \cdot & \cdot & b_{n_b} & & & \\ & & & & & & & & & & \\ & & & & 0 & b_0 & b_1 & \cdot & \cdot & \cdot & b_{n_b} \\ 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n_a} & & & & \\ & 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n_a} & & & \\ & & & & & & & & & & \\ & & & & 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n_a} \end{bmatrix},$$

is an $(m + 1) \mid (m + 1)$ Sylvester matrix, which is nonsingular. By [48,lemma 4.7], the second matrix in (B.1) is of rank r when $(n_b+1)-n_c \geqslant 0$ since $m \geqslant r$. Therefore, it follows that $\gamma_i = 0, i = 0, 1, ..., r$, or equivalently,

$$D(q^{-1})A(q^{-1}) \equiv B(q^{-1})C(q^{-1}). \tag{B.2}$$

Since A and B are relatively prime, we must have

$$A(q^{-1}) = C(q^{-1}),$$

$$B(q^{-1}) = D(q^{-1}),$$

which means that

$$a_{n_c+1} = \cdots = a_{n_a} = 0,$$

$$b_{n_d+1} = \cdots = b_{n_b} = 0.$$

Unless $n^* = \min(n_a - n_c, n_b - n_d) = 0$, this is not possible since in that case S(B,A) would become singular, resulting in a contradiction.

(b) When $A(q^{-1})$ and $B(q^{-1})$ are of *true* order N and M, respectively, with $N < n_a$ and $M < n_b$, Equation (2.2.2) can be expressed by

$$S^*(B,A)\, E \begin{bmatrix} \frac{1}{A(q^{-1})A(q^{-1})} x(n) \\ \vdots \\ \frac{1}{A(q^{-1})A(q^{-1})} x(n-m^*) \end{bmatrix} \left[\frac{1}{C(q^{-1})A(q^{-1})} x(n) \cdots \frac{1}{C(q^{-1})A(q^{-1})} x(n-r^*) \right] \begin{bmatrix} \xi_0 \\ \vdots \\ \xi_{r^*} \end{bmatrix} = 0 , \tag{B.3}$$

where

$$m^* = \max(n_a + M , n_b + N) , \; r^* = \max(N + n_d , n_c + M) ,$$

$$\sum_{i=0}^{r^*} \xi_i q^{-i} \triangleq D(q^{-1})A(q^{-1}) - B(q^{-1})C(q^{-1}) ,$$

and for the case when $(n_a + M) = (n_b + N)$

$$S^*(B,A) = \begin{bmatrix} 0 & b_0 & b_1 & \cdots & b_M & & & \\ & 0 & b_0 & b_1 & \cdots & b_M & & \\ & & & & & & & \\ & & & & 0 & b_0 & b_1 & \cdots & b_M \\ 1 & a_1 & a_2 & \cdots & a_N & & & \\ & 1 & a_1 & a_2 & \cdots & a_N & & \\ & & & & & & & \\ & & & & 1 & a_1 & a_2 & \cdots & a_N \end{bmatrix} .$$

but other cases are easily followed. Obviously, the $(m + 1) \mid (m^* + 1)$ Sylvester matrix $S^*(B,A)$, obtained by removing zero columns of $S(B,A)$, is of rank $(m^* + 1)$ and the second matrix is of rank r^*. Therefore, we conclude that $\xi_i = 0, i = 0, 1, ..., r^*$, or equivalently,

$$D(q^{-1})A(q^{-1}) \equiv B(q^{-1})C(q^{-1}) . \tag{B.4}$$

Since A and B as well as C and D are relatively prime, it follows that this case is possible if $\min(N - n_c , M - n_d) = 0$; otherwise, contradiction is the result.

Case 2: When A and B are not coprime, there is a polynomial $L(q^{-1})$ of order n_L such that

$$A(q^{-1}) = \bar{A}(q^{-1})\, L(q^{-1}) ,$$

$$B(q^{-1}) = \bar{B}(q^{-1})\, L(q^{-1}), \tag{B.5}$$

where $\bar{A}(q^{-1})$ and $\bar{B}(q^{-1})$ are coprime polynomials given by

$$\bar{A}(q^{-1}) = 1 + \bar{a}_1 q^{-1} + \dots + \bar{a}_{\bar{n}_a} q^{-\bar{n}_a},$$

$$\bar{B}(q^{-1}) = \bar{b}_0 + \bar{b}_1 q^{-1} + \dots + \bar{b}_{\bar{n}_b} q^{-\bar{n}_b}, \tag{B.6}$$

in which $\bar{n}_a = n_a - n_L$ and $\bar{n}_b = n_b - n_L$. Consequently, (2.2.2) is reduced to

$$E \begin{bmatrix} \dfrac{\bar{B}(q-1)}{A(q^{-1})\bar{A}(q^{-1})}\, x(n-i) \\ \dfrac{1}{A(q^{-1})}\, x(n-j) \end{bmatrix} \left[\frac{D(q^{-1})\bar{A}(q^{-1}) - \bar{B}(q^{-1})C(q^{-1})}{C(q^{-1})\bar{A}(q^{-1})}\, x(n) \right] = 0,$$

$$1 \leqslant i \leqslant n_a\,,\ 0 \leqslant j \leqslant n_b\,. \tag{B.7}$$

Similar to case 1, when the *true* order of $\bar{A}$ is $\bar{n}_a$ or the *true* order of $\bar{B}$ is $\bar{n}_b$, the left-hand side of the equality in (B.7) can be expressed by

$$\bar{S}(\bar{B},\bar{A})\, E \begin{bmatrix} \dfrac{1}{A(q^{-1})\bar{A}(q^{-1})} x(n) \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{1}{A(q^{-1})\bar{A}(q^{-1})} x(n-m+n_L) \end{bmatrix} \left[\frac{1}{C(q^{-1})\bar{A}(q^{-1})}\, x(n) \dots \frac{1}{C(q^{-1})\bar{A}(q^{-1})}\, x(n-r+n_L) \right] \begin{bmatrix} \bar{\gamma}_0 \\ \cdot \\ \cdot \\ \cdot \\ \bar{\gamma}_{r-n_L} \end{bmatrix}, \tag{B.8}$$

where

$$\sum_{i=0}^{r-n_L} \bar{\gamma}_i\, q^{-1} \triangleq D(q^{-1})\bar{A}(q^{-1}) - \bar{B}(q^{-1})C(q^{-1}),$$

and $\bar{S}(\bar{B},\bar{A})$ is an $(m+1)|(m-n_L+1)$ Sylvester matrix. (i.e., the same as $S(\bar{B},\bar{A})$ with the last n_L columns removed.) Since $\bar{A}$ and $\bar{B}$ are coprime, $\bar{S}(\bar{B},\bar{A})$ is of rank $(m-n_L+1)$ and the second matrix is of full rank $(r-n_L+1)$ by [48, lemma 4.7]. Thus, we conclude again that $\bar{\gamma}_i = 0$, $i = 0, 1, \dots, r-n_L$, or equivalently,

$$D(q^{-1})\bar{A}(q^{-1}) \equiv \bar{B}(q^{-1})C(q^{-1}). \tag{B.9}$$

Now, since C and D are coprime as well, then, as in part 1, $\min(\bar{n}_a - n_c\,,\ \bar{n}_b - n_d) = 0$ (or equivalently $n_L = n^*$) and

$$\bar{A}(q^{-1}) = C(q^{-1}),$$

$$\bar{B}(q^{-1}) = D(q-1). \tag{B.10}$$

The case in which the *true* orders of $\bar{A}$ and $\bar{B}$ are not $\bar{n}_a$ and $\bar{n}_b$, respectively, follows from the same set of arguments presented in case 1. This proves the conjecture when (2.2.14) is satisfied. □

APPENDIX C

DETERMINANT OF THE JACOBIAN OF THE PARALLEL REALIZATION

We have

$$\alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_N z^{-N} \equiv \sum_{i=1}^{N} (\beta_i z^{-1}) \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 + \beta_{N+j} z^{-1})$$

$$1 + \alpha_{N+1} z^{-1} + \ldots + \alpha_{2N} z^{-N} \equiv \prod_{j=1}^{N} (1 + \beta_{N+j} z^{-1}) = (1 + \beta_{N+i} z^{-1}) \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 + \beta_{N+j} z^{-1}). \tag{C.1}$$

Notice that

$$\frac{\partial \alpha_{N+j}}{\partial \beta_k} = 0 \text{ for } j, k = 1, 2, \ldots, N .$$

Also

$$\frac{\partial \alpha_{N+k}}{\partial \beta_{N+i}} = \text{the coefficient of } z^{-k} \text{ in } \left\{ z^{-1} \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 + \beta_{N+j} z^{-1}) \right\}, \quad i, k = 1, 2, \ldots, N$$

$$\frac{\partial \alpha_k}{\partial \beta_i} = \text{the coefficient of } z^{-k} \text{ in } \left\{ z^{-1} \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 + \beta_{N+j} z^{-1}) \right\}, \quad i, k = 1, 2, \ldots, N. \tag{C.2}$$

Therefore $\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\beta}}$ is

$$\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\beta}} = \left[\begin{array}{c|c} \Gamma & P \\ \hline 0 & \Gamma \end{array} \right] \tag{C.3}$$

where Γ and P are $N \times N$ matrices. Since $\det \left| \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\beta}} \right| = [\det \Gamma]^2$, then P does not play any role in the singularity of $\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\beta}}$ and the matrix Γ is the one to be concerned with.

Now, if we define $F_i(z^{-1}) \triangleq z^{-1}\prod_{\substack{j=1 \\ j\neq i}}^{N}(1 + \beta_{N+j}z^{-1})$, then

$\gamma_{1i} \triangleq$ The coefficient of z^{-1} (i.e., $k = 1$) in $F_i(z^{-1}) = 1$

$\gamma_{2i} \triangleq$ The coefficient of z^{-2} (i.e., $k = 2$) in $F_i(z^{-1}) = \sum_{\substack{m_1=N+1 \\ m_1\neq N+i}}^{2N} \beta_{m_1}$

$\gamma_{3i} \triangleq$ The coefficients of z^{-3}(i.e., $k = 3$) in $F_i(z^{-1}) = \sum_{\substack{m_1,\, m_2 = N+1 \\ m_1\neq m_2\neq N+i}}^{2N} \beta_{m_1}\beta_{m_2}$

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$\gamma_{ki} \triangleq$ The coefficients of z^{-k} in $F_i(z^{-1}) = \sum_{\substack{m_1,\, m_2,\, \ldots,\, m_{k-1} = N+1 \\ m_1\neq m_2\neq \ldots\neq m_{k-1}\neq N+i}}^{2N} \beta_{m_1}.\beta_{m_2} \ldots\, \beta_{m_{k-1}}$

Thus $\Gamma_{N\times N}$ is given as follows

$$\Gamma = \begin{bmatrix} 1 & 1 & \ldots & 1 & \ldots & 1 \\ \gamma_{21} & \gamma_{22} & \ldots & \gamma_{2i} & \ldots & \gamma_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_{k1} & \gamma_{k2} & \cdots & \gamma_{ki} & \cdots & \gamma_{kN} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{Ni} & \cdots & \gamma_{NN} \end{bmatrix}. \tag{C.4}$$

Now, to find $|\Gamma| \triangleq \det(\Gamma)$ we proceed as follows:

1. Subtract the first column from every other column. Note that

$$\gamma_{ki} - \gamma_{k1} = \sum_{\substack{m_1, m_2, \ldots, m_{k-1} = N+1 \\ m_1 \neq m_2 \neq \ldots \neq m_{k-1} \neq N+i}}^{2N} \beta_{m_1} . \beta_{m_2} \ldots . \beta_{m_{k-1}} - \sum_{\substack{m_1, m_2, \ldots, m_{k-1} = N+1 \\ m_1 \neq m_2 \neq \ldots \neq m_{k-1} \neq N+1}}^{2N} \beta_{m_1} \beta_{m_2} \ldots . \beta_{m_{k-1}}$$

$$= \beta_{N+1} \sum_{\substack{m_1, m_2, \ldots, m_{k-2} = N+1 \\ m_1 \neq m_2 \neq \ldots \neq m_{k-1} \neq N+i \neq N+1}}^{2N} \beta_{m_1} . \beta_{m_2} \ldots \beta_{m_{k-2}} - \beta_{N+i} \sum_{\substack{m_1, m_2, \ldots, m_{k-2} = N+1 \\ m_1 \neq m_2 \neq \ldots \neq m_{k-1} \neq N+i \neq N+1}}^{2N} \beta_{m_1} . \beta_{m_2} \ldots \beta_{m_{k-2}}$$

$$= (\beta_{N+1} - \beta_{N+i}) \sum_{\substack{m_1, m_2, \ldots, m_{k-2} = N+1 \\ m_1 \neq m_2 \neq \ldots \neq m_{k-1} \neq N+i \neq N+1}}^{2N} \beta_{m_1} . \beta_{m_2} \ldots \beta_{m_{k-2}}$$

$$= (\beta_{N+1} - \beta_{N+i}) . \left\{ \text{Coeff. of } z^{-k+1} \text{in the polynominal } z^{-1} \prod_{\substack{j=2 \\ j \neq i}}^{N} (1 + \beta_{N+j} z^{-1}) = \frac{F_i(z^{-1})}{1 + \beta_{N+1} z^{-1}} . \right\}$$

$$\triangleq (\beta_{N+1} - \beta_{N+i}) . \delta_{(k-1)(i-1)} .$$

Then, $|\Gamma|$ is equivalent to the determinant of

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \gamma_{21} & (\beta_{N+1} - \beta_{N+2}) & \cdots & (\beta_{N+1} - \beta_{N+i}) & \cdots & (\beta_{N+1} - \beta_{2N}) \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \gamma_{k1} & (\beta_{N+1} - \beta_{N+2}) \delta_{(k-1)1} & \cdots & (\beta_{N+1} - \beta_{N+i}) \delta_{(k-1)(i-1)} & \cdots & (\beta_{N+1} - \beta_{2N}) . \delta_{(k-1)(N-1)} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \gamma_{N1} & (\beta_{N+1} - \beta_{N+2}) \delta_{(N-1)1} & \cdots & (\beta_{N+1} - \beta_{N+i}) \delta_{(N-1)(i-1)} & \cdots & (\beta_{N+1} - \beta_{2N}) \delta_{(N-1)(N-1)} \end{vmatrix} .$$

2. Factor out $(\beta_{N+1} - \beta_{N+i})$ terms, $i = 2, \ldots, N$ and then expand with respect to the first row

$$|\Gamma| = \left\{ \prod_{i=2}^{N} (\beta_i - \beta_1) \right\} \cdot \det \underbrace{\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \delta_{(k-1)1} & \cdots & \delta_{(k-1)(i-1)} & \cdots & \delta_{(k-1)(N-1)} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \delta_{(N-1)1} & \cdots & \delta_{(N-1)(i-1)} & \cdots & \delta_{(N-1)(N-1)} \end{vmatrix}}$$

reduced version of Γ when $F_i(z^{-1})$ is replaced by $\dfrac{F_i(z^{-1})}{1 + \beta_{N+1} z^{-1}}$

3. Continue this process until Γ has been reduced to a 1×1 unit matrix. Then we see that

$$|\Gamma| = \prod_{\substack{i,j=1 \\ i<j}}^{N} (\beta_{N+i} - \beta_{N+j}) \tag{C.5}$$

Therefore, $\left|\frac{\partial \alpha}{\partial \beta}\right| = \left[\prod_{\substack{i,j=1 \\ i<j}}^{N} (\beta_{N+i} - \beta_{N+j})\right]^2$. □

REFERENCES

[1] B. Widrow and M. E. Hoff, "Adaptive switching circuits," in *IRE WESCON Conv. Rec.*, pt. 4, pp. 96-104, 1960.

[2] B. Widrow, "Adaptive filters," in *Aspects of Network and System Theory*, R. Kalman and N. DeClaris, Eds. New York: Holt, Rinehart, and Winston, pp. 563-587, 1971.

[3] B. Widrow, "Adaptive filters I: fundamentals," *Rept. SEL-66-126 (TR 6764-6)*, Stanford Electronics Laboratories, Stanford, CA, Dec. 1966.

[4] B. Widrow, J. R. Glover, Jr., J. M. McCool, J. Kaunitz, C. S. Williams, R. H. Hearn, J. R. Zeidler, E. Dong, Jr., and R. C. Goodlin, "Adaptive noise cancelling: Principles and applications," *Proc. IEEE*, vol. 63, no. 12, pp. 1692- 1716, Dec. 1975.

[5] B. Widrow, J. M. McCool, M. G. Larimore, and C. R. Johnson, Jr., "Stationary and nonstationary learning characteristics of the LMS adaptive filter," *Proc. IEEE*, vol. 64, no. 8, pp. 1151-1162, Aug. 1976.

[6] B. Widrow and J. M. McCool, "A comparison of adaptive algorithms based on the method of steepest descent and random search," *IEEE Trans. Antennas Propag.*, vol. AP-24, no. 5, pp. 615-637, Sept. 1976.

[7] B. Widrow and E. Walach, "On the statistical efficiency of the LMS algorithm with nonstationary inputs," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 2, pp. 211-221, Mar. 1984.

[8] S. S. Narayan, A. M. Peterson, and M. J. Narasimha, "Transform domain LMS algorithm," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, no. 3, pp. 609-615, June 1983.

[9] N. J. Bershad and P. L. Feintuch, "Adaptive filtering in the frequency domain," *Proc. IEEE*, vol. 67, no. 12, pp. 1658-1659, Dec. 1978.

[10] J. J. Shynk and R. P. Gooch, "Frequency-domain adaptive pole-zero filtering," *Proc. IEEE*, vol. 73, no. 10, pp. 1526-1528, Oct. 1985.

[11] J. J. Shynk and B. Widrow, " Bandpass adaptive pole-zero filtering," *Proc. IEEE Intl. Conf. Acoust., Speech, Signal Processing*, Tokyo, Japan, pp. 2107-2110, Apr. 1986.

[12] S. D. Stearns, "Error surfaces of recursive adaptive filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, no. 3, pp. 763-766, June 1981.

[13] S. D. Stearns, G. R. Elliott, and N. Ahmed, "On adaptive recursive filtering," *Proc. 10th Asilomar Conf. Circuit., Syst., Comput.*, pp. 5-10, Pacific Grove, CA, Nov. 1976.

[14] P. L. Feintuch, "An adaptive recursive LMS filter," *Proc. IEEE*, vol. 64, no. 11, pp. 1622-1624, Nov. 1976.

[15] C. R. Johnson, Jr., and M. G. Larimore, "Comments on and additions to 'An adaptive recursive LMS filter'," *Proc. IEEE*, vol. 65, no. 9, pp. 1399-1402, Sept. 1977.

[16] B. Widrow and J. M. McCool, "Comments on and additions to 'An adaptive recursive LMS filter'," *Proc. IEEE*, vol. 65, no. 9, pp. 1402-1404, Sept. 1977.

[17] C. R. Johnson, Jr., "A convergence proof for a hyperstable adaptive recursive filter," *IEEE Trans. Inf. Theory*, vol. IT-25, no. 6, pp. 745-749, Nov. 1978.

[18] M. G. Larimore, J. R. Treichler, and C. R. Johnson, Jr., "SHARF: An algorithm for adapting IIR digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, no. 4, pp. 428-440, Aug. 1980.

[19] C. R. Johnson, Jr., M. G. Larimore, J. R. Treichler, and B. D. O. Anderson, "SHARF convergence properties," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, no. 3, pp. 659-670, June 1981.

[20] C. R. Johnson, Jr., "Adaptive IIR filtering: current results and open issues," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 2, pp. 237-250, Mar. 1984.

[21] D. Parikh, S. C. Sinha, and N. Ahmed, "On a modification of the SHARF algorithm," *Proc. 22nd Midwest Symp. Circuit. Syst.*, pp. 362-366, Philadelphia, PA, June 1979.

[22] H. Fan and W. K. Jenkins, "A new adaptive IIR filter," *IEEE Trans. Circuit. Syst.*, vol. CAS-33, no. 10, pp. 939-947, Oct. 1986.

[23] M. Nayeri, H. Fan, and W. K. Jenkins, "Characteristics of error surfaces for IIR adaptive filters," Submitted to *IEEE Trans. Acoust., Speech, Signal Processing*, 1987.

[24] M. Nayeri and W. K. Jenkins, "Analysis of alternate realizations of adaptive IIR filters," Submitted to *IEEE Trans. Circuit. Syst.*, 1987.

[25] W. K. Jenkins and M. Nayeri, "Adaptive filters realized with second order sections," *Proc. IEEE Intl. Conf. Acoust., Speech, Signal Processing*, Tokyo, Japan, Apr. 7-11, 1986.

[26] M. Nayeri, D. Marshall, and W. K. Jenkins, "Relationships between error surfaces and network architectures in adaptive filters," *Proc. IEEE Intl. Symp. Circuit. Syst.*, vol. 3, pp. 777-780, Philadelphia, PA, May 4-7, 1987.

[27] M. M. Sondhi, "An adaptive echo canceler," *Bell Syst. Tech. J.*, vol. 46, no. 3, pp. 497-511, Mar. 1967.

[28] J. R. Rosenberger and E. J. Thomas, "Performance of an adaptive echo canceler operating in a noisy, linear, time-invariant environment," *Bell Syst. Tech. J.*, vol. 50, no. 3, pp. 785-813, Mar. 1971.

[29] W. B. Mikhael and F. F. Yassa-Greiss, "A frequency domain adaptive echo cancellation algorithm," *Proc. Intl. Symp. Circuit. Syst.*, Chicago, IL, May 1981.

[30] R. D. Gitlin and J. S. Thompson, "A phase adaptive structure for echo cancellation," *IEEE Trans. Commun.*, vol. COM-26, no. 8, pp. 1211-1220, Aug. 1978.

[31] R. D. Gitlin and J. S. Thompson, "A new structure for adaptive digital echo cancellation," *Proc. Natl. Telecomm. Conf.*, Dallas, TX, Dec. 1976.

[32] R. Wehrmann, J. Van der List, and P. Meissner, "A noise-insensitive compromise gradient method for the adjustment of adaptive echo cancelers," *IEEE Trans. on Commun.*, vol. COM-28, no. 5, pp. 753-759, May 1980.

[33] R. W. Lucky, "Automatic equalization for digital communication," *Bell Syst. Tech. J.*, vol. 44, no. 4, pp. 547-589, Apr. 1965.

[34] R. W. Lucky, "Techniques for adaptive equalization of digital communication systems," *Bell Syst. Tech. J.*, vol. 45, no. 2, pp. 255-297, Feb. 1966.

[35] R. D. Gitlin and F. R. Magee, "Self-orthogonalizing adaptive equalization algorithms," *IEEE Trans. Commun.*, vol. COM-25, no. 7, pp. 666-672, July 1977.

[36] T. Walzman and M. Schwartz, "Automatic equalization using discrete frequency domain," *IEEE Trans. Inf. Theory*, vol. IT-29, no. 1, pp. 59-68, Jan. 1973.

[37] A. Benveniste, M. Goursat, and G. Ruget, "Robust identification of a nonminimum phase system: Blind adjustment of a linear equalizer in data communications," *IEEE Trans. Automat. Contr.*, vol. AC-25, no. 3, pp. 385-398, June 1980.

[38] I. D. Landau, "A feedback system approach to adaptive filtering," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 2, pp. 251-262, Mar. 1984.

[39] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete time multivariable adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 449-456, June 1980.

[40] I. D. Landau and H. M. Silveira, "A stability theorem with application to adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 305-312, Apr. 1979.

[41] T. Soderstrom, "On the uniqueness of maximum likelihood identification," *Automatica*, vol. 11, no. 2, pp. 193-197, 1975.

[42] T. Soderstrom, L. Ljung, and I. Gustavsson, "A theoretical analysis of recursive identification methods," *Automatica*, vol. 14, pp. 231-244, 1978.

[43] T. Soderstrom and P. Stoica, "Some properties of the output error method," *Automatica*, vol. 18, no. 12, pp. 1692-1716, Dec. 1982.

[44] K. J. Astrom and T. Soderstrom, "Uniqueness of the maximum likelihood estimation of the parameters of an ARMA model," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 769-773, 1974.

[45] P. Stoica and T. Soderstrom, "The Steiglitz-McBride algorithm revisited - Convergence analysis and accuracy aspects," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 3, June 1981.

[46] L. Ljung, "On the positive real transfer functions and the convergence of some recursive schemes," *IEEE Trans. Automat. Contr.*, vol. AC-22, no. 4, pp. 539-550, Aug. 1977.

[47] B. Friedlander, "System identification techniques for adaptive signal processing," *Circuit., Syst., Signal Processing*, vol. 1, no. 1, pp. 3-41, 1982.

[48] L. Ljung and T. Soderstrom, *Theory and Practice of Recursive Identification*. Cambridge, MA: The MIT Press, 1983.

[49] L. Ljung, *System Identification - Theory for The User*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1987.

[50] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction, and Control*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1984.

[51] M. Honig and D. G. Messerschmidt, *Adaptive Filters Structure, Algorithms, and Applications*. Hingham, MA: Kluwer Academic Publishers, 1984.

[52] B. Widrow and S. D. Stearns, *Adaptive Signal Processing*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1985.

[53] S. Haykin, *Adaptive Filter Theory*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1986.

[54] R. R. Bitmead, "Persistence of excitation conditions and the convergence of adaptive schemes," *IEEE Trans. Inf. Theory*, vol. IT-30, no.2, pp. 183-191, Mar. 1983.

[55] A. Benveniste, M. Goursat, and G. Ruget, "Analysis of stochastic approximation schemes with discontinuous and dependent forcing terms with applications to data communication algorithms," *IEEE Trans. Automat. Contr.*, vol. AC-25, no. 6, pp. 1042-1058, Dec. 1980.

[56] H. Fan, "Application of Benveniste's convergence results in the study of adaptive IIR filtering algorithms," *IEEE Trans. Inf. Theory*, to appear.

[57] A. Oppenheim and R. Schafer, *Digital Signal Processing*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1974.

[58] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1975.

[59] M. Nayeri and H. Fan, "A counter example to Stearns' conjecture," *Proc. 21st Asilomar Conf. Signal, System, and Computers*, Pacific Grove, CA, Nov. 1987.

[60] H. Fan and M. Nayeri, "Counter examples to Stearns' conjecture on error surfaces of adaptive IIR filters," *Proc. IEEE Intl. Conf. Acoust., Speech, Signal Processing*, New York, NY, Apr. 11-14, 1988.

[61] J. Ackermann, "Parameter space design of robust control systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 1058-1072, Dec. 1980.

[62] H. Fan, Personal communication, May 1987.

[63] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1980.

VITA

Majid Nayeri was born on June 1, 1960, in Tehran, Iran. He received the B.S. and M.S. degrees in Electrical Engineering from the University of Illinois, in 1982 and 1983, respectively.

From January of 1982 until August of 1983, Mr. Nayeri was a research assistant in the Communication Group at the Coordinated Science Laboratory on the Illinois campus. He then joined the Signal and Image Processing Group in August of 1984 as a research assistant. Mr. Nayeri also served as a teaching assistant in the Department of Electrical and Computer Engineering. He has accepted a position as an assistant professor in the Department of Electrical Engineering and Systems Science at Michigan State University. His research interests are in digital signal processing, control, and communication.