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This paper discusses typical applications of singular perturbation techniques to control problems in the last fifteen years. The first three sections are devoted to the standard model and its time-scale, stability and controllability properties. The next two sections deal with linear-quadratic optimal control and one with cheap (near-singular) control. Then the composite control and trajectory optimization are considered in two sections, and stochastic control in one section. The last section returns to the problem of modeling, this time in the context of large scale systems. The bibliography contains more then 250 titles.

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APPLICATIONS OF SINGULAR PERTURBATION TECHNIQUES TO CONTROL PROBLEMS*

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Abstract

This paper discusses typical applications of singular perturbation techniques to control problems in the last fifteen years. The first three sections are devoted to the standard model and its time-scale, stability and controllability properties. The next two sections deal with linear-quadratic optimal control and one with cheap (near-singular) control. Then the composite control and trajectory optimization are considered in two sections, and stochastic control in one section. The last section returns to the problem of modeling, this time in the context of large scale systems. The bibliography contains more than 250 titles.

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Introduction

This guided tour of the applications of singular perturbation techniques to control theory begins with a glance at the origins of the two disciplines. In a recent book review, O'Malley (1982) gives an erudite outline of the history of singular perturbations, starting from Prandtl's 1904 paper on fluid dynamical boundary layers. The benchmark works of Tichonov (1948) and Levinson (1950) which appeared almost half a century later, were to have a major impact on control applications in the 1960's and 1970's. Vasileva's (1963) continuation of Tichonov's work and Wasow's (1965) book finally placed singular perturbations within the framework of the analytic theory of differential equations. These texts, along with more recent books by Vasileva and Butuzov (1973), O'Malley (1974), and a paper by Hoppensteadt (1971), remain the most readable sources on asymptotic methods for ordinary differential equations.

Although control concepts appear in some 19th century papers, the origins of present day control theory are more recent. They are, first, the foundations of feedback theory, laid in the 1930's by Nyquist and Bode, and, second, the stability theory of nonlinear regulators developed by Lurie and Krasovski in the 1940's. The two approaches to the same feedback control problem differed in their use of frequency domain (complex variable) versus time domain (o.d.e. state variable) techniques. The control theory of the 1980's is a harmonious merger of these two methodologies with Pontryagin's maximum principle and Bellman's dynamic programming. The elegant synthesis was accomplished by Kalman in the early 1960's and further expanded by Wonham's geometric method.

In this paper a control engineer looks at singular perturbation methods as tools to solve problems in his field. The first and foremost problem is modeling, that is, how to mathematically describe the system to be controlled. Modeling for control is parsimonious and implicit. It is parsimonious, because the model should not be more detailed than required by the specific control task. It is implicit, because the extent of necessary detail is not known before the control task is accomplished. Typical control tasks are optimal regulation, tracking and guidance. Since these tasks are to be accomplished in the presence of unknown disturbances, parameter variations and other uncertainties, the control system must possess a sufficient degree of insensitivity and robustness.

How do the singular perturbation techniques respond to this challenge? Their key contribution, from which all other benefits follow, is at the level of modeling. Control engineers, like Molière's character Mr. Jourdain, had been simplifying their models long before they were told that what they were doing was a singular perturbation. As our bibliography shows, they became aware of the new tool about fifteen years ago and have been increasingly interested in it ever since.

For the control engineer, singular perturbations legitimize his ad hoc simplifications of dynamic models. One of them is to neglect some "small" time constants, masses, capacitances, and similar "parasitic" parameters which increase the dynamic order of the model. However, the design based on a simplified model may result in a system far from its desired performance or even an unstable system. If this happens, the control engineer needs a tool which will help him to improve his oversimplified design. He wants to treat the simplified design as a first step, which captures the dominant phenomena. The disregarded phenomena, if important, are to be treated in the second step.

It turns out that asymptotic expansions into reduced ("outer") and boundary layer ("inner") series, which are the main characteristic of singular perturbation techniques, coincide with the outlined design stages. Because most control systems are dynamic, the decomposition into stages is dictated by a separation of time scales. Typically, the reduced model represents the slowest (average) phenomena which in most applications are dominant. Boundary layer[†] (and sublayer) models evolve in faster time scales and represent deviations from the predicted slow behavior. The goal of the second, third, and later, design stages is to make the boundary layers and sublayers asymptotically stable, so that the deviations rapidly decay. The separation of time scales also eliminates stiffness difficulties and prepares for a more efficient hardware and software implementation of the controller.

This paper is a tutorial presentation of typical, but not all, applications of singular perturbation techniques to control problems. The focus is on systems modeled by ordinary differential equations and most topics discussed are deterministic. Only one out of ten sections is dedicated to stochastic problems because of the existence of two excellent surveys of singular perturbation methods in stochastic differential equations, Blankenship (1979) and Schuss (1980). A further bias in the choice of topics is personal. The author has greatly benefitted from the doctoral research of his former students P. Sannuti (1968), C. Hadlock (1970), R. Yackel (1971), R. Wilde (1972),

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[†]Although we are now talking about fast transients of control systems we continue to use the traditional boundary layer terminology.

J. Chow (1977), H. Javid (1977), K.-K. D. Young (1977), H. Khalil (1978), R. Phillips (1980), B. Avramovic (1980), P. Ioannou (1982), and G. Peponides (1982), and from many colleagues among whom particularly influential were A. Haddad, R. O'Malley, F. Hoppensteadt, G. Blankenship, A. Bensoussan, J. P. Quadrat, F. Delebecque, H. Kelley, A. Calise, M. Ardema, and P. Habets. As a consequence, their results are discussed in greater detail. However, this does not imply that the contributions of other authors are less significant. An attempt was made to compile an extensive bibliography, including additional titles not explicitly referenced in the text. Generous help by V. Saksena and J. O'Reilly made this task much easier. Special thanks are due to our Soviet colleagues A. B. Vasileva, M. G. Dmitriev, A. Pervozvanski, and V. Utkin, who sent us their lists of references.

The remaining text is organized into ten sections and concluding remarks. Sections 1 and 2 introduce a standard model and discuss its properties. Sections 3, 4, 5, and 6 deal with linear control problems in open-loop and feedback form. Sections 7 and 8 are devoted to nonlinear, and Section 9 to stochastic problems. In Section 10 we return to the issue of modeling by examining nonstandard models common in networks and other large scale systems. Although some results are quoted as theorems, they are spelled out in a less technical form than that in the referenced works, which should be consulted for more rigorous formulations. Whenever convenient, simple examples are inserted to illustrate basic concepts.

1. The Standard Singular Perturbation Model

The singular perturbation model of finite dimensional dynamic systems extensively studied in mathematical literature by Tichonov (1948,1952), Levinson (1950), Vasileva (1963), Wasow (1965), Hoppensteadt (1967,1971), O'Malley (1971, 1973), Lagerstrom and Casten (1972), etc., was also the first model to be used in control and systems theory. This model is in the explicit state variable form in which the derivatives of some of the states are multiplied by a small positive scalar ε, that is,

$$\dot{x} = f(x,z,u,\varepsilon,t), \qquad x \in R^n \tag{1.1}$$

$$\varepsilon\dot{z} = g(x,z,u,\varepsilon,t), \qquad z \in R^m \tag{1.2}$$

where $u = u(t)$ is the control vector and a dot denotes a derivative with respect to time t. It is assumed that f and g are sufficiently many times continuously differentiable functions of their arguments x,z,u,ε,t. The scalar ε represents all the small parameters to be neglected. In most applications having a single parameter is not a restriction. For example, if T_1 and T_2 are small time constants of the same order of magnitude, $O(T_1) = O(T_2)$, then one of them can be taken as ε and the other expressed as its multiple, say $T_1 = \varepsilon$, $T_2 = \alpha\varepsilon$, where $\alpha = T_2/T_1$ is a known constant.

In control and systems theory, the model (1.1), (1.2) is a convenient tool for "reduced order modeling," a common engineering task. The order reduction is converted into a parameter perturbation, called "singular." When we set $\varepsilon = 0$ the dimension of the state space of (1.1)-(1.2) reduces from n+m to n because the differential equation (1.2) degenerates into an algebraic or a transcendental equation

$$0 = g(\bar{x},\bar{z},\bar{u},0,t), \tag{1.3}$$

where the bar is used to indicate that the variables belong to a system with $\varepsilon=0$. We will say that the model (1.1)-(1.3) is in standard form if and only if the following crucial assumption concerning (1.3) is satisfied.

Assumption 1.1

In a domain of interest equation (1.3) has $k \geq 1$ distinct ("isolated") real roots

$$\bar{z} = \varphi_i(\bar{x},\bar{u},t), \qquad i = 1,2,...,k. \tag{1.4}$$

This assumption assures that a well-defined n-dimensional reduced model will correspond to each root (1.4). To obtain the i-th reduced model we substitute (1.4) into (1.1), so

$$\dot{\bar{x}} = f(\bar{x},\varphi_i(\bar{x},\bar{u},t),\bar{u},0,t). \tag{1.5}$$

In the sequel we will drop the subscript i and rewrite (1.5) more compactly as

$$\dot{\bar{x}} = \bar{f}(\bar{x},\bar{u},t). \tag{1.6}$$

This model is sometimes called a quasi-steady-state model, because z, whose velocity $\dot{z} = \frac{g}{\varepsilon}$ can be large when ε is small, may rapidly converge to a root of (1.3), which is the quasi-steady-state form of (1.2). We will discuss this two-time-scale property of (1.1), (1.2) in the next section.

The convenience of using a parameter to achieve order reduction has also a drawback: it is not always clear how to pick the parameters to be considered as small. Fortunately, in many applications our knowledge of physical processes and components of the system set us on the right track. Let us illustrate this by a couple of examples.

Example 1.1

A well-known model of an armature controlled DC-motor is

$$\dot{x} = az \tag{1.7}$$

$$L\dot{z} = bx - Rz + u \tag{1.8}$$

where x, z, and u are, respectively, speed, current, and voltage, R and L are armature resistance and inductance, and a and b are some motor constants. In most DC-motors L is a "small parameter" which is often neglected, so we set $\varepsilon = L$. In this case equation (1.3) is

$$0 = b\bar{x} - R\bar{z} + \bar{u} \tag{1.9}$$

and has only one root

$$\bar{z} = (\bar{u} + b\bar{x})/R. \tag{1.10}$$

Thus the reduced model (1.6) is

$$\dot{\bar{x}} = \frac{a}{R}(\bar{u} + b\bar{x}). \tag{1.11}$$

It is frequently used in the design of servosystems.

Example 1.2

In a feedback system, Fig. 1.1a, with a high-gain amplifier K, where the nonlinear block N is tanz, the choice of ε is not as obvious. However, any student of feedback systems would pick $\varepsilon = \frac{1}{K}$, where K is the amplifier gain, and obtain

$$\dot{x} = z \tag{1.12}$$

$$\varepsilon\dot{z} = -x - \varepsilon z - \tan z + u. \tag{1.13}$$

In this case (1.3) and (1.4) yield

$$0 = -\bar{x} - 0 - \tan\bar{z} + \bar{u} \tag{1.14}$$

$$\bar{z} = \tan^{-1}(\bar{u} - \bar{x}) \tag{1.15}$$

and hence the reduced model (1.6) is

$$\dot{\bar{x}} = \tan^{-1}(\bar{u} - \bar{x}). \tag{1.16}$$

This model is represented by the block diagram in Fig. 1.1b in which the loop with infinite gain $\varepsilon = 0$ is replaced by the inverse of the operator in the feedback path.

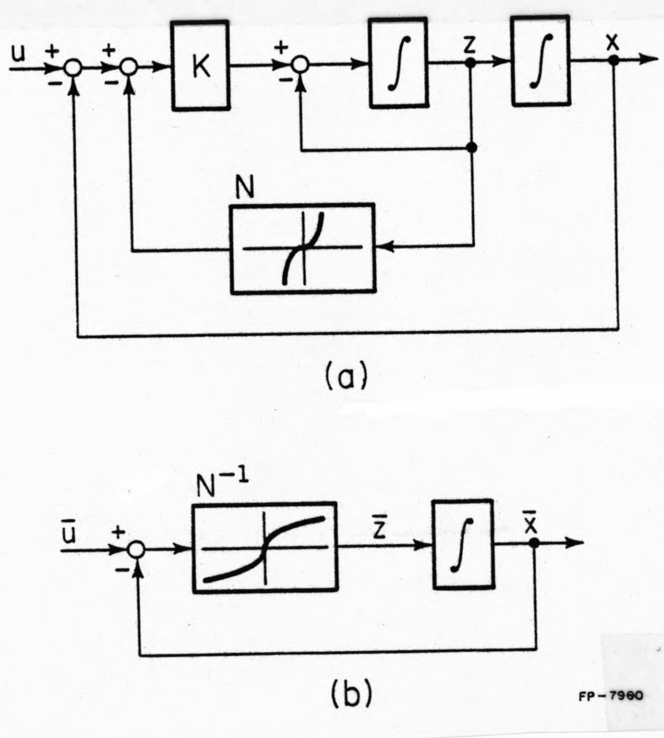

Fig. 1.1. System with a high gain amplifier: (a) full model, (b) reduced model.

It is easily seen that both (1.9) and (1.14) satisfy Assumption 1.1, that is, both models (1.7), (1.8) and (1.12), (1.13) appear in the standard form and their reduced models can be obtained by setting $\varepsilon = 0$. To avoid a misleading conclusion that this is always the case, let us consider another simple example in which the original model is not in the standard form.

Example 1.3

In the RC-network in Fig. 1.2a the capacitances C_1 and C_2 are $O(1)$, while one of the resistances, r, is much smaller than the other one, R. Letting $r = \varepsilon$ and using the capacitor voltages as the state variables and the input voltage u as the control, the model of this network is

$$\varepsilon\dot{v}_1 = \frac{1}{C_1}[-v_1 + v_2] \tag{1.17}$$

$$\varepsilon\dot{v}_2 = \frac{1}{C_2}[v_1 - (1+\frac{\varepsilon}{R})v_2 + \frac{\varepsilon}{R}u]. \tag{1.18}$$

If this model were in the form (1.1)-(1.2), both v_1 and v_2 would be considered as z-variables and (1.3) would be

$$0 = -\bar{v}_1 + \bar{v}_2 \tag{1.19}$$

However, Assumption 1.1 would then be violated because the roots of (1.3), in this case $\bar{v}_1 = \bar{v}_2$, are not distinct. The question remains whether the model of this RC-network can be simplified by setting $\varepsilon = 0$, that is, by neglecting the small parasitic resistance r? For a simple answer we multiply (1.17) by C_1, (1.18) by C_2, add them together and obtain an equation without ε, that is,

$$C_1\dot{v}_1 + C_2\dot{v}_2 = -\frac{1}{R}v_2 + \frac{1}{R}u. \tag{1.20}$$

This suggests that instead of v_1 we can use

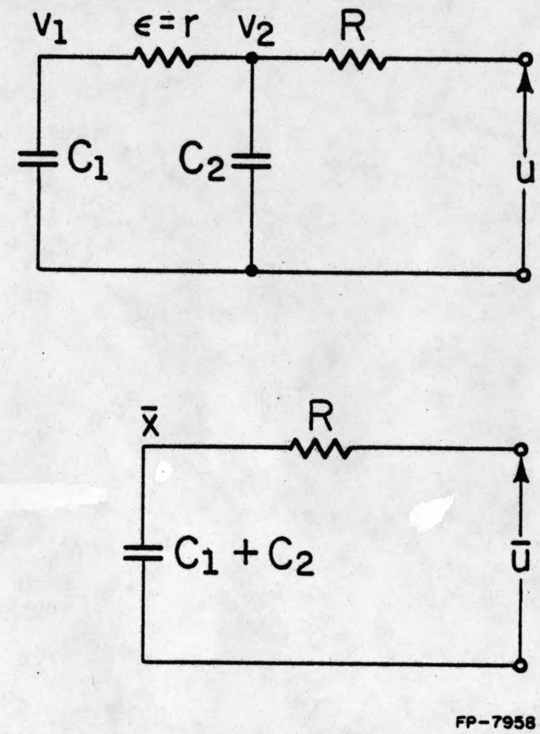

Fig. 1.2. (a) full model, (b) reduced model.

$$x = \frac{C_1 v_1 + C_2 v_2}{C_1 + C_2} \tag{1.21}$$

as a new voltage variable, which, along with $v_2 = z$, transforms (1.17)-(1.18) into

$$\dot{x} = \frac{1}{R(C_1 + C_2)} [-z + u] \tag{1.22}$$

$$\varepsilon \dot{z} = (\frac{1}{C_1} + \frac{1}{C_2}) x - (\frac{1}{C_1} + \frac{1}{C_2} - \frac{\varepsilon}{RC_2}) z + \frac{\varepsilon}{RC_2} u \tag{1.23}$$

Now (1.3) becomes

$$0 = (\frac{1}{C_1} + \frac{1}{C_2}) \bar{x} - (\frac{1}{C_1} + \frac{1}{C_2}) \bar{z} \tag{1.24}$$

and it satisfies Assumption 1.1. The substitution of $\bar{z}=\bar{x}$ into (1.22) results in the reduced model

$$\dot{\bar{x}} = \frac{1}{R(C_1+C_2)} [-\bar{x}+\bar{u}] \tag{1.25}$$

describing the circuit in Fig. 1.2b. Every electrical engineer would propose this circuit as a "low-frequency equivalent" of the circuit in Fig. 1.2a when $r=\varepsilon$ is small.

Most of the quoted singular perturbation literature assumes that model (1.1)-(1.2) is in the standard form, that is, it satisfies Assumption 1.1. The importance of Example 1.3 is that it points out the dependence of Assumption 1.1 on the choice of state variables. In most applications a goal of modeling is to remain close to the original "physical" variables. This was possible in our Examples 1.1 and 1.2, but not in Example 1.3, where a new voltage variable (1.21) had to be introduced. However, few engineers, accustomed to the simplified "equivalent" circuit in Fig. 1.2b, would question the "physicalness" of this new variable. On the contrary, physical properties of the circuit in Fig. 1.2a are more clearly displayed by the standard form (1.22)-(1.23). Nevertheless the problem of presenting and analyzing singular perturbation properties in a coordinate-free form is of fundamental importance. A geometric approach to this problem has recently been developed by Fenichel (1979) and Kopell (1979). More common are indirect approaches which deal with singular singularly perturbed problems, such as in O'Malley (1979), or transform the original "nonstandard" model into the standard form (1.1)-(1.2), such as in Peponides, Kokotovic, and Chow (1982), or Campbell (1980,1982). We will return to this modeling issue in Section 10.

2. Time-Scale Properties of the Standard Model

Singular perturbations cause a multi-time-scale behavior of dynamic systems characterized by the presence of both slow and fast transients in the system response to external stimuli. Loosely speaking, the slow response, or the "quasi-steady-state," is approximated by the reduced model (1.6), while the discrepancy between the response of the reduced model (1.6) and that of the full model (1.1)-(1.2) is the fast transient. To see this let us return to (1.1)-(1.6) and examine variable z which has been excluded from the reduced model (1.6) and substituted by its "quasi-steady-state" $\bar{z}$. In contrast to the original variable z, starting at t_o from a prescribed z^o, the quasi-steady-state $\bar{z}$ is not free to start from z^o and there may be a large discrepancy between its initial value

$$\bar{z}(t_o) = \varphi(\bar{x}(t_o),\bar{u}(t_o),t_o) \tag{2.1}$$

and the prescribed initial condition z^o. Thus $\bar{z}$ cannot be a uniform approximation of z. The best we can expect is that the approximation

$$z = \bar{z}(t) + O(\varepsilon) \tag{2.2}$$

will hold on an interval excluding t_o, that is, for $t \in [t_1,T]$ where $t_1 > t_o$. However, we can constrain the quasi-steady-state $\bar{x}$ to start from the prescribed initial condition x^o and, hence the approximation of x by $\bar{x}$ may be uniform. In other words,

$$x = \bar{x}(t) + O(\varepsilon) \tag{2.3}$$

may hold on an interval including t_o, that is, for all t in the interval $[t_o,T]$ on which $\bar{x}(t)$ exists.

The approximation (2.2) establishes that during an initial ("boundary layer") interval $[t_o,t_1]$ the original variable z approaches $\bar{z}$ and then, during $[t_1,T]$, remains close to $\bar{z}$. Let us remember that the speed of z can be large, $\dot{z} = g/\varepsilon$. In fact, having set ε equal to zero in (1.2) we have made the transient of z instantaneous wherever $g \neq 0$. Will z escape to infinity during this transient or converge to its quasi-steady-state $\bar{z}$?

To answer this question let us analyze $\varepsilon\dot{z}$, which may remain finite, even when ε tends to zero and $\dot{z}$ tends to infinity. We set

$$\varepsilon \frac{dz}{dt} = \frac{dz}{d\tau}, \qquad \text{hence } \frac{d\tau}{dt} = \frac{1}{\varepsilon}, \tag{2.4}$$

and use $\tau = 0$ as the initial value at $t = t_o$. The new time variable

$$\tau = \frac{t-t_o}{\varepsilon}; \qquad \tau = 0 \text{ at } t = t_o, \tag{2.5}$$

is "stretched," that is, if ε tends to zero, τ tends to infinity even for fixed t only slightly larger than t_o. On the other hand, while z and τ almost instantaneously change, x remains very near its initial value x^o. To describe the behavior of z as a function of τ we use the so-called "boundary layer system"

$$\frac{d\hat{z}}{d\tau} = g(x^o,\hat{z}(\tau),u,0,t_o), \tag{2.6}$$

with z^o as the initial condition for $\hat{z}(\tau)$, and x^o, t_o as fixed parameters. The solution $\hat{z}(\tau)$ of this initial value problem is used as a "boundary layer" correction of (2.2) to form a possibly uniform approximation of z,

$$z = \bar{z}(t) + \hat{z}(\tau) - \bar{z}(t_o) + O(\varepsilon). \tag{2.7}$$

Clearly $\bar{z}(t)$ is the slow, and $\hat{z}(\tau) - \bar{z}(t_o)$ is the fast transient of z.

To control these two transients the control u can also be composed of a slow control $\bar{u}(t)$, already assumed in the reduced model (1.6), and a fast control $\hat{u}(\tau)$ for the boundary layer system (2.6). The design of such a two-time-scale composite control is the main topic of several subsequent sections. In this section we concentrate on the assumptions under which the approximations (2.3) and (2.7) are valid.

Assumption 2.1

The equilibrium $\bar{z}(t_o)$ of (2.6) is asymptotically stable uniformly in x^o and t_o, and z^o belongs to its domain of attraction, so $\hat{z}(\tau)$ exists for $\tau \geq 0$.

If this assumption is satisfied,

$$\lim_{\tau\to\infty} \hat{z}(\tau) = \bar{z}(t_o), \tag{2.8}$$

uniformly in x^o, t_o, then z will come close to its quasi-steady-state $\bar{z}$ at some time $t_1 > t_o$. Interval $[t_o,t_1]$ can be made arbitrarily short by making ε sufficiently small. To assure that z stays close to $\bar{z}$, we think as if any instant $t \in [t_1,T]$ can be the initial instant. At such an instant z is already close to $\bar{z}$, which motivates the following assumption about the linearization of (2.6).

Assumption 2.2

The eigenvalues of $\partial g/\partial z$ evaluated along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ for all $t \in [t_o,T]$ have real parts smaller than a fixed negative number, i.e.,

$$\text{Re}\lambda\{\frac{\partial g}{\partial z}\} \leq -c < 0. \tag{2.9}$$

Both assumptions describe a strong stability property of the boundary layer system (2.6). If z^o is assumed to be sufficiently close to $\bar{z}(t_o)$, then Assumption 2.2 encompasses Assumption 2.1. We also note from (2.9) that the nonsingularity of $\partial g/\partial z$ along $\bar{x}(t)$, $\bar{z}(t)$ implies that the root $\bar{z}(t)$ is distinct as required by Assumption 1.1. These assumptions are common in much of the singular perturbation literature, Tichonov (1948,1952), Levinson (1950), Vasileva (1963), Hoppensteadt (1971), et al. These references contain the proof and refinements of the following result, frequently referred to as Tichonov's theorem.

Theorem 2.1:

If Assumptions 2.1 and 2.2 are satisfied, then (2.3) and (2.7) hold for all $t \in [t_o, T]$, while (2.2) holds for all $t \in [t_1, T]$, where the "thickness of the boundary layer" $t_1 - t_o$ can be made arbitrarily small by choosing small enough ε.

As we shall see, many control applications of singular perturbations make use of this theorem. In the remainder of this section we first illustrate Theorem 2.1 by a simple nonlinear example and then specialize it to linear systems.

Example 2.1

Let the system

$$\dot{x} = \frac{x^2 t}{z}, \qquad x(t_o) = x^o = 1, \qquad t_o = 0 \tag{2.10}$$

$$\varepsilon\dot{z} = -(z + xu)(z-2)(z-4), \qquad z(t_o) = z^o \tag{2.11}$$

be controlled by $u(t) = \bar{u}(t) = t$. In this case (1.3) is

$$0 = -(\bar{z} + \bar{x}t)(\bar{z}-2)(\bar{z}-4) \tag{2.12}$$

and has three distinct roots

$$\bar{z} = -\bar{x}t, \qquad \bar{z} = 2, \qquad \bar{z} = 4; \tag{2.13}$$

that is, there can be three reduced models. Analyzing the boundary layer system

$$\frac{d\hat{z}}{d\tau} = -(\hat{z} + \bar{x}\bar{u})(\hat{z}-2)(\hat{z}-4) \tag{2.14}$$

we see that Assumptions 2.1 and 2.2 hold for $\bar{z} = -\bar{x}\,\bar{u}$ if $z^o < 2$, and for $\bar{z} = 4$ if $z^o > 2$. Both assumptions are violated by $z = 2$ which is an unstable equilibrium of (2.14). Hence there are only two reduced models

$$\dot{\bar{x}} = -\bar{x}, \qquad \text{if} \quad z^o < 2 \tag{2.16}$$

$$\dot{\bar{x}} = \frac{\bar{x}^2 t}{4}, \qquad \text{if} \quad z^o > 2. \tag{2.17}$$

It is interesting to note that the solution $\bar{x} = (1 - \frac{t^2}{8})^{-1}$ of (2.17) escapes to infinity at $t = 2\sqrt{2}$. However, Theorem 2.1 still holds for $t \in [t_o, T]$ with $T < 2\sqrt{2}$. This is illustrated by simulation results in Fig. 2.1 for four different values of z^o, two for each reduced model. The approximate (dotted) and the exact (solid) trajectories for z are virtually indistinguishable, although $\varepsilon = 0.3$.

Let us now specialize (1.1)-(1.2) to linear systems

$$\dot{x} = Ax + Bz, \qquad x \in R^n \tag{2.18}$$

$$\varepsilon\dot{z} = Cx + Dz, \qquad z \in R^m \tag{2.19}$$

assuming first that A, B, C, and D are constant matrices. Clearly, Theorem 2.1 holds if $\text{Re}\lambda\{D\} < 0$. The root of (1.3),

$$\bar{z} = -D^{-1}C\bar{x} \tag{2.20}$$

substituted in (2.18) yields the reduced model

$$\dot{\bar{x}} = (A-BD^{-1}C)\bar{x}. \tag{2.21}$$

Introducing the fast variable η as the difference between z and its quasi-steady-state $\bar{z}$,

$$\eta = z + D^{-1}Cx \tag{2.22}$$

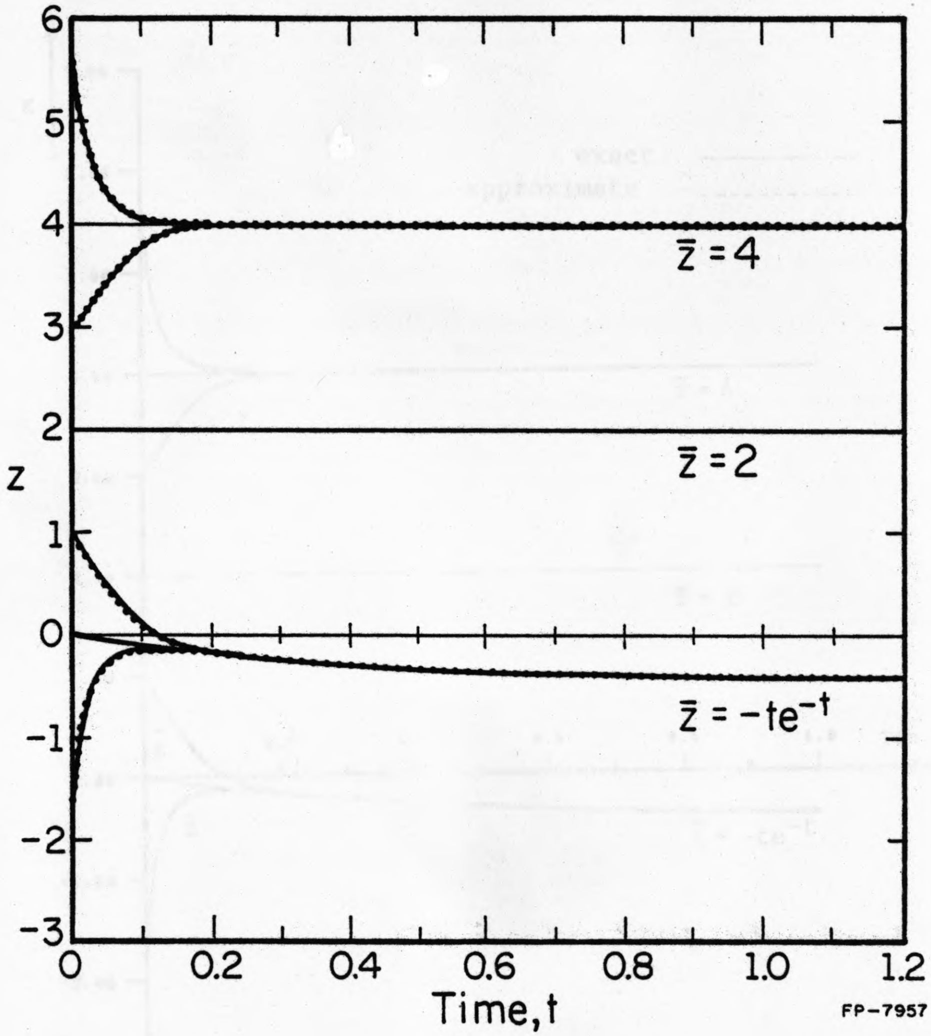

Fig. 2.1. Approximate and exact solutions for variable z of the system (2.10)-(2.11), with $\varepsilon = 0.3$.

the boundary layer system of (2.19) is simply

$$\frac{d\eta}{d\tau} = D\eta, \qquad \eta(0) = z^o + D^{-1}Cx^o \tag{2.23}$$

and its initial condition $\eta(0)$ is sometimes called the "boundary layer jump." In terms of η the approximation (2.7) becomes

$$z = \bar{z}(t) + \eta(\tau) + O(\varepsilon) \tag{2.24}$$

and its explicit expression is

$$z = -D^{-1}Ce^{(A-BD^{-1}C)t}x^o + e^{\frac{Dt}{\varepsilon}}(z^o + D^{-1}Cx^o) + O(\varepsilon). \tag{2.25}$$

This expression illustrates the following eigenvalue property of (2.18)-(2.19).

Corollary 2.1

For ε sufficiently small the eigenvalues of (2.18)-(2.19) are clustered into two groups, the n "slow" (small) eigenvalues λ_s, and the m "fast" (large) eigenvalues λ_f such that

$$\lambda_s = \lambda\{A-BD^{-1}C + O(\varepsilon)\}; \qquad \lambda_f = \frac{1}{\varepsilon}\lambda\{D + O(\varepsilon)\}. \tag{2.26}$$

As expected, the two-time-scale property of linear time invariant systems caused by the singular perturbation $\varepsilon \to 0$ is equivalent to the separation of the spectrum into its fast and slow parts. It is of interest to obtain this result algebraically. This is done by using a definition of the fast variable η more general than (2.22), namely

$$\eta = z - Lx \tag{2.27}$$

and requiring that L be a root of the matrix quadratic ("Riccati") equation

$$DL - \varepsilon LA + \varepsilon LBL - C = 0. \tag{2.28}$$

Then (2.27) transforms the original singular perturbation system (2.18)-(2.19) into a block-triangular form

$$\dot{x} = (A-BL)x + B\eta \tag{2.29}$$

$$\varepsilon\dot{\eta} = (D+\varepsilon LB)\eta \tag{2.30}$$

where eigenvalues are the eigenvalues of the blocks. An application of the implicit function theorem to (2.28) shows that

$$L = D^{-1}C + 0(\varepsilon) \tag{2.31}$$

and, hence, the result (2.26). More details on this algebraic approach to time scale modeling can be found in Kokotovic (1975), Anderson (1978), O'Malley and Anderson (1978), and Avramovic (1979).

In linear-time-varying systems the time-scale properties also depend on the speed of parameter variations. A well-known example is the stability of

$$\varepsilon \frac{dz}{dt} = D(t)z. \tag{2.32}$$

For $\varepsilon = 1$, even if

$$\mathrm{Re}\lambda\{D(t)\} \le -c_1 < 0, \qquad \forall t \ge t_o \tag{2.33}$$

system (2.32) can be unstable. However, when ε is small, the following result holds.

Theorem 2.2

If, in addition to (2.33), the derivative $\dot{D}(t)$ of $D(t)$ is bounded, say $\|\dot{D}(t)\| \le c_2$ for all $t \ge t_o$, then there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \le \varepsilon_1$ the system (2.32) is uniformly asymptotically stable.

To prove this theorem we define $M(t)$ for all $t \ge t_o$ by

$$D'(t)M(t) + M(t)D(t) = -I. \tag{2.34}$$

In view of (2.33), $M(t)$ is positive definite and its derivative $\dot{M}(t)$ is bounded, that is,

$$z'\dot{M}(t)z \leq c_3 z'z. \tag{2.35}$$

Theorem 2.2 follows from the fact that the derivative $\dot{v}$ of the Lyapunov function

$$v = z'M(t)z \tag{2.36}$$

for (2.32) is

$$\dot{v} \leq -(\frac{1}{\varepsilon} - c_3). \tag{2.37}$$

This analysis reveals the meaning of the boundary layer stability assumption of Theorem 2.1. For ε sufficiently small, the "frozen" spectrum of $\frac{1}{\varepsilon}\frac{\partial g}{\partial z}$, in this case $\frac{1}{\varepsilon}D(t)$, is sufficiently faster than the variations of the entries of $\frac{\partial g}{\partial z}$ and the "frozen" stability condition (2.33) applies.

We are now in a position to generalize the transformation (2.27) to the time-varying system (2.18)-(2.19), that is when

$$A = A(t), \quad B = B(t), \quad C = C(t), \quad D = D(t). \tag{2.38}$$

If the transformation matrix $L = L(t)$ in (2.27) satisfies the matrix ("Riccati") differential equation

$$\varepsilon\dot{L} = D(t) - \varepsilon LA(t) + \varepsilon LB(t)L - C(t) \tag{2.39}$$

then the time-varying system is in the form (2.29), (2.30). Equation (2.39) has been analyzed by Chang (1969,1972) who proved the following result.

<u>Theorem 2.3</u>

If the matrices (2.38) are bounded and (2.33) holds for all $t \in [t_o,T]$, then there exists $\varepsilon_2 > 0$ such that for all $t \in [t_o,T]$, $\varepsilon \in (0,\varepsilon_2]$, a bounded, continuously differentiable solution $L = L(t)$ of (2.39) exists and can be uniformly approximated by

$$L(t) = D^{-1}(t)C(t) + O(\varepsilon) \tag{2.40}$$

This theorem furnishes a simple proof of Theorem 2.1 for linear problems. The validity of the approximation (2.3) of x by $\bar{x}$ follows from replacing $L(t)$ by $D^{-1}(t)C(t)$ in (2.29) and neglecting $B(t)\eta$, because $\|\eta\| \leq c_4 \exp(-c_5 \frac{t-t_o}{\varepsilon})$, where $c_4, c_5 > 0$. The approximation of z by (2.24) follows by the same argument.

While the approximations (2.3) and (2.7) are within an $O(\varepsilon)$ error, expressions in two-time-scale asymptotic series can improve the accuracy up to any desired order. The details of construction and validation of asymptotic series are presented in Vasileva (1963), Hoppensteadt (1971), Vasileva and Butuzov (1973), and O'Malley (1974). In addition to these direct expansions, formal series can also be formed indirectly by expanding the transformation matrix L in (2.27), through its defining equation (2.28) or (2.39). This leads to a convenient numerical procedure, because L can be computed iteratively, as in Kokotovic (1975), Anderson (1978), and Avramovic (1979). An alternative procedure for the expansion of the state equation was presented in Kokotovic, Allemong, Winkelman, and Chow (1980). The validation of indirect and iterative procedures was given by Phillips (1983) who proved that they produce the terms of the asymptotic series in Vasileva and Butuzov (1973).

In conclusion, the solution $x(t,\varepsilon)$, $z(t,\varepsilon)$ of a singularly perturbed system satisfying Theorem 2.1 can be approximated by the solutions of two lower order systems like (2.29) and (2.30) in two separate time scales. In the remaining sections we will show how this result can be used to simplify the analysis and design of control systems.

3. Controllability and Stability

It is of conceptual and practical importance that many properties of singular perturbation systems can be deduced from the same properties of simpler slow and fast subsystems defined in separate time scales. In this section we concentrate on controllability and stability properties. We begin with the linear time varying control system

$$\dot{x} = A_{11}(t)x + A_{12}(t)z + B_1(t)u, \quad x \in R^n, \tag{3.1}$$

$$\varepsilon\dot{z} = A_{12}(t)x + A_{22}(t)z + B_2(t)u, \quad z \in R^m, \tag{3.2}$$

with a change of notation suitable for control applications. This system is said to be controllable if for any two points (x^1, z^1) and (x^2, z^2) in R^{n+m} there exists a bounded control $u(t)$ such that if $x(t_1) = x^1$, $z(t_1) = z^1$, then $x(t_2) = x^2$, $z(t_2) = z^2$ for some finite $t_2 > t_1$. Precise definitions of this and other control concepts can be found in most recent texts, such as Kailath (1980). Since the controllability and stability properties of (3.1)-(3.2) are invariant with respect to a nonsingular linear transformation, we will exhibit their dependence on time scales using a transformation proposed by Chang (1972). We let $L(t)$ satisfy (2.28) in the new notation and we also define $H(t)$ as a solution of

$$\varepsilon\dot{H} = H(A_{22} + \varepsilon LA_{12}) - \varepsilon(A_{11} - A_{12}L)H - A_{12} \tag{3.3}$$

which can be approximated by

$$H(t) = A_{12}(t)A_{22}^{-1}(t) + O(\varepsilon). \tag{3.4}$$

Denoting by I_k the $k \times k$ identity matrix, we introduce the transformation

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I_n - \varepsilon HL & -\varepsilon H \\ -L & I_m \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{3.5}$$

Henceforth the argument t is omitted whenever convenient. The inverse of (3.5) is

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I_n & \varepsilon H \\ L & I_m - \varepsilon LH \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

In the new coordinates ξ, η, the system (3.1)-(3.2) separates into two subsystems

$$\dot{\xi} = (A_{11} - A_{12}L)\xi + (B_1 + HB_2)u \tag{3.7}$$

$$\varepsilon\dot{\eta} = (A_{22} + \varepsilon LA_{12})\eta + (B_2 - \varepsilon LB_1 - \varepsilon LHB_2)u. \tag{3.8}$$

Taking into account (2.39) and (3.4) and noting from (3.5) that $\bar{x} = \bar{\xi}$, we arrive at the following conclusion.

Theorem 3.1

There exists $\varepsilon^* > 0$ such that the controllability of the slow (reduced) subsystem

$$\dot{\bar{x}} = (A_{11} - A_{12}A_{22}^{-1}A_{21})\bar{x} + (B_1 + A_{12}A_{22}^{-1}B_2)u \tag{3.9}$$

and the boundary layer controllability condition

$$\text{rank}[B_2(t), A_{22}(t)B_2(t), \ldots, A_{22}^{m-1}(t)B_2(t)] = m, \qquad \forall t \geq t_o \tag{3.10}$$

imply the controllability of the full system (3.1)-(3.2) for all $\varepsilon \in (0, \varepsilon^*]$.

We note that (3.10) has the form of a well-known controllability condition for linear time invariant systems, Kailath (1980). Here it is applied to the fast (boundary layer) subsystem

$$\frac{d\bar{\eta}}{d\tau} = A_{22}(t)\bar{\eta} + B_2(t)u \tag{3.11}$$

where t is treated as a fixed parameter and $u = u(\tau)$. This condition appeared in Kokotovic and Yackel (1972) and has been extended by Sannuti (1977) and used for time-optimal control in Kokotovic and Haddad (1975), Javid and Kokotovic (1977), and Javid (1978).

The sufficient condition of Theorem 3.1 is not necessary. As shown by Chow (1977), this condition excludes the controllable case when the fast subsystem is controlled through the slow subsystem.

Example 3.1

The two systems in Fig.3.1 have the same modes: one slow mode $s=-1$ and one fast mode $s=-\frac{1}{\varepsilon}$. However, they are structurally different. While in system (a) the slow mode is controlled through the fast mode, in system (b) the fast mode is controlled through the slow mode, which is a cause for its weak controllability. To illustrate this we give two minimal realizations of these systems

$$
\begin{array}{llll}
(a') & \dot{x} = -x+z & (a'') & \dot{\xi} = -\xi + \frac{1}{1-\varepsilon}u \\
 & \varepsilon\dot{z} = -z+u & & \varepsilon\dot{\eta} = -\eta+u \\
(b') & \dot{x} = -x+u & (b'') & \dot{\xi} = -\xi+u \\
 & \varepsilon\dot{z} = -z+x & & \varepsilon\dot{\eta} = -\eta+\frac{\varepsilon}{1-\varepsilon}u
\end{array}
\tag{3.12}
$$

where (a',b') are as in Fig.3.1 and (a",b") are the modal realizations, in this case identical to (3.7)-(3.8).

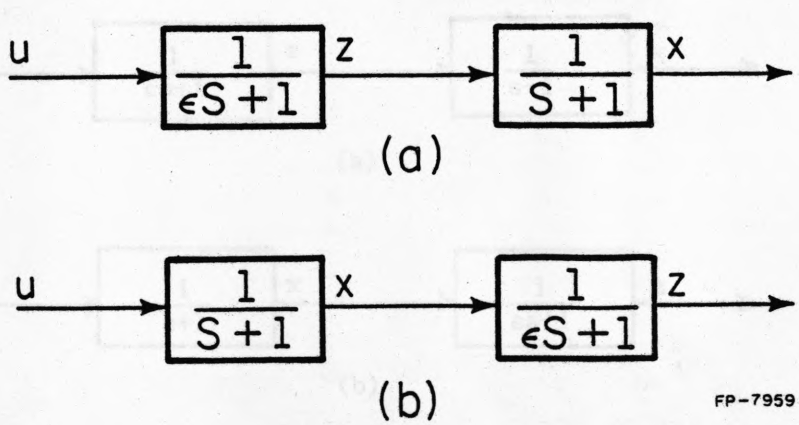

Fig.3.1. Controllable (a) and weakly controllable (b) singularly perturbed systems, where s is the Laplace transform variable.

Although for $\varepsilon > 0$ both systems are controllable, system (b) does not satisfy Theorem 3.1. We see from (b") that the fast mode in (b) is controlled through an ε-term. Thus it ceases to be controllable as $\varepsilon \to 0$ which is a type of weak controllability. Analogous observability results can be obtained by duality. For example, if for system (a) the output is $y = x$, then (a) is weakly observable.

The fact that the term "weakly" applies to the fast mode is of practical importance. When the fast modes are neglected as "parasitics," their weak controllability and observability contribute to the robustness of the simplified design. Although the problems of observability and robust observer design for singularly perturbed systems have attracted the attention of several authors, Porter (1974,1977), Balas (1978), O'Reilly (1979,1980), Javid (1980, 1982), Khalil (1981), Saksena and Cruz (1981), more work remains to be done on this important problem.

We proceed to the stability properties. Being invariant with respect to the transformation (3.5), these properties can be inferred from the properties of the two separate systems (3.7) and (3.8). Since the reduced system (3.9) and the boundary layer system (3.11) are regular perturbations of (3.7) and (3.8), respectively, the following result is immediate.

Theorem 3.2

If Theorem 2.2 holds for $A_{22}(t) = D(t)$ and the reduced system (3.9) is uniformly asymptotically stable, then there exists $\varepsilon^* > 0$ such that the original system (3.1)-(3.2) is uniformly asymptotically stable for all $\varepsilon \in (0,\varepsilon^*]$.

This theorem also follows as a corollary from more general results by Klimushev and Krasovski (1962), Wilde and Kokotovic (1972), Hoppensteadt (1974), and Habets (1974). The time-invariant version of the Theorem 3.2 was applied to networks with parasitics by Desoer and Shensa (1970) and to control systems by B. Porter (1974). A more detailed stability analysis leads to an estimate of ε^* in terms of bounds on system matrices and their derivatives. For the linear time-invariant case a bound was derived by Zien (1973) and for the time-varying case by Javid (1978). A robustness bound for linear time-invariant systems uses the Laplace transform of (3.1)-(3.2) expressed in a feedback form with u=0 as

$$x(s) = (sI-A_{11})^{-1}A_{12}z(s) \tag{3.13}$$

$$z(s) = [I-\varepsilon s(\varepsilon sI-A_{22})^{-1}](-A_{22}^{-1}A_{21})x(s). \tag{3.14}$$

Defining the transfer function matrices G and ΔG

$$G(s) = A_{22}^{-1}A_{21}(sI-A_{11})^{-1}A_{12} \tag{3.15}$$

$$\Delta G(s,\varepsilon) = -\varepsilon s(\varepsilon sI-A_{22})^{-1} \tag{3.16}$$

and denoting by $\bar{\sigma}(M)$ and $\underline{\sigma}(M)$ the largest and the smallest singular values of M, respectively, the robustness condition due to Sandell (1979) is stated as follows.

<u>Theorem 3.3</u>

If the reduced system (3.9) is stable, the full system (3.1), (3.2) remains stable for all $\varepsilon > 0$ satisfying

$$\bar{\sigma}(\Delta G(j\omega,\varepsilon)) \leq \underline{\sigma}(I+G^{-1}(j\omega)) \tag{3.17}$$

for all $\omega > 0$.

For nonlinear singularly perturbed systems the stability is frequently analyzed using separate Lyapunov functions for the reduced system and the boundary layer system and composing them into a single Lyapunov function for the full system. Let us first illustrate this on a nonlinear system which is linear in z,

$$\dot{x} = f(x) + F(x)z \tag{3.18}$$

$$\varepsilon\dot{z} = g(x) + G(x)z \tag{3.19}$$

assuming that $G^{-1}(x)$ exists for all x. The Lyapunov function introduced by Chow (1978) consists of two functions. The first function

$$v = a'(x)Q(x)a(x) \tag{3.19}$$

establishes the asymptotic stability of the reduced system $\dot{\bar{x}} = a(\bar{x})$, where

$$a(x) = f(x) - F(x)G^{-1}(x)g(x) \tag{3.20}$$

and $Q(x) > 0$ satisfies, for some differentiable $C(x) > 0$,

$$Q(x)a_x(x) + a_x'(x)Q(x) = -C(x), \quad a_x = \frac{\partial a}{\partial x} \tag{3.21}$$

where prime denotes a transpose. The second function

$$w = (z + \Gamma g - P^{-1}\Gamma'F'v_x')'P(z + \Gamma g - P^{-1}\Gamma'F'v_x'), \tag{3.22}$$

where $\Gamma = G^{-1}(x)$ and $P(x)$ satisfies

$$P(x)G(x) + G'(x)P(x) = -I, \tag{3.23}$$

establishes the asymptotic stability (uniform in x) of the fast (boundary layer) subsystem

$$\frac{d\eta}{d\tau} = G(x)\eta + g(x). \tag{3.24}$$

The Lyapunov function $V(x,z,\varepsilon)$ for the full system (3.18)-(3.19) is composed from v and w as follows

$$V(x,z,\varepsilon) = v(x) + \frac{\varepsilon}{2}w(x,z). \tag{3.25}$$

It can be used to estimate the dependence of the domain of attraction of $x = 0$, $z = 0$ on ε as illustrated by the following example.

Example 3.2

For the system

$$\dot{x} = x - x^3 + z, \qquad \varepsilon\dot{z} = -x - z, \tag{3.26}$$

the Lyapunov function (3.25) is

$$V(x,z,\varepsilon) = \frac{x^4}{4} + \frac{\varepsilon}{4}(z + x + 2x^3)^2. \tag{3.27}$$

When $\varepsilon < 0.01$ it is found that the region of attraction of $x = 0$, $z = 0$ includes $|x| \leq 1$, $|z| \leq 10$, while for $\varepsilon < 0.005$ the z bound is extended to $|z| \leq 20$.

Among the stability results obtained by Klimushev and Krasovski (1962), Hoppensteadt (1967,1974), Habets (1974), Grujic (1979,1981), and Saberi and Khalil (1981,1982) for the more general nonlinear system

$$\dot{x} = f(x,z,t) \tag{3.28}$$

$$\varepsilon\dot{z} = g(x,z,t) \tag{3.29}$$

we briefly outline the result by Habets. The reduced system (1.5) of (3.28)-(3.29) is

$$\dot{x} = f(x,\varphi(x,t),t), \tag{3.30}$$

where $^-$ is omitted and $\varphi(x,t)$ satisfies

$$g(x,\varphi(x,t),t) = 0, \tag{3.31}$$

while the boundary layer system is

$$\frac{dz}{d\tau} = g(x,z,t). \tag{3.32}$$

For simplicity let $f(0,0,t) = 0$, $g(0,0,t) = 0$ and hence $\varphi(0,t) = 0$.

Theorem 3.4

Suppose that there exist Lyapunov functions $v(x,t)$ for (3.30) and $w(x,z,t)$ for (3.32) such that

$$a(\|x\|) \leq v(t,x) \leq b(\|x\|) \tag{3.33}$$

$$a(\|z-\varphi(x,t)\|) \leq w(x,z,t) \leq b(\|z-\varphi(x,t)\|) \tag{3.34}$$

where a and b are positive nondecreasing scalar functions. Furthermore, suppose that positive constants k_1 and k_2 exist such that

$$\dot{v}(x,t) \leq -k_1\|x\|^2, \tag{3.35}$$

$$\left\|\frac{\partial v}{\partial x}\right\| \leq k_2\|x\|, \tag{3.36}$$

$$\dot{w}(x,z,t) \leq k_1\|z-\varphi(x,t)\|^2, \tag{3.37}$$

$$\left|\frac{\partial w}{\partial t}\right| \leq k_2\|z-\varphi(x,t)\|(\|z-\varphi(x,t)\| + \|x\|), \tag{3.38}$$

$$\left\|\frac{\partial w}{\partial x}\right\| \leq k_2\|z-\varphi(x,t)\|, \tag{3.39}$$

$$\|f(x,z,t)-f(x,\varphi(x,t),t)\| \leq k_2\|z-\varphi(x,t)\|, \tag{3.40}$$

$$\|f(x,z,t)\| \leq k_2(\|x\| + \|z-\varphi(x,t)\|), \tag{3.41}$$

$$\|\varphi(x,t)\| \leq b(\|x\|), \tag{3.42}$$

where $\dot{v}$ in (3.35) denotes the t-derivative for the reduced system (3.30), while $\dot{w}$ in (3.37) denotes the τ-derivative for the boundary layer system (3.32). If (3.36) to (3.42) are satisfied, then there exists ε^* such that for all $\varepsilon \in (0,\varepsilon^*]$ the equilibrium $x=0$, $z=0$ of (3.28), (3.29) is uniformly asymptotically stable.

Obtaining more easily verifiable stability conditions remains an active research topic. Further progress has been reported by Grujic (1979,1981) and Saberi and Khalil (1981,1983). The relevance of this topic has recently increased due to the interest in robustness of adaptive systems, as discussed in Ioannou and Kokotovic (1982,1983). A possible approach to these problems is to investigate the perturbations of the absolute stability property as in

Siljak (1972), Ioannou (1981), and Saksena and Kokotovic (1981). Further stability, stabilizability, and robustness issues will be discussed in the sections dealing with regulator design.

4. Optimal Linear State Regulators

One of the basic results of control theory is the solution of the optimal linear state regulator problem by Kalman (1960), which reduces the problem to the solution of a matrix Riccati equation. For the linear singularly perturbed system (3.1)-(3.2) this equation is also singularly perturbed. It was investigated by Sannuti and Kokotovic (1969), Yackel (1971), Haddad and Kokotovic (1971), Kokotovic and Yackel (1972), O'Malley (1972), and O'Malley and Kung (1974). Another form of the regulator solution is obtained via a Hamiltonian boundary value problem which in this case is singularly perturbed. This approach was taken by O'Malley (1972b), Wilde (1972), Wilde and Kokotovic (1973), Asatani (1974), and others. A comparison of the two approaches was given by O'Malley and Kung (1974) and O'Malley (1975). In this section we give an outline of the Riccati equation approach.

The problem considered is to find a control u which, for the system (3.1)-(3.2), minimizes the quadratic cost

$$J = \frac{1}{2}\int_{t_o}^{t_f} (y'y + u'R(t)u)dt \tag{4.1}$$

where $y = C_1(t)x + C_2(t)z$ is the system output and $x(t_f)$, $z(t_f)$ are free. For $\varepsilon > 0$ the optimal state feedback control for this problem is

$$u(t) = -R(t)^{-1}[B_1'(t) \quad \frac{1}{\varepsilon} B_2'(t)]K \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{4.2}$$

where K is the positive definite solution of a Riccati equation, Kailath (1980), i.e.,

$$\frac{dK}{dt} = -KA - A'K + KBR^{-1}B'K - C'C, \tag{4.3}$$

with the end condition $K(t_f) = 0$. The matrices A and B are as in (4.4) below and C is the output matrix $C = [C_1 \quad C_2]$. A few terminological remarks will be helpful. The term "state feedback control" refers to the fact that the control law in (4.2) is an explicit function (in this case linear) of the state $x(t)$, $z(t)$. The term "state regulator" expresses the tendency of (4.1) to diminish the output y, and, under the complete observability condition, also the state $x(t)$, $z(t)$. This tendency can be enhanced by adding a terminal cost term to (4.1). As t_f increases, $x(t_f)$, $z(t_f)$ decrease and as $t_f \to \infty$, the state is "regulated" to zero, $x(t_f)$, $z(t_f) \to 0$. However, this is an asymptotic stability requirement which should not be confused with the fixed end-problem in which $x(t_f), z(t_f) = 0$ is a hard constraint. This latter problem is treated in the next section.

The singularity of (4.3) is due to the fact that the system matrices

$$A = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \dfrac{A_{21}(t)}{\varepsilon} & \dfrac{A_{22}(t)}{\varepsilon} \end{bmatrix}, \quad B = \begin{bmatrix} B_1(t) \\ \dfrac{B_2(t)}{\varepsilon} \end{bmatrix}, \tag{4.4}$$

are unbounded as $\varepsilon \to 0$. It is not obvious that (4.3) is a singularly perturbed system in the form (1.1), (1.2). However, the search for a solution in the form

$$K = \begin{bmatrix} K_{11} & \varepsilon K_{12} \\ \varepsilon K_{12}' & \varepsilon K_{22} \end{bmatrix} \tag{4.5}$$

makes the singular perturbation form explicit, Sannuti (1968). Denoting $S_{11} = B_1 R^{-1} B_1'$, $S_{22} = B_2 R^{-1} B_2'$, $S_{12} = B_1 R^{-1} B_2'$ and substituting (4.5) into (4.3) we obtain

$$\frac{dK_{11}}{dt} = -K_{11}A_{11} - A_{11}'K_{11} - K_{12}A_{21} - A_{21}'K_{12}' + K_{11}S_{11}K_{11} + K_{11}S_{12}K_{12}'$$
$$+ K_{12}S_{12}'K_{11} + K_{12}S_{22}K_{12}' - C_1'C_1 \tag{4.6}$$

$$\varepsilon\frac{dK_{12}}{dt} = -K_{11}A_{12} - K_{12}A_{22} - \varepsilon A_{11}'K_{12} - A_{21}'K_{22} + \varepsilon K_{11}S_{11}K_{12} + K_{11}S_{12}K_{21}$$
$$+ \varepsilon K_{12}S_{12}'K_{12} + K_{12}S_{22}K_{22} - C_1'C_2 \tag{4.7}$$

$$\varepsilon\frac{dK_{22}}{dt} = -\varepsilon K_{12}'A_{12} - \varepsilon A_{12}'K_{12} - K_{22}A_{22} - A_{22}'K_{22} + \varepsilon^2 K_{12}'S_{11}K_{12} + \varepsilon K_{12}'S_{12}K_{22}$$
$$+ \varepsilon K_{22}S_{12}'K_{12} + K_{22}S_{22}K_{22} - C_2'C_2 \tag{4.8}$$

with the end condition

$$K_{11}(t_f) = 0, \qquad K_{12}(t_f) = 0, \qquad K_{22}(t_f) = 0. \tag{4.9}$$

This is clearly a singularly perturbed system of the type (1.1)-(1.2) and we can apply Theorem 1.1. When we set $\varepsilon = 0$, we get

$$\frac{d\bar{K}_{11}}{dt} = -\bar{K}_{11}(A_{11}-S_{12}\bar{K}_{12}') - (A_{11}-S_{12}\bar{K}_{12}')'\bar{K}_{11} + \bar{K}_{11}S_{11}\bar{K}_{11} - \bar{K}_{12}A_{21} - A_{21}'\bar{K}_{12}'$$
$$+ \bar{K}_{12}S_{22}\bar{K}_{12}' - C_1'C_1 \tag{4.10}$$

$$0 = -\bar{K}_{12}(A_{22}-S_{22}\bar{K}_{22}) - \bar{K}_{11}A_{12} - A_{21}'\bar{K}_{22} + \bar{K}_{11}S_{12}\bar{K}_{22} - C_1'C_2 \tag{4.11}$$

$$0 = -\bar{K}_{22}A_{22} - A_{22}'\bar{K}_{22} + \bar{K}_{22}S_{22}\bar{K}_{22} - C_2'C_2. \tag{4.12}$$

The only end condition to be imposed on this algebraic-differential system is $\bar{K}_{11}(t_f) = 0$, while (4.11) and (4.12) now play the role of (1.3). A crucial property of this system is that (4.12) is independent of (4.10) and (4.11). To satisfy Assumption 1.1 we need a unique positive definite solution $\bar{K}_{22}$ of (4.12) to exist.

Assumption 4.1

For each fixed $t \in [t_o, t_f]$ the pair $A_{22}(t)$, $B_2(t)$ is stabilizable and pair $A_{22}(t)$, $C_2(t)$ is detectable.

For this assumption to hold it is sufficient that the controllability condition (3.11) and

$$\operatorname{rank}[C_2'(t), A_{22}'(t)C_2'(t), \ldots, A_{22}'(t)^{m-1}C_2'(t)] = m \tag{4.13}$$

hold for all $t \in [t_o, t_f]$. Under Assumption 4.1 eigenvalues of $A_{22}-S_{22}K_{22}$ all have negative real parts and (4.11) can be solved for $\bar{K}_{12}$ in terms of $\bar{K}_{22}$, known from (4.12), and $\bar{K}_{11}$. Thus, the root (1.4) of interest in this case is distinct (isolated). The boundary layer system at t_f corresponding to (4.11) and (4.12), in the reverse fast time $\tau = \frac{t-t_f}{\varepsilon}$, is

$$\frac{d\hat{K}_{12}(\tau)}{d\tau} = -\hat{K}_{12}(\tau)[A_{22}(t)-S_{22}(t)\hat{K}_{22}(\tau)] - [A_{21}(t)-S_{12}(t)\bar{K}_{11}(t)]'\hat{K}_{22}(\tau)$$
$$-\bar{K}_{11}(t)A_{12}(t) - C_1'(t)C_2(t) \tag{4.14}$$

$$\frac{d\hat{K}_{22}(\tau)}{d\tau} = -\hat{K}_{22}(\tau)A_{22}(t) - A_{22}'(t)\hat{K}_{22}(\tau) + \hat{K}_{22}(\tau)S_{22}(t)\hat{K}_{22}(\tau) - C_2'(t)C_2(t) \tag{4.15}$$

with $\hat{K}_{12} = 0$ and $\hat{K}_{22} = 0$ at $\tau = 0$. For fixed t and $\varepsilon \to 0$ the limit (2.8) of Assumption 2.1 is to be taken as $\tau \to -\infty$. It follows from the regulator theory that Assumption 4.1 guarantees that, as $\tau \to -\infty$, the solution $\hat{K}_{22}(\tau)$ of (4.15) converges uniformly to the positive definite root $\bar{K}_{22}(t)$ of (4.12), that is, to the solution of a "boundary layer" regulator problem for each fixed $t \in [t_o, t_f]$. The uniform asymptotic stability of equation (4.14), which is linear in $\hat{K}_{12}(\tau)$, follows from standard stability theorems. Thus (4.14) and (4.15) satisfy Assumption 2.1. Furthermore, matrix $\partial g/\partial z$ of Assumption 2.2 for (4.14)-(4.15) is block upper triangular with the eigenvalues identical to the eigenvalues of

$-[A_{22}(t)-S_{22}(t)\bar{K}_{22}(t)]$. Thus the uniform asymptotic stability of the boundary layer regulator also guarantees that Assumption 2.2 is satisfied. Hence the following result.

Theorem 4.1

If Assumption 4.1 is satisfied then for all $t \in [t_o, t_f]$ the solution of the full Riccati equation (4.3) is approximated by

$$K_{11}(t) = \bar{K}_{11}(t) + 0(\varepsilon) \tag{4.16}$$

$$K_{12}(t) = \bar{K}_{12}(t) + \hat{K}_{12}(\tau) - \bar{K}_{12}(t_f) + 0(\varepsilon) \tag{4.17}$$

$$K_{22}(t) = \bar{K}_{22}(t) + \hat{K}_{22}(\tau) - \bar{K}_{22}(t_f) + 0(\varepsilon) \tag{4.18}$$

that is, by separately solving the slow ("reduced") and the fast ("boundary layer") Riccati systems. Excluding the boundary layer correction terms the approximation

$$K_{11}(t) = \bar{K}_{11}(t) + 0(\varepsilon) \tag{4.19}$$

$$K_{12}(t) = \bar{K}_{12}(t) + 0(\varepsilon) \tag{4.20}$$

$$K_{22}(t) = \bar{K}_{22}(t) + 0(\varepsilon) \tag{4.21}$$

is valid for all $t \in [t_o, t_1]$, where $t_1 < t_f$ can be made arbitrarily close to t_f by choosing ε small enough.

Higher order approximations are given in Yackel and Kokotovic (1973) and O'Malley and Kung (1974). Theorem 4.1 has important practical implications. First, we note that (4.15) represents the time-invariant Riccati equation depending on the fixed parameter t, which is, in fact, an independent optimality condition for the boundary layer regulator problem (3.10) in fast time scale τ.

Then the resulting feedback matrix $A_{22}-S_{22}K_{22}$ satisfies Theorem 2.2, that is, it guarantees the uniform asymptotic stability of the boundary layer. This demonstrates the stabilizing role of the fast regulator feedback K_{22}. We reiterate that the weakly controllable (stabilizable) case is excluded, that is, Theorem 4.1 requires that the fast modes be controlled directly, rather than through the slow subsystem. Although not necessary for stability, this requirement is needed for a robust design. The slow regulator is defined by the reduced system (4.10), (4.11), (4.12). At the first glance it appears that it depends on the quasi-steady-state solution $\bar{K}_{22}$ of the fast regulator. This would allow it to differ from the regulator solution for the problem in which ε is neglected already in the system (3.1)-(3.2) and in the cost (4.1), rather than later in the Riccati equation. The difference between the two reduced solutions would indicate nonrobustness, because the result would depend on when ε is neglected.

Let us investigate the reduced solution when, instead of neglecting ε in the Riccati equation, we neglect it in the model (3.1), (3.2) by substituting

$$z_r = -A_{22}^{-1}(A_{21}x_r + B_2u_r) \tag{4.27}$$

into (3.1)-(3.2) to obtain

$$\dot{x}_r = A_r(t)x_r + B_r(t)u_r \tag{4.23}$$

$$y_r = C_r(t)x_r + D_r(t)u_r \tag{4.24}$$

$$J_r = \frac{1}{2}\int_{t_o}^{t_f}(x_r'C_r'C_rx_r + 2x_r'C_rD_ru_r + u_r'R_ru_r)dt \tag{4.25}$$

where

$$A_r = A_{11}-A_{12}A_{22}^{-1}A_{21}, \qquad B_r = B_1-A_{12}A_{22}^{-1}B_2 \tag{4.26}$$

$$C_r = C_1C_2A_{22}^{-1}A_{21}, \qquad D_r = -C_2A_{22}^{-1}B_2, \qquad R_r = R+D_r'D_r. \tag{4.27}$$

The problem with the model reduced will be indicated by a subscript "r," to distinguish it from the reduced Riccati problem indicated by a bar. The Riccati equation corresponding to (4.23)-(4.25) is

$$\frac{dK_r}{dt} = -K_r(A_r - B_r R_r^{-1} D_r' C_r) - (A_r - B_r R_r^{-1} D_r' C_r)' K_r + K_r B_r R_r^{-1} B_r K_r - C_r'(I + D_r R_r D_r')^{-1} C_r \tag{4.28}$$

with the end condition $K_r(t_f) = 0$. We observe that this equation, in contrast to (4.10), (4.11), (4.12), does not contain K_{12} or K_{22}. The relationship between this Riccati system for the reduced model, and (4.10), (4.11), (4.12), which is the reduced Riccati system for the full model, is established by Haddad and Kokotovic (1971) as follows.

Theorem 4.2

If Theorem 4.1 holds and $A_{22}^{-1}(t)$ exists for all $t \in [t_o, t_f]$, then the solution $K_r(t)$ of (4.28) is identical to the solution $\bar{K}_{11}(t)$ of (4.10).

For $C_2 = 0$ and hence $\bar{K}_{22} = 0$, $\bar{K}_{12} = -\bar{K}_{11} A_{12} A_{22}^{-1}$, the identity of (4.10) and (4.27) is clear by inspection. The proof for any C_2 satisfying Assumption 4.1 involves more calculation and leads to the important conclusion that $\bar{K}_{11}(t)$ is independent of $\bar{K}_{22}(t)$.

This result demonstrates the robustness of the optimal state regulator problem with respect to singular perturbations. The same robustness property is not automatic in other feedback designs. Khalil (1981) gives examples of non-robust feedback designs using reduced order observers or static output feedback. Gardner and Cruz (1978) show that, even with the state feedback, Nash games are non-robust with respect to singular perturbations.

Once the robustness of the optimal state regulator is established, we can proceed with the design which consists of implementing the control law (4.2) with approximate feedback gains (4.16), (4.17), (4.18). This is a two-time-scale design because the feedback gains depending on t and τ are obtained separately. However, an equivalent, but more direct approach is possible which uses only $K_r(t)$ from (4.28) and $K_{22}(\tau)$ from (4.15). This is the so-called composite control approach developed by Suzuki and Miura (1976) and Chow and Kokotovic (1976). We do not discuss the composite control of linear systems here because it follows as a special case of the nonlinear composite control presented in Section 7.

Further extensions of the results presented in this section are due to Glizer and Dmitriev (1975a,b,1977), Gaitsgori and Pervozvanski (1979) and Pervozvanski and Gaitsgori (1981). The singularly perturbed optimal regulator problem for linear difference (rather than differential) equations was solved by Blankenship (1981), and Litkouhi and Khalil (1983).

5. Linear Optimal Control

Although convenient for the feedback solution of linear optimal control problems with free endpoints, the Riccati equation approach must be modified in order to apply to problems with fixed endpoints. Two such modifications were developed by Wilde and Kokotovic (1973) and Asatani (1976). In general endpoint constraints require the solution of Hamiltonian boundary value problems, which are in our case singularly perturbed. Various forms of singularly perturbed boundary value problems, not directly related to control applications, were studied earlier by Levin (1957), Vishik and Liusternik (1958), Harris (1960), Vasileva (1963), Wasow (1965), Chang (1972), O'Malley (1974), and others. Most of these works develop "inner" (in τ and σ) and "outer" (in t) asymptotic expansions. This approach to the boundary value problem arising in linear optimal control was taken by O'Malley (1972b,1975), O'Malley and Kung (1974), and Sannuti (1974). The results are based on hypotheses assuring the matching at both ends of the optimal trajectory.

Another approach, more in the spirit of the regulator theory, is that of Wilde (1972) and Wilde and Kokotovic (1973). It exploits the stabilizing properties of both the positive definite and the negative definite solutions of the same Riccati equation appearing in the regulator problem. These solutions correspond to the exponential dichotomy of the Hamiltonian systems, Wilde and Kokotovic (1972). They split the original boundary value problem into two initial value problems, one of which is in reverse time. We present this approach by considering the same linear optimal control problem (3.1), (3.2), and (4.1), but this time with fixed endpoints

$$x(t_o) = x^o, \quad z(t_o) = z^o; \quad x(t_f) = x^f, \quad z(t_f) = z^f. \tag{5.1}$$

Using p and εq as the adjoint variables corresponding to x and z, respectively, the optimal control is obtained as

$$u = -R^{-1}(B_1'p + B_2'q). \tag{5.2}$$

The standard necessary optimality conditions, Kailath (1980), yield the singularly perturbed boundary value problem (5.1) for the Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \varepsilon\dot{z} \\ \dot{p} \\ \varepsilon\dot{q} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & -S_{11} & -S_{12} \\ A_{21} & A_{22} & -S_{12}' & -S_{22} \\ -C_1'C_1 & -C_1'C_2 & -A_{11}' & -A_{12}' \\ -C_2'C_1 & -C_2'C_1 & -A_{21}' & -A_{22}' \end{bmatrix} \begin{bmatrix} x \\ z \\ p \\ q \end{bmatrix}. \tag{5.3}$$

The reduced problem is

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{p}} \end{bmatrix} = \left\{ \begin{bmatrix} A_{11} & -S_{11} \\ -C_1'C_1 & -A_{11}' \end{bmatrix} - \begin{bmatrix} A_{12} & -S_{12} \\ -C_1'C_2 & -A_{21}' \end{bmatrix} \begin{bmatrix} A_{22} & -S_{22} \\ -C_2'C_2 & -A_{22}' \end{bmatrix}^{-1} \begin{bmatrix} A_{21} & -S_{12}' \\ -Q_{12}' & -A_{12}' \end{bmatrix} \right\} \begin{bmatrix} \bar{x} \\ \bar{p} \end{bmatrix} \tag{5.4}$$

with the end conditions

$$\bar{x}(t_o) = x^o, \qquad \bar{x}(t_f) = x^f. \tag{5.5}$$

Conditions will be imposed to guarantee existence of the inverse indicated. The end conditions on z had to be dropped because the slow parts $\bar{z}$ and $\bar{q}$ of z and q are obtained from the linear algebraic equations when $\varepsilon\dot{z} = 0$ and $\varepsilon\dot{q} = 0$ is set in (5.3). Hence, $\bar{z}$ and $\bar{q}$ in general do not satisfy the end conditions (5.1) and "boundary layers" appear at both ends of the optimal trajectory. The layer at the left end point must be uniformly asymptotically stable in the forward, and the layer at the right end point in the reverse time.

The two-time-scale design of a near optimal trajectory is summarized in the following theorem.

Theorem 5.1

Suppose that Assumption 4.1 is satisfied and $\bar{x}(t)$ and $\bar{p}(t)$ uniquely solve (5.4) and (5.5). Denote by P_{22} the positive definite root of the Riccati equation (4.12) at $t=t_o$ and by N_{22} its negative definite root at $t=t_f$. Let $L(\tau)$ and $R(\sigma)$ be the solutions of two mutually independent time-invariant initial value problems

$$\frac{dL(\tau)}{d\tau} = [A_{22}(t_o)-S_{22}(t_o)P_{22}]L(\tau) \tag{5.6}$$

$$L(0) = z^o - \bar{z}(t_o) \tag{5.7}$$

and

$$\frac{dR(\sigma)}{d\sigma} = [A_{22}(t_f)-S_{22}(t_f)N_{22}]R(\sigma) \tag{5.8}$$

$$R(0) = z^f - \bar{z}(t_f) \tag{5.9}$$

where $\tau = (t-t_o)/\varepsilon$ and $\sigma = (t-t_f)/\varepsilon$ are the "stretched" time scales. Then there exists $\varepsilon^* > 0$ such that for all $t \in [t_o, t_f]$, $\varepsilon \in (0,\varepsilon^*]$

$$x(t,\varepsilon) = \bar{x}(t) + 0(\varepsilon) \tag{5.10}$$

$$z(t,\varepsilon) = \bar{z}(t) + L(\tau) + R(\sigma) + 0(\varepsilon) \tag{5.11}$$

$$p(t,\varepsilon) = \bar{p}(t) + 0(\varepsilon) \tag{5.12}$$

$$q(t,\varepsilon) = \bar{q}(t) + P_{22}L(\tau) + N_{22}R(\sigma) + 0(\varepsilon) \tag{5.13}$$

$$u(t,\varepsilon) = \bar{u}(t) + U_L(\tau) + U_R(\sigma) + 0(\varepsilon) \tag{5.14}$$

where

$$\bar{u}(t) = -R^{-1}(B_1'\bar{p} + B_2'\bar{q}) \tag{5.15}$$

$$U_L(\tau) = -R^{-1}(t_o)B_2'(t_o)P_{22}L(\tau) \tag{5.16}$$

$$U_R(\sigma) = -R^{-1}(t_f)B_2'(t_f)N_{22}R(\sigma). \tag{5.17}$$

The time scales for these two operations can be selected to be independent. For the reduced problem, a standard two point boundary value technique is used. The advantage over the original problem is that the order is lower, and the fast phenomena due to ε are eliminated.

Example 5.1

We illustrate the procedure using the system and the cost

$$\dot{x} = z$$
$$\varepsilon\dot{z} = tz + u \tag{5.18}$$

$$J = \frac{1}{2}\int_1^2 [x^2 + (9-t^2)z^2 + u^2]dt \tag{5.19}$$

with end conditions as in (5.1). Since $A_{22} = t$, $B_2 = 1$, and $C_2'C_2 = 9-t^2$, Assumption 4.1 holds for $0 < t < 3$. The exact optimal solution must satisfy

$$\begin{aligned} \dot{x} &= z \\ \varepsilon\dot{z} &= tz - q \\ \dot{p} &= -x \\ \varepsilon\dot{q} &= -(9-t^2)z - p - tq \end{aligned} \tag{5.20}$$

subject to (5.1). When ε is set equal to zero, the reduced problem is

$$\begin{aligned} \dot{\bar{x}} &= -\frac{1}{9}\bar{p} \\ \dot{\bar{p}} &= -\bar{x}. \end{aligned} \tag{5.21}$$

Its solution $\bar{x}(t)$, $\bar{p}(t)$ is easily found using the eigenvalues $\frac{1}{3}$ and $-\frac{1}{3}$ of the system matrix in (5.21), while $\bar{z}$ and $\bar{q}$ are evaluated from

$$\bar{z} = -\frac{1}{9}\bar{p}, \qquad \bar{q} = -\frac{t}{9}\bar{p}. \tag{5.22}$$

Then the roots of the Riccati equation

$$2K_{22}t - K_{22}^2 + (9-t^2) = 0 \tag{5.23}$$

are

$$P_{22} = t_o + 3 = 4, \qquad N_{22}(t) = t_f - 3 = -1 \tag{5.24}$$

and are used in (5.6), (5.8)

$$\frac{dL}{d\tau} = -3L; \qquad \frac{dR}{d\sigma} = 3R \tag{5.25}$$

to obtain the layer correction terms

$$L = [z_o - \bar{z}(1)]e^{-3(t-1)/\varepsilon}$$

$$R = [z_f - \bar{z}(2)]e^{3(t-2)/\varepsilon}. \tag{5.26}$$

Thus the corrections $L(\tau)$ and $R(\sigma)$ are the solutions of the left and the right "boundary layer regulators," respectively. It is the right regulator (5.26) that allows us to automatically satisfy the end point matching condition for jump $z^f - \bar{z}(2)$. It is totally unstable in real time t, that is, asymptotically stable in the reverse time $t_f - t$. Typical exact and reduced trajectories, z and $\bar{z}$, and the correction terms are sketched in Fig. 5.1.

We can use the same example to illustrate the more common approach by O'Malley (1972b). Starting with (5.20) an asymptotic series in t, τ, and σ would be substituted for each of the variables and the terms with like powers of ε are identified. The first terms $\bar{x}(t)$, $\bar{z}(t)$, $\bar{p}(t)$, $\bar{q}(t)$ in the t-series are obtained from (5.21) and (5.22), as in this approach. However, instead of using the Riccati and the boundary layer systems, (5.23) and (5.25), the first terms $z(\tau)$, $q(\tau)$, $z(\sigma)$, $q(\sigma)$ in the τ- and the σ-series would be obtained from the τ- and the σ-form of (5.20), subject to appropriate matching of their

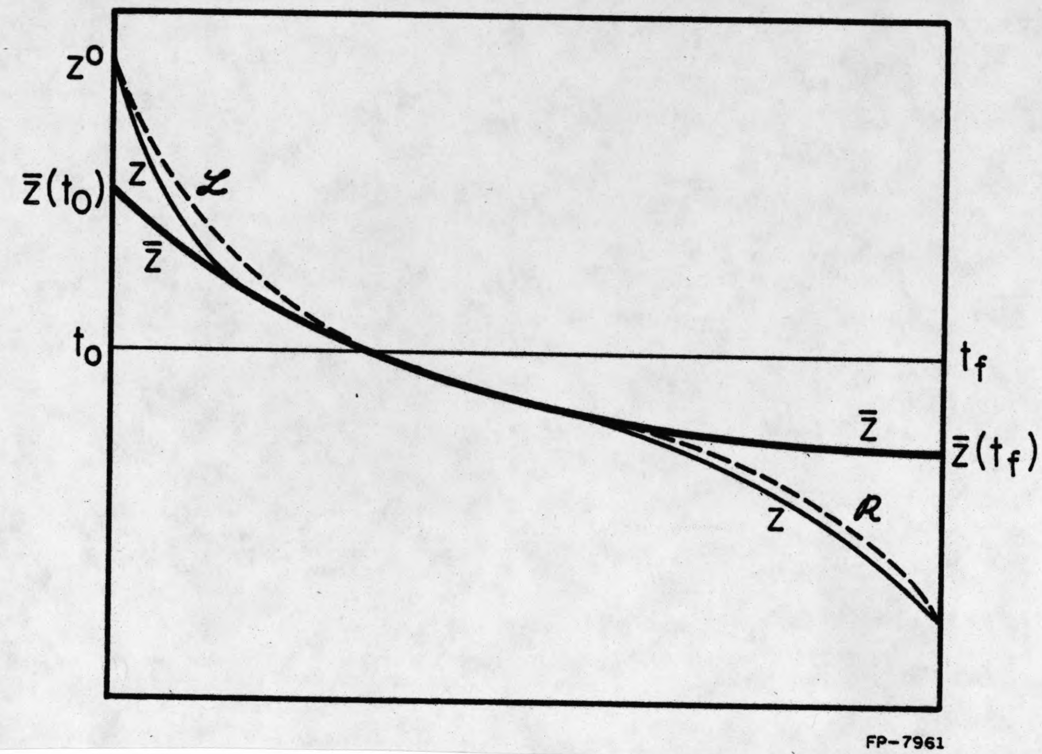

Fig. 5.1. Typical trajectories: optimal z, reduced $\bar{z}$, and boundary layer corrections L and R.

initial and end conditions. This approach can handle other types of consistent initial and end-conditions. Both approaches lead to the same asymptotic solution, but under different hypotheses. The relationship of the hypotheses was investigated by O'Malley (1975).

The structure of the solution of a singularly perturbed boundary value problem is always as in Fig. 5.1. The left boundary layer must be asymptotically stable in the time t and the right boundary layer in the reverse time t_f-t, and hence the complete solution can no longer be stable. It possesses both stable and unstable manifolds. The initial point is on the stable manifold, while the end-point is on the unstable manifold (stable in reverse time). Hence, the fixed end-point problem is not robust with respect to singular perturbations, because the right boundary layer does not have a stable implementation in real time. It is important to point out that this is not a property of the asymptotic procedure, but of the problem itself.

Let us return to the optimal control problem with a free end-point and assume that, instead of an integral cost like (4.1), the control is now to minimize a terminal cost, that is, a function of $x(t_f)$ and $z(t_f)$, such as

$$J = x^2(t_f) + z_1^2(t_f) + (z_2(t_f) + 0.25)^2. \tag{5.27}$$

The absence of an integral cost for u may lead to its impulsive behavior near t_f even if the system is not singularly perturbed. A more realistic behavior will result if u is constrained, $u \in U$, for example

$$u \in [-1, 1]. \tag{5.28}$$

Singular perturbation problems with terminal cost and control constraints have recently been studied by Dontchev and Veliov (1983) and Dontchev (1983). They show that the usual reduced problem is not valid and construct a new reduced problem which yields the desired approximation. As an illustration, we consider the minimization of (5.27), subject to (5.28), for the system

$$\begin{aligned} \dot{x} &= z_1 - z_2, && x(0) = 0.5 \\ \varepsilon \dot{z}_1 &= -z_1 + u, && z_1(0) = z_2(0) = 0 \\ \varepsilon \dot{z}_2 &= -2z_2 + u, && t_o = 0, \quad t_f = 1. \end{aligned} \tag{5.29}$$

For the usual reduced problem $\bar{z}_1 = \bar{u}$, $\bar{z}_2 = 0.5\bar{u}$ and

$$\dot{\bar{x}} = -0.5\bar{u}, \qquad \bar{J} = \bar{x}^2(t_f) + \bar{u}^2(t_f) + [0.5\bar{u}(t_f) + 0.25]^2 \tag{5.30}$$

the optimal control is

$$\bar{u}(t) = 1 \quad \text{for} \quad 0 \le t < 1, \quad \text{and } \bar{u}(t) = -0.1 \quad \text{for } t = 1. \tag{5.31}$$

However, this control yields $\bar{J} = 0.05$, which is not the limit of the exact optimal cost J^* as $\varepsilon \to 0$, because $J^* \to 0$. The reason for this discrepancy is the presence of the term $\bar{u}(t_f)$ in $\bar{J}$. This term contributes to the cost, but does not affect the slow trajectory. For this example the new limit problem proposed in Dontchev and Veliov (1983) consists of the fast problem over a reachable set $\mathcal{R}$, that is,

$$\inf\{z_1^2 + (z_2 + 0.25)^2;\quad z_1, z_2 \in \mathcal{R}\}, \tag{5.32}$$

and the slow problem in which the cost is $\bar{x}^2(t_f)$, rather than $\bar{J}$ in (5.30). Although a difficulty in this approach is the determination of the set $\mathcal{R}$, it is an important step in the study of singularly perturbed problems with control constraints.

6. Singular, Cheap, and High Gain Control

In our discussions thus far the singular perturbation properties of the system to be controlled were not altered by the control law. However, even if the original system is not singularly perturbed, a strong control action can force it to have fast and slow transients, that is, to behave like a singularly perturbed system. In feedback systems, the strong control action is achieved by high feedback gain. For a high gain system to emerge as a result of an optimal control problem, the control should be "cheap," that is, instead of u'Ru, its cost in (4.1) should be only $\varepsilon^2 u'Ru$, where $\varepsilon > 0$ is very small. On the other hand, an optimal control problem (3.1) – (3.2), and (4.1) with det R=0 is singular in the sense that the standard optimality conditions do not provide adequate information for its solution. Singular optimal controls and resulting singular arcs have been a control theory topic of considerable research interest, see for example Bell and Jacobson (1975). By formulating and analyzing the cheap control problem as a singular perturbation problem O'Malley and Jameson (1975,1977), Jameson and O'Malley (1975), and O'Malley (1976) have developed a new tool for a study of singular controls as the limits of cheap controls. The application of these results to the design of high gain and variable structure systems was discussed in Young, Kokotovic, and Utkin (1977). Here we closely follow a presentation in O'Malley (1978).

The cheap (near-singular) control problem for a linear system

$$\dot{x} = A(t)x + B(t)u, \qquad x \in R^n, \quad u \in R^r \tag{6.1}$$

is characterized by the presence of ε in the cost functional

$$J = \frac{1}{2}\int_{t_o}^{t_f} [x'Q(t)x + \varepsilon^2 u'R(t)u]dt \tag{6.2}$$

where Q and R are positive semidefinite and positive definite, respectively. For $\varepsilon > 0$ the standard optimality conditions hold,

$$u = -\frac{1}{\varepsilon^2} R^{-1}B'p \tag{6.3}$$

$$\varepsilon^2\dot{x} = \varepsilon^2 Ax - BR^{-1}B'p, \qquad x(t_o) = x^o \tag{6.4}$$

$$\dot{p} = -Qx - A'p, \qquad p(t_f) = 0 \tag{6.5}$$

but they are not defined for $\varepsilon = 0$. The singular perturbation method of the preceding sections does not apply because (6.4) is not in the standard form. On the other hand, singular control theory establishes that the optimal singular arcs satisfy $B'p = 0$ which is consistent with the formal reduced system $BR^{-1}B'\bar{p} = 0$ obtained from (6.4). The results of O'Malley and Jameson treat a hierarchy of cases, where Case ℓ is defined by requiring that for $j = 0,1,\ldots,\ell-2$ and all $t \in [t_o,t_f]$

$$B_j'QB_j = 0, \qquad B_{\ell-1}'QB_{\ell-1} > 0 \tag{6.6}$$

where

$$B_o = B, \qquad B_j = AB_{j-1} - \dot{B}_{j-1}. \tag{6.7}$$

(There are also problems beyond all cases and those where the case changes with t.) For Case ℓ the fast time variables are defined as

$$\tau = \frac{t-t_o}{\mu}, \qquad \sigma = \frac{t_f-t}{\mu}, \qquad \mu = \varepsilon^{\frac{1}{\ell}} \tag{6.8}$$

and the control and the corresponding trajectory are of the form

$$u = \bar{u}(t,\mu) + \frac{1}{\mu^{\ell}} v(\tau,\mu) + w(\sigma,\mu) \tag{6.9}$$

$$x = \bar{x}(t,\mu) + \frac{1}{\mu^{\ell-1}} \eta(\tau,\mu) + \mu\rho(\sigma,\mu) \tag{6.10}$$

where the slow limiting control is $\bar{u}(t) = \bar{u}(t,0)$ and the slow trajectory $\bar{x}(t) = \bar{x}(t,0)$ lies on a manifold of dimension $n-\ell r$. A crucial property of the control (6.9) is its term $\frac{1}{\mu^{\ell}} v(\tau,\mu)$, which allows a rapid transfer from the given initial state to the singular arc. In the limit as $\varepsilon \to 0$ the control behavior is impulsive and can be analyzed by distributions, Francis and Glover (1978) and Francis (1979, 1982). The trajectory will feature impulsive behavior at $t = t_o$ whenever $\ell > 1$.

Applying the Riccati approach to (6.3), (6.4), (6.5), that is, setting $p = Kx$ we get

$$u = -\frac{1}{\varepsilon^2} R^{-1}B'Kx \tag{6.11}$$

where K satisfies

$$\varepsilon^2(\dot{K} + KA + A'K + Q) = KBR^{-1}B'K, \qquad K(t_f) = 0. \tag{6.12}$$

This equation is in the standard form only if r=n and $\det B \neq 0$ which is a very special and unlikely situation. For $r < n$ the r.h.s. term of (6.12) is singular. Hence this equation is not in the standard form and the procedure in Section 4 does not apply. We see, however, that a reduced solution $\bar{K}_o$ satisfies

$$B'\bar{K}_o = 0 \tag{6.13}$$

but this $\bar{K}_o$ is not fully defined. Since B'KB can be nonsingular, we pre- and post-multiply (6.12) by B' and B, respectively,

$$\varepsilon^2[B'(\dot{K}B) + B'KAB + B'A'KB + B'QB] = B'KBR^{-1}B'KB. \tag{6.14}$$

Substituting K for εK_1 and equating lowest order terms we obtain the reduced equation

$$B'QB = (B'\bar{K}_1B)R^{-1}(B'\bar{K}_1B) \tag{6.15}$$

which in Case 1, when $B'QB > 0$, has a unique solution

$$B'\bar{K}_1B = \sqrt{R^{\frac{1}{2}}(B'QB)R^{\frac{1}{2}}} > 0. \tag{6.16}$$

Such an analysis suggests that K be sought in the form

$$K = \bar{K}_o(t) + \varepsilon\bar{K}_1(t) + \varepsilon\hat{K}_1(\sigma) + 0(\varepsilon^2) \tag{6.17}$$

where $\sigma = (t_f - t)/\varepsilon$ and $\hat{K}_1(\sigma)$ is the boundary layer correction at $t = t_f$. Substituting (6.17) into (6.12) and equating the terms of like powers in ε, we obtain at $\varepsilon = 0$

$$\dot{\bar{K}}_o + \bar{K}_o A_1 + A_1'\bar{K}_o - \bar{K}_o S\bar{K}_o + Q = 0, \qquad \bar{K}_o(t_f) = 0 \tag{6.18}$$

where

$$A_1 = A - B_1(B'QB)^{-1}B'Q \tag{6.19}$$

$$Q_1 = Q - QB(B'QB)^{-1}B'Q \tag{6.20}$$

$$S_1 = B_1(B'QB)^{-1}B_1' \geq 0 \tag{6.21}$$

and

$$\frac{d\hat{K}_1}{d\sigma} = -\hat{K}_1 S_f \bar{K}_1(t_f) - \bar{K}_1(t_f) S_f \hat{K}_1 - \hat{K}_1 S_f \hat{K}_1 \tag{6.22}$$

where $S_f = B(t_f)R^{-1}(t_f)B'(t_f)$. It can be shown that $\bar{K}_o(t)$ is defined by (6.13) and (6.18) and that (6.22) and $B'(t_f)\hat{K}_1(0) + B'(t_f)\bar{K}_1(t_f) = 0$ uniquely define $\hat{K}_1(\sigma)$ in terms of $\bar{K}_1(t_f)$. These facts and (6.16) allow us to form the control law (6.11) with the approximation (6.17) which, in view of $B'\bar{K}_o = 0$, becomes

$$u = -\frac{1}{\varepsilon} R^{-1}B'(\bar{K}_1 + \hat{K}_1)x. \tag{6.23}$$

With this high-gain feedback control the system (6.1) is

$$\varepsilon\dot{x} = [\varepsilon A - BR^{-1}B'(\bar{K}_1 + \hat{K}_1)]x. \tag{6.24}$$

Although it is not in the standard form we can expect that the reduced solution $\bar{x}$ satisfies $B'\bar{K}_1\bar{x} = 0$, that is, the corresponding singular arc is in the null space of $B'\bar{K}_1$. Since the prescribed initial condition $x(0) = x^o$ in general does not satisfy $B'\bar{K}_1 x^o = 0$, there will be a boundary layer at $t = t_o$, the rapid transition of x from x^o to $\bar{x}$. Another boundary layer will exist at $t = t_f$ because of the presence of $\hat{K}_1(\sigma)$ in (6.23).

We see that the analysis of singular perturbation problems which are not in the standard form is more complex than those which are. It is often useful to transform the problem into the standard form. The time-invariant problem (6.1) can, after a change of variables, always be written as

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{6.25}$$

where $x \in R^n$, $y \in R^r$, $u \in R^r$, and B_2 is a nonsingular $r \times r$ matrix. With a high-gain feedback control

$$u = \frac{1}{\varepsilon}(F_1 y + F_2 z) \tag{6.26}$$

where F_1 and F_2 are unspecified constant matrices, the system (6.25) becomes

$$\dot{y} = A_{11}y + A_{12}z \tag{6.27}$$

$$\varepsilon\dot{z} = (\varepsilon A_{21} + BF_1)y + (\varepsilon A_{22} + B_2F_2)z. \tag{6.28}$$

If F_2 is chosen such that

$$\mathrm{Re}\lambda\{B_2F_2\} < 0, \tag{6.29}$$

Theorem 2.1 holds and a two-time-scale design is possible by designing the reduced (slow) subsystem

$$\dot{\bar{x}} = [A_{11} - A_{12}(B_2F_2)^{-1}B_2F_1]\bar{x} \tag{6.30}$$

and the boundary layer (fast) subsystem

$$\frac{d\eta}{d\tau} = B_2F_2\eta. \tag{6.31}$$

Taking $F_1 = F_2F_s$ (6.30) becomes

$$\dot{\bar{x}} = (A_{11} - A_{12}F_s)\bar{x}. \tag{6.32}$$

By Corollary 2.1 the feedback matrices can be separately chosen, F_2 to place the eigenvalues of B_2F_2, and F_s to place the eigenvalues of $A_{11}-A_{12}F_s$. Such a design procedure was proposed by Young, Kokotovic, and Utkin (1977).

High gain systems have good disturbance rejection properties. They have been extensively studied in the control literature. Typical references include Shaked (1976,1978), Kouvaritakis (1978), Kouvaritakis and J. M. Edmunds (1979), and more recently Sastry and Desoer (1983). A geometric approach was developed by Willems (1981,1982). Insensitivity and disturbance decoupling properties are analyzed by Young (1976,1982a,b). High-gain systems may suffer because of neglected high frequency parasitics. This aspect was addressed by Young and Kokotovic (1982). Variable structure systems, Utkin (1977a,b), in most situations behave similarly to high-gain systems. Since they are described by differential equations with discontinuous right-hand-sides, their solutions are typically defined in the sense of Filipov. Grishin and Utkin (1980) show that this definition is well-posed with respect to singular perturbations. This and other current topics in variable structure systems are discussed in the most recent survey by Utkin (1983).

7. Composite Feedback Control of Nonlinear Systems

In the preceding three sections approximations of both the optimal feedback control and the optimal trajectory consisted of slow and fast parts. They were obtained from singularly perturbed Riccati equations or two-point boundary value problems. These optimality conditions also consisted of slow and fast parts. A further step toward a final decomposition of the two-time-scale design has been made which decomposes the optimal control problem itself into a slow subproblem and a fast subproblem. Separate solutions of these subproblems are then composed into a composite feedback control which is applied to the original system. As an engineering tool the composite control approach has both conceptual and practical advantages. The fast and the slow controllers appear as recognizable entities which can be implemented in separate hardware or software.

The composite control was first developed for time-invariant optimal linear state regulators by Suzuki and Miura (1976), Chow (1977), and Chow and Kokotovic (1976), and then for nonlinear systems by Chow and Kokotovic (1978a,b, 1981) and Suzuki (1981). A frequency domain composite design was developed by Fossard and Magni (1980). Extensions to stochastic control problems are due to Bensoussan (1981) and Khalil and Gajic (1982). The composite control has also been applied to large scale systems, as will be discussed in a subsequent section. The composite control approach is now presented following Chow and Kokotovic (1981).

The optimal control problems in the preceding sections were linear and over a finite time interval. We consider now a nonlinear infinite interval problem in which the system is

$$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u, \qquad x(0) = x^o \tag{7.1}$$

$$\varepsilon\dot{z} = a_2(x) + A_2(x)z + B_2(x)u, \qquad z(0) = z^o \tag{7.2}$$

where $x \in R^n$, $z \in R^m$, $u \in R^r$ and the cost to be optimized is

$$J = \int_0^\infty [p(x) + s'(x)z + z'Q(x)z + u'R(x)u]dt. \qquad (7.3)$$

Assumption 7.1

There exists a domain $D \subset R^n$, containing the origin as an interior point, such that for all $x \in D$ functions a_1, a_2, A_1, A_2, A_2^{-1}, B_1, B_2, p, s, r, and Q are differentiable with respect to x; a_1, a_2, p, and s are zero only at $x = 0$; Q and R are positive-definite matrices for all $x \in D$; the scalar $p+s'z+a'Qz$ is a positive-definite function of its arguments x and z, that is, it is positive except for $x = 0$, $z = 0$ where it is zero.

The usual approach would be to assume that a differentiable optimal value function $V(x,z,\varepsilon)$ exists satisfying

$$0 = \min_u [p + s'z + z'Qz + u'Ru + V_x(a_1+A_1z+B_1u) + \frac{1}{\varepsilon} V_z(a_2+A_2z+B_2u)] \qquad (7.4)$$

where V_x, V_z denote the partial derivatives of V. Since the control minimizing (7.4) is

$$u = -\frac{1}{2} R^{-1}(B_1'V_x' + \frac{1}{\varepsilon} B_2'V_2'), \qquad (7.5)$$

the problem would consist of solving the Hamilton-Jacobi equation

$$0 = p + s'z + z'Qz + V_x(a_1+A_1z) + \frac{1}{\varepsilon} V_z(a_2+A_2z)$$
$$-\frac{1}{4} (V_xB_1 + \frac{1}{\varepsilon} V_zB_2)R^{-1}(B_1'V_x' + \frac{1}{\varepsilon} B_2'V_2'), \quad V(0,0,\mu) = 0. \qquad (7.6)$$

To solve (7.6) is difficult even for well-behaved nonlinear systems. The presence of $\frac{1}{\varepsilon}$ terms increases the difficulties. To avoid the difficulties we do not deal with the full problem directly. In contrast, we take advantage of the fact that as $\varepsilon \to 0$ the slow and the fast phenomena separate, and define two separate lower dimensional subproblems. The solutions of the two subproblems are combined into a composite control whose stabilizing and near optimal properties can be guaranteed.

For the slow subproblem, denoted by subscript "s," the fast transient is neglected, that is

$$\dot{x}_s = a_1(x_s) + A_1(x_s)z_s + B_1(x_s)u_s, \qquad x_s(0) = x^o \tag{7.7}$$

$$0 = a_2(x_s) + A_2(x_s)z_s + B_2(x_s)u_s \tag{7.8}$$

and, since A_2^{-1} is assumed to exist,

$$z_s(x_s) = -A_2^{-1}(a_2 + B_2u_s) \tag{7.9}$$

is eliminated from (7.7) and (7.3). Then the slow subproblem is to optimally control the "slow subsystem"

$$\dot{x}_s = a_o(x_s) + B_o(x_s)u_s, \qquad x_s(0) = x^o \tag{7.10}$$

with respect to "slow cost"

$$J_s = \int_0^\infty [p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s]dt \tag{7.11}$$

where

$$\begin{aligned}
a_o &= a_1 - A_1A_2^{-1}a_2 \\
B_o &= B_1 - A_1A_2^{-1}B_2 \\
p_o &= p = s'A_2^{-1}a_2 + a_2'A_2'^{-1}QA_2^{-1}a_2 \\
s_o &= B_2'A_2'^{-1}(QA_2^{-1}a_2 - \tfrac{1}{2}s) \\
R_o &= R + B_2'A_2'^{-1}QA_2^{-1}B_2.
\end{aligned} \tag{7.12}$$

We note that, in view of Assumption 7.1, the equilibrium of the slow subsystem (7.10) is $x_s = 0$ and

$$p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s > 0, \quad \forall x_s \neq 0, \quad x_s \in D; \quad \forall u_s \neq 0. \tag{7.13}$$

Our crucial Assumption 7.2 concerns the existence of the optimal value function $L(x_s)$ satisfying the optimality principle

$$0 = \min_{u_s}[p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s + L_x(a_o(x_s) + B_o(x_s)u_s)] \tag{7.14}$$

where L_x denotes the derivative of L with respect to its argument x_s. The elimination of the minimizing control

$$u_s = -R_o^{-1}(s_o + \frac{1}{2} B_o'L_x') \tag{7.15}$$

from (7.14) results in the Hamilton-Jacobi equation

$$0 = (p_o - s_o'R_o^{-1}s_o) + L_x(a_o - B_oR_o^{-1}s_o) - \frac{1}{2} L_xB_oR_o^{-1}B_o'L_x', \qquad L(0) = 0, \tag{7.16}$$

where $p_o - s_o'R_o^{-1}s_o$ is positive definite in D.

Assumption 7.2

For all $x_s \in D$, (7.16) has a unique differentiable positive-definite solution $L(x_s)$ with the property that positive constants k_1, k_2, k_3, k_4 exist such that

$$k_1L_xL_x' \leq -L_x\bar{a}_o \leq k_2L_xL_x' \tag{7.17}$$

$$k_3\bar{a}_o'\bar{a}_o \leq -L_x\bar{a}_o \leq k_4\bar{a}_o'\bar{a}_o. \tag{7.18}$$

Then $L(x_s)$ is a Lyapunov function guaranteeing the asymptotic stability of $x_s = 0$ for the slow subsystem (7.10) controlled by (7.15), that is, for the feedback system

$$\dot{x}_s = a_o - B_o R_o^{-1}(s_o + \frac{1}{2} B_o' L_x') = \bar{a}_o(x_s). \qquad (7.19)$$

It also guarantees that D belongs to the region of attraction of $x_s = 0$.

For the fast subproblem, denoted by subscript "f," we recall that only an $O(\varepsilon)$ error is made by replacing x with x_s, or z with z_s. Thus we subtract (7.8) from (7.2), introduce $z_f = z - z_s$, $u_f = u - u_s$, neglect $O(\varepsilon)$ terms, and define the fast subproblem as

$$\varepsilon \dot{z}_f = A_2(x) z_f + B_2(x) u_f, \qquad z_f(0) = z^o - z_s(0), \qquad (7.20)$$

$$J_f = \int_0^\infty (z_f' Q(x) z_f + u_f' R(x) u_f) dt. \qquad (7.21)$$

This problem is to be solved for every fixed $x \in D$. It has the familiar linear quadratic form and a controllability assumption is natural.

<u>Assumption 7.3</u>

For every fixed $x \in D$,

$$\text{rank}[B_2, A_2 B_2, \ldots, A_2^{m-1} B_2] = m. \qquad (7.22)$$

Alternatively, a less demanding stabilizability assumption can be made. For each $x \in D$ the optimal solution of the fast subproblem is

$$u_f(z_f, x) = -R^{-1}(x) B_2'(x) K_f(x) z_f \qquad (7.23)$$

where $K_f(x)$ is the positive-definite solution of the x-dependent Riccati equation

$$0 = K_f A_2 + A_2' K_f - K_f B_2 R^{-1} B_2' K_f + Q. \qquad (7.24)$$

The control (7.23) is stabilizing in the sense that the fast feedback system

$$\varepsilon \dot{z}_f = (A_2 - B_2 R^{-1} B_2' K) z_f \triangleq \bar{A}_2(x) z_f \tag{7.25}$$

has the property that $\mathrm{Re}\lambda[\bar{A}_2(x)] < 0, \quad \forall x \in D.$

We now form a "composite" control $u_c = u_s + u_f$, in which x_s is replaced by x and z_f by $z + A_2^{-1}(a_2 + B_2 u_s(x))$, that is

$$\begin{aligned} u_c(x,z) &= u_s(x) - R^{-1} B_2' K_f (z + A_2^{-1}(a_2 - B_2 u_s(x))) \\ &= -R_o^{-1}(s_o + \frac{1}{2} B_o' L_x') - R^{-1} B_2' K_f (z + \bar{A}_2^{-1} \bar{a}_2) \end{aligned} \tag{7.26}$$

where

$$\begin{aligned} \bar{a}_2(x) &= a_2 - \frac{1}{2} B_2 R^{-1} (b_1' L_x' + B_2' V_1), \qquad \bar{a}_2(0) = 0 \\ V_1' &= -(s' + 2a_2' K_f + L_x \bar{A}_1) \bar{A}_2^{-1} \\ \bar{A}_1 &= A_1 - B_1 R^{-1} B_2' K_f . \end{aligned} \tag{7.27}$$

The properties of the system controlled by the composite control are summarized in the following theorem.

Theorem 7.1

When Assumptions 7.1, 7.2, and 7.3 are satisfied then there exists ε^* such that $\forall \varepsilon \in (0, \varepsilon^*]$, the composite control u_c defined by (7.26) stabilizes the full system (7.1),(7.2) in a sphere centered at x=0, z=0. The corresponding cost J_c is bounded. Moreover, J_c is near optimal in the sense that $J_c \to J_s$ as $\varepsilon \to 0$.

This theorem shows that the considered nonlinear regulator problem is well-posed with respect to ε. It is the basis for a two-time scale design procedure whose steps are illustrated by the following example.

Example 7.1

The system and the cost are

$$\dot{x} = -\frac{3}{4} x^3 + z \tag{7.28}$$

$$\varepsilon\dot{z} = -z + u \tag{7.29}$$

$$J = \int_0^\infty (x^6 + \frac{3}{4} z^2 + \frac{1}{4} u^2)dt. \tag{7.30}$$

<u>Step 1</u>. The slow subproblem

$$\dot{x}_s = -\frac{3}{4} x_s^3 + u_s \tag{7.31}$$

$$J_s = \int_0^\infty (x_s^6 + u_s^2)dt \tag{7.32}$$

consists in solving the Hamilton-Jacobi equation

$$L_x = \frac{dL}{dx_s} = x_s^3, \qquad L(0) = 0 \tag{7.33}$$

which yields

$$L = \frac{1}{4} x_s^4, \qquad u_s = -\frac{1}{2} x_s^3, \qquad \dot{x}_s = -\frac{5}{4} x_s^3. \tag{7.34}$$

<u>Step 2</u>. Assumption 7.2

$$k_1 x_s^6 \le \frac{5}{4} x_s^6 \le k_2 x_s^6, \tag{7.35}$$

$$\frac{25}{16} k_3 x_s^6 \le \frac{5}{4} x_s^6 \le \frac{25}{16} k_4 x_s^6, \tag{7.36}$$

is satisfied by

$$k_1 = k_2 = \frac{5}{4}, \qquad k_3 = k_4 = \frac{4}{5}. \tag{7.37}$$

<u>Step 3</u>. The fast subproblem

$$\varepsilon\dot{z}_f = -z_f + u_f \tag{7.38}$$

$$J_f = \int_0^\infty (\frac{3}{4} z_f^2 + \frac{1}{4} u_f^2)dt \tag{7.39}$$

is, in this case, independent of x and its solution is

$$K_f = \frac{1}{4}, \qquad u_f = -z_f, \qquad \varepsilon\dot{z}_f = -2z_f. \tag{7.40}$$

Step 4. The design is completed by forming the composite control

$$u_c = -x^3 - z \tag{7.41}$$

and applying it to the full system (7.28), (7.29), that is,

$$\dot{x} = -\frac{3}{4}x^3 + z \tag{7.42}$$

$$\varepsilon\dot{z} = -x^3 - 2z. \tag{7.43}$$

It should be noted that this system could not have been designed by methods based on linearization since its linearized model at $x = 0$, $z = 0$ has a zero eigenvalue. However, Theorem 7.1 guarantees that the equilibrium $x = 0$, $z = 0$ is asymptotically stable for ε sufficiently small.

8. Nonlinear Trajectory Optimization

We now consider a more general class of nonlinear optimal control problems on a finite interval $[t_o, t_f]$, frequently encountered in flight dynamics and start-up or shut-down operations for industrial plants. In Section 5 we have discussed such problems for linear systems and quadratic functionals. In this section we deal with nonlinear systems in the form

$$\dot{x} = f(x,z,u), \qquad x \in R^n \tag{8.1}$$

$$\varepsilon \dot{z} = g(x,z,u), \qquad z \in R^m \tag{8.2}$$

and the functional to be minimized

$$J = \int_{t_o}^{t_f} v(x,z,u)\,dt \tag{8.3}$$

where for simplicity of notation we do not show the dependence of f, g, and v on ε and t. The Hamiltonian function for this problem is defined as

$$H = v + p'f + q'g \tag{8.4}$$

that is, the second adjoint variable εq is scaled for g/ε. This was the problem that attracted control engineers to singular perturbations, Kokotovic and Sannuti (1968), Sannuti and Kokotovic (1969), Kelley and Edelbaum (1970), and Kelley (1970a,b,c,1971a,b), and singular perturbationists to control, Bagirova, Vasileva, Imanaliev (1967), O'Malley (1972,1974). In Bagirova, et al. (1967) the system was only the fast part (8.2), while in Kokotovic and Sannuti (1968) and Sannuti and Kokotovic (1969) only the reduced problem was considered. Papers by Kelley (1970a,b,c,1971a,b,1973) demonstrated the relevance of singular perturbations and boundary layer approximation for aircraft maneuver optimization and similar flight dynamics problems. These applications were further advanced by Ardema (1976,1979,1980), Calise (1976, 1978,1979,1980,1981), Mehra, et al. (1979), Sridhar and Gupta (1980), and Shinar (1981,1983). An application to nuclear reactors was reported in Reddy and Sannuti (1975). Asymptotic expansions and their validity were investigated by Hadlock (1970,1973), O'Malley (1974), Sannuti (1974a,b,1975), Freedman and Granoff (1976), Freedman and Kaplan (1976), Kurina (1977), Vasileva and Dmitriev (1980), Vasileva and Faminskaya (1981). A methodology similar to that of Sections 5 and 7 was developed by Chow (1979).

A different methodology was developed for linear time-optimal controls by Collins (1973), Kokotovic and Haddad (1975a,b), Javid and Kokotovic (1977), Javid (1978), and Halanay and Mirica (1979), in which case the bang-bang control exhibits outer low-frequency and inner high-frequency switches.

Using Pontryagin's principle, or $\frac{\partial H}{\partial u} = 0$ if the control is unconstrained, u is eliminated in terms of the state and adjoint variables. The result is a nonlinear singularly perturbed (2n+2m)-dimensional boundary value problem

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{8.5}$$

$$\varepsilon\dot{z} = \frac{\partial H}{\partial q} \qquad \varepsilon\dot{q} = -\frac{\partial H}{\partial z}\,. \tag{8.6}$$

In general, the initial and final states are required to be on some lower dimensional manifolds M_o at $t = t_o$ and M_f at $t = t_f$, that is, the boundary conditions for (8.5), (8.6) are

$$x(t_o), z(t_o) \in M_o, \qquad p(t_o), q(t_o) \perp M_o \tag{8.7}$$

$$x(t_f), z(t_f) \in M_f, \qquad p(t_f), q(t_f) \perp M_f. \tag{8.8}$$

From general properties of singularly perturbed boundary value problems, Wasow (1965), Chang (1972), Vasileva and Butuzov (1973), we know that an optimal trajectory consists of a slow "outer" part with "boundary layers" at the ends. In the limit as $\varepsilon \to 0$ the problem decomposes into one slow and two fast subproblems. The slow ("outer") subproblem

$$\dot{x}_s = \frac{\partial H_s}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial H_s}{\partial x_s} \tag{8.9}$$

is 2n-dimensional. To satisfy the remaining 2m boundary conditions, the layer ("inner") corrections $z_L(\tau_L)$, $z_R(\tau_R)$ for z, and $q_L(\tau_L)$, $q_R(\tau_R)$ for q are determined from the initial (L) and final (R) boundary layer systems with appropriately defined Hamiltonians H^L and H^R, that is,

$$\frac{dz_L}{d\tau_L} = \frac{\partial H^L}{\partial q_L}, \qquad \frac{dq_L}{d\tau_L} = -\frac{\partial H^L}{\partial z_L} \tag{8.10}$$

$$\frac{dz_R}{d\tau_R} = \frac{\partial H^R}{\partial q_R}, \qquad \frac{dq_R}{d\tau_R} = -\frac{\partial H^R}{\partial z_R} \tag{8.11}$$

where $\tau_L = \frac{t-t_o}{\varepsilon}$, while $\tau_R = \frac{t_f-t}{\varepsilon}$ is the reversed fast time scale. The results of these subproblems are used to form approximations

$$u = u_s(t) + u_L(\tau_L) + u_R(\tau_R) + 0(\varepsilon) \tag{8.12}$$

$$x = x_s(t) + 0(\varepsilon) \tag{8.13}$$

$$z = z_s(t) + z_L(\tau_L) + z_R(\tau_R) + 0(\varepsilon) \tag{8.14}$$

As was already discussed in Section 5, the L-layer must asymptotically decay, that is, the initial condition at $\tau_L = 0$ for (8.10) must be on a stable manifold. In Kelley's terminology, this initial condition is chosen to "suppress the unstable fast modes" at $t = t_o$, that is, at $\tau_L = 0$. At $t = t_f$, that is, at $\tau_R = 0$, the situation is opposite, (or the same, in the reverse time). The endlayer $z_R(\tau_R)$ must asymptotically decay as $\tau_R \rightarrow \infty$, that is, as $t \rightarrow -\infty$ and hence $z_R(0)$ must lie on a totally unstable manifold of (8.11).

In realistic nonlinear problems the matching of layers and reduced solutions is not an easy task. It is more complex if the control is constrained and if singular arcs occur. For this reason practical approaches are problem-dependent and based on prior experience. This is particularly true in flight dynamics, where reduced order approximations based on "energy state" or "point mass" and "rigid body" models are common. In flight dynamics singular perturbations are used to legitimize such "ruthless order-reductions which facilitate numerical computations in two ways: first, they reduce the number of costate initial values that must be determined simultaneously, and, second, they improve the conditioning of the boundary value problem. For example, the wild, undamped phugoid-like oscillations characteristic of the system (8.5), (8.6) for lifting atmospheric flight is avoided for the most part, being relegated to boundary layer corrections, where it may be no more docile, but, at least, can be dealt with separately over shorter lengths of arc." This quote is from Kelley (1973) which is still the clearest presentation of the subject.

The already quoted works of Kelley, Calise and Ardema contain details of several applications containing the layers not only at the ends, but also at some inner points where the reduced trajectory is permitted to be discontinuous. Another difficulty in these applications is a proper choice of fast and slow variables, and the selection of one or several small parameters.

Time scales differ in low thrust (aircraft) and high thrust (missile) applications. We will only give a simple example of a long range flight from the papers of Calise.

Example 8.1

The point mass equations of motion for two-dimensional flight using the sum of kinetic and potential energy

$$E = h + \frac{v^2}{2g} \tag{8.15}$$

as a state variable, can be written as

$$\dot{x} = v \cos \gamma, \qquad v = \sqrt{(E-h)/2g} \tag{8.16}$$

$$\varepsilon \dot{E} = \frac{(T-D)v}{W} \tag{8.17}$$

$$\varepsilon^2 \dot{h} = v \sin \gamma \tag{8.18}$$

$$\varepsilon^3 \dot{\gamma} = g \frac{L-W \cos \gamma}{Wv} \tag{8.19}$$

where T is thrust, D is drag, L is lift, W is weight, γ is the flight path angle, x is down range position, h is altitude, g is the gravitation constant and v is velocity, in this case not a state variable. Cost functional J is formulated to penalize fuel consumption and the time of flight. The "total" scaling with ε, ε^2, and ε^3 leads not only to a boundary layer, but also to a sublayer and a subsublayer. This scaling is introduced to reduce each subproblem to a scalar problem which can be solved in an explicit feedback form, thus simplifying the on-board implementation of the control law. However, this "total" scaling is limited to systems without oscillatory modes. Indeed, in the long range flight example, the "total" scaling gave the results, closely approximating the actual optimal trajectory, which is made up of climb, cruise, and descent arcs, Fig. 8.1. During the cruise $L = W$ and $\gamma = 0$ and E and h are chosen to minimize J. The change of weight W is periodically updated, such that the reduced (outer) solution is the cruise-climb path A-B.

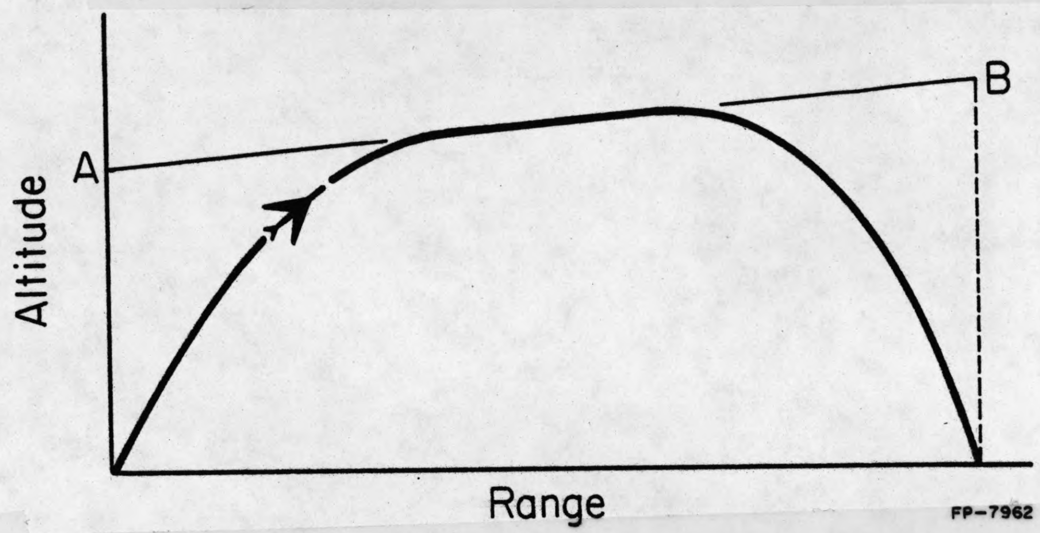

Fig. 8.1. Trajectory of long-range flight.

The first boundary layer with $\tau_1 = \frac{t}{\varepsilon}$ deals with the energy variable during the climb, while the sub- and subsublayers $\tau_2 = \frac{t}{\varepsilon^2}$ and $\tau_3 = \frac{t}{\varepsilon^3}$ deal with h and γ dynamics. If they are coupled as in some fighter aircraft, it is more appropriate to treat them in the same time scale τ_2, that is, to multiply $\dot{\gamma}$ by ε^2 rather than ε^3. For a transport aircraft both h and γ dynamics can be neglected. Their quasi-steady state obtained in the τ_1-layer is a good approximation in this case.

9. Stochastic Filtering and Control

Research in singular perturbation of filtering and stochastic control problems with white noise inputs has revealed difficulties not present in deterministic problems. This is due to the fact that the input white noise process "fluctuates" faster than the fast dynamic variables, which as $\varepsilon \to 0$, themselves tend to white noise processes. In their surveys of stochastic differential equations and diffusion models Blankenship (1979) and Shuss (1980) and Kushner (1982), (in a note), stress the importance of attaching clear probabilistic meaning to time scales.

To illustrate the problems arising in the singularly perturbed formulation of systems with white noise problems, we note that setting $\varepsilon = 0$ in the linear system

$$\dot{x} = A_{11}x + A_{12}z + B_1u + G_1w \tag{9.1}$$

$$\varepsilon\dot{z} = A_{21}x + A_{11}z + B_2u + G_2w \tag{9.2}$$

where w(t) is white Gaussian noise, is inadequate, since

$$\bar{z} = -A_{22}^{-1}(A_{21}\bar{x} + B_2\bar{u} + G_1w) \tag{9.3}$$

has a white noise component and, therefore, has infinite variance. As shown by Haddad (1976), variable $\bar{z}$ from (9.3) may be substituted for z in defining a reduced (slow) subsystem, but $\bar{z}$ cannot serve as an approximation for z in the mean square sense.

For the linear filtering of (9.1), (9.2) with respect to the observations

$$y = C_1x + C_2z + v \tag{9.4}$$

where v(t) is a white Gaussian noise independent of w(t), Haddad (1976) demonstrated that the Kalman filter can be approximately decomposed into two filters in different time scales. Similar results are obtained for near-optimal smoothing by Altshuler and Haddad (1978), and state estimation with uncertain singular perturbation parameter by Sebald and Haddad (1978).

For the control problem of the system (9.1), (9.2) and (9.4) with respect to the cost functional

$$J = E\{x'(T)\Gamma_1 x(T) + 2\varepsilon x'(T)\Gamma_{12} z(T) + \varepsilon z'(T)\Gamma_2 z(T) + \int_0^T (x'L_1 x + 2x'L_{12} z + z'L_2 z + u'Ru)\,dt\} \tag{9.5}$$

it was demonstrated by Teneketzis and Sandell (1977) and Haddad and Kokotovic (1977) that the optimal solution may be approximated by the solutions of two reduced order stochastic control problems in the slow and fast time scale. However, to avoid divergence $J \sim \frac{1}{\varepsilon}$, it is required that $L_2 \sim \varepsilon$, and $\Gamma_2 \sim \varepsilon^{\frac{1}{2}}$. A more general scaling is discussed by Khalil, Haddad, and Blankenship (1978). More recently Khalil and Gajic (1982) approached this problem via singularly perturbed Lyapunov equations. Razevig (1978) and Singh and Ram-Nandan (1982) have established the weak convergence, as $\varepsilon \to 0$, of the fast stochastic variable z which satisfies

$$\varepsilon \dot{z} = Az + \sqrt{\varepsilon}\, G w, \qquad \mathrm{Re}\lambda(A) < 0 \tag{9.6}$$

where w(t) is Gaussian white noise with covariance W, that is,

$$\lim_{\varepsilon \to 0} z(t;\varepsilon) = \bar{z} \quad \text{weakly} \tag{9.7}$$

where $\bar{z}$ is a constant Gaussian random vector with covariance P satisfying the Lyapunov equation

$$AP + PA' + GWG' = 0. \tag{9.8}$$

Alternative formulations of the linear stochastic regulator problem have been reported by Tsai (1978) and Khalil (1978). Khalil assumes a colored noise disturbance in the fast subsystem to account for situations when the correlation time of the input stochastic process is longer than the time constants of fast variables. Thus the optimal solution to the stochastic regulator problem can be approximated by the optimal solution of the slow subproblem and optimal cost J does not diverge.

A composite control approach to a class of nonlinear systems driven by white noise disturbances, as a stochastic version of the results reviewed in Section 8, was developed by Bensoussan (1981). He considered

$$\dot{x} = c(x)z + d(x) + 2\beta(x)u + \sqrt{2}\; w_1 \tag{9.9}$$

$$\varepsilon\dot{z} = a(x)z + b(x) + 2\alpha(x)u + \varepsilon\sqrt{2}\; w_2 \tag{9.10}$$

$$J^{\varepsilon}_{x,z}(u(\cdot)) = E\int_0^{\infty} e^{-\nu t}[(f(x) + h(x)z)^2 + u^2]dt \tag{9.11}$$

where $w_1(t)$, $w_2(t)$ are white noise processes independent of each other.

The optimal feedback law is obtained as

$$u^{\varepsilon}(x,z) = -\beta(x)V^{\varepsilon}_{x}(x,z) - \frac{\alpha(x)V^{\varepsilon}_{z}(x,z)}{\varepsilon} \tag{9.12}$$

where $V^{\varepsilon}(x,z)$ is the Bellman function. As $\varepsilon \to 0$, the optimal solution converges to the solutions of the two subproblems. The slow subproblem is

$$\dot{x} = -\frac{c}{a}(b + 2\alpha u_s) + d + 2\beta u_s + \sqrt{2}\; w \tag{9.13}$$

$$J^{o}_{x}(u_s(\cdot)) = E\int_0^{\infty} e^{-\nu t}[(f - \frac{h}{a}(b + 2\alpha u_s)^2 + y_s^2]dt. \tag{9.14}$$

The fast subproblem is an x-dependent deterministic optimal control problem given by

$$\varepsilon \dot{z}_f = az_f + 2\alpha u_f \tag{9.15}$$

$$J^o_{z_f}(u_f(\cdot)) = \int_0^\infty (h^2 z_f^2 + u_f^2)dt. \tag{9.16}$$

The composite control is formed as in Section 8, namely,

$$u_c(x,z) = u_s(x) + u_f(x,z) \tag{9.17}$$

where $u_s(x)$ is the optimal control for (9.13), (9.14) and $u_f(x,z)$ is the optimal control for (9.15), (9.16).

Singular perturbations of quasi-variational inequalities arising in optimal stochastic scheduling problems are investigated by Hopkins and Blankenship (1981). Results for wide-band noise disturbances are obtained by Blankenship and Meyer (1977), Blankenship and Papanicolaou (1978), and El-Ansary and Khalil (1982). Time scales in stochastic differential equations are studied by Blankenship and Sachs (1977) and Blankenship (1978). Singular perturbations of stochastic filtering and control are an active research topic which has some common features with problems of mathematical physics, surveyed by Blankenship (1979) and Schuss (1980).

10. Time Scale Modeling of Networks

In the last several years time scale modeling and singular perturbation techniques have been extensively used in the study of large scale systems. We first give an overview of the main topics and then concentrate on modeling issues. A time scale modeling methodology was developed for Markov chains with weak interactions by Gaitsgori and Pervozvanski (1975,1979,1980), Delebecque and Quadrat (1981), Phillips and Kokotovic (1981), Delebecque (1983), and Coderich et al. (1983) and for networks with weak connections by Avramovic, et al. (1980), Kokotovic (1981), Kokotovic, et al. (1982), Peponides, et al. (1982), and Peponides and Kokotovic (1983), summarized in a monograph by Chow, et al. (1982). This methodology has been applied to energy and power systems for management of dams, as in Delebecque and Quadrat (1978), and for network equivalencing as in Chow, et al. (1982). The models of large scale systems obtained by this methodology consist of a slow "core" which represents the only coupling of otherwise decoupled fast models of local subsystems. This model structure motivated a "multimodeling" approach to the decentralized control by Khalil and Kokotovic (1978,1979a,b) further developed by Khalil (1979,1980,1981), Saksena and Cruz (1981,1982), and Saksena and Basar (1983). The characteristic of the multimodeling approach is that each local controller has a different model of the same large scale system which agrees with the models of other controllers only in the model of the slow core. A multi-parameter singular perturbation model with one slow core and N local fast subsystems captures this situation

$$\dot{x} = A_{00}x + \sum_{i=1}^{N} A_{0i}z_i + \sum_{i=1}^{N} B_{0i}u_i \tag{10.1}$$

$$\varepsilon_i\dot{z}_i = A_{i0}x + A_{ii}z_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \varepsilon_{ij}A_{ij}z_j + B_{ii}u_i \tag{10.2}$$

This model allows us to assume that each controller neglects all other fast subsystems and concentrates on its own subsystem, plus the interaction with others through the slow core. For the i-th controller, this is simply effected by setting all ε-parameters to zero, except for ε_i. The i-th controller's simplified model is then

$$\dot{x}^i = A_i x^i + A_{0i} z_i + B_{0i} u_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} B_{ij} u_j \tag{10.3}$$

$$\varepsilon_i \dot{z}_i = A_{i0} x^i + A_{ii} z_i + B_{ii} u_i \tag{10.4}$$

which is often all the i-th controller knows about the whole system. The k-th controller, on the other hand, has a different model of the same large scale system. Control u_i can be divided into a slow part, which contributes to the control of the core, and a fast part controlling only its own fast subsystem. The multiparameter perturbation problem has been solved under rather restrictive D-stability assumptions, Khalil and Kokotovic (1979), Ozguner (1979), and Khalil (1981). Stochastic multimodeling problems are even more complex, because of the so-called nonclassical information patterns, Saksena and Basar (1982).

Singular perturbation problems for multiple controllers with different cost functionals (e.g., differential games) are complex even with a single perturbation parameter. We have already mentioned the ill-posedness of linear Nash games with respect to singular perturbations, Gardner and Cruz (1978). Singularly perturbed differential games were further investigated by Salman and Cruz (1979), Khalil and Kokotovic (1979), and Khalil and Medanic (1980). A singularly perturbed pursuit-evasion problem was studied by Farber and Shinar (1980) and Shinar (1981).

Let us conclude this section and the whole survey with a closer look at a fundamental property of large scale systems--the fact that the time scales are caused by weak connections, Kokotovic (1981). Although this is a property of a wide class of nonlinear systems, such as power systems, Peponides, Kokotovic, and Chow (1982), and multimarket economies, Peponides and Kokotovic (1983), we restrict our discussion to linear time-invariant systems in the form

$$\varepsilon\dot{v} = [A + \varepsilon B(\varepsilon)]v, \qquad v \in R^n, \tag{10.5}$$

where A represents strong internal connections within a subsystem while εB are weak external connections among subsystems. If A is singular, this is not a standard form (1.1), (1.2) because the crucial Assumption (1.1) is violated. Of the rich literature dealing with (10.5) and its generalizations we mention only a few representative references. Vasileva in (1975,1976) and in her monograph with Butuzov (1978) treats (10.5) as a "critical case" of singular perturbations. For O'Malley (1978,1979) and O'Malley and Flaherty (1977, 1980) these are "singular singularly perturbed" problems. In the monographs by Campbell (1980,1982) they are special cases of "singular systems" of differential equations. These terminological differences refer to different levels of complexity implied by different assumptions about A and B in (10.5). Denoting by R(A) and N(A) the range space and the null space of A, respectively, the simplest case treated in Peponides, et al. (1982) is when

$$\mathcal{R}(A) \oplus \mathcal{N}(A) = R^n \tag{10.6}$$

$$\dim \mathcal{R}(A) = \rho, \ \dim \mathcal{N}(A) = \nu \tag{10.7}$$

and hence

$$\rho + \nu = n. \tag{10.8}$$

Then $\mathcal{N}(A)$ is the equilibrium manifold of

$$\frac{dv(\tau)}{d\tau} = Av(\tau) \tag{10.9}$$

and there exists a $\rho \times n$ matrix Q such that

$$Qv = 0 \iff v \in N(A). \tag{10.10}$$

Moreover, let the rows of an $\nu \times n$ matrix P span the left null space of A, then

$$P \frac{dv}{d\tau} = PAv = 0 \tag{10.11}$$

represents a <u>conservation manifold</u> of (10.9) because

$$Pv(0) = Pv(\tau), \qquad \forall \tau > 0. \tag{10.12}$$

The time scales of (10.5) are clear from (10.9) and (10.12) which represent the "near-equilibrium" and "near-conservation" properties of (10.5).

<u>Theorem 10.1</u>

The slow and fast variables of (10.5) are x and z, respectively,

$$x = Pv, \qquad z = Qv, \qquad v = Sx + Tz, \tag{10.13}$$

and this change of variables transforms (10.5) into

$$\dot{x} = PB(\varepsilon)Sx + PB(\varepsilon)Tz \tag{10.14}$$

$$\varepsilon\dot{z} = \varepsilon QB(\varepsilon)Sx + [QAT + \varepsilon QB(\varepsilon)T]z \tag{10.15}$$

which is a standard form because QAT is nonsingular due to (10.7).

This defines the fastest time scale $\tau = \frac{t}{\varepsilon}$ and

$$\dot{\bar{x}} = PB(0)S\bar{x} \tag{10.16}$$

is the slow (reduced) subsystem of (10.14)-(10.15). If a decomposition similar to (10.7) applies to PB(0)S, there will be time-scales slower than t which can be determined by a nested procedure, Peponides (1982). More general approaches to the determination of time-scales are due to Coderch, et al. (1983) and Delebecque (1983).

Example 10.1

Let us re-examine the RC-network in Fig. 1.2 and its model (1.17)-(1.18). In this case

$$A = \begin{bmatrix} \frac{-1}{C_1} & \frac{1}{C_1} \\ \frac{1}{C_2} & \frac{-1}{C_2} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{RC_2} \end{bmatrix} \tag{10.17}$$

and hence

$$Q = [1 \quad -1], \qquad P = \begin{bmatrix} \frac{C_1}{C_1+C_2} & \frac{C_2}{C_1+C_2} \end{bmatrix}. \tag{10.18}$$

We see therefore that the near-conservation property (10.12) of the network in Fig. 1.2 reflects the fact that in the limit $R \to \infty$, the total charge on the capacitors is constant and the "aggregate" voltage x is the voltage on the sum of the capacitors with that charge. During the fast transient this voltage remains essentially constant, while the actual voltages v_1 and v_2 converge to their quasi-steady state $v_1 = v_2$. Their difference

$$z = Qv = v_1 - v_2 \tag{10.19}$$

is the fast variable. Its substitution into (1.22)-(1.23) would put the network model in the form (10.15).

In networks and Markov chains, A is often block-diagonal and each of its N blocks A_i represents a local network or Markov chain with the property that

$$\det A_i = 0, \qquad i = 1,\ldots,N. \tag{10.20}$$

The most interesting case is when $\dim N(A_i) = 1$ for all $i = 1,\ldots,N$ and hence $\nu = N$. Then P is an N×n dimensional <u>aggregation</u> matrix and $x = Pv$ defines one aggregate variable for each subsystem. In mass-spring networks and electromechanical models of power systems, see Chow, et al. (1982), this variable is the familiar "center of mass,"

$$x_i = \frac{\Sigma\, m_j v_j}{\Sigma\, m_j}, \qquad i = 1,\ldots,N \tag{10.21}$$

where the summation is restricted to the i-th subsystem. In Markov chains the aggregate variable x_i is the probability for the Markov process to be in the class i of the strongly interacting states, which corresponds to (10.21) with $m_j = 1$ for all j. For the multimodeling approach to decentralized control it is of crucial importance that for such networks and Markov chains QAT is block diagonal, that is, the fast subsystems are indeed "local." An interesting interpretation of the fast variables, which in the case of networks appear as differences between local variables, as in (1.19), is that they express a <u>coherency</u> property. This property has been experimentally observed and extensively used in power systems. The variables in the same subsystem are coherent because their response to the excitation of system-wide slow modes is identical. This is why for slow phenomena all the variables of the same subsystem can be aggregated into one variable. Aggregation and coherency are generalized to nonlinear networks in Peponides (1982), and Peponides, et al. (1982) and extended to modeling of multimarket economies in Peponides and Kokotovic (1983). The relationship of aggregability and weak coupling was investigated in early aggregation works by Simon and Ando (1961) and Simon (1962). These concepts can now be further analyzed by singular perturbation techniques.

In applications, an inverse problem is of even greater importance. We have seen that weak connections imply the time-scales. The inverse question is how to use the knowledge of time-scales to find the weak connections and decompose a large network into weakly connected subnetworks ("areas"). For linear systems an efficient computer algorithm was developed by Avramovic (1980), Avramovic, et al. (1980), and Chow, et al. (1982), which from the slow eigenspace of A determines the areas, that is, decides which connections to consider to be of ε-order. An example is the decomposition of the 42-machine power network of the Western portion of the United States into ten areas, shown in Fig. 10.1.

For an application to queueing networks see Phillips and Kokotovic (1981). These first experiences show that singular perturbations and time-scales will plan an important role in computer-assisted modeling of large scale systems.

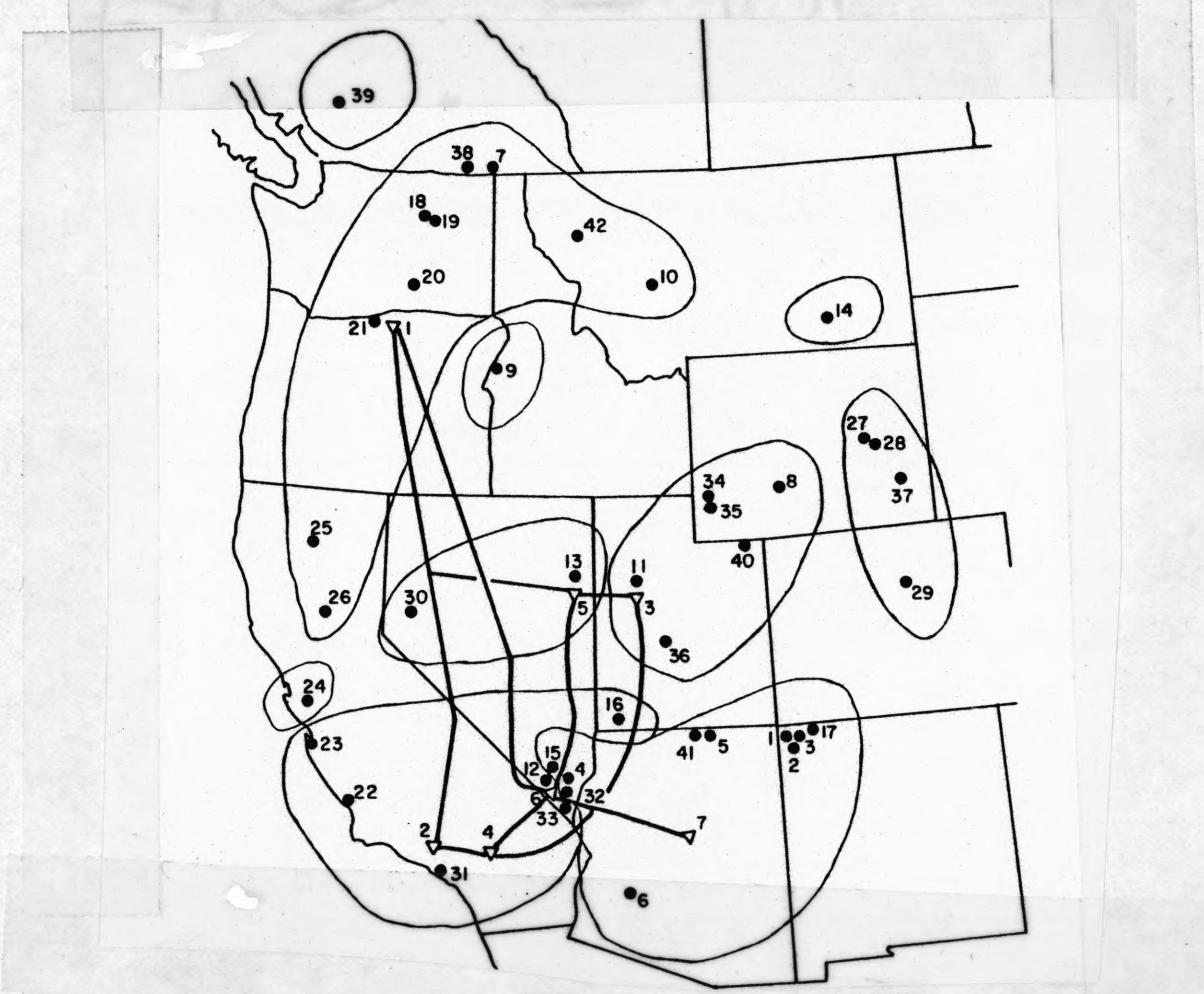

Fig. 10.1. Partition of the Western U.S. power network into ten areas, see Chow (1982).

Concluding Remarks: Future Topics

Instead of conclusions, let us predict some future topics. Several results discussed in this paper have already been extended to distributed parameter systems. Typical references are Lions (1973), Asatani (1976), Desoer (1977), and Balas (1982). It is clear that more work will be done in this area. Averaging and homogenization (Bensoussan, Lions, and Papanicolaou (1978), Blankenship (1979)) are a related class of time-scale methods which have not been discussed. We expect to see more control applications of these methods. Our discussion of stochastic control, with the help of Blankenship (1979), and Schuss (1980), indicates that most of the major problems are still open for an efficient time-scale asymptotic treatment.

This is not to say that all is quiet on the deterministic front. The composite control approach is still restricted to special classes of systems. Trajectory optimization problems with singular arcs and state and control constraints have so far been treated in a semi-heuristic way and are in need of theoretical support. Time scaling of nonlinear models is a crucial unsolved problem. Will geometric methods help?

The developments in modeling and control of large scale systems, (Chow, et al. (1982)) are extremely encouraging and are expected to continue at a rapid rate. When the relationship between weak or sparse connections and time scales is fully understood, the time-scale asymptotic methods will be one of the most powerful tools for analysis and design of large scale systems. Let us not forget that one of the advantages of time-scale methods is that they do not depend on linearity and should apply to most nonlinear models.

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