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# SINGULAR PERTURBATIONS AND ORDER REDUCTION IN CONTROL THEORY - AN OVERVIEW

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by

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# SINGULAR PERTURBATIONS AND ORDER REDUCTION IN CONTROL THEORY - AN OVERVIEW*

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ABSTRACT

Recent results on singular perturbations are surveyed as a tool for model order reduction and separation of time scales in control system design. Conceptual and computational simplifications of design procedures are examined by a discussion of their basic assumptions. Over one hundred references are organized into several problem areas. The content of main theorems is presented in a tutorial form aimed at a broad audience of engineers and applied mathematicians interested in control, estimation and optimization of dynamic systems.

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# INTRODUCTION

Although many control theory concepts are valid for any system order, their actual use is limited to low order models. In optimization of dynamic systems the "curse of dimensionality" is not only in a formidable amount of computation, but also in the ill-conditioned initial and two point boundary value problems. The interaction of fast and slow phenomena in high-order systems results in "stiff" numerical problems which require expensive integration routines.

The singular perturbation approach outlined in this survey alleviates both dimensionality and stiffness difficulties. It lowers the model order by first neglecting the fast phenomena. It then improves the approximation by reintroducing their effect as "boundary layer" corrections calculated in separate time scales. Further improvements are possible by asymptotic expansion methods. In addition to being helpful in design procedures, the singular perturbation approach is an indispensable tool for analytical investigations of robustness of system properties, behavior of optimal controls near singular arcs, and other effects of intentional or unintentional changes of system order.

This paper is a tutorial survey of recent works on singular perturbations in control and estimation theory. Only several other references are mentioned to establish mathematical background and illustrate related approaches. Among surveys and monographs providing a broader view of the field are [A1-10].

## ORDER REDUCTION

Suppose that a dynamic system is modeled by

$$\dot{x} = f(x,z,u,t,\mu) \tag{1}$$

$$\mu\dot{z} = g(x,z,u,t,\mu) \tag{2}$$

where $\mu > 0$ is a scalar and x, z and u are n-, m-, and r-dimensional vectors, respectively. For $\mu = 0$, the order n+m of (1), (2) reduces to n, that is (2) becomes

$$0 = g(\bar{x},\bar{z},\bar{u},t,0) \tag{3}$$

and the substitution of a root of (3),

$$\bar{z} = \varphi(\bar{x},\bar{u},t), \tag{4}$$

into (1) yields a "reduced" model

$$\dot{\bar{x}} = f[\bar{x},\varphi(\bar{x},\bar{u},t),\bar{u},t,0] \equiv \bar{f}(\bar{x},\bar{u},t). \tag{5}$$

The use of $\mu = 0$ is formal since then $\dot{z} = \frac{g}{\mu}$ in (2) may be unbounded for $g \neq 0$. An analysis validating this order reduction procedure is outlined in the next section where it also becomes apparent that a reduced model (4) represents slow and neglects fast phenomena in (1), (2). In this respect the singular perturbation approach is related to familiar "dominant mode" techniques [B2,E4] which neglect "high-frequency" parts and retain "low-frequency" parts of models.

We note that (3) may have several roots each resulting in a different reduced model (4). Most of the available theory is restricted to models (4) corresponding to real and distinct roots of (3), along which $\frac{\partial g}{\partial z}$ is nonsingular.

In the special case when g is linear in z the reduced model (4) is unique. For a linear system

$$\dot{x} = A_{11}x + A_{12}z + B_1u \tag{6}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_2u \tag{7}$$

the root (4) is

$$\bar{z} = -A_{22}^{-1} A_{21}\bar{x} - A_{22}^{-1} B_2\bar{u}, \tag{8}$$

yielding the reduced model

$$\dot{\bar{x}} = (A_{11}-A_{12}A_{22}^{-1}A_{21})\bar{x} + (B_1-A_{12}A_{22}^{-1}B_2)\bar{u}. \tag{9}$$

In applications, models of various physical systems are put in form (1), (2) by expressing small time constants $T_i$, small masses $m_j$, large gains $K_\ell$ etc., as $T_i = c_i\mu$, $m_j = c_j\mu$, $K_\ell = \frac{c_\ell}{\mu}$ etc., where $c_i$, $c_j$, $c_\ell$ are known coefficients [A5,B5]. In power system models $\mu$ can represent machine reactances or transients in voltage regulators [B8], in industrial control systems it may represent time-constants of drives and actuators [B11], in biochemical models $\mu$ can indicate a small quantity of an enzyme [B4], in a flexible booster model $\mu$ is due to bending modes [B3], and in nuclear reactor models it is due to fast neutrons [B7,9,12]. Singular perturbations are extensively used in aircraft and rocket flight models [B6,10,13,14].

## INITIAL VALUE PROBLEMS

When does a reduced solution $\bar{x}$, $\bar{z}$ approximate the original solution x, z and in what sense? For clarity we begin with the linear system (6), (7), assuming that it is time-invariant and that $u = 0$. To exhibit the error

$z - \bar{z} = z + A_{22}^{-1}A_{21}\bar{x}$ let

$$\eta = z + A_{22}^{-1}A_{21}x + \mu M_1 x \tag{10}$$

and choose $M_1$ such that the substitution of (10) into (6), (7) separates the $\eta$-subsystem, as

$$\dot{x} = (A_{11} - A_{12}A_{22}^{-1}A_{21} + \mu M_2)x + A_{12}\eta \tag{11}$$

$$\mu\dot{\eta} = (A_{22} + \mu M_3)\eta. \tag{12}$$

It is easily shown that there exists $\mu^* > 0$ such that $M_i = M_i(\mu)$, $i = 1,2,3$, are bounded for all $\mu \in [0,\mu^*]$. For $\mu \to 0$ the eigenvalues of the independent $\eta$-subsystem (12) tend to infinity like the eigenvalues of $\frac{1}{\mu} A_{22}$. Thus (12) is the "fast" part of (6), (7). It can be written as

$$\frac{d\eta(\tau)}{d\tau} = (A_{22} + \mu M_3)\eta(\tau) \tag{13}$$

where $\tau$ is the "stretched time scale" defined for all $\mu \geq 0$,

$$\tau = \frac{t - t_o}{\mu}, \quad \tau = 0 \quad \text{at} \quad t = t_o. \tag{14}$$

The system (13) depends continuously on $\mu$ and at $\mu = 0$ it becomes

$$\frac{d\eta(\tau)}{d\tau} = A_{22}\eta(\tau). \tag{15}$$

From (8) and (10) at $\mu = 0$ the initial condition for (15) is

$$\eta(0) = z(t_o) - \bar{z}(t_o). \tag{16}$$

The solution $\eta(\tau)$ of the "fast" subsystem (13) is the input to the "slow" subsystem (11). The homogeneous part of (11) is an $0(\mu)$ perturbation[†] of the reduced model (9) with $u=0$. If the eigenvalues of $A_{22}$ all have negative real parts, then $\eta(\tau)\to 0$ as $\tau\to\infty$, that is for $\mu$ small $\eta$ as a function of t rapidly decays away from $t_o$. Under this condition, integration by parts in the variation of parameters formula for the solution of (11) yields

$$x(t) = \bar{x}(t) + 0(\mu) \tag{17}$$

and, on substitution into (10),

$$z(t) = \bar{z}(t) + \eta(\tau) + 0(\mu). \tag{18}$$

Thus the reduced model state $\bar{x}(t)$ approximates the x-part of the actual state, while to approximate its z-part we need both $\bar{z}(t)$ from (8) and $\eta(\tau)$ from (14). The "boundary layer" correction $\eta(\tau)$ is significant only during a short interval $[t_o, t_1]$ after which

$$z(t) = \bar{z}(t) + 0(\mu). \tag{19}$$

A remarkable property of the singularly perturbed model (1), (2) is that the structure of the approximation (17), (18) remains the same for time-varying and nonlinear systems. This is established by a fundamental theorem due to Tihonov [C1], whose essential conditions are imposed on a "boundary layer" system for $\eta = z - \bar{z}$

$$\frac{d\eta}{d\tau} = g[\bar{x}, \bar{z} + \eta(\tau), \bar{u}, t, 0], \tag{20}$$

---

[†] A function of $\mu$ is denoted by $0(\mu^k)$ when for all $\mu\in[0,\mu^*]$ its norm is less than $c\mu^k$, where $c>0$, $\mu^*>0$ and k are some constants.

a nonlinear analog of (15). By virtue of (3) an equilibrium of (20) is $\eta = 0$. Assuming the existence and smoothness of $\bar{x}(t)$, $\bar{z}(t)$ for $t \in [t_o, T]$, the conditions imposed on (20) are, first, that $\eta = 0$ be an asymptotically stable equilibrium of (20) at $\bar{x}(t_o)$, $\bar{z}(t_o)$, $\bar{u}(t_o)$, $t_o$, with $\eta(0) = z(t_o) - \bar{z}(t_o)$ belonging to its domain of attraction; second, that for all $t \in [t_o, T]$ the eigenvalues of $\frac{\partial g}{\partial z}$ along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ all have real parts less than a fixed negative number. Then (17), (18) hold for all $t \in [t_o, T]$ and (19) holds for all $t \in [t_1, T]$.

The proof of this theorem is found in [A1,8;C1,2,3] and, under slightly weaker conditions, in [C4]. The separation of time scales is exemplified by the fact thatin the boundary layer system the variables $\bar{x}$, $\bar{z}$, $\bar{u}$ and t are fixed parameters. The boundary layer correction $\eta(\tau)$ used in (18) is the solution of (20) with (16), where $\bar{x}$, $\bar{z}$, $\bar{u}$ and t are fixed at their values for $t = t_o$.

Expressions (17) and (18) represent $O(\mu)$ approximations of $x(t)$, $z(t)$. If f and g in (1), (2) possess k+2 derivatives in their arguments, then $x(t)$, $z(t)$ can be approximated up to $O(\mu^k)$ using series with terms depending on t and terms depending on $\tau$. These terms can be generated by methods in [A4,8,10;C4,5].

## BOUNDARY VALUE PROBLEMS

In boundary value problems when $z(t)$ is specified at both $t = t_o$ and $t = T$, two boundary layer correction terms $\eta_L$ and $\eta_R$ are needed to compensate for $z(t_o) - \bar{z}(t_o)$ and $z(T) - \bar{z}(T)$, respectively. The correction $\eta_L$ is the same as $\eta$ in the initial value problems. To define $\eta_R$ an additional stretched variable is introduced for all $\mu \geq 0$,

$$\sigma = \frac{t-T}{\mu} \quad , \sigma = 0 \text{ at } t = T, \tag{21}$$

and (20) is rewritten in $\sigma$-scale with $\bar{x}$, $\bar{z}$, $\bar{u}$ and t fixed at their values for $t = T$. Then $\eta_R = \eta_R(\sigma)$ is its solution for $\eta_R(0) = z(T) - \bar{z}(T)$. The approximation of z(t) is sought in the form

$$z(t) = \bar{z}(t) + \eta_L(\tau) + \eta_R(\sigma) + 0(\mu) \tag{22}$$

such that $\eta_L$ and $\eta_R$ decay exponentially as $\tau \to \infty$ and $\sigma \to -\infty$, that is their norms satisfy the "dichotomy condition"

$$\begin{aligned} \|\eta_L\| &\leq c_1 e^{-c_2\tau} , \quad \text{for } 0 \leq \tau < \infty \\ \|\eta_R\| &\leq c_3 e^{c_4\sigma} , \quad \text{for } -\infty < \sigma \leq 0 \end{aligned} \tag{23}$$

where $c_1, \ldots, c_4$ are positive constants. A simple illustration is again the linear system (12). Its solutions in $\tau$ and $\sigma$ scales at $\mu = 0$ are

$$\eta_L(\tau) = e^{A_{22}\tau} \eta_L(0), \quad \eta_R(\sigma) = e^{A_{22}\sigma} \eta_R(0). \tag{24}$$

Let the first k eigenvalues of $A_{22}$ have negative real parts and the remaining m-k eigenvalues positive real parts. Then (23) will result if $\eta_L(0) = z(t_o) - \bar{z}(t_o)$ belongs to the eigenspace corresponding to the first k-eigenvalues of $A_{22}$, and if $\eta_R(0) = z(T) - \bar{z}(T)$ belongs to the eigenspace corresponding to the remaining m-k eigenvalues of $A_{22}$. Under this condition (17) and (22) hold for all $t \in [t_o, T]$, while (19) holds for $t_o < t_1 < t < t_2 < T$.

In nonlinear problems $\frac{\partial g}{\partial z}$ along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$, is assumed to possess the above eigenvalue distribution throughout the interval $[t_o, T]$.

Also $z(t_o) - \overline{z}(t_o)$ and $z(T) - \overline{z}(T)$ are restricted to be on manifolds for which the equilibrium $\eta = 0$ of (20) is attractive in forward and reverse directions of t, respectively. Then (17) and (22) hold for all $t \in [t_o, T]$. Higher order approximations are possible by asymptotic expansions [A4,8,10;C4].

In a wider class of "matched" expansion methods [A3,9] other conditions for "matching" of "outer" (slow) and "inner" (fast) terms are used. They are often motivated by specific applications, such as in interplanetary guidance problems [D6]. The conditions outlined here originate from [A1;D1-5] and can be found in more recent works [A8,10;D7,8] and, in a compact form, in [D9]. These conditions are particularly suitable for optimal control problems whose Hamiltonian symmetry is related to the dichotomy (23). Practical implications of this relationship are discussed in the section on "Trajectory Optimization."

## STABILITY AND STABILIZABILITY

In approximations discussed so far stability requirements were imposed only on (20), and the reduced solution $\overline{x}(t)$ was permitted to be unstable. In infinite time-interval problems it is of interest to establish stability properties of $x(t)$, $z(t)$ from stability properties of $\overline{x}(t)$ and $\eta(\tau)$. Several such results are available.

For linear time-invariant systems a stability result immediately follows from the upper triangular form of the system (11), (12). Its m+n eigenvalues are perturbations of the n eigenvalues of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ and of the m eigenvalues of $\frac{1}{\mu} A_{22}$. If the real parts of these eigenvalues are negative,

$$\mathrm{Re}\lambda\{A_{22}\} < 0, \quad \mathrm{Re}\lambda\{A_{11}-A_{12}A_{22}^{-1}A_{21}\} < 0, \tag{25}$$

that is, if the reduced solution $\bar{x}(t)$ and the boundary layer correction $\eta(\tau)$ are asymptotically stable, then there exists $\mu^* > 0$ such that the original solution $x(t)$, $z(t)$ is asymptotically stable for all $\mu \in (0,\mu^*]$. For linear time-varying systems a similar condition is derived in [E1,6], assuming that the reduced model be uniformly asymptotically stable and that for $t \geq t_o$ the eigenvalues of $A_{22}(t)$ have real parts less than a fixed negative number.

In nonlinear systems the first requirement of (25) is imposed on the eigenvalues of $\frac{\partial g}{\partial z}$ evaluated along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ for all $t \geq t_o$. In addition, $\bar{x}(t)$, $\bar{z}(t)$ and

$$F(t) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}^{-1}\frac{\partial g}{\partial x} \tag{26}$$

evaluated along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$, are assumed to have finite limits $\bar{x}(\infty)$, $\bar{z}(\infty)$ and $F(\infty)$ as $t\to\infty$, where $\mathrm{Re}\lambda\{F(\infty)\} < 0$. Then, if $x(t_o)$ and $z(t_o)$ are in the appropriate domain of attraction, the limits $x(t)$ and $z(t)$ as $t\to\infty$ are

$$x(t) \to \bar{x}(\infty) + 0(\mu), \quad z(t) \to \bar{z}(\infty) + 0(\mu). \tag{27}$$

This is the content of the stability theorem in [E9], whose proof (along with an estimate of the domain of attraction) is given in [E10]. Alternative sets of conditions are given in [E1]. In [E4,5] similar conditions are employed to analyze stability of networks with parasitics, while a problem of absolute stability is discussed in [E7] and stability bounds for $\mu$ are estimated in [E8]. Some early results on stability of control systems with infinite gain coefficients are found in [B1]. Certain theorems on linear systems with slowly varying coefficients, such as [E2] and [E3, Sect. 32], are related to

[E1,6]. A general stabilizability condition for linear time-varying systems is formulated in [G6]. Two special cases for linear time-invariant systems, are discussed in [E11,12].

## REGULATORS AND RICCATI EQUATIONS

Among the most actively investigated singularly perturbed optimal control problems is the general linear-quadratic regulator problem. For brevity we consider only the time-invariant case. When the system (6), (7) is optimized with respect to

$$J = \frac{1}{2}\int_0^\infty (y'y + u'Ru)dt \tag{28}$$

where $y = C_1x + C_2z$ and $R > 0$, then to implement the optimal control

$$u = -R^{-1}[B_1' \quad \frac{1}{\mu} B_2']K \begin{bmatrix} x \\ z \end{bmatrix} \tag{29}$$

we have to solve

$$K \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\mu} & \frac{A_{22}}{\mu} \end{bmatrix} + \begin{bmatrix} A_{11}' & \frac{A_{21}'}{\mu} \\ A_{12}' & \frac{A_{22}'}{\mu} \end{bmatrix} K - K \begin{bmatrix} B_1 \\ \frac{B_2}{\mu} \end{bmatrix} R^{-1}[B_1' \quad \frac{B_2'}{\mu}]K + C'C = 0, \tag{30}$$

where $C = [C_1 \quad C_2]$. To avoid unboundedness as $\mu \to 0$ the solution is sought in the form

$$K = K(\mu) = \begin{bmatrix} K_{11}(\mu) & \mu K_{12}(\mu) \\ \mu K_{12}'(\mu) & \mu K_{22}(\mu) \end{bmatrix} \tag{31}$$

which permits us to set $\mu = 0$ in (30). At $\mu = 0$ an $m \times m$ equation for $\overline{K}_{22}$,

$$\overline{K}_{22}A_{22} + A_{22}'\overline{K}_{22} - \overline{K}_{22}S_2\overline{K}_{22} + C_2'C_2 = 0, \tag{32}$$

where $S_2 = B_2R^{-1}B_2'$, separates from the $(n+m)\times(n+m)$ equation (30). If $A_{22}$, $B_2$ is a stabilizable pair, and if $A_{22}$, $C_2$ is a detectable pair, then a unique positive semidefinite solution $\overline{K}_{22}$ exists and the eigenvalues of $A_{22} - S_2\overline{K}_{22}$ have negative real parts. Another result of the substitution of (31) into (30) is that at $\mu = 0$ it is possible to express $\overline{K}_{12}$ in terms of $\overline{K}_{11}$ and $\overline{K}_{22}$, and to obtain an $n\times n$ equation for $\overline{K}_{11}$,

$$\overline{K}_{11}\hat{A} + \hat{A}'\overline{K}_{11} - \overline{K}_{11}\hat{B}R^{-1}\hat{B}'\overline{K}_{11} + \hat{C}'\hat{C} = 0. \tag{33}$$

The expressions for $\hat{A}$, $\hat{B}$ and $\hat{C}$ are given in [F5]. An interpretation of (32) and (33) is that (32) yields a "boundary layer regulator" for the fast variable $\eta(\tau)$, and (33) yields the regulator for the reduced state variable $\overline{x}(t)$. For $\hat{A},\hat{B}$ stabilizable and $\hat{A},\hat{C}$ detectable, the implicit function theorem applied to (30) with (31) shows that

$$K_{ij} = \overline{K}_{ij} + 0(\mu) \qquad i,j = 1,2. \tag{34}$$

Not only are the approximations $\overline{K}_{ij}$ calculated from lower order equations, but in addition the ill-conditioning of (30) has been removed.

If $\overline{K}_{ij}$ are used instead of $K_{ij}$ the system (6), (7) with feedback control (29) becomes

$$\dot{x} = (A_{11}-S_1\overline{K}_{11}-S\overline{K}_{12}')x + (A_{12}-S\overline{K}_{22})z \tag{35}$$

$$\mu\dot{z} = (A_{21}-S'\overline{K}_{11}-S_2\overline{K}_{12}')x + (A_{22}-S_2\overline{K}_{22})z \tag{36}$$

where $S_1 = B_1R^{-1}B_1'$ and $S = B_1R^{-1}B_2'$. If this system is asymptotically stable, then because of (34), its solution x(t), z(t) is within $O(\mu)$ of the optimal solution. The stability condition (25) can now be applied to the feedback system (35), (36). The boundary layer stability condition is satisfied by $A_{22}-S_2\bar{K}_{22}$. The condition for the reduced system is satisfied by the solution of (33). Thus (35), (36) is a near-optimal system.

The singularly perturbed regulator problem was posed in [F1] with $C_2 = 0$ and $A_{22}$ stable, which gave $\bar{K}_{22} = 0$. The general time-varying problem was treated in [F3,5] using the notion of boundary layer controllability-observability. These results and extensions [F6,7,9,10] are based on the singularly perturbed differential Riccati equation. An alternative approach via boundary value problems is presented in [G8,19], its relationship with the Riccati approach is analyzed in [F12]. In [F2] it was shown that the reduced Riccati equation (33) can also be obtained from the reduced model (9). Asymptotic expansions are constructed in [F6,7] and applied to a 17th order power station model in [F8]. Two other order reduction techniques [F4,11] lead to equations similar to (33) and it would be of interest to investigate their relationship with the singular perturbation approach.

## TRAJECTORY OPTIMIZATION

In trajectory optimization problems for the system (1), (2) some conditions are imposed on x,z at both $t = t_o$ and $t = T$, and a control u(t) is sought to minimize the performance index

$$J = \int_{t_o}^{T} V(x,z,u,t)dt. \tag{37}$$

An optimal solution must satisfy $H_u = 0$ and

$$\dot{x} = H_p, \qquad \dot{p} = -H_x \tag{38}$$

$$\mu\dot{z} = H_q, \qquad \mu\dot{q} = -H_z, \tag{39}$$

with $2n+2m$ boundary conditions. Here $H_u$, $H_x$, $H_z$, $H_p = f$, $H_q = g$, denote the partial derivatives of the Hamiltonian $H = V+p'f+q'g$, and the adjoint variables for (1) and (2) are p and $\mu q$, respectively. At $\mu = 0$ we use $H_q = 0$ and $H_z = 0$ to eliminate z and q from (38) and to get the reduced system

$$\dot{\bar{x}} = \bar{H}_p, \qquad \dot{\bar{p}} = -\bar{H}_x \tag{40}$$

for which only 2n conditions can be imposed. Suppose that they are uniquely satisfied by a continuously differentiable reduced solution $\bar{x}(t),\bar{p}(t)$. Since the reduced variables $\bar{z}(t),\bar{q}(t)$ obtained from $H_q = 0$, $H_z = 0$ may not satisfy the remaining 2m conditions, corrections $\eta_L(\tau),\eta_R(\sigma)$ for z, and $\rho_L(\tau),\rho_R(\sigma)$ for q, are to be determined from appropriately defined layer systems

$$\frac{d\eta_L}{d\tau} = \tilde{H}_q(\eta_L,\rho_L), \qquad \frac{d\rho_L}{d\tau} = -\tilde{H}_z(\eta_L,\rho_L) \tag{41}$$

$$\frac{d\eta_R}{d\sigma} = \tilde{H}_q(\eta_R,\rho_R), \qquad \frac{d\rho_R}{d\sigma} = -\tilde{H}_z(\eta_R,\rho_R) \tag{42}$$

where (41) is used at $t = t_o$ and (42) at $t = T$. To be specific consider the problem with fixed end points,

$$z(t_o) = z^o, \qquad z(T) = z^T. \tag{43}$$

Then the initial values for $\eta_L$ and $\eta_R$ are

$$\eta_L(0) = z^o - \bar{z}(t_o), \qquad \eta_R(0) = z^T - \bar{z}(T) \tag{44}$$

and the additional boundary conditions are

$$\eta_L, \rho_L \to 0, \quad \tau \to \infty; \qquad \eta_R, \rho_R \to 0, \quad \sigma \to -\infty. \tag{45}$$

Existence of optimal solutions and their approximation by reduced solutions have been investigated in [G1,3,9] and extended in [G16,17] by a construction of asymptotic expansions. Unfortunately the applicability of these results is restricted by the requirement that $\eta_L(0)$ and $\eta_R(0)$ be sufficiently small. To what extent such restrictions can be avoided in a general nonlinear problem (1), (2) and (37) is still an open question. Results without restrictions on $z^o$, $z^T$ are available for linear time-varying systems [G6,8,13,19] and for a special class of nonlinear systems [G14,15,20]. They are breifly outlined here.

Let theperformance index be (28), but on the interval $[t_o,T]$, and consider the trajectory optimization problem for (6),(7) allowing that the matrices in (6), (7) and (28) be time-varying. Using a "dichotomy transformation" proposed in [G6]

$$x_1 = \ell_1 + r_1, \quad z = \ell_2 + r_2 \tag{46}$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = P(t) \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} + N(t) \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \tag{47}$$

where P(t) is a positive definite and N(t) is a negative definite solution of a differential equation analogous to (30), we transform (41),(42) into two separate "layer regulator systems"

$$\frac{d\eta_L}{d\tau} = [A_{22}(t_o) - S_{22}(t_o)P_{22}(t_o)]\eta_L \qquad (48)$$

$$\frac{d\eta_R}{d\sigma} = [A_{22}(T) - S_{22}(T)N_{22}(T)]\eta_R \qquad (49)$$

where $\eta_L = \ell_2-\bar{\ell}_2$, $\eta_R = r_2-\bar{r}_2$ and $P_{22}(t_o),N_{22}(T)$ are the positive and the negative definite roots of (32) at $t_o$ and T. If for all $t\in[t_o,T]$

$$\text{rank}[B_{22},A_{22}B_2,\ldots,A_{22}^{m-1}B_2] = m \qquad (50)$$

$$\text{rank}[C_2',A_{22}'C_2',\ldots,A_{22}'^{m-1}C_2'] = m \qquad (51)$$

then the approximations (17),(22) and

$$p(t) = \bar{p}(t) + O(\mu) \qquad (52)$$

$$q(t) = \bar{q}(t) + P_{22}(t_o)\eta_L + N_{22}(T)\eta_R + O(\mu) \qquad (53)$$

hold for arbitrary boundary values $z^o$, $z^T$ since (48),(49) satisfy the dichotomy condition (23). A less restrictive stabilizability-detectability condition can be used instead of (50),(51). This is the main content of [G13]. An "inverse" Riccati approach to the linear fixed end-point problem is developed in [F9]. In [G8] a different set of conditions is derived and asymptotic expansions are constructed for the linear boundary value problem.

In [G14,15] the above results have been extended to the nonlinear problem

$$\dot{x} = f_1(x,t) + A_{12}(x,t)z + B_1(x,t)u \tag{54}$$

$$\mu\dot{z} = g_1(x,t) + A_{22}(x,t)z + B_2(x,t)u \tag{55}$$

$$J = \frac{1}{2}\int_t^T [v_1(x,t) + z'C_2'(x,t)C_2(x,t)z + U'R(x,t)u]dt. \tag{56}$$

It is shown in [G15] that, if the matrices in (32), (48), (49), (50), (51) are interpreted as the matrices of (54), (55), (56) evaluated along $\bar{x}(t)$, then (50), (51) are sufficient for the approximation (17), (22), (52), (53) to hold for (54), (55), (56) with arbitrary $z^o$, $z^T$. The conditions derived in [G14] extend the results of [G8] to (54), (55), (56). Among other works on trajectory optimization, [G18] shows that (40) can also be obtained from the reduced system, [G1] analyzes the scalar problem, [G2,B5] give approximations without layer corrections and [G10] makes an attempt to include control inequality constraints. Applications to aircraft control problems are discussed in [G4,5,11,12] and in [B6,10,13,14].

## CONTROLLABILITY AND TIME OPTIMAL CONTROL

In the design of time-optimal controls difficulties with high-order systems are considerable even in the linear time-invariant problems. A simplified design procedure has been developed in [H1,2,3]. The discussion here is based on [H2], where also the following controllability result is obtained. The use of (10) and

$$\xi = x - \mu A_{12}A_{22}^{-1}\eta + 0(\mu^2) \tag{57}$$

transforms (6), (7) into

$$\dot{\xi} = [\bar{A}+O(\mu)]\xi + [\bar{B}+O(\mu)]u \tag{58}$$

$$\mu\dot{\eta} = [A_{22}+O(\mu)]\eta + [B_2+O(\mu)]u \tag{59}$$

where $\bar{A} = A_{11}-A_{12}A_{22}^{-1}A_{21}$, $\bar{B} = B_1-A_{12}A_{12}^{-1}B_2$, see (9).

It follows from (58), (59) that for $\mu$ small the controllability of the reduced and the boundary layer systems, that is of the pairs $\bar{A}$, $\bar{B}$ and $A_{22}$, $B_2$, implies the controllability of the original system (6), (7).

In the time-optimal control problem a control u, subject to constraint $|u_i| \leq 1$, $i=1,\ldots,r$, is to transfer the state of (6), (7) from $x(0) = x^o$, $z(0) = z^o$ to $x(T) = 0$, $z(T) = 0$ in minimum time T. Equivalently the problem can be solved in terms of $\xi$ and $\eta$. A control steering $\xi,\eta$ to zero in minimum time is of the form

$$u = -\mathrm{Sgn}\{\bar{B}'e^{\bar{A}'(T-t)}p + B_2'e^{-A_{22}\sigma}q\}, \tag{60}$$

where $\sigma$ is as in (21), p and q are constant vectors and $O(\mu)$ terms have been neglected. When the eigenvalues of $A_{22}$ all have negative real parts, the term depending on $\sigma$ is significant only near T. For some $\sigma^* < 0$ and $0 \leq t \leq T+\mu\sigma^*$ the control (60) can be approximated by

$$\bar{u} = -\mathrm{Sgn}\{\bar{B}'e^{\bar{A}'(\bar{T}-t)}\bar{p}\}, \tag{61}$$

which is interpreted as a time-optimal control for the reduced system (9), steering $\bar{x}$ to zero. For $T+\mu\sigma^* < t \leq T$ the control (60) is approximated by

$$u_\sigma = -\mathrm{Sgn}\{\bar{B}'\bar{p} + B_2'e^{-A_{22}\sigma}q\}. \tag{62}$$

We note from (8) that, after the last switching of $\bar{u}$, z may be far from the origin and the boundary layer control $u_\sigma$ is needed to correct this error.

This separation of slow and fast switchings was first analyzed for single-input systems in [H1], and then generalized in [H2]. A special case when (7) is due to actuator dynamics is discussed in [H3].

## FILTERING AND SMOOTHING

Results on singular perturbation of linear-quadratic regulator problems should have their counterparts in the linear-quadratic-Gaussian filtering and smoothing problems. Preliminary investigations along this line have been reported in [I1,3-6]. The analysis in [I6] shows that the duality is not complete and the singularly perturbed filtering and smoothing problems require separate treatment and cautious interpretation. The analysis is more complicated since the white noise input process u in (58), (59) "fluctuates" faster than the fast part $\eta$ of the state no matter how small $\mu > 0$ is. In the limit, $\eta$ becomes a white noise process whose covariance is the same as the covariance of the reduced solution $\bar{\eta}$, and the integral error covariance of $\eta(t) - \bar{\eta}(t)$ tends to zero. Thus, as an input to a slow system, $\bar{\eta}(t)$ can replace $\eta(t)$, but not as an approximation for each t. Pursuing such considerations it is shown in [I6] that a filtering (or smoothing) problem for the system (6), (7) can be obtained by solving two lower order problems in separate time scales.

An example given in [I2] indicates that deterministic observers also can be approached as singular perturbations. Control problems with small noise are treated in [I7,8].

## CHEAP CONTROL AND SINGULAR ARCS

In singular perturbation problems considered so far a small parameter $\mu$ multiplies derivatives and the differential order is reduced when $\mu = 0$. Another sign of singular perturbation phenomena is a characteristic lowering of dimensionality for the limiting problem, such as in limit approaches to singular optimal controls [J1]. An example of these problems is

$$\dot{x} = Ax + Bu, \quad x(0) = x^{o} \tag{63}$$

$$J = \frac{1}{2}\int_{0}^{T} (x'Qx + \mu^{2}u'Ru)dt \tag{64}$$

where J is to be minimized for $\mu$ small. In [L2] analogous problems for systems governed by partial differential equations are called "cheap control" problems since the cost of the control u is cheap relative to that of the state x (for $Q > 0$). Other applications include study of limiting possibilities for regulators and filters [J2,5;I8].

When $\mu = 0$, the resulting problem is a well-known singular problem [J3] whose solution satisfies the singular arc condition

$$B'K_{o} = 0 \tag{65}$$

for $t > 0$ and the appropriate Riccati gain $K_o$. Motion is thereby restricted to a manifold of dimension at most n-r. By obtaining the asymptotic solution of (63), (64) as $\mu \to 0$, we show how this reduction in order comes about and, simultaneously, discover the nature of the initial control impulse. For $\mu > 0$, the feedback control is

$$u = -\frac{1}{\mu^2} R^{-1} B'Kx \tag{66}$$

where $K \geq 0$ satisfies the singularly perturbed problem

$$\mu^2 \frac{dK}{dt} + \mu^2 (KA + A'K + Q) = KBR^{-1}B'K, \qquad K(T) = 0. \tag{67}$$

The limiting solution $K_o$ of (67) within (O,T) satisfies the singular arc condition (65). Asymptotic solution of (67) is complicated and considerably different however, in a hierarchy of cases (Case 1 where $B'QB > 0$, Case 2 where $B'QB = 0$ and $B_1'QB_1 > 0$ for $B_1 = AB-\dot{B}$). This reflects the situation for the singular arc problem [J3,4] where the initial optimal control successively becomes increasingly impulsive and the singular arc increasingly restrictive. A singular perturbation analysis in [J6-9] reveals the detailed structure of these phenomena.

## TIME-DELAY SYSTEMS

The difficulties incumbent with control systems having time delays have motovated various approximations. When the delay is small, it is often neglected and a tractable "nominal" problem is solved. Such design procedures can be justified in terms of singular perturbation methods. Boundary layer phenomena do occur, although they are not of lowest order importance. Interesting and significant extensions are to problems with both small parameters multiplying derivatives and small delays. Discussions with applications to nuclear reactor models occur in [K1-4]. In [K5] a method is proposed replacing several small time constants by a single time-delay.

## DISTRIBUTED PARAMETER SYSTEMS

From the results of [L1,2,3] it can be expected that the singular perturbation techniques will be among the main tools for asymptotic analysis and design of optimal control of distributed parameter systems. Several generalizations of the finite dimensional linear-quadratic problems are available. In particular, a distributed parameter analog of the method [F5,7] is developed in [L3] for systems described by singularly perturbed parabolic differential equations.

## CONCLUSION

It seems that, instead of giving a short summary of solved problems, the conclusion of a survey of a new direction of research should concentrate on missing links, restrictive assumptions, and hints of new problems. Starting with Order Reduction the need for a systematic modeling procedure to formulate the model (1),(2) is apparent. Conversely, this model is expected to interpret other order reduction procedures as limit processes. In Initial and Boundary Value Problems, controllability and stabilizability studies may relax the restrictions of stable initial and final manifolds. Although Optimal Regulators seem a solved problem, there remains a desire to reduce the dimensionality of the feedback matrix. In Trajectory Optimization, restrictions on norms of boundary layer jumps should be, and very likely can be, removed for a wider class of Hamiltonian systems. The only result with constrained control is the linear time-optimal control. Various generalizations to other bang-bang controls are visible.

In addition to linear regulators, other optimum feedback design problems need to be solved. Order reduction in dynamic programming and Hamilton-Jacobi optimization approaches would result in even bigger conceptual and computational simplifications. Singularly Perturbed Filtering, Smoothing, Singular Arc, Distributed Systems and Time-Delay Problems require further exploration. More work on numerical aspects of these problems is also needed. What has been surveyed here is only a first step.

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