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SYSTEMATIC DESIGN OF ADAPTIVE CONTROLLERS FOR FEEDBACK LINEARIZABLE SYSTEMS

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A systematic procedure is developed for the design of adaptive regulation schemes for a class of feedback linearizable nonlinear systems. The coordinate-free geometric conditions, which characterize this class of systems, neither restrict the location of the unknown parameters, nor constrain in any way the growth of the nonlinearities. Instead, they require that the nonlinear system be transformable into the so-called pure-feedback form. This becomes now the broadest class of nonlinear systems for which an adaptive regulation scheme can be systematically designed. We also show that, under additional conditions, global stability properties can be established.

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Systematic Design of Adaptive Controllers for Feedback Linearizable Systems*

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Abstract

A systematic procedure is developed for the design of adaptive regulation and tracking schemes for a class of feedback linearizable nonlinear systems. The coordinate-free geometric conditions, which characterize this class of systems, neither restrict the location of the unknown parameters, nor constrain the growth of the nonlinearities. Instead, they require that the nonlinear system be transformable into the so-called pure-feedback form. When this form is "strict", the proposed scheme guarantees global regulation and tracking properties. This result substantially enlarges the class of nonlinear systems for which global stabilization can be achieved. Apart from the geometric conditions, this paper uses simple analytical tools, familiar to most control engineers.

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Contents

1 Introduction

Most of the research activity on adaptive control of nonlinear systems [1–15] is still focused on the full-state feedback case [1–13], although output-feedback results are beginning to appear [14,15]. The full-state feedback case continues to be a challenge because of the severe restrictions of the two currently available types of schemes: the *uncertainty-constrained schemes* [1,2,3,4,10,11] restrict the *location of unknown parameters*, and the *nonlinearity-constrained schemes* [5,6,7,8,9,12] impose restrictions on the *type of nonlinearities.*

The systems to which *uncertainty-constrained schemes* can be applied may contain all types of smooth nonlinearities and are fully characterized by coordinate-free geometric conditions [2,3,11], which, unfortunately, are quite restrictive. On the other hand, the applicability of *nonlinearity-constrained schemes* is restricted by coordinate-dependent growth conditions on the nonlinearities, which may exclude even certain *linear* systems [13]. The nonlinearity-constrained schemes based on the "*Control Lyapunov Function*" approach [6,7,8], are applicable to the class of systems for which a Lyapunov function with prespecified growth properties is known. Unfortunately, the existence of such a Lyapunov function can not be ascertained *a priori.*

The new adaptive control scheme developed in this paper combines the main advantages of earlier schemes without most of their disadvantages. It significantly extends the class of nonlinear systems for which adaptive controllers can be systematically designed. At each step of the new design procedure, the change of coordinates is interlaced with the construction of a parameter update law. The main idea of this nonlinear procedure evolved from an early linear result of Feuer and Morse [16].

Among the advantages of the new scheme are its conceptual clarity and wide applicability. Its stability proof, based on a straightforward Lyapunov argument, is particularly simple. The coordinate-free geometric conditions, characterizing the class of systems to which the new scheme is applicable, neither restrict the location of the unknown parameters, nor constrain the growth of the nonlinearities. Instead, they require that the nonlinear system be transformable into the so-called *pure-feedback form.* Furthermore, in the case of systems transformable into the more restrictive *strict-feedback form*, the new adaptive scheme guarantees *global* regulation and tracking properties. This is now the broadest class of nonlinear systems for which an adaptive control scheme can be systematically designed to achieve global regulation or tracking without growth constraints.

The presentation is organized as follows: First, we address the regulation problem. In Section 2 we characterize the class of single-input nonlinear systems to which the new scheme is applicable. The design procedure is presented in Section 3, and the simple proof of stability is given in Section 4. In Section 5 we give the conditions under which the stability of the closed-loop system is global. The design procedure is extended to multi-input systems in Section 6. Then, in Section 7, we use the design procedure to solve the tracking problem for a class of input-output linearizable systems with exponentially stable zero dynamics. In Section 8 we illustrate this procedure on some "benchmark" examples, and discuss its properties in comparison with previous results. Finally, some concluding remarks are given in Section 9. The reader unfamiliar with differential geometric results for nonlinear systems can follow the presentation starting with Section 3 and then omitting Propositions 5.3, 6.1 and 7.3.

2 The Class of Nonlinear Systems

The adaptive regulation problem will first be solved for single-input feedback linearizable systems that are linear in the unknown parameters:

$$\dot{\zeta} = f_0(\zeta) + \sum_{i=1}^{p} \theta_i f_i(\zeta) + \left[g_0(\zeta) + \sum_{i=1}^{p} \theta_i g_i(\zeta) \right] u \,, \tag{2.1}$$

where $\zeta \in I\!R^n$ is the state, $u \in I\!R$ is the input, $\theta = [\theta_1, \ldots, \theta_p]^{\mathrm{T}}$ is the vector of constant unknown parameters, and f_i, g_i, $0 \leq i \leq p$, are smooth vector fields in a neighborhood of the origin $\zeta = 0$ with $f_i(0) = 0$, $0 \leq i \leq p$, $g(0) \neq 0$.

The design of the adaptive scheme assumes that the system (2.1) can be transformed into the pure-feedback form via a parameter-independent diffeomorphism. Necessary and sufficient conditions for the existence of such a diffeomorphism are given in the following proposition.

Proposition 2.1. *Consider a parameter-independent diffeomorphism $z = \phi(\zeta)$, with $\phi(0) = 0$, that transforms, in a neighborhood B_z of the origin, the system (2.1) into the so-called pure-feedback form*

$$\begin{aligned}
\dot{z}_1 &= z_2 + \theta^{\mathrm{T}}\gamma_1(z_1, z_2) \\
\dot{z}_2 &= z_3 + \theta^{\mathrm{T}}\gamma_2(z_1, z_2, z_3) \\
&\vdots \\
\dot{z}_{n-1} &= z_n + \theta^{\mathrm{T}}\gamma_{n-1}(z_1, \ldots, z_n) \\
\dot{z}_n &= \gamma_0(z) + \theta^{\mathrm{T}}\gamma_n(z) + \left[\beta_0(z) + \theta^{\mathrm{T}}\beta(z)\right] u \,,
\end{aligned} \tag{2.2}$$

with

$$\gamma_i(0) = 0 \,,\ 0 \leq i \leq n \,,\ \beta_0(0) \neq 0 \,. \tag{2.3}$$

Such a diffeomorphism exists if and only if the following conditions are satisfied in a neighborhood U of the origin:

(i) **Feedback linearization condition**. *The distributions*

$$\mathcal{G}^i = \operatorname{span}\left\{g_0, ad_{f_0}g_0, \ldots, ad^i_{f_0}g_0\right\}, \quad 0 \leq i \leq n-1 \tag{2.4}$$

are involutive and of constant rank $i+1$.

(ii) **Pure-feedback condition.**

$$\left.\begin{aligned}
g_i &\in \mathcal{G}^0 \,, \\
[X, f_i] &\in \mathcal{G}^{j+1} \,,\ \forall X \in \mathcal{G}^j \,, \quad 0 \leq j \leq n-2 \,,
\end{aligned}\right. \quad 1 \leq i \leq p \,. \tag{2.5}$$

Proof. *Sufficiency.* As proved in [17], condition (i) is sufficient for the existence of a diffeomorphism $z = \phi(\zeta)$ that transforms the system

$$\dot{\zeta} = f_0(\zeta) + g_0(\zeta)u \,,\ f_0(0) = 0 \,,\ g_0(0) \neq 0 \tag{2.6}$$

into the system

$$\begin{aligned}
\dot{z}_i &= z_{i+1} \,, 1 \leq i \leq n-1 \\
\dot{z}_n &= \gamma_0(z) + \beta_0(z)u \,,
\end{aligned} \tag{2.7}$$

with

$$\gamma_0(0) = 0\,,\ \beta_0(0) \neq 0\,. \tag{2.8}$$

Hence, in the coordinates of (2.7) we have

$$\begin{aligned} f_0(\phi^{-1}(z)) &= [z_2 \ldots z_n\, \gamma_0(z)]^{\mathrm{T}} && (2.9)\\ g_0(\phi^{-1}(z)) &= [0 \ldots 0\, \beta_0(z)]^{\mathrm{T}} && (2.10)\\ \mathcal{G}^i &= \operatorname{span}\left\{\frac{\partial}{\partial z_n}, \ldots, \frac{\partial}{\partial z_{n-i}}\right\}, \quad 0 \leq i \leq n-1\,. && (2.11) \end{aligned}$$

Because of (2.11), the pure-feedback condition (2.5), expressed in the z-coordinates, states that

$$\left.\begin{aligned} g_i &\in \operatorname{span}\left\{\frac{\partial}{\partial z_n}\right\}, \\ \left[\frac{\partial}{\partial z_j}, f_i\right] &\in \operatorname{span}\left\{\frac{\partial}{\partial z_n}, \ldots, \frac{\partial}{\partial z_{j-1}}\right\}, \quad 2 \leq j \leq n\,, \end{aligned}\right. \quad 1 \leq i \leq p\,. \tag{2.12}$$

But (2.12) can be equivalently rewritten as

$$g_i(\phi^{-1}(z)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_i(z) \end{pmatrix}, \quad f_i(\phi^{-1}(z)) = \begin{pmatrix} \gamma_1(z_1) \\ \gamma_2(z_1, z_2) \\ \vdots \\ \gamma_{n-1,i}(z_1, \ldots, z_n) \\ \gamma_{n,i}(z_1, \ldots, z_n) \end{pmatrix}, \quad 1 \leq i \leq p\,. \tag{2.13}$$

Furthermore, since $\phi(0) = 0$ and $f_i(0) = 0, 1 \leq i \leq p$, we conlude from (2.13) that

$$\gamma_j(0) = 0\,,\ 1 \leq j \leq n\,. \tag{2.14}$$

Combining (2.9), (2.10), (2.13) and (2.14), we see that in the z-coordinates the system (2.1) becomes (2.2).

Necessity. If there exists a diffeomorphism $z = \phi(\zeta)$ that transforms (2.1) into (2.2), one can directly verify that the coordinate-free conditions (i) and (ii) are satisfied for the system (2.2), and hence for the system (2.1). □

Remark 2.2. The "extended-matching" condition, introduced in [2,3] and used in [1] in the equivalent form of a "strong linearizability" condition, is a special case of the "pure-feedback" condition (2.5). This is easily seen by noting that if the system (2.1) satisfies the feedback linearization condition (2.4) and the extended-matching condition

$$g_i \in \mathcal{G}^0\,, \;\; f_i \in \mathcal{G}^1\,, \quad 1 \leq i \leq p\,, \tag{2.15}$$

then it is transformable into the pure-feedback form (2.2) with $\gamma_1 \equiv 0, \ldots, \gamma_{n-2} \equiv 0$. □

3 Adaptive Scheme Design

The conditions of Proposition 2.1 give a precise geometric characterization of the class of nonlinear systems to which the new adaptive scheme is applicable. We now design the new adaptive scheme for systems of the form (2.2):

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \theta^{\mathrm{T}}\gamma_i(z_1, \ldots, z_{i+1})\,, \quad 1 \leq i \leq n-1 \\ \dot{z}_n &= \gamma_0(z) + \theta^{\mathrm{T}}\gamma_n(z) + \left[\beta_0(z) + \theta^{\mathrm{T}}\beta(z)\right] u\,, \end{aligned} \tag{3.1}$$

with

$$\gamma_i(0) = 0\,,\; 0 \leq i \leq n\,, \beta_0(0) \neq 0\,. \tag{3.2}$$

Recall that θ is the vector of unknown parameters, and γ_0, β_0, and the components of β and γ_i, $1 \leq i \leq n$, are smooth nonlinear functions in B_z, a neighborhood of the origin $z = 0$.

Using an idea similar to those exploited by Feuer and Morse [16] for adaptive control of linear systems, the design procedure interlaces, at each step, a change of coordinates with the construction of a parameter update law. Not only is the design procedure systematic and conceptually clear, but also the stability proof is a straightforward Lyapunov argument.

The new adaptive scheme for the system (3.1) is designed step-by-step as follows:

Step 0. Define $x_1 = z_1$, and denote by $c_1, c_2, \ldots, c_n$ constant coefficients to be chosen later.

Step 1. Starting with

$$\dot{x}_1 = z_2 + \theta^{\mathrm{T}}\gamma_1(z_1, z_2)\,, \tag{3.3}$$

let ϑ_1 be an estimate of θ and define the new state x_2 as

$$x_2 = c_1 x_1 + z_2 + \vartheta_1^{\mathrm{T}}\gamma_1(z_1, z_2)\,. \tag{3.4}$$

Substitute (3.4) into (3.1) to obtain

$$
\begin{aligned}
\dot{x}_1 &= -c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} \gamma_1(z_1, z_2) \\
&= -c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(x_1, x_2, \vartheta_1)\,.
\end{aligned} \tag{3.5}
$$

Then, let the update law for the parameter estimate ϑ_1 be

$$
\dot{\vartheta}_1 = x_1\, w_1(x_1, x_2, \vartheta_1). \tag{3.6}
$$

Step 2. Using the definitions for x_1, x_2 and $\dot{\vartheta}_1$, write $\dot{x}_2$ as

$$
\begin{aligned}
\dot{x}_2 &= c_1[-c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(x_1, x_2, \vartheta_1)] + z_3 + \theta^{\mathrm{T}} \gamma_2(z_1, z_2, z_3) \\
&\quad + x_1 w_1(x_1, x_2, \vartheta_1)^{\mathrm{T}} \gamma_1(z_1, z_2) + \vartheta_1^{\mathrm{T}} \left[\frac{\partial \gamma_1}{\partial z_1}(z_2 + \theta^{\mathrm{T}} \gamma_1) + \frac{\partial \gamma_1}{\partial z_2}(z_3 + \theta^{\mathrm{T}} \gamma_2) \right] \\
&= \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \left[z_3 + \theta^{\mathrm{T}} \gamma_2(z_1, z_2, z_3) \right] + \varphi_2(x_1, x_2, \vartheta_1) + \theta^{\mathrm{T}} \psi_2(x_1, x_2, \vartheta_1)\,.
\end{aligned} \tag{3.7}
$$

Let ϑ_2 be a *new* estimate of θ and define the new state x_3 as

$$
\begin{aligned}
x_3 &= c_2 x_2 + \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \left[z_3 + \vartheta_2^{\mathrm{T}} \gamma_2(z_1, z_2, z_3) \right] \\
&\quad + \varphi_2(x_1, x_2, \vartheta_1) + \vartheta_2^{\mathrm{T}} \psi_2(x_1, x_2, \vartheta_1)\,.
\end{aligned} \tag{3.8}
$$

Substitute (3.8) into (3.7) to obtain

$$
\begin{aligned}
\dot{x}_2 &= -c_2 x_2 + x_3 \\
&\quad + (\theta - \vartheta_2)^{\mathrm{T}} \left[\psi_2(x_1, x_2, \vartheta_1) + \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1(z_1, z_2)}{\partial z_2}\right) \gamma_2(z_1, z_2, z_3) \right] \\
&= -c_2 x_2 + x_3 + (\theta - \vartheta_2)^{\mathrm{T}} w_2(x_1, x_2, x_3, \vartheta_1, \vartheta_2)\,.
\end{aligned} \tag{3.9}
$$

Then, let the update law for the new estimate ϑ_2 be

$$
\dot{\vartheta}_2 = x_2\, w_2(x_1, x_2, x_3, \vartheta_1, \vartheta_2)\,. \tag{3.10}
$$

Step i ($2 \leq i \leq n-1$) Using the definitions for $x_1, \ldots, x_i$ and $\dot{\vartheta}_1, \ldots, \dot{\vartheta}_{i-1}$, express the derivative of x_i as

$$
\begin{aligned}
\dot{x}_i &= \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \cdots \left(1 + \vartheta_{i-1}^{\mathrm{T}} \frac{\partial \gamma_{i-1}}{\partial z_i}\right) \left[z_{i+1} + \theta^{\mathrm{T}} \gamma_i(z_1, \ldots, z_{i+1}) \right] \\
&\quad + \varphi_i(x_1, \ldots, x_i, \vartheta_1, \ldots, \vartheta_{i-1}) + \theta^{\mathrm{T}} \psi_i(x_1, \ldots, x_i, \vartheta_1, \ldots, \vartheta_{i-1})\,.
\end{aligned} \tag{3.11}
$$

Let ϑ_i be a *new* estimate of θ and define the new state x_{i+1} as

$$\begin{aligned} x_{i+1} &= c_i x_i + \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \cdots \left(1 + \vartheta_{i-1}^{\mathrm{T}} \frac{\partial \gamma_{i-1}}{\partial z_i}\right) \left[z_{i+1} + \vartheta_i^{\mathrm{T}} \gamma_i(z_1, \ldots, z_{i+1})\right] \\ &\quad + \varphi_i(x_1, \ldots, x_i, \vartheta_1, \ldots, \vartheta_{i-1}) + \vartheta_i^{\mathrm{T}} \psi_i(x_1, \ldots, x_i, \vartheta_1, \ldots, \vartheta_{i-1}) . \end{aligned} \tag{3.12}$$

Substitute (3.12) into (3.11) to obtain

$$\begin{aligned} \dot{x}_i &= -c_i x_i + x_{i+1} + (\theta - \vartheta_i)^{\mathrm{T}} \left[\psi_i + \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \cdots \left(1 + \vartheta_{i-1}^{\mathrm{T}} \frac{\partial \gamma_{i-1}}{\partial z_i}\right) \gamma_i\right] \\ &= -c_i x_i + x_{i+1} + (\theta - \vartheta_i)^{\mathrm{T}} w_i(x_1, \ldots, x_{i+1}, \vartheta_1, \ldots, \vartheta_i) . \end{aligned} \tag{3.13}$$

Then, let the update law for ϑ_i be

$$\dot{\vartheta}_i = x_i \, w_i(x_1, \ldots, x_{i+1}, \vartheta_1, \ldots, \vartheta_i) . \tag{3.14}$$

<u>Step n</u>. Using the definitions for $x_1, \ldots, x_n$ and $\dot{\vartheta}_1, \ldots, \dot{\vartheta}_{n-1}$, express the derivative of x_n as

$$\begin{aligned} \dot{x}_n &= \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \cdots \left(1 + \vartheta_{n-1}^{\mathrm{T}} \frac{\partial \gamma_{n-1}}{\partial z_n}\right) \left[\beta_0(z) + \theta^{\mathrm{T}} \beta(z)\right] u \\ &\quad + \varphi_n(x, \vartheta_1, \ldots, \vartheta_{n-1}) + \theta^{\mathrm{T}} \psi_n(x, \vartheta_1, \ldots, \vartheta_{n-1}) . \end{aligned} \tag{3.15}$$

Let ϑ_n be a *new* estimate of θ and define the control u as

$$u = \frac{1}{\bar{\beta}(z, \vartheta_1, \ldots, \vartheta_n)} \left[-c_n x_n - \varphi_n - \vartheta_n^{\mathrm{T}} \psi_n\right] , \tag{3.16}$$

where

$$\bar{\beta}(z, \vartheta_1, \ldots, \vartheta_n) = \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \cdots \left(1 + \vartheta_{n-1}^{\mathrm{T}} \frac{\partial \gamma_{n-1}}{\partial z_n}\right) \left[\beta_0(z) + \vartheta_n^{\mathrm{T}} \beta(z)\right] . \tag{3.17}$$

Substitute (3.16) into (3.15) to obtain

$$\begin{aligned} \dot{x}_n &= -c_n x_n + (\theta - \vartheta_n)^{\mathrm{T}} \left[\psi_n + \left(1 + \vartheta_1^{\mathrm{T}} \frac{\partial \gamma_1}{\partial z_2}\right) \cdots \left(1 + \vartheta_{n-1}^{\mathrm{T}} \frac{\partial \gamma_{n-1}}{\partial z_n}\right) \beta(z) u\right] \\ &= -c_n x_n + (\theta - \vartheta_n)^{\mathrm{T}} w_n(x, \vartheta_1, \ldots, \vartheta_n) , \end{aligned} \tag{3.18}$$

where (3.16) is used in the definition of w_n. Finally, let the update law for the estimate ϑ_n be

$$\dot{\vartheta}_n = x_n \, w_n(x, \vartheta_1, \ldots, \vartheta_n) . \tag{3.19}$$

The above steps complete the formal development of the new design procedure. Its feasibility and the stability of the resulting closed-loop system are analyzed in the next section.

4 Feasibility and Stability

The above design procedure has introduced a set of new coordinates $x_1, \ldots, x_n$ defined by

$$
\begin{aligned}
x_1 &= z_1 \\
x_{i+1} &= \left(1+\vartheta_1^{\mathrm{T}}\frac{\partial\gamma_1}{\partial z_2}\right)\cdots\left(1+\vartheta_{i-1}^{\mathrm{T}}\frac{\partial\gamma_{i-1}}{\partial z_i}\right)\left[z_{i+1}+\vartheta_i^{\mathrm{T}}\gamma_i(z_1,\ldots,z_{i+1})\right]+c_i x_i \\
&\quad +\varphi_i(x_1,\ldots,x_i,\vartheta_1,\ldots\vartheta_{i-1})+\vartheta_i^{\mathrm{T}}\psi_i^{\mathrm{T}}(x_1,\ldots,x_i,\vartheta_1,\ldots,\vartheta_{i-1}),\ 1\le i\le n-1\,.
\end{aligned}
\tag{4.1}
$$

In order to ensure that the procedure is *feasible*, we construct in Proposition 4.1 an estimate $\mathcal{F} \subset I\!R^{n(1+p)}$ of the *feasibility region* such that for all $(z, \vartheta_1, \ldots, \vartheta_n) \in \mathcal{F}$ the coordinate change (4.1) is one-to-one, onto, continous and has a continuous inverse, and the denominator in (3.16) is nonzero.

Proposition 4.1. *Let B_z be defined as in Proposition 2.1 and $B_\vartheta \subset I\!R^p$ be an open set such that*

$$
\left|1+\vartheta_i^{\mathrm{T}}\frac{\partial\gamma_i(z)}{\partial z_{i+1}}\right| > 0,\ \forall z\in B_z,\ \forall\vartheta_i\in B_\vartheta,\ 1\le i\le n-1 \tag{4.2}
$$

$$
\left|\beta_0(z)+\vartheta_n^{\mathrm{T}}\beta(z)\right| > 0,\ \forall z\in B_z,\ \forall\vartheta_n\in B_\vartheta\,. \tag{4.3}
$$

Then, the set $\mathcal{F} = B_z \times B_\vartheta^n$ is a subset of the feasibility region.

Proof. Obvious, since (4.2) and (4.3) guarantee that in $B_z \times B_\vartheta^n$ (4.1) is *uniquely* solvable for z and the denominator in (3.16) is nonzero. □

Remark 4.2. The nonglobal nature of the feasibility region is not due to the adaptive scheme, because, even when the parameters θ are known, the feedback linearization of the system (3.1) can only be guaranteed for $\theta \in B_\theta$, with $B_\theta \subset I\!R^p$ an open set such that

$$
\left|1+\theta^{\mathrm{T}}\frac{\partial\gamma_i(z)}{\partial z_{i+1}}\right| > 0,\ \forall z\in B_z,\ \forall\theta\in B_\theta,\ 1\le i\le n-1 \tag{4.4}
$$

$$
\left|\beta_0(z)+\theta^{\mathrm{T}}\beta(z)\right| > 0,\ \forall z\in B_z,\ \forall\theta\in B_\theta\,. \tag{4.5}
$$

In the feasibility region, the adaptive system resulting from the design procedure can be expressed in the x-coordinates as

$$
\begin{aligned}
\dot{x}_1 &= -c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(x_1, x_2, \vartheta_1) \\
&\vdots \\
\dot{x}_{n-1} &= -c_{n-1} x_{n-1} + x_n + (\theta - \vartheta_{n-1})^{\mathrm{T}} w_{n-1}(x_1, \ldots, x_n, \vartheta_1, \ldots, \vartheta_{n-1}) \\
\dot{x}_n &= -c_n x_n + (\theta - \vartheta_n)^{\mathrm{T}} w_n(x, \vartheta_1, \ldots, \vartheta_n) \\
\dot{\vartheta}_i &= x_i\, w_i(x, \vartheta_1, \ldots, \vartheta_i), \quad 1 \leq i \leq n\,.
\end{aligned}
\tag{4.6}
$$

The stability properties of this system are now established using the quadratic Lyapunov function

$$
V(x, \vartheta_1, \ldots, \vartheta_n) = \frac{1}{2} x^{\mathrm{T}} x + \frac{1}{2} \sum_{i=1}^{n} (\theta - \vartheta_i)^{\mathrm{T}} (\theta - \vartheta_i)\,. \tag{4.7}
$$

The derivative of $V(x, \vartheta_1, \ldots, \vartheta_n)$ along the solutions of (4.6) is

$$
\begin{aligned}
\dot{V} &= -\sum_{i=1}^{n} \left[c_i x_i^2 + (\theta - \vartheta_i)^{\mathrm{T}} (x_i w_i - \dot{\vartheta}_i) \right] + \sum_{i=1}^{n-1} x_i x_{i+1} \\
&= -\sum_{i=1}^{n} c_i x_i^2 + \sum_{i=1}^{n-1} x_i x_{i+1}\,.
\end{aligned}
\tag{4.8}
$$

At this point we can choose the coefficients $c_1, \ldots, c_n$ that were left free in the design procedure. The choice $c_i \geq 2$, for all $i = 1, \ldots, n$, guarantees that $\dot{V}$ is negative semidefinite:

$$
\dot{V} \leq -\|x\|^2\,. \tag{4.9}
$$

This proves the uniform stability of the equilibrium

$$
x = 0\,,\ \vartheta_i = \theta\,,\ 1 \leq i \leq n \tag{4.10}
$$

of the adaptive system (4.6). To give an estimate Ω of the region of attraction of this equilibrium, we note that Ω must be a subset of our estimate $\mathcal{F}$ of the feasibility region. Let $\Omega(c)$ be the invariant set of (4.6) defined by $\{V < c\}$, and let c^* be the largest constant c such that $\Omega(c) \subset \mathcal{F}$. Then, an estimate Ω of the region of attraction is

$$
\Omega = \Omega(c^*) = \{(x, \vartheta_1, \ldots, \vartheta_n) : V(x, \vartheta_1, \ldots, \vartheta_n) < c^*\}\ ,\ c^* = \arg \sup_{\Omega(c) \subset \mathcal{F}} \{c\}\,. \tag{4.11}
$$

Remark 4.3. It can be expected that the above estimate is not the tightest possible one, because the choice of the unity gains in the update laws was made for simplicity. With some *a priori* knowledge about the shape of $\mathcal{F}$, different adaptation gains can be found so that Ω is maxixized by a better fit of $\mathcal{F}$. $\square$

Next, we use the invariance theorem of LaSalle to establish that for all initial conditions $(x, \vartheta_1, \ldots, \vartheta_n)_{t=0} \in \Omega$, the adaptive system (4.6) has the following regulation properties:

$$\lim_{t\to\infty} x(t) = 0\,, \quad \lim_{t\to\infty} \dot{x}(t) = 0\,, \quad \lim_{t\to\infty} \dot{\vartheta}_i(t) = 0\,,\ 1 \le i \le n\,. \tag{4.12}$$

In order to return to the original coordinates ζ, we note that, because of (4.2), the solution $z_2 = \cdots = z_n = 0$ of the system of equations

$$z_{i+1} + \theta^{\mathrm{T}} \gamma_i(0, z_2, \ldots, z_{i+1}) = 0\,, \quad 1 \le i \le n-1\,, \tag{4.13}$$

is *unique* in $B_z \times B_\vartheta$, and that $z_i, 1 \le i \le n$ can be expressed as smooth functions of $x, \vartheta_i, 1 \le i \le n$ using (4.1). Combining these facts with (4.12), we obtain

$$\lim_{t\to\infty} z_1(t) = 0\,, \quad \lim_{t\to\infty} \dot{z}_i(t) = 0\,, \quad 1 \le i \le n\,. \tag{4.14}$$

Using an induction argument, it is now shown that $z_i(t) \to 0$ as $t \to \infty$, $1 \le i \le n$:

- For $i = 1$, we have $z_1(t) \to 0$ as $t \to \infty$.
- For $i = k$, $2 \le k \le n$, we assume that $z_j(t) \to 0$ as $t \to \infty$, $1 \le j \le k-1$. Then, from (4.14) we have

$$\lim_{t\to\infty} \dot{z}_{k-1}(t) = \lim_{t\to\infty} \left\{ z_{k+1} + \theta^{\mathrm{T}} \gamma_{k-1}(z_1, \ldots, z_{k-1}, z_k) \right\} = 0\,, \tag{4.15}$$

  and from the uniqueness of solutions of (4.13) we conclude that $z_k(t) \to 0$ as $t \to \infty$.

Hence, $z(t) \to 0$ as $t \to \infty$. Finally, since $z = \phi(\zeta)$ is a diffeomorphism with $\phi(0) = 0$, regulation is achieved in the original coordinates ζ, namely

$$\lim_{t\to\infty} \zeta(t) = 0\,. \tag{4.16}$$

The above facts prove the following result:

Theorem 4.4. *When the design procedure of Section 3 is applied to a system of the form (2.1) that satisfies conditions (i) and (ii) of Proposition 2.1, the resulting adaptive system has a stable equilibrium at $\zeta = 0$, $\vartheta_i = \theta$, $1 \leq i \leq n$, whose region of attraction includes the set Ω defined in (4.11). Furthermore, regulation of the state $\zeta(t)$ is achieved:*

$$\lim_{t\to\infty} \zeta(t) = 0\,, \tag{4.17}$$

for all initial conditions in Ω. □

5 Global Stability

There are strong theoretical and practical reasons for investigating whether the stability properties of an adaptive system can be made global in the space of the states and parameter estimates. Systems with a finite region of attraction may not possess a wide enough robustness margin for disturbances, unmodeled dynamics, and other model imperfections. Furthermore, for nonglobal results it is usually hard to find nonconservative verifiable estimates of the region of attraction.

Another aspect of the global stability issue is the comparison of the proposed adaptive controller with its deterministic counterpart, that is, the controller that would be used if the parameter vector θ were known. Suppose that for all values of θ there exists a deterministic controller that achieves global stabilization and regulation of the system. If, with θ unknown, the proposed adaptive controller does not achieve the same global stability, this loss is clearly due to adaptation.

The stability result of Theorem 4.4 is not global, but, as pointed out in Remark 4.2, this is not due to adaptation. For pure-feedback systems, global stability may not be achievable even with θ known. We now consider "strict-feedback" systems for which a globally stabilizing controller exists when θ is known, and prove that our adaptive scheme guarantees global stability when θ is unknown.

In order to characterize the class of "strict-feedback" systems, we use the following assumption about the part of the system (2.1) that does not contain unknown parameters:

Assumption 5.1. *There exists a* **global** *diffeomorphism $z = \phi(\zeta)$, with $\phi(0) = 0$, that transforms the system*

$$\dot{\zeta} = f_0(\zeta) + g_0(\zeta)u\,, \tag{5.1}$$

into the system

$$\begin{aligned} \dot{z}_i &= z_{i+1}\,, 1 \le i \le n-1 \\ \dot{z}_n &= \gamma_0(z) + \beta_0(z)u\,, \end{aligned} \tag{5.2}$$

with

$$\gamma_0(0) = 0\,,\ \beta_0(z) \neq 0\ \forall z \in I\!R^n\,. \tag{5.3}$$

Remark 5.2. The *local* existence of such a diffeomorphism is equivalent to the feedback linearization condition (2.4). However, at present there are no necessary and sufficient conditions that can verify the *global* validity of this assumption. Sufficient conditions for Assumption 5.1 are given in [18], while necessary and sufficient conditions for the case where $\beta_0(z) \equiv$ const. can be found in [19,20]. □

Proposition 5.3. *Under Assumption 5.1, the system (2.1) is globally diffeomorphically equivalent to the "strict-feedback" system*

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \theta^{\mathrm{T}}\gamma_i(z_1,\ldots,z_i)\,, \quad 1 \le i \le n-1 \\ \dot{z}_n &= \gamma_0(z) + \theta^{\mathrm{T}}\gamma_n(z) + \beta_0(z)u \end{aligned} \tag{5.4}$$

if and only if the following condition holds globally:

Strict-feedback condition.

$$\left.\begin{aligned} g_i &\equiv 0\,, \\ [X, f_i] &\in \mathcal{G}^j, \quad \forall X \in \mathcal{G}^j, \quad 0 \le j \le n-2\,, \end{aligned}\right\} \quad 1 \le i \le p\,, \tag{5.5}$$

with $\mathcal{G}^j, 0 \le j \le n-1$, as defined in (2.4).

Proof. The proof is very similar to that of Proposition 2.1. First note that because of the assumptions that the diffeomorphism $z = \phi(\zeta)$ is global and that $\beta_0(z) \neq 0\ \forall z \in I\!R^n$, the distributions $\mathcal{G}^j$, $0 \leq j \leq n-1$, are globally defined and can be expressed in the z-coordinates as

$$\mathcal{G}^i = \operatorname{span}\left\{\frac{\partial}{\partial z_n}, \cdots, \frac{\partial}{\partial z_{n-i}}\right\}, \quad 0 \leq i \leq n-1. \tag{5.6}$$

To prove the sufficiency part of the proposition, note that if the pure-feedback condition (2.5) of Proposition 2.1 is replaced by the strict-feedback condition (5.5), then (2.12) and (2.14) are replaced by

$$\left.\begin{aligned} g_i &\equiv 0, \\ \left[\frac{\partial}{\partial z_j}, f_i\right] &\in \operatorname{span}\left\{\frac{\partial}{\partial z_n}, \ldots, \frac{\partial}{\partial z_j}\right\}, \quad 2 \leq j \leq n, \end{aligned}\right. \quad 1 \leq i \leq p. \tag{5.7}$$

Thus, the expression for $f_i(\phi^{-1}(z))$ in (2.13) becomes

$$f_i(\phi^{-1}(z)) = \begin{pmatrix} \gamma_{1,i}(z_1) \\ \gamma_{2,i}(z_1, z_2) \\ \vdots \\ \gamma_{n-1,i}(z_1, \ldots, z_{n-1}) \\ \gamma_{n,i}(z_1, \ldots, z_n) \end{pmatrix}, \quad 1 \leq i \leq p. \tag{5.8}$$

The necessity part is again straightforward. □

The above proposition gives a geometric characterization of the class of systems for which the following global properties can be achieved.

Theorem 5.4. *Under the conditions of Proposition 5.3 the stability and regulation results of Theorem 4.4 become* **global**, *i.e., they are valid for any initial conditions in* $\Omega = \mathbb{R}^{n(1+p)}$.

Proof. When the adaptive design procedure (3.3)-(3.19) is applied to the system (5.4), then for all $\vartheta_i \in \mathbb{R}^p$, $1 \leq i \leq n$, the change of coordinates (4.1) is one-to-one, onto, continuous and has a continuous inverse, and the control (3.16) is well defined, since

$$\frac{\partial \gamma_i}{\partial z_{i+1}}(z) \equiv 0, \quad \beta(z) \equiv 0, \quad \beta_0(z) \neq 0 \quad \forall z \in \mathbb{R}^n. \tag{5.9}$$

Hence (4.2)–(4.3) are trivially satisfied on $\mathcal{F} = B_z \times B_\vartheta^n = \mathbb{R}^{n(1+p)}$, and from (4.11) we conclude that $\Omega = \mathbb{R}^{n(1+p)}$. □

6 Multi-input Systems

The design procedure of Section 3 can be easily extended to multi-input nonlinear systems of the form

$$\dot{\zeta} = f_0(\zeta) + \sum_{i=1}^{p} \theta_i f_i(\zeta) + \sum_{j=1}^{m} \left[g_0^j(\zeta) + \sum_{i=1}^{p} \theta_i g_i^j(\zeta) \right] u_j \,, \tag{6.1}$$

with

$$f_i(0) = 0\,,\ 0 \le i \le p\,,\ \operatorname{rank} G_0(0) = m\,,\ G_0 = [g_0^1 \cdots g_0^m]\,, \tag{6.2}$$

that can be transformed into

$$\begin{aligned} \dot{z}_i^j &= z_{i+1}^j + \theta^{\mathrm{T}} \gamma_i^j \left(z_1^1, \ldots, z_{k_1-k_j+2}^1, \ldots, z_1^m, \ldots, z_{k_m-k_j+2}^m \right)\,,\ 1 \le i \le k_j - 1\,,\ 1 \le j \le m \\ \dot{z}_{k_j}^j &= \gamma_0^j(z) + \theta^{\mathrm{T}} \gamma_{k_j}^j(z) + \left[\beta_0^j(z) + \sum_{\ell=1}^{p} \theta_\ell \beta_\ell^j(z) \right]^{\mathrm{T}} u\,,\ 1 \le j \le m\,, \end{aligned} \tag{6.3}$$

with

$$\gamma_\ell^j(0) = 0\,,\ 0 \le i \le k_j\,,\ 1 \le j \le m\,,\ \det B_0(0) \ne 0\,, \tag{6.4}$$

where $B_0 = [\beta_0^1, \ldots, \beta_0^m]^{\mathrm{T}}$, and $\sum_{j=1}^m k_j = n$.

Proposition 6.1. *There exists a parameter-independent diffeomorphism $z = \phi(\zeta)$, with $\phi(0) = 0$, valid in a neighborhood B_z of the origin, that transforms the system (6.1) into the system (6.3) if and only if the following conditions are satisfied in a neighborhood of the origin:*

(i) **Feedback linearization condition.** *The distributions*

$$\mathcal{G}^i = \operatorname{span}\left\{ g_0^j, ad_{f_0} g_0^j, \ldots, ad_{f_0}^i g_0^j,\ 1 \le j \le m \right\}, \quad 0 \le i \le n-1 \tag{6.5}$$

are involutive and of constant rank r_i, with $r_{n-1} = n$.

(ii) **Pure-feedback condition.**

$$\left.\begin{aligned} g_i^j &\in \mathcal{G}^0,\ \ 1 \le j \le m\,, \\ [X, f_i] &\in \mathcal{G}^{k+1}\,,\ \forall X \in \mathcal{G}^k\,,\ 0 \le k \le n-2\,, \end{aligned}\right\} \quad 1 \le i \le p\,. \tag{6.6}$$

Proof. As proved in [21,22], condition (i) is necessary and sufficient for the existence of a diffeomorphism $z = \phi(\zeta)$ such that in the z-coordinates we have

$$\begin{aligned} f_0(\phi^{-1}(z)) &= \left[z_1^1 \dots z_{k_1-1}^1\, \gamma_0^1(z) \dots\, z_1^m \dots z_{k_m-1}^m\, \gamma_0^m(z)\right]^{\mathrm{T}} && (6.7)\\ G_0(\phi^{-1}(z)) &= \left[0 \dots 0\, \beta_0^1(z)\, \dots 0 \dots 0\, \beta_0^m(z)\right]^{\mathrm{T}} && (6.8)\\ \mathcal{G}^i &= \operatorname{span}\left\{\frac{\partial}{\partial z_{k_j}}, \dots, \frac{\partial}{\partial z_{k_j-i}}, 1 \le j \le m\right\}, \quad 0 \le i \le n-1\,. && (6.9) \end{aligned}$$

It is now a tedious but straightforward task to verify that condition (ii) is equivalent to

$$g_i^j(\phi^{-1}(z)) = \left[0 \dots 0\, \beta_{j,i}^1(z) \dots 0 \dots 0\, \beta_{j,i}^m(z)\right]^{\mathrm{T}}, \quad 1 \le i \le p, \quad 1 \le j \le m \tag{6.10}$$

$$f_i(\phi^{-1}(z)) = \begin{bmatrix} \gamma_{1,i}^1(z_1^1, z_2^1, \dots, z_1^m, \dots, z_{k_m-k_1+2}^m) \\ \vdots \\ \gamma_{k_1,i}^1(z) \\ \vdots \\ \gamma_{1,i}^m(z_1^1, \dots, z_{k_1-k_m+2}^1, \dots, z_1^m, z_2^m) \\ \vdots \\ \gamma_{k_m,i}^m(z) \end{bmatrix}, \quad 1 \le i \le p\,. \tag{6.11}$$

□

The design procedure for the system (6.3) is the following:

Steps 0 through $(n-m)$: Apply steps 0 through $(k_j - 1)$ of the single-input procedure to the first $(k_j - 1)$ equations of each of the m subsystems of (6.3), to obtain the system:

$$\begin{aligned} \dot{x}_i^j &= -c_i^j x_i^j + x_{i+1}^j + (\theta - \vartheta_\ell)^{\mathrm{T}} w_i^j(x, \vartheta_1, \dots, \vartheta_{\ell-1})\,,\ c_i^j \ge 2\,, \\ &\qquad \ell = \sum_{\rho=1}^{j-1}(k_\rho - 1) + i\,, \quad 1 \le i \le k_j\,, \quad 1 \le j \le m \\ \dot{\vartheta}_\ell &= x_i^j\, w_i^j(x, \vartheta_1, \dots, \vartheta_\ell), \quad 1 \le \ell \le n-m && (6.12)\\ \frac{d}{dt}\begin{bmatrix} x_{k_1}^1 \\ \vdots \\ x_{k_m}^m \end{bmatrix} &= \left[\bar{B}_0(z, \vartheta_1, \dots, \vartheta_{n-m}) + \sum_{i=1}^{p} \bar{B}_i(z, \vartheta_1, \dots, \vartheta_{n-m})\theta_i\right] u \\ &\quad + \Phi(x, \vartheta_1, \dots, \vartheta_{n-m}) + W^{\mathrm{T}}(x, \vartheta_1, \dots, \vartheta_{n-m})\theta\,, \end{aligned}$$

where

$$\bar{B}_i(z,\vartheta_1,\dots,\vartheta_{n-m}) = \begin{bmatrix} \left(1+\vartheta_1^{\mathrm{T}}\frac{\partial\gamma_1^1}{\partial z_2^1}\right)\cdots\left(1+\vartheta_{k_1-1}^{\mathrm{T}}\frac{\partial\gamma_{k_1-1}^1}{\partial z_{k_1}^1}\right)\beta_i^{1\,\mathrm{T}}(z) \\ \vdots \\ \left(1+\vartheta_{n-m-k_m+1}^{\mathrm{T}}\frac{\partial\gamma_1^m}{\partial z_2^m}\right)\cdots\left(1+\vartheta_{n-m}^{\mathrm{T}}\frac{\partial\gamma_{k_m-1}^m}{\partial z_{k_m}^m}\right)\beta_i^{m\,\mathrm{T}}(z) \end{bmatrix}. \quad (6.13)$$

Step $n-m+1$: Let ϑ_{n-m+1} be a new estimate of θ and define the control u as

$$\begin{aligned} u \;=\; & \left[\bar{B}_0(z,\vartheta_1,\dots,\vartheta_{n-m})+\sum_{i=1}^{p}\bar{B}_i(z,\vartheta_1,\dots,\vartheta_{n-m})\vartheta_{n-m+1,i}\right]^{-1}\left\{-\left[c_{k_1}^1 x_{k_1}^1\cdots c_{k_m}^m x_{k_m}^m\right]^{\mathrm{T}}\right. \\ & \left.-\Phi(x,\vartheta_1,\dots,\vartheta_{n-m})-W^{\mathrm{T}}(x,\vartheta_1,\dots,\vartheta_{n-m})\vartheta_{n-m+1}\right\},\; c_{k_j}^j\geq 2,\; 1\leq j\leq m. \quad (6.14)\end{aligned}$$

Substitute (6.14) into (6.12) and rewrite the last m equations of (6.12) as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_{k_1}^1 \\ \vdots \\ x_{k_m}^m \end{bmatrix} &= -\begin{bmatrix} c_{k_1}^1 x_{k_1}^1 \\ \vdots \\ c_{k_m}^1 x_{k_m}^m \end{bmatrix} + \left\{W+\left[\bar{B}_1 u\dots\bar{B}_p u\right]\right\}(\theta-\vartheta_{n-m+1}) \\ &= -\begin{bmatrix} c_{k_1}^1 x_{k_1}^1 \\ \vdots \\ c_{k_m}^m x_{k_m}^m \end{bmatrix} + W_{n-m+1}^{\mathrm{T}}(x,\vartheta_1,\dots,\vartheta_{n-m+1})(\theta-\vartheta_{n-m+1}), \quad (6.15)\end{aligned}$$

where (6.14) was used in the definition of W_{n-m+1}. Finally, let the update law for the estimate ϑ_{n-m+1} be

$$\dot{\vartheta}_{n-m+1} = W_{n-m+1}(x,\vartheta_1,\dots,\vartheta_{n-m+1})\begin{bmatrix} x_{k_1}^1 \\ \vdots \\ x_{k_m}^m \end{bmatrix}. \quad (6.16)$$

Note that this procedure will again be feasible only in a certain feasibility region, which can be defined as the region in which the matrix $\bar{B}=\bar{B}_0+\sum_{i=1}^{p}\bar{B}_i\vartheta_{n-m+1,i}$ is invertible. The stability properties of the resulting closed-loop system are analogous to those listed in Theorem 4.4, and can be similarly established using the Lyapunov function

$$V(x,\vartheta_1,\dots,\vartheta_{n-m+1}) = \frac{1}{2}x^{\mathrm{T}}x+\frac{1}{2}\sum_{i=1}^{n-m+1}(\theta-\vartheta_i)^{\mathrm{T}}(\theta-\vartheta_i). \quad (6.17)$$

7 A Global Tracking Result

We now turn our attention to the tracking problem for a class of input-output linearizable systems characterized by structural conditions analogous to those in Propositions 2.1 and 5.3. Every regulation result in Sections 2–5 has its tracking counterpart. For brevity, we restrict our presentation to the tracking version of the global regulation result in Section 5. The counterparts of nonglobal regulation results can be obtained using the same Lyapunov function argument as in this section to determine an invariant set in which asymptotic tracking is guaranteed.

Consider the nonlinear system

$$\begin{aligned} \dot{\zeta} &= f_0(\zeta) + \sum_{i=1}^{p} \theta_i f_i(\zeta) + g_0(\zeta)u \\ y &= h(\zeta)\,, \end{aligned} \tag{7.1}$$

where $\zeta \in I\!R^n$ is the state, $u \in I\!R$ is the input, $y \in I\!R$ is the output, $\theta = [\theta_1, \ldots, \theta_p]^{\mathrm{T}}$ is the vector of constant unknown parameters, h is a smooth function on $I\!R^n$ with $h(0) = 0$, and the vector fields g_0, f_i, $0 \leq i \leq p$, are smooth on $I\!R^n$ with $g(\zeta) \neq 0\,, \forall \zeta \in I\!R^n$, $f_i(0) = 0$, $0 \leq i \leq p$. We first formulate the input-output counterpart of Assumption 5.1:

Assumption 7.1. *There exist $n - \rho$ smooth functions $\phi_i(\zeta)$, $\rho + 1 \leq i \leq n$, such that the change of coordinates*

$$\begin{aligned} z_1 &= h(\zeta) \\ z_2 &= L_{f_0} h(\zeta) \\ z_3 &= L_{f_0}^2 h(\zeta) \\ &\vdots \\ z_\rho &= L_{f_0}^{\rho-1} h(\zeta) \\ z_i &= \phi_i(\zeta)\,, \quad \rho + 1 \leq i \leq n \end{aligned} \tag{7.2}$$

is a **global** *diffeomorphism $z = \phi(\zeta)$ that transforms the system*

$$\begin{aligned} \dot{\zeta} &= f_0(\zeta) + g_0(\zeta)u \\ y &= h(\zeta) \end{aligned} \tag{7.3}$$

into the special normal form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= \gamma_0(z) + \beta_0(z)u \\
\dot{z}^{\mathrm{r}} &= \Phi_0(y, z^{\mathrm{r}}) \\
y &= z_1 ,
\end{aligned} \tag{7.4}$$

with

$$\gamma_0(0) = L_{f_0}^{\rho} h(0) = 0\,, \ \Phi_0(0,0) = 0 \tag{7.5}$$

$$\beta_0(z) = L_{g_0} L_{f_0}^{\rho-1} h(\zeta) \neq 0 \ \forall z \in I\!R^n\,. \tag{7.6}$$

Remark 7.2. In order for (7.3) to be *locally* equivalent to (7.4), it is necessary and sufficient that the following conditions hold in a neighborhood of the origin $\zeta = 0$:

$$L_{g_0} L_{f_0}^{i} h \equiv 0\,, \ 0 \leq i \leq \rho - 2 \tag{7.7}$$

$$L_{g_0} L_{f_0}^{\rho-1} h(0) \neq 0 \tag{7.8}$$

$$\mathcal{G}^i \text{ is involutive and of constant rank } i+1\,, \ \ 0 \leq i \leq \rho - 1\,. \tag{7.9}$$

The sufficiency of these conditions is a consequence of Proposition 10 in [23]. The necessity can be easily established by verifying that (7.7)–(7.9) hold in the coordinates of (7.4). However, at present there are no necessary and sufficient conditions that can verify the *global* validity of this assumption. □

We are now ready to formulate the input-output counterpart of Proposition 5.3:

Proposition 7.3. *Under Assumption 7.1, the system (7.1) is globally diffeomorphically equivalent to the "strict-feedback" normal form*

$$\begin{aligned}
\dot{z}_i &= z_{i+1} + \theta^{\mathrm{T}} \gamma_i(z_1, \ldots, z_i, z^{\mathrm{r}})\,, \quad 1 \leq i \leq \rho - 1 \\
\dot{z}_\rho &= \gamma_0(z) + \theta^{\mathrm{T}} \gamma_\rho(z) + \beta_0(z)u \\
\dot{z}^{\mathrm{r}} &= \Phi_0(y, z^{\mathrm{r}}) + \sum_{i=1}^{p} \theta_i \Phi_i(y, z^{\mathrm{r}}) \\
y &= z_1 ,
\end{aligned} \tag{7.10}$$

if and only if the following condition holds globally:

Strict-feedback condition.

$$[X, f_i] \in \mathcal{G}^j\,, \quad \forall X \in \mathcal{G}^j, \quad 0 \le j \le \rho - 2\,, \quad 1 \le i \le p\,, \tag{7.11}$$

with $\mathcal{G}^j, 0 \le j \le \rho - 1$, as defined in (2.4).

Proof. The proof follows closely that of Proposition 5.3. First, because of the assumptions that the diffeomorphism $z = \phi(\zeta)$ defined in (7.2) is global and that $\beta_0(z) \neq 0\ \forall z \in I\!R^n$, the distributions $\mathcal{G}^j\,, 0 \le j \le \rho-1$, are globally defined and can be expressed in the z-coordinates as

$$\mathcal{G}^i \;=\; \operatorname{span}\left\{\frac{\partial}{\partial z_\rho}, \cdots, \frac{\partial}{\partial z_{\rho-i}}\right\}, \quad 0 \le i \le \rho - 1\,. \tag{7.12}$$

The sufficiency follows from the fact that, by (7.11) and (7.12),

$$\left[\frac{\partial}{\partial z_j}, f_i\right] \in \operatorname{span}\left\{\frac{\partial}{\partial z_\rho}, \ldots, \frac{\partial}{\partial z_j}\right\}, \quad 2 \le j \le \rho\,, \quad 1 \le i \le p\,. \tag{7.13}$$

Thus, the expression for $f_i(\phi^{-1}(z))$ is

$$f_i(\phi^{-1}(z)) = \begin{pmatrix} \gamma_{1,i}(z_1, z^{\mathrm{r}}) \\ \gamma_{2,i}(z_1, z_2, z^{\mathrm{r}}) \\ \vdots \\ \gamma_{\rho-1,i}(z_1, \ldots, z_{\rho-1}, z^{\mathrm{r}}) \\ \gamma_{\rho,i}(z_1, \ldots, z_\rho, z^{\mathrm{r}}) \\ \Phi_i(z_1, z^{\mathrm{r}}) \end{pmatrix}, \quad 1 \le i \le p. \tag{7.14}$$

The necessity part is again straightforward. □

Remark 7.4. To obtain the input-output counterpart of Proposition 2.1, one just needs to replace condition (2.4) (feedback linearization condition) of Proposition 2.1 with conditions (7.7)–(7.9) and condition (2.5) (pure-feedback condition) with

$$\begin{aligned} g_i \;&\in\; \mathcal{G}^0\,, \\ [X, f_i] \;&\in\; \mathcal{G}^{j+1}\,,\ \forall X \in \mathcal{G}^j\,, \quad 0 \le j \le \rho - 2\,, \end{aligned} \qquad 1 \le i \le p\,. \tag{7.15}$$

□

As in most tracking problems, we need an assumption about the stability of the zero-dynamics of (7.10):

Assumption 7.5. *The z^{r}-subsystem of (7.10) has the bounded-input-bounded-state (BIBS) property with respect to y as its input.*

It was shown in [9, Proposition 2.1] that the following conditions are sufficient for Assumption 7.5 to be satisfied:

(i) the zero dynamics of (7.1) are globally exponentially stable, and

(ii) the vector field $\Phi = \Phi_0 + \sum_{i=1}^{p} \theta_i \Phi_i$ in (7.10) is globally Lipschitz in z.

However, they are too restrictive for our purposes. For example, the system $\dot{z}^{\mathrm{r}} = -(z^{\mathrm{r}})^3 + y^2$ violates both these conditions, but is easily seen to satisfy Assumption 7.5. On the other hand, for nonglobal results it is convenient to use the assumption of exponential stability of the zero dynamics in order to estimate the region of attraction using a converse Lyapunov theorem.

The control objective is to force the output y of the system (7.1) to *asymptotically track* a known reference signal $y_{\mathrm{r}}(t)$, while keeping all the closed-loop signals bounded.

Assumption 7.6. *The reference signal $y_{\mathrm{r}}(t)$ and its first ρ derivatives are known and bounded.*

To achieve the asymptotic tracking objective, the design procedure presented in Section 3 is modified as follows:

Step 0. Define

$$x_1 = z_1 - y_{\mathrm{r}} \,. \tag{7.16}$$

Step 1. Starting with

$$\dot{x}_1 = z_2 + \theta^{\mathrm{T}} \gamma_1(z_1, z^{\mathrm{r}}) - \dot{y}_{\mathrm{r}} \,, \tag{7.17}$$

let ϑ_1 be an estimate of θ and define the new state x_2 as

$$\begin{aligned} x_2 &= c_1 x_1 + z_2 + \vartheta_1^{\mathrm{T}} \gamma_1(z_1, z^{\mathrm{r}}) - \dot{y}_{\mathrm{r}} \\ &= c_1 x_1 + z_2 + \vartheta_1^{\mathrm{T}} w_1(x_1, z^{\mathrm{r}}, y_{\mathrm{r}}) - \dot{y}_{\mathrm{r}} \,, \; c_1 \geq 2 \,. \end{aligned} \tag{7.18}$$

Substitute (7.18) into (7.17) to obtain

$$\dot{x}_1 = -c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(x_1, z^{\mathrm{r}}, y_{\mathrm{r}}) . \tag{7.19}$$

Then, let the update law for the parameter estimate ϑ_1 be

$$\dot{\vartheta}_1 = x_1\, w_1(x_1, z^{\mathrm{r}}, y_{\mathrm{r}}) . \tag{7.20}$$

Step 2. Using the definitions for x_1, x_2 and $\dot{\vartheta}_1$, write $\dot{x}_2$ as

$$\begin{aligned}
\dot{x}_2 &= c_1[-c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(x_1, z^{\mathrm{r}}, y_{\mathrm{r}})] + z_3 + \theta^{\mathrm{T}} \gamma_2(z_1, z_2, z^{\mathrm{r}}) \\
&\quad + x_1 w_1(x_1, z^{\mathrm{r}}, y_{\mathrm{r}})^{\mathrm{T}} \gamma_1(z_1, z^{\mathrm{r}}) + \vartheta_1^{\mathrm{T}} \left[\frac{\partial \gamma_1(z_1, z^{\mathrm{r}})}{\partial z_1} \left(z_2 + \theta^{\mathrm{T}} \gamma_1(z_1, z^{\mathrm{r}}) \right) \right. \\
&\quad \left. + \frac{\partial \gamma_1(z_1, z^{\mathrm{r}})}{\partial z^{\mathrm{r}}} \left(\Phi_0(z_1, z^{\mathrm{r}}) + \sum_{i=1}^{p} \theta_i \Phi_i(z_1, z^{\mathrm{r}}) \right) \right] - \ddot{y}_{\mathrm{r}} \\
&= z_3 + \varphi_2(x_1, x_2, z^{\mathrm{r}}, \vartheta_1, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ddot{y}_{\mathrm{r}}) + \theta^{\mathrm{T}} w_2(x_1, x_2, z^{\mathrm{r}}, \vartheta_1, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}) .
\end{aligned} \tag{7.21}$$

Let ϑ_2 be a *new* estimate of θ and define the new state x_3 as

$$x_3 = c_2 x_2 + z_3 + \varphi_2(x_1, x_2, z^{\mathrm{r}}, \vartheta_1, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ddot{y}_{\mathrm{r}}) + \vartheta_2^{\mathrm{T}} w_2(x_1, x_2, z^{\mathrm{r}}, \vartheta_1, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}), \quad c_2 \geq 2 . \tag{7.22}$$

Substitute (7.22) into (7.21) to obtain

$$\dot{x}_2 = -c_2 x_2 + x_3 + (\theta - \vartheta_2)^{\mathrm{T}} w_2(x_1, x_2, z^{\mathrm{r}}, \vartheta_1, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}) , \tag{7.23}$$

Then, let the update law for the new estimate ϑ_2 be

$$\dot{\vartheta}_2 = x_2\, w_2(x_1, x_2, z^{\mathrm{r}}, \vartheta_1, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}) . \tag{7.24}$$

Step i $(2 \leq i \leq \rho - 1)$ Using the definitions for $x_1, \ldots, x_i$ and $\dot{\vartheta}_1, \ldots, \dot{\vartheta}_{i-1}$, express the derivative of x_i as

$$\begin{aligned}
\dot{x}_i &= z_{i+1} + \varphi_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i)}) \\
&\quad + \theta^{\mathrm{T}} w_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}) .
\end{aligned} \tag{7.25}$$

Let ϑ_i be a *new* estimate of θ and define the new state x_{i+1} as

$$\begin{aligned}
x_{i+1} &= c_i x_i + z_{i+1} + \varphi_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i)}) \\
&\quad + \vartheta_i^{\mathrm{T}} w_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}), \quad c_i \geq 2 .
\end{aligned} \tag{7.26}$$

Substitute (7.26) into (7.25) to obtain

$$\dot{x}_i = -c_i x_i + x_{i+1} + (\theta - \vartheta_i)^{\mathrm{T}} w_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}) . \tag{7.27}$$

Then, let the update law for ϑ_i be

$$\dot{\vartheta}_i = x_i\, w_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}) . \tag{7.28}$$

Step ρ. Using the definitions for $x_1, \ldots, x_n$ and $\dot{\vartheta}_1, \ldots, \dot{\vartheta}_{\rho-1}$, express the derivative of x_n as

$$\begin{aligned} \dot{x}_\rho &= \beta_0(z)u + \varphi_\rho(x_1, \ldots, x_\rho, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho)}) \\ &\quad + \theta^{\mathrm{T}} w_\rho(x_1, \ldots, x_\rho, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-1)}) . \end{aligned} \tag{7.29}$$

Let ϑ_ρ be a *new* estimate of θ and define the control u as

$$u = \frac{1}{\beta_0(z)} \left[-c_\rho x_\rho - \varphi_\rho - \vartheta_\rho^{\mathrm{T}} w_\rho \right], \ c_\rho \geq 2 . \tag{7.30}$$

Substitute (7.30) into (7.29) to obtain

$$\dot{x}_\rho = -c_\rho x_\rho + (\theta - \vartheta_\rho)^{\mathrm{T}} w_\rho(x_1, \ldots, x_\rho, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-1)}) . \tag{7.31}$$

Finally, let the update law for the estimate ϑ_ρ be

$$\dot{\vartheta}_\rho = x_\rho\, w_\rho(x_1, \ldots, x_\rho, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-1)}) . \tag{7.32}$$

As was the case in the regulation result of Section 5, the assumptions of Proposition 7.3 guarantee that the design procedure (7.16)–(7.32) is globally feasible. The resulting closed-loop adaptive system is given by

$$\begin{aligned} \dot{x}_1 &= -c_1 x_1 + x_2 + (\theta - \vartheta_1)^{\mathrm{T}} w_1(x_1, z^{\mathrm{r}}, y_{\mathrm{r}}) \\ &\vdots \\ \dot{x}_{\rho-1} &= -c_{\rho-1} x_{\rho-1} + x_\rho + (\theta - \vartheta_{\rho-1})^{\mathrm{T}} w_{\rho-1}(x_1, \ldots, x_{\rho-1}, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-2)}) \\ \dot{x}_\rho &= -c_\rho x_\rho + (\theta - \vartheta_\rho)^{\mathrm{T}} w_\rho(x_1, \ldots, x_\rho, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{\rho-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-1)}) \\ \dot{z}^{\mathrm{r}} &= \Phi_0(y, z^{\mathrm{r}}) + \sum_{i=1}^{p} \theta_i \Phi_i(y, z^{\mathrm{r}}) \\ \dot{\vartheta}_i &= x_i\, w_i(x_1, \ldots, x_i, z^{\mathrm{r}}, \vartheta_1, \ldots, \vartheta_{i-1}, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}), \quad 1 \leq i \leq \rho \\ y &= x_1 + y_{\mathrm{r}} . \end{aligned} \tag{7.33}$$

The stability and tracking properties of (7.33) will be established using the quadratic function

$$V_t(x_1,\ldots,x_\rho,\vartheta_1,\ldots,\vartheta_\rho) = \frac{1}{2}\sum_{i=1}^{\rho}\left[x_i^2 + (\theta-\vartheta_i)^{\mathrm{T}}(\theta-\vartheta_i)\right]. \tag{7.34}$$

The derivative of V_t along the solutions of (7.33), with $c_i \geq 2$, $1 \leq i \leq \rho$, is

$$\begin{aligned}
\dot{V}_t &= -\sum_{i=1}^{\rho}\left[c_i x_i^2 + (\theta-\vartheta_i)^{\mathrm{T}}(x_i w_i - \dot{\vartheta}_i)\right] + \sum_{i=1}^{\rho-1} x_i x_{i+1} \\
&= -\sum_{i=1}^{\rho} c_i x_i^2 + \sum_{i=1}^{\rho-1} x_i x_{i+1} \\
&\leq -\sum_{i=1}^{\rho} x_i^2 \leq 0\,.
\end{aligned} \tag{7.35}$$

This proves that V_t is bounded. Hence $x_1,\ldots,x_\rho$ and $\vartheta_1,\ldots,\vartheta_\rho$ are bounded. The boundedness of x_1 and y_r implies that y is bounded. Combining this with Assumption 7.5 proves that z^r is bounded. Therfore, the state vector of (7.33) is bounded. This fact, combined with Assumption 7.6, implies the boundedness of z, ζ and u. Thus, the derivatives $\dot{x}_1,\ldots,\dot{x}_\rho$ are bounded. Now (7.34) and (7.35) imply that $\dot{V}_t$ is bounded and integrable. Moreover, the boundedness of $x_1,\ldots,x_\rho$ and $\dot{x}_1,\ldots,\dot{x}_\rho$ implies that $\ddot{V}_t$ is bounded. Hence, $\dot{V}_t \to 0$ as $t \to \infty$, which, combined with (7.35), proves that

$$\lim_{t\to\infty} x_i(t) = 0\,,\quad 1 \leq i \leq \rho\,. \tag{7.36}$$

In particular, this means that asymptotic tracking is achieved:

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty}\left[y(t) - y_r(t)\right] = 0\,. \tag{7.37}$$

These results are summarized as:

Theorem 7.7. *Under Assumptions 7.1, 7.5 and 7.6, and the strict-feedback condition (7.11), the adaptive design procedure (7.16)–(7.32), applied to the nonlinear system (7.1), yields global asymptotic tracking and boundedness of all the closed-loop signals.* □

8 Discussion and Examples

With the help of two simple examples, we now discuss some of the main features of the new adaptive scheme. The first example illustrates the systematic nature of the design procedure, while the second one compares the stability properties of the new scheme with those of the nonlinearity-constrained scheme of [9].

Example 8.1 (Regulation). We first consider a "benchmark" example of adaptive nonlinear regulation:

$$
\begin{aligned}
\dot{z}_1 &= z_2 + \theta z_1^2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= u \,,
\end{aligned}
\tag{8.1}
$$

where θ is an unknown constant parameter. This system violates both the geometric conditions of the schemes proposed in [1,2,3] and the growth assumptions of [5,6,9,12]. In fact, the only available global result for this example was obtained in [7].

The system (8.1) is already in the form of (5.4) with $\beta_0 \equiv 1$. Hence, this system satisfies the conditions of Theorem 5.4, which guarantees that the point $z = 0$, $\vartheta_1 = \vartheta_2 = \vartheta_3 = \theta$ is a globally stable equilibrium of the adaptive system. Moreover, for any initial conditions $z(0) \in I\!R^3$, $(\vartheta_1(0), \vartheta_2(0), \vartheta_3(0)) \in I\!R^3$, the regulation of the state $z(t)$ is achieved:

$$\lim_{t\to\infty} z(t) = 0\,. \tag{8.2}$$

The design procedure of Section 4, applied to (8.1), is as follows:

Step 0. Define $x_1 = z_1$.

Step 1. Let ϑ_1 be an estimate of θ and define the new state x_2 as

$$x_2 = 2x_1 + z_2 + \vartheta_1 x_1^2\,. \tag{8.3}$$

Substitute (8.3) into (8.1) to obtain

$$\dot{x}_1 = -2x_1 + x_2 + x_1^2(\theta - \vartheta_1)\,. \tag{8.4}$$

Then, let the update law for ϑ_1 be

$$\dot{\vartheta}_1 = x_1^3\,. \tag{8.5}$$

Step 2. Using (8.3) and (8.5), write $\dot{x}_2$ as

$$\dot{x}_2 = 2(z_2 + \theta z_1^2) + z_3 + \vartheta_1 2x_1(z_2 + \theta z_1^2) + x_1^5\,. \tag{8.6}$$

Let ϑ_2 be a new estimate of θ, and define the new state

$$x_3 = 2x_2 + 2(z_2 + \vartheta_2 z_1^2)(1 + \vartheta_1 x_1) + x_1^5 + z_3 \,. \tag{8.7}$$

Substitute (8.7) into (8.6) to obtain

$$\dot{x}_2 = -2x_2 + x_3 + 2x_1^2(1 + \vartheta_1 x_1)(\theta - \vartheta_2) \,. \tag{8.8}$$

Then, let the update law for ϑ_2 be

$$\dot{\vartheta}_2 = 2x_2 x_1^2(1 + \vartheta_1 x_1) \,. \tag{8.9}$$

Step 3. Using (8.3), (8.5), (8.7) and (8.8), write $\dot{x}_3$ as

$$\begin{aligned} \dot{x}_3 \;=\;& 2\left[-2x_2 + x_3 + 2x_1^2(1 + \vartheta_1 x_1)(\theta - \vartheta_2)\right] + 2\left[z_3 + 2z_1\vartheta_2(z_2 + \theta z_1^2)\right. \\ & \left. +2z_1^2 x_2 x_1^2(1 + \vartheta_1 x_1)\right](1 + \vartheta_1 x_1) + 2(z_2 + \vartheta_2 z_1^2)\left[x_1^4 + \vartheta_1(z_2 + \theta z_1^2)\right] \\ & +5x_1^4(z_2 + \theta z_1^2) + u \,. \end{aligned} \tag{8.10}$$

Let ϑ_3 be a new estimate of θ, and define the control u as

$$\begin{aligned} u \;=\;& -2x_3 - 2\left[-2x_2 + x_3 + 2x_1^2(1 + \vartheta_1 x_1)(\theta - \vartheta_2)\right] - 2\left[z_3 + 2z_1\vartheta_2(z_2 + \theta z_1^2)\right. \\ & \left. +2z_1^2 x_2 x_1^2(1 + \vartheta_1 x_1)\right](1 + \vartheta_1 x_1) - 2(z_2 + \vartheta_2 z_1^2)\left[x_1^4 + \vartheta_1(z_2 + \theta z_1^2)\right] \\ & -5x_1^4(z_2 + \theta z_1^2) \,. \end{aligned} \tag{8.11}$$

Substitute (8.11) into (8.10) to obtain

$$\dot{x}_3 = -2x_3 + \left[2x_1^2(1 + 2\vartheta_1 x_1) + 4z_1^3\vartheta_2 + 2\vartheta_1(z_2 + \vartheta_2 z_1^2)z_1^2 + 5x_1^6\right](\theta - \vartheta_3) \,. \tag{8.12}$$

Finally, let the parameter update law for ϑ_3 be

$$\dot{\vartheta}_3 = x_3\left[2x_1^2(1 + 2\vartheta_1 x_1) + 4z_1^3\vartheta_2 + 2\vartheta_1(z_2 + \vartheta_2 z_1^2)z_1^2 + 5x_1^6\right] \,. \tag{8.13}$$

The resulting adaptive system is

$$
\begin{aligned}
\dot{x}_1 &= -2x_1 + x_2 + x_1^2(\theta - \vartheta_1) \\
\dot{x}_2 &= -2x_2 + x_3 + 2x_1^2(1 + \vartheta_1 x_1)(\theta - \vartheta_2) \\
\dot{x}_3 &= -2x_3 + \left[2x_1^2(1 + \vartheta_1 x_1) + 4z_1^3\vartheta_2 + 2\vartheta_1(z_2 + \vartheta_2 z_1^2)z_1^2 + 5x_1^6\right](\theta - \vartheta_3) \\
\dot{\vartheta}_1 &= x_1^3 \\
\dot{\vartheta}_2 &= 2x_2 x_1^2(1 + \vartheta_1 x_1) \\
\dot{\vartheta}_3 &= x_3\left[2x_1^2(1 + \vartheta_1 x_1) + 4z_1^3\vartheta_2 + 2\vartheta_1(z_2 + \vartheta_2 z_1^2)z_1^2 + 5x_1^6\right] .
\end{aligned}
\tag{8.14}
$$

Using the Lyapunov function

$$
V = \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2 + (\theta - \vartheta_1)^2 + (\theta - \vartheta_2)^2 + (\theta - \vartheta_3)^2\right] , \tag{8.15}
$$

it is straightforward to establish the above mentioned global stability properties. □

Example 8.2 (Tracking). Consider now the problem in which the output y of the nonlinear system

$$
\begin{aligned}
\dot{z}_1 &= z_2 + \theta z_1^2 \\
\dot{z}_2 &= u + z_3 \\
\dot{z}_3 &= -z_3 + y \\
y &= z_1 ,
\end{aligned}
\tag{8.16}
$$

is required to asymptotically track the reference signal $y_r = 0.1 \sin t$.

For the sake of comparison, let us first solve this problem using the scheme of [9]. This scheme employs the control

$$
u = -z_3 + k_1(z_1 - y_r) + k_2(z_2 + \hat{\theta}_1 z_1^2 - \dot{y}_r) + \ddot{y}_r - 2\hat{\theta}_1 z_1 z_2 - 2\hat{\theta}_2 z_1^3 , \tag{8.17}
$$

where $\hat{\theta}_1$, $\hat{\theta}_2$, the estimates of θ, θ^2, respectively, are obtained from the update laws:

$$
\dot{\hat{\theta}}_1 = \frac{e_1\xi_1}{1 + \xi_1^2 + \xi_2^2} , \quad \dot{\hat{\theta}}_2 = \frac{e_1\xi_2}{1 + \xi_1^2 + \xi_2^2} . \tag{8.18}
$$

Using a relative-degree-two stable filter $M(s)$, the variables e_1, ξ_1, ξ_2 in (8.18) are defined as

$$
e_1 = y - y_r + \omega - \hat{\theta}_1\xi_1 - \hat{\theta}_2\xi_2 \tag{8.19}
$$

$$
\xi_1 = M(s)\left[2z_1z_2 + k_2 z_1^2\right] \tag{8.20}
$$

$$
\xi_2 = M(s)\left[2z_1^3\right] \tag{8.21}
$$

$$
\omega = M(s)\left[\hat{\theta}_1\left(2z_1z_2 + k_2z_1^2\right) + \hat{\theta}_2\left(2z_1^3\right)\right] . \tag{8.22}
$$

Simulations of this system were performed with

$$M(s) = \frac{1}{s^2 + 5s + 6}, \; \theta = 1\,, \; k_1 = -6\,, \; k_2 = -5\,, \tag{8.23}$$

and all the initial conditions zero, except for $z_1(0)$, which was varied between 0 and 0.45. The results of these simulations are shown in Fig. 1. The response of the closed-loop system is bounded for $z_1(0)$ sufficiently small, that is, for $z_1(0) < 0.45$. However, for larger $z_1(0)$, the response is unbounded. This behavior is consistent with the proof of Theorem 3.3 in [9], which guarantees boundedness for all initial conditions only under a global Lipschitz assumption. In the above system, the presence of the term z_1^2 leads to the violation of this assumption.

The unbounded behavior in Fig. 1 is avoided by the new scheme, which results in a globally stable adaptive system. This is illustrated by simulations in Fig. 2. The design procedure of Section 7, applied to the system (8.16), results in the change of coordinates

$$\begin{aligned} x_1 &= z_1 - y_r \\ x_2 &= 2(z_1 - y_r) + z_2 + \vartheta_1 z_1^2 - \dot{y}_r\,, \end{aligned} \tag{8.24}$$

the control

$$u = -z_3 - 3x_2 - 2(z_2 + \vartheta_2 z_1^2)(1 + \vartheta_1 z_1) - x_1 z_1^4 + 2\dot{y}_r + \ddot{y}_r\,, \tag{8.25}$$

and the update laws

$$\dot{\vartheta}_1 = x_1 z_1^2\,, \quad \dot{\vartheta}_2 = 2x_2 z_1^2(1 + \vartheta_1 x_1)\,. \tag{8.26}$$

□

The above example illustrates an obvious advantage of the new scheme in the case of strict-feedback systems: it guarantees global stability for all types of smooth nonlinearities. Its advantages are less obvious, but still important, in the case of pure-feedback systems, when the feedback linearization is not global. In this case, the new scheme provides an estimate of the region of attraction, which is not the case with the schemes of [5,9,12]. On the other hand, the schemes of [1,6] guarantee local results and give stability region estimates for larger classes of systems than the scheme presented in this paper. In the case of pure-feedback systems, it would be of interest to compare the sizes of stability regions obtained with

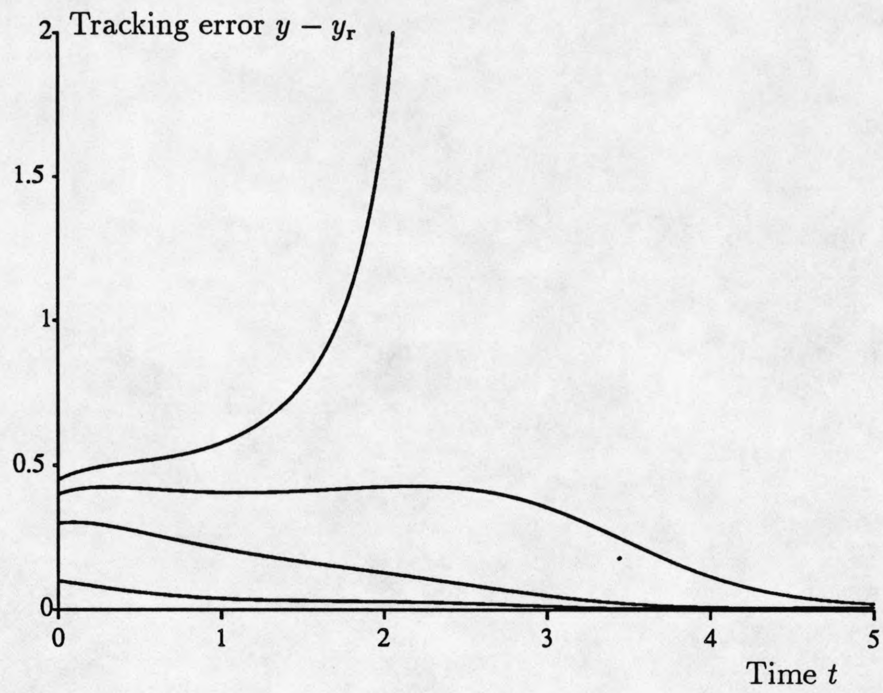

Figure 1: Locally stable tracking with the adaptive scheme of [9].

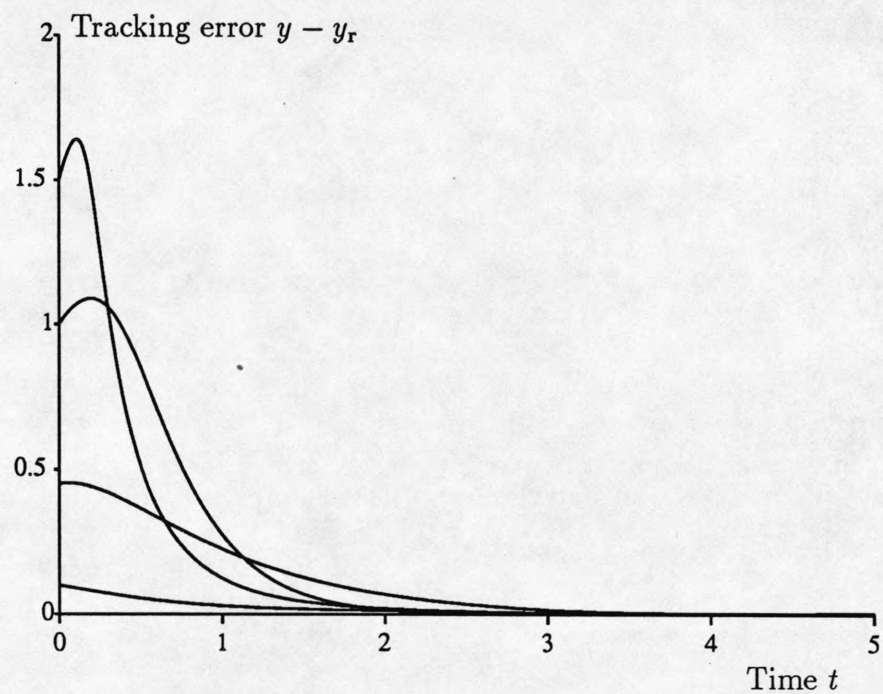

Figure 2: Globally stable tracking with the new adaptive scheme.

different schemes. Another significant task would be to compare their robustness properties. However, such tasks are beyond the scope of this paper.

9 Conclusions

The results of this paper have advanced in several directions our ability to control nonlinear systems with unknown constant parameters. The most significant progress has been made in solving the *global* adaptive regulation and tracking problems. The class of nonlinear systems for which these problems can be solved systematically is now much larger than ever before. The strict-feedback condition precisely characterizes the class of systems for which the global results hold with any type of smooth nonlinearities. For the broader class of systems satisfying the pure-feedback condition, the regulation and stability results may not be global, but are guaranteed in regions for which *a priori* estimates are given. It is crucial that the loss of globality, when it occurs, is not due to adaptation, but is inherited from the deterministic part of the problem. All these results are obtained using a step-by-step procedure which, at each step, interlaces a change of coordinates with the construction of an update law. Apart from the geometric conditions, this paper uses simple analytical tools, familiar to most control engineers.

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