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ON CERTAIN RELAXATION OSCILLATIONS: ASYMPTOTIC SOLUTIONS

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ABSTRACT

Periodic solutions of the generalized Liénard equation $\ddot{x} + \mu f(x)\dot{x} + g(x) = o$ ($\cdot \equiv d/dt$), with $\mu >> 1$, are investigated in the phase and Liénard planes. Certain comparison equations are obtained by modifying the functions $f(x)$, $g(x)$, $F(x) = \int_o^x f(u)du$, so that the resulting equations may be integrated to within an error of order μ^{-2}. The comparison solutions are used to approximate the solution trajectories of the orignal equations, and, in particular, the periodic orbits. The result is an analytic description of the trajectoris, in both planes, to order μ^{-2}. The asymptotic form of the amplitude and period, for a periodic orbit, is obtained with errors of order μ^{-2} and μ^{-1} respectively. For the particular case of the van der Pol equation, $\ddot{x} + \mu(x^2-1)\dot{x} + x = o$, the amplitude agrees with the expressions previously obtained by other workers, while the period does not. The results of a numerical study of the expression for the period given here support the present work.

1.1. Introduction

Various geometrical and analytical methods have been used to determine the existence of a stable periodic solution of the generalized Liénard equation

$$\ddot{x} + \mu f(x)\dot{x} + g(x) = 0 \qquad (\dot{} \equiv d/dt) \tag{1.1}$$

when $\mu \gg 1$. The methods used, and the results obtained, depend largely on the assumptions made concerning the functions f(x) and g(x).

It is convenient, not only for geometrical but also for analytical procedures, to study the systems equivalent to Eq.(1.1):

$$\dot{x} = \mu v, \ \dot{v} = -\mu f(x)v - g(x)/\mu \tag{1.2}$$

and

$$\dot{x} = \mu\left[y - F(x)\right], \ \dot{y} = -g(x)/\mu, \ F(x) = \int_0^x f(\xi)d\xi \tag{1.3}$$

which define the "scaled" phase (x,v) and Liénard (x,y) coordinates, respectively.

Any initial condition $(x(t_0), \dot{x}(t_0))$ prescribed for the solution of Eq. (1.1) defines a unique point in the phase and Lienard planes. The subsequent development in time of x(t), $\dot{x}(t)$ corresponds to a motion of the point in these planes. The curves traced by such motions will be referred to as trajectories, orbits, or integral curves, interchangeably. Periodic orbits are simple closed curves in the planes.

In the geometrical treatment of Eq. (1.1), it is usual to construct an annulus, in either the phase or Liénard planes, which contains the periodic orbit in its interior. The narrower the annulus, the more precisely is the periodic orbit isolated. La Salle [1], treating the Liénard equation ($g(x) = x$) constructed an annulus of maximum width $O(\mu^{-1})$ and obtained asymptotic forms for the period and amplitude of the periodic solution. Recently, Ponzo and Wax [2] have found an annulus of maximum width $O(\mu^{-4/3})$ enclosing the periodic solution of Eq. (1.1), for a different class of functions $f(x)$, and a wider class of $g(x)$ than those considered by La Salle.

The analytical discussions of Eq. (1.1), or its specilizations, have frequently employed particular methods for solving either Eq's. (1.2) or (1.3). Haag [3], in a series of long and difficult papers, has derived the asymptotic form of the periodic solution, for the special case of Eq. (1.1) when $g(x) = x$, and for what appears to be quite general $f(x)$, using Liénard coordinates. Dorodnicyn [4] has given an elaborate treatment of the van der Pol equation ($f(x) = x^2-1$, $g(x) =x$), in which he used various asymptotic expansions of the solution, each valid in some region of the phase plane. By matching the expansions at points of common convergence, he was able to give asymptotic expressions for the amplitude and period of the non-zero periodic solution, with errors $O(\mu^{-8/3})$ and $O(\mu^{-4/3})$ respectively. Cartwright [5] studied the van der Pol equation, primarily in the form of Eq. (1.1). She obtained results much less precise than those of Dorodnicyn, determining the amplitude to $o(\mu^{-4/3})$ and the period to $o(\mu^{-1/3})$. Her treatment, however, employs a comparison equation in the phase plane, and it is by a systematic use of comparison equations and their solutions that the present results concerning Eq. (1.1) are achieved.

We work in both the phase and Liénard planes, and show that the periodic orbit can be located with an error not greater than $O(\mu^{-2})$, when μ is large. This leads to a determination of the amplitude to $O(\mu^{-2})$ and the period to $O(\mu^{-1})$.

1.2. It is assumed throughout this paper that 1) there exists an $a<o$ and $b>o$ such that $f(a) = f(b) = o$; $F(a)>o$, $F(b)<o$; that $A<o$ is the root of $F(x) = F(b)$, and $B>o$ the root of $F(x) = F(a)$, of smallest modulus; 2) there exists an $\epsilon>o$ such that $g(x)$ and $f'(x)$ satisfy a Lipschitz condition ($' = d/dx$), with $xg(x)>o$ for $x \neq o$, in the interval $A - \epsilon \leq x \leq B + \epsilon$; 3) $g'(x)$ and $f''(x)$ satisfy a Lipschitz condition in the interval $a \leq x \leq b$; 4) there exist constants $K_1 > o$ and $K_2 > o$ such that $f(x) > K_1 (a-x)$ for $A - \epsilon \leq x < a$, and $f(x) > K_2 (x-b)$ for $b < x \leq B + \epsilon$; 5) there exist positive constants L_1, L_2, L_3, L_4 such that

$$L_2(a-x)^2 \leq F(a) - F(x) \leq L_1(a-x)^2$$

$$L_4(x-b)^2 \leq F(x) - F(b) \leq L_3(b-x)^2$$

in $a \leq x \leq b$. (See Figure 1).

The assumptions (1) - (5) are sufficient to guarantee the existence of a stable non-zero periodic solution of Eq. (1.1) [2]. This solution is illustrated in the Liénard plane in Figure 2 (labelled $y_p(x)$). Figure 2 also shows the contour Γ_o: $y = F(x) - g(x)/\mu^2 f(x)$, which, in the phase plane, is $v = v_o(x) = -g(x)/\mu^2 f(x)$, the zero-slope isocline (where $dv/dx = -f(x) - g(x)/\mu^2 v = o$). It will be shown that the periodic orbit, once having crossed $y = F(x)$ near $x = A$ (or $x = B$) approaches Γ_o to within $O(\mu^{-4})$ in the Liénard plane. This places the orbit a distance $O(\mu^{-2})$ from $y = F(x)$. The periodic orbit then moves to a distance $O(\mu^{-4/3})$ from $y = F(x)$ at $x = a$ (or $x = b$). The time taken to describe that portion of the trajectory in $A \leq x \leq a$ is $O(\mu)$. The periodic orbit, $y_p(x)$, remains nearly horizontal, in the upper half of the Liénard plane, from $x = a$ until its intersection with $y = F(x)$, near $x = B$. The time of traversal of this arc, which deviates from the horizontal by $O(\mu^{-4/3})$, is $O(\mu^{-1/3})$. Both the time of traversal, and the vertical variations, are largely associated with the region near $x = a$. Particular attention is paid, therefore, to the range $a - O(\mu^{-2/3}) \leq x \leq a + O(\mu^{-2/3})$ in the analysis. Once the trajectory is out of this range, the remaining time and y - deviation, until $y = F(x)$ is reached, are $O(\log \mu/\mu)$ and $O(\log \mu/\mu^2)$ respectively.

1.3 In the treatment which follows, we discuss various comparison equations whose solutions approximate the trajectories of Eq's. (1.2) or (1.3), to at least order μ^{-2}. The comparison equations are obtained by modifying the functions $f(x)$, $F(x)$, and $g(x)$ in such a way that the resulting equations may be integrated to the required degree of approximation. In the region $A - \epsilon \leq x \leq o$, $y \geq F(x)$ (or $v \geq o$), for example, it is sufficient to consider Eq's. (1.2) and (1.3) with $f(x)$, $F(x)$, and $g(x)$ replaced by Taylor series approximations about $x = a$. It will be noted, however, that the periodic orbit, in the Liénard plane, follows $y = F(x)$ very closely in $A - \epsilon \leq x \leq a$, so that truncating the Taylor series would produce an error of order 1 in $F(x)$ (near $x = A$, say), and consequently, a comparable error in the comparison solution. To avoid this situation, we make the comparison of solutions for $A - \epsilon \leq x \leq a$ in the phase plane; the phase plane ordinate measures the deviation of $y(x)$ from $F(x)$, $v = y - F(x)$, and the comparison solution is adequate for this variable, in the region R_1: $A - \epsilon \leq x \leq a$, $v \geq o$.

The remainder of the orbit is considered, in the Liénard plane in two parts: R_2: $a \leq x \leq o$ and R_3: $o \leq x \leq B + \epsilon$, for $y \geq F(x)$. Similar arguments apply to the lower half-orbit, in $y \leq F(x)$.

It should be emphasized that our discussion is concerned with the approximate location of the periodic orbit, to within an error $O(\mu^{-2})$. Consequently, we regard the periodic orbit as being embedded in a bundle of trajectories, of width $O(\mu^{-2})$, and the results we obtain refer to the entire bundle. Nothing in our treatment distinguishes any one of these integral curves from the periodic orbit. This set of orbits will be termed "the bundle", hereafter.

In Sec. 2 we consider the region R_1, in the phase plane. First, it is shown that one hyperbolic contour provides an upper bound on v, in all of R_1, and that another hyperbolic contour is a lower bound on v, over most of R_1. These contours help to isolate the bundle of orbits. Special attention is focused on the ends of the interval, in particular, near $x = a$ where the hyperbolic bounds are singular (in the manner of Γ_o, Figure 2). The major result is given in Lemma I where it is shown that a comparison solution which begins within the bundle (hence $O(\mu^{-2})$ from the periodic orbit), remains within $O(\mu^{-2})$ of the bundle, until $x = a$. In the corollary to Lemma I we obtain the time of traversal of

R_1, to within an error of $O(\mu^{-1})$. In obtaining this result, however, it is necessary to locate the periodic orbit more precisely than $O(\mu^{-2})$. For this reason, it is shown that the zero-slope isocline,

$$v_o(x) = -g(x)/\mu^2 f(x),$$

lies $O(\mu^{-4})$ from the periodic orbit over most of R_1 (except near the ends).

In Sec. 3 we treat the region R_2, in the Liénard plane. As before, Lemma II deals with a comparison equation whose solution approximates the periodic orbit to $O(\mu^{-2})$. The specific choice of comparison equation, for the region R_2, is made with some care. It would be sufficient to use a comparison equation which contained a cubic approximation to $F(x)$ and a linear approximation to $g(x)$. The resulting equation then becomes a generalized van der Pol equation, whose solutions are largely unknown. Nevertheless, an $O(\mu^{-2})$ approximation to this comparison solution may be obtained by iteration. The solution of an equation which uses a quadratic approximation to $F(x)$, and a constant in place of $g(x)$, is taken as a first approximation. Indeed, this latter solution, which is found from a Riccati equation and known exactly, provides an $O(\mu^{-2})$ comparison solution for the bundle of trajectories in R_1, when the transformation to the phase plane is made. In R_2, however, it approximates the periodic orbit to $O(\log \mu/\mu^2)$. It is the second approximation, found from the first, which is within $O(\mu^{-2})$ of the periodic orbit.

The time of traversal of R_2, for any member of the bundle, is estimated, in the corollary to Lemma II, to $O(\mu^{-1})$.

In Sec. 4 we treat the region R_3, in the Liénard plane. The comparison equation used involves a linear approximation to F(x), near x = B, and a constant in place of g(x). The resulting equation is easily integrated; Lemma II, together with its corollary, yields the location and time of traversal for the periodic orbit in R_3.

The results of previous sections are summarized in Sec. 5, in the form of a theorem which gives the explicit form of the amplitude, and period, of the non-zero periodic solution of Eq. (1.1), to $O(\mu^{-2})$ and $O(\mu^{-1})$ respectively. These results are specialized to the case of the van der Pol equation, and our disagreement with Dorodnicyn's [4] and Urabe's [6] expressions for the period is discussed.

In Sec. 6 we briefly consider the possibility of periodic solutions lying entirely in $a < x < b$, $F(b) < y < F(a)$.

1.4 In the sections which follow, we will frequently refer to the periodic orbit, when, in fact, there may be several periodic solutions of the equations we consider. We always refer, however, to the orbit, $y_p(x)$, shown schematically in Figure 2. Indeed, an annulus of maximum width $O(\mu^{-4/3})$ may be constructed about $y_p(x)$ [2]; this periodic orbit is unique within the annulus.

We also mention that the subscript "1" will be used consistently, in the following sections, to denote functions and trajectories associated with a comparison equation. The symbol "O" is always to denote orders of magnitude when $\mu >> 1$; O(1) is a constant.

2.1 The Region R_1: $A - \epsilon \leq x \leq a$.

As mentioned in Sec. 1.3, trajectories in R_1 are most conveniently discussed in the phase plane. Our first objective is to obtain upper and lower bounding arcs which isolate a bundle of trajectories within which the periodic orbit is embedded. These bounds will be used in Lemma I and its corollary.

Note that the periodic orbit crosses the x - axis to the left of α [2], where $\alpha < o$ is the unique root of $F(x) = o$, in $A \leq x \leq a$ (see Figure 1). Thus it is convenient to consider trajectories which start on the x - axis at $x = x_1$, where $A - \epsilon \leq x_1 \leq \alpha$. From Eq. (1.2) the trajectories satisfy

$$dv/dx = f(x) - g(x)/\mu^2 v \tag{2.1}$$

Hence the contour of zero-slope, $v_o(x)$, is given by

$$v_o(x) = -g(x)/\mu^2 f(x). \tag{2.2}$$

Observe that any curve, $v = v_u(x)$, which is monotone increasing and lies on or above $v_o(x)$, in $x < a$, is an upper bound on trajectories which start at $x = x_1$, $v = o$. This follows from the fact that integral curves can only cross $v_u(x)$ from above, since $dv/dx \leq o$ for $v = v_u(x) \geq v_o(x)$. Using $f(x) \geq K_1(a - x)$ and the continuity of $g(x)$, so that $- g(x) \leq C_1$ in R_1, we have

$$v_o(x) = -g(x)/\mu^2 f(x) \leq C_1/\mu^2 K_1(a - x) = M/\mu^2(a - x).$$

Thus, the contour $v_u(x) = M/\mu^2(a - x)$ is an upper bound, in R_1 for all trajectories starting at $x = x_1$, $v = o$.

A trajectory which starts at $x = x_1$, $v = o$ rises vertically at first. When it has attained an ordinate of $O(\mu^{-2})$, at $x = x_2$ say, then it will remain above a hyperbolic contour $v_L = m/\mu^2(a - x)$ $(m > o)$ in the interval $x_2 \leq x \leq a-O(\mu^{-2/3})$. To show this, note that if v_L is a lower bound, then trajectories on v_L must rise above it. It is sufficient to have

$$dv_L/dx = m/\mu^2(a - x)^2 \leq -f(x) - g(x)/\mu^2 v_L$$

in $x_2 \leq x \leq a - O(\mu^{-2/3})$.

One has, therefore,

$$m^2 + \mu^2(a - x)^2 f(x)m + \mu^2(a - x)^3 g(x) \leq o.$$

Since $f(x)$ is continuous in R_1, and $g(x)$ is continuous and negative, then there exist positive constants C_2, C_3 such that $f(x) \leq C_2(a - x)$ and $g(x) \leq - C_3$. Then m should satisfy

$$m^2 - \mu^2(a - x)^3 \quad (C_3 - C_2 m) \leq o$$

for $x_2 \leq x \leq a - \Delta\mu^{-2/3}$, where $\Delta = \text{const.} > o$. The last inequality is surely satisfied if one chooses $\Delta^3 = 4C_3/C_2^2$, and $o < m \leq C_3/2C_2$.

Thus, for any trajectory at $x = x_2$, $v(x_2) = O(\mu^{-2})$, there exist constants $m > o$, $M > o$ such that its subsequent velocity, $v(x)$, satisfies

$$m/\mu^2(a - x) \leq v(x) \leq M/\mu^2(a - x) \qquad (2.3)$$

for $x_2 \leq x \leq a - O(\mu^{-2/3})$.

Note that, for $x = x_3 = a - O(\mu^{-2/3})$, one has from Eq. (2.3)

$$C_4\mu^{-4/3} \leq v \leq C_5\mu^{-4/3} \qquad (2.4)$$

for some positive constants C_4, C_5. We now wish to show that Eq. (2.4) remains valid in the interval $x_3 \leq x \leq a$. This will extend our bounds to the end of R_1.

To obtain the lower bound in Eq. (2.4), we note that $v_o(x)$ is monotone increasing, for sufficiently large μ, and greater than $O(\mu^{-4/3})$, in $x_3 \leq x \leq a$. Thus, if $v(x_3) \geq v_o(x_3)$, trajectories descend until they intersect $v_o(x)$, and then rise monotonically in this interval. If $v(x_3) < v_o(x_3)$, then trajectories rise monotonically so that $v(x) \geq v(x_3) = O(\mu^{-4/3})$. In either case, the lower bound in Eq. (2.4) is established.

For the upper bound, we have from Eq. (2.1), $dv/dx \leq C_1/\mu^2 v$, since $f(x) \leq o$ and $-g(x) \leq C_1$ in R_1. Setting $v \geq C_4\mu^{-4/3}$ and integrating gives $v(x) \leq v(x_3) + \mu^{-2/3}\ C_1\ (x - x_3)/C_4 \geq O(\mu^{-4/3})$.

2.2 The upper bound in Eq. (2.3) allows us to show that trajectories which start at $x = x_1$, $v = o$, follow $v_o(x)$ very closely in R_1.

On writing Eq. (2.1) as $vdv/dx = - f(x)v - g(x)/\mu^2$ and letting $v = v_o(x) - z$, one obtains $dz/dx + f(x)z/v = v_o'(x)$. Therefore,

$$|z(x)| \leq |z(x_1)|\exp\left[-\int_{x_1}^{x} fdu/v\right] + \int_{x_1}^{x}|v_o'(\xi)|\exp\left[-\int_{\xi}^{x} fdu/v\right]d\xi.$$

Since $f(x) \geq K_1(a - x)$, then

$$|v_o'(\xi)| = \frac{|fg' - gf'|}{\mu^2 f^2} \leq \frac{O(1)}{\mu^2 K_1^2(a - \xi)^2} \leq C_6/\mu^2(a - x)^2$$

(C_6 = const. > o). Further, the exponential factors can be bounded by using the inequality

$$\int_\xi^x f du/v \geq \frac{\mu^2 K_1}{M} \int_\xi^x (a - u)^2 du \geq \frac{\mu^2 K_1}{M} (a - x)^2 (x - \xi)$$

in the exponents (where we have used $f(x) \geq K_1(a - x)$, and Eq. (2.3)). Since $|z(x_1)| = O(\mu^{-2})$, we now have

$$|z(x)| \leq O(\mu^{-2}) \exp\left[-O[\mu^2(x-x_1)]\right] + \frac{C_6}{\mu^2(a-x)^2} \int_{x_1}^x \exp\left[-\frac{\mu^2 K_1 (a-x)^2 (x-\xi)}{M}\right] d\xi .$$

The first term in this inequality is reduced to $O(\mu^{-4})$ for

$$x \geq x_1 + K \log \mu/\mu^2$$

and K sufficiently large; the second term, on performing the integration, is $O\left[1/\mu^4 (a - x)^4\right]$.

Thus, trajectories which start at $x = x_1$, within $O(\mu^{-2})$ of $v_o(x)$, will satisfy

$$|v(x) - v_o(x)| \leq O\left[\frac{1}{\mu^4 (a - x)^4}\right] \tag{2.5}$$

after moving to the right a distance $O(\log \mu/\mu^2)$. Note, however, that if $x = a - O(\mu^{-2/3})$, then one infers from Eq. (2.5) that

$$|v(x) - v_o(x)| \leq O(\mu^{-4/3}).$$

We use the method of comparison equations, in Lemma I, to help sharpen the estimate of v(x); in particular, we require a better estimate for v(x) in $a - O(\mu^{-2/3}) \leq x \leq a$.

2.3 Lemma I

Let f, g and f_1, g_1 be two pairs of functions which satisfy the assumptions of Sec. 1, in R_1, and furthermore, let

$$\text{(a)} \quad |f(x) - f_1(x)| \leq d_1|x - a|^2$$

$$\text{(b)} \quad |g(x) - g_1(x)| \leq d_2|x - a|, \text{ in } R_1.$$

Let the solutions of $dv/dx = -f(x) - g(x)/\mu^2 v$, $dv_1/dx = -f_1(x) - g_1(x)/\mu^2 v_1$, which begin at $x = x_1$, be $v(x, x_1, c)$ and $v_1(x, x_1, c_1)$ respectively, with $v(x_1, x_1, c) = c$ and $v_1(x_1, x_1, c_1) = c_1$.

If $|c - c_1| \leq O(\mu^{-2})$, then $|v(x, x_1, c) - v_1(x, x_1, c_1)| \leq O(\mu^{-2})$ in R_1.

Proof:

Let $\zeta = v - v_1$. On subtracting the equations for v and v_1, one gets

$$d\zeta/dx + \frac{|g_1|}{\mu^2 vv_1}\zeta = f_1 - f + (g_1 - g)/\mu^2 v,$$

where $-g_1(x) = |g_1(x)| > \partial$ in R_1.

$$\zeta(x) = \zeta(x_1)\exp\left[-\frac{1}{\mu^2}\int_{x_1}^{x}\frac{|g_1|du}{vv_1}\right] + \int_{x_1}^{x}\left[f_1 - f + \frac{g_1 - g}{\mu^2 v}\right]\exp\left[-\frac{1}{\mu^2}\int_{\xi}^{x}\frac{|g_1|d\mu}{vv_1}\right]d\xi$$

Using assumptions (a) and (b) above, together with $|g_1(x)| \leq C_1$, $|\zeta(x_1)| \leq O(\mu^{-2})$, and the bound v, $v_1 \leq O(\mu^{-4/3})$, in R_1, we have

$$|\zeta(x)| \leq O(\mu^{-2}) + \int_{x_1}^{x}\left[d_1(a-\xi)^2 + \frac{d_2(a-\xi)}{\mu^2 v}\right]\exp\left[-\mu^{2/3} C_7(x-\xi)\right]d\xi,$$

On setting $x = x_3 = a - \Delta\mu^{-2/3}$ and $\xi = a - s\mu^{-2/3}$, one gets

$$|\zeta(x)| \leq O(\mu^{-2}) + (d_1 + d_2/m) \frac{e^{C_7\Delta}}{\mu^2} \int_{\Delta}^{\infty} s^2 e^{-C_7 s} ds \leq O(\mu^{-2}).$$ Thus the $O(\mu^{-2})$ agreement between $v(x)$ and $v_1(x)$ is preserved to $x = x_3$.

For $x_3 \leq x \leq a$, recall that $v, v_1 = O(\mu^{-4/3})$. Then, using Eq. (2.6) with x_1 replaced by x_3, we have

$$|\zeta(x)| \leq O(\mu^{-2}) + \int_{x_3}^{a} \left[d_1(a-\xi)^2 + \mu^{-2/3} d_2(a-\xi)\right] d\xi \leq O(\mu^{-2}),$$

since $a - x_3 = O(\mu^{-2/3})$, and the exponential factor is less than unity.

This completes the proof of Lemma I.

Note that the equation for $\zeta(x) = v(x) - v_1(x)$ also applies to the deviation between two solutions of the <u>same</u> differential equation (i.e. when $f \equiv f_1$ and $g \equiv g_1$). Two solutions of Eq. (2.1) which differ by $O(\mu^{-2})$ at $x = x_1$, will differ, according to Eq. (2.6), by

$$\zeta(x) = \zeta(x_1) \exp\left[-\frac{1}{\mu^2} \int_{x_1}^{x} \frac{|g| du}{v v_1}\right] \leq O(\mu^{-2}) \exp\left[-O(\mu^{-2})\right]$$

after moving a distance O(1), in R_1; the rapid convergence of neighboring trajectories to each other, and to the periodic orbit, is evident.

In the corollary which follows, we use a comparison solution, $v_1(x)$, to compute the time taken to describe an arc of a trajectory in R_1.

The time required to traverse any portion of an orbit is $\mu^{-1}\int dx/v$ in the phase plane, and $-\mu\int dy/g(x)$ in the Liénard plane; the integrals are to be evaluated along the particular trajectory.

2.4 Corollary to Lemma I

Let $T(x_1,v)$ be the time required for a phase plane trajectory, which starts at $x=x_1$, $v=o$, to traverse the region R_1. Then

$$T(x_1,v) = T(x_1,v_o) - \mu v_1(a)/g(a) - \frac{2}{3}\frac{g'(a)}{g(a)f'(a)}\frac{\log \mu}{\mu} + O(\mu^{-1}),$$

where $v_1(x)$ is a comparison solution, as in Lemma I, and

$$T(x_1,v_o) = \frac{1}{\mu}\int_{x_1}^{a} \frac{dx}{v_o(x)} = -\mu \int_{x_1}^{a} \frac{f(x)}{g(x)}\,dx$$

is the time integral evaluated along the path $v=v_o(x)$.

Proof:

Since $T(x_1,v) = -\mu \int_{x_1}^{a} \frac{dy}{g(x)}$, in the Liénard plane, and $y=v + F(x)$, then one has

$$T(x_1,v) = -\mu \int_{x_1}^{a} \frac{dv}{g(x)} - \mu \int_{x_1}^{a} \frac{f(x)}{g(x)}\,dx.$$

The second integral is $T(x_1,v_o)$. The first integral may be written as

$$-\mu \int_{x_1}^{x_2} \frac{dv}{g(x)} - \mu \int_{x_2}^{a} \frac{dv}{g(x)} .$$

On choosing $x_2 = x_1 + O(\log \mu/\mu^2)$, where $\left|v(x_2) - v_o(x_2)\right| \leq O(\mu^{-4})$ from the discussion in Sec. 2.2, then, since $1/g(x) = O(1)$ and since $\int_{x_1}^{x_2} dv \leq O(\mu^{-2})$, one has

$$T(x_1,v) = T(x_1,v_o) - \mu \int_{x_2}^{a} \frac{dv}{g(x)} + 0(\mu^{-1}).$$

On integrating by parts, one gets

$$T(x_1,v) = T(x_1,v_o) - \frac{\mu v(a)}{g(a)} + \frac{\mu v(x_2)}{g(x_2)} + \mu \int_{x_2}^{a} v \frac{d}{dx} [\frac{1}{g(x)}] dx + 0(\mu^{-1}).$$

Now $v(x_2) = 0(\mu^{-2})$, and if $v_1(x)$ is a comparison solution for which $\left| v(x_2) - v_1(x_2) \right| \leq 0(\mu^{-2})$, then $v(a) = v_1(a) + 0(\mu^{-2})$, from Lemma I. Therefore

$$T(x_1,v) = T(x_1,v_o) - \mu v_1(a)/g(a) + \mu \int_{x_2}^{a} v [\frac{1}{g(x)}]' dx + 0(\mu^{-1}).$$

We now divide the interval (x_2,a) into two parts: (x_2,x_3) and (x_3,a), where $x_3 = a - \Delta\mu^{-2/3}$. In $x_3 \leq x \leq a$, one has $v = 0(\mu^{-4/3})$ and $[\frac{1}{g(x)}]' = 0(1)$, so that $\mu \int_{x_3}^{a} v [\frac{1}{g(x)}]' dx = 0(\mu^{-1})$. Furthermore, in $x_2 \leq x \leq x_3$, one has that $v(x) = v_o(x) + 0 [1/\mu^4(a-x)^4]$ from Eq. (2.5), and $[\frac{1}{g(x)}]' = - g'(x)/g^2(x) = - g'(a)/g^2(a) + 0(a-x)$. Then

$$\mu \int_{x_2}^{x_3} v \frac{d}{dx} [\frac{1}{g(x)}] dx = - \frac{\mu g'(a)}{g^2(a)} \int_{x_2}^{x_3} v_o(x) dx + \mu \int_{x_2}^{x_3} v_o(x) 0(a-x) dx + 0(\mu^{-1}).$$

Now $v_o(x) = - g(x)/\mu^2 f(x) = - g(a)/[\mu^2 f'(a)(a-x)] + 0(\mu^{-2})$, in $x_2 \leq x \leq x_3$, therefore

$$- \mu \frac{g'(a)}{g^2(a)} \int_{x_2}^{x_3} v_o(x)\, dx = \frac{g'(a)}{g(a)f'(a)} \frac{\log(\triangle\mu^{-2/3})}{\mu} + 0(\mu^{-1})$$

$$= - \frac{2}{3} \frac{g'(a)}{g(a)f'(a)} \frac{\log \mu}{\mu} + 0(\mu^{-1}) ,$$

and

$$\mu \int_{x_2}^{x_3} v_o(x)\, 0(a-x)\, dx = 0(\mu^{-1}).$$

Collecting results, one has, finally

$$T(x_1,v) = T(x_1,v_o) - \mu v_1(a)/g(a) - \frac{2}{3} \frac{g'(a)}{g(a)f'(a)} \frac{\log \mu}{\mu} + 0(\mu^{-1}),$$

which completes the proof of the corollary.

Note that an explicit representation has been given for all orbits which start at x_1, with $v(x_1) = o$. These orbits all follow $v_o(x)$, to $0[1/\mu^4(a-x)^4]$, in $x_2 \leq x \leq a$. Furthermore, a comparison solution provides an orbit approximation, to $0(\mu^{-2})$, throughout $x_1 \leq x \leq a$. The time taken to describe any one of these trajectories, in R_1, has also been given in the corollary to Lemma I, to $0(\mu^{-1})$, in terms of $v_o(x)$ and a comparison orbit, $v_1(x)$. The particular choice of comparison equation will be left until the next section.

It should be emphasized that a specific choice of x_1, defines a particular trajectory, and that the description provided above applies equally well to a bundle of trajectories within $0(\mu^{-2})$ of this trajectory. Eventually we shall choose x_1, so that the bundle contains the periodic orbit.

3.1 The region R_2: $a \leq x \leq b$

We continue our description of the bundle of orbits which started in $A - \epsilon \leq x \leq \alpha$, with $\dot{x}=0$, in terms of comparison equations and their solutions. Our first result, Lemma II, shows that the $O(\mu^{-2})$ agreement is maintained, in R_2, if the comparison equation satisfies certain conditions. Secondly, we choose a specific comparison equation which suffices to describe the bundle of trajectories in both R_1 and R_2, and find that a solution of a particular Riccati equation may be used to obtain an explicit description of this bundle. Finally, for any member of the bundle, we compute the time taken to traverse R_2.

Before proceeding to Lemma II, we determine bounds for the bundle of trajectories. From Sec. 2, we have that $y(a) \geq F(a) + O(\mu^{-4/3})$, and since $y' \geq 0$, then $y(x) \geq F(a) + O(\mu^{-4/3})$, in R_2. We use this lower bound in $dy/dx = - g(x)/\mu^2[y-F(x)]$, together with $-g(x) \leq C_8$ (C_8 = const. > 0) and $F(x) \leq F(a) - L_2(x-a)^2$, and thus obtain, upon integrating, the upper bound $y(x) \leq F(a) + O(\mu^{-4/3})$. Consequently, in R_2, we have that

$$y(x) = F(a) + O(\mu^{-4/3}) \tag{3.1}$$

for the bundle of trajectories which we consider.

Lemma II:

Let $F(x)$, $g(x)$ and $F_1(x)$, $g_1(x)$ be two pairs of functions which satisfy the assumptions of Sec. 1, as they apply to R_2, and let $\ell_3, \ell_4 > 0$ exist such that

(a) $$\left|F(x) - F_1(x)\right| \leq \ell_3 \left|x-a\right|^4 ,$$

and

(b) $$\left|g(x) - g_1(x)\right| \leq \ell_4 \left|x-a\right|^2, \text{ in } R_2 .$$

If the solutions of

$$dy/dx = \frac{-g(x)}{\mu^2[y-F(x)]} \tag{3.2}$$

and

$$dy_1/dx = \frac{-g_1(x)}{\mu^2[y_1-F_1(x)]} , \tag{3.3}$$

which start at x=a, be $y(x,c)$ and $y_1(x,c_1)$ respectively, where $y(a,c) = c$ and $y_1(a,c_1) = c_1$, if $c \geq F(a) + \delta$, where $\delta \geq O(\mu^{-4/3})$, and if $\left|c - c_1\right| \leq O(\mu^{-2})$, then $\left|y(x,c) - y_1(x,c_1)\right| \leq O(\mu^{-2})$, in R_2.

Proof:

Let $z = y - y_1$. On subtracting Eq. (3.3) from Eq. (3.2), one has

$$\frac{dz}{dx} + \frac{\left|g_1\right| z}{\mu^2 vv_1} = \frac{\left|g_1\right|(F-F_1)}{\mu^2 vv_1} + \frac{g_1- g}{\mu^2 v} ,$$

where $v = y-F$ and $v_1 = y_1- F_1$.

Then

$$z(x) = z(a) \exp\left[- \frac{1}{\mu^2}\int_a^x \frac{\left|g_1\right| du}{vv_1}\right] + \frac{1}{\mu^2}\int_a^x \left\{\left[\frac{\left|g_1\right|(F-F_1)}{vv_1} + \frac{g_1- g}{v}\right] \exp\left[- \frac{1}{\mu^2}\int_\xi^x \frac{\left|g_1\right| du}{vv_1}\right]\right\} d\xi .$$

On using the assumptions of Sec. 1, and (a), (b) above, one has

$$\left|z(x)\right| \leq \left|z(a)\right| + \frac{1}{\mu^2}\int_a^x \left\{\frac{C_8 \ell_3 (\xi-a)^4}{[\delta + L(\xi-a)^2]^2} + \frac{\ell_4 (\xi-a)^2}{\delta+L(\xi-a)^2}\right\} d\xi, \qquad (3.4)$$

where $v = y - F \geq c - [F(a) - L(x-a)^2] \geq \delta + L(x-a)^2$ for some $L > 0$, and similarly for v_1.

Since $\delta \geq 0$ and $\left|z(a)\right| \leq 0(\mu^{-2})$, by assumption, then Eq. (3.4) gives $\left|z(x)\right| \leq 0(\mu^{-2})$, in R_2. This completes the proof of Lemma II.

3.2 We make a specific choice of comparison equation here, and derive from it the appropriate comparison solution.

Consider $F_1(x) = F(a) + f'(a)(x-a)^2/2 + f''(a)(x-a)^3/6$ and $g_1(x) = g(a) + g'(a)(x-a)$. The conditions of Lemma II are then satisfied. Putting $y_1 = F(a) + [-2g^2(a)/f'(a)]^{1/3}\eta$ and $x = a + [-4g(a)/f'^2(a)]^{1/3}\xi$ into $\mu^2 dy_1/dx = -g_1(x)/[y_1 - F_1(x)]$, one obtains

$$\mu^2 \frac{d\eta}{d\xi} = \frac{1 - N_1 \xi}{\eta + \xi^2 - N_2 \xi^3}, \qquad (3.5)$$

where $N_1 = [\frac{2}{g(a)f'(a)}]^{1/3} g'(a)$ and $N_2 = [\frac{4g(a)^5}{f'(a)}]^{1/3} \frac{f''(a)}{3}$.

Eq. (3.5) does not appear to be integrable in terms of the known classical functions. However, an $0(\mu^{-2})$ approximation to the solutions of Eq. (3.5) may be obtained as follows.

Note that Eq. (3.5) has $f_1(x) = f'(a)(x-a) + f''(a)(x-a)^2/2$, consequently, $f_1(x)$ and $g_1(x)$ satisfy the conditions of Lemma I, for the region R_1. Indeed, the conditions of Lemma I will be satisfied simply by choosing $f_1(x) = f'(a)(x-a)$ and $g_1(x) = g(a)$, which corresponds to setting $N_1 = N_2 = o$ in Eq. (3.5). The equation which results, when $N_1 = N_2 = o$,

$$\mu^2 \, d\eta/d\xi = \frac{1}{\eta + \xi^2} , \tag{3.6}$$

can be integrated exactly and will provide adequate comparison solutions for R_1, when the transformation is made to the phase plane. Furthermore, Eq. (3.5), our comparison equation for R_2, may be estimated to $O(\mu^{-2})$, by making use of the solutions of Eq. (3.6). The following manipulations of Eq. (3.5) help to achieve this estimation.

On using the identity $1/(1-x) = 1 + x + x^2/(1-x)$, and rearranging, Eq. (3.5) may be written as

$$\mu^2 \, d\eta/d\xi = \frac{1}{\eta + \xi^2} - \frac{N_1 \xi}{\eta + \xi^2} + \frac{N_2 \xi^3}{(\eta + \xi^2)^2} + E_1 + E_2 \tag{3.7}$$

where $\left| E_1 \right| = \dfrac{N_1 N_2 \xi^4}{(\eta + \xi^2)^2} \leq N_1 N_2 = O(1)$ since $\eta \geq o$ in R_2. Now

$$\left| E_2 \right| = \frac{(1 - N_1 \xi) N_2^2 \xi^6}{(\eta + \xi^2)^2 (\eta + \xi^2 - N_2 \xi^3)} \leq O\left[\frac{\xi^2}{\eta + \xi^2 - N_2 \xi^3} \right]$$

since $1 - N_1 \xi = O(1)$ and $\xi^2/(\eta + {}^2) \leq 1$, in R_2.

On using assumption (5) and shifting to the (ξ, η) variables, one finds that $\eta + \xi^2 - N_2 \xi^3 \geq \eta + C\xi^2$ for some $C = O(1) > o$. Therefore $\left| E_2 \right| \leq O(1)$, and Eq. (3.7) can thus be written as

$$\mu^2 \, d\eta/d\xi = \frac{1}{\eta + \xi^2} - \frac{N_1 \xi}{\eta + \xi^2} + \frac{N_2 \xi^3}{(\eta + \xi^2)^2} + O(1). \tag{3.8}$$

Now

$$\frac{N_2\,\xi^3}{(\eta+\xi^2)^2} = \frac{N_2\,\xi}{\eta+\xi^2} - \frac{N_2\,\xi\eta}{(\eta+\xi^2)^2}$$

and

$$\frac{\xi}{\eta+\xi^2} = \frac{1}{2}\,\frac{\eta'+2\xi}{\eta+\xi^2} - \frac{\eta'\xi}{2(\eta+\xi^2)}$$

$$= \frac{1}{2}\,\frac{d}{d\xi}\,\log(\eta+\xi^2) - \frac{1}{2\mu^2}\,\frac{1-N_1\xi}{(\eta+\xi^2)(\eta+\xi^2-N_2\xi^3)}\,,$$

on using Eq. (3.5), so that Eq. (3.8) becomes

$$\mu^2\,\frac{d\eta}{d\xi} = \frac{1}{\eta+\xi^2} - \frac{(N_1-N_2)}{2}\,\frac{d}{d\xi}\,\log(\eta+\xi^2) - \frac{N_2\xi\eta}{(\eta+\xi^2)^2} - \frac{1}{2\mu^2}\,\frac{1-N_1\xi}{(\eta+\xi^2)(\eta+\xi^2-N_2\xi^3)} + 0(1).$$

Therefore

$$\eta(\xi) = \eta(0) + \frac{1}{\mu^2}\int_o^{\xi}\frac{dz}{\eta+z^2} - \frac{(N_1-N_2)}{2\mu^2}\,\log\left[\frac{\eta+\xi^2}{\eta(o)}\right] + E_3$$

where

$$\left|E_3\right| \le \frac{\left|N_2\right|}{\mu^2}\int_o^{\xi}\frac{z\eta\,dz}{(\eta+z^2)^2} + \frac{1}{2\mu^4}\int_o^{\xi}\frac{\left|1-N_1 z\right|dz}{(\eta+C\,z^2)^2} + 0(\mu^{-2}),$$

on using $\eta+\xi^2 \ge \eta+\xi^2-N_2\,\xi^3 \ge \eta+C\xi^2$ again.

From Eq. (3.1) it follows that $\eta = 0(\mu^{-4/3})$, in R_2. Using this in the bounding expression for E_3, and integrating, one finds that $\left|E_3\right| \le 0(\mu^{-2})$.

Thus, corresponding solutions of Eq. (3.5) and of the integral equation

$$\eta(\xi) = \eta(o) + \frac{1}{\mu^2}\int_o^\xi \frac{dz}{\eta + z^2} - \frac{(N_1 - N_2)}{2\mu^2} \log \left[\frac{\eta + \xi^2}{\eta(o)}\right], \tag{3.9}$$

will differ by $O(\mu^{-2})$ at most, in R_2.

In order to complete our partial integration of Eq. (3.5), to $O(\mu^{-2})$, we now make use of the solutions of Eq. (3.6), as mentioned above. Let $\eta_R(\xi)$ be the solution of Eq. (3.6) which satisfies $\eta_R(o) = \eta(o)$. Then

$$\eta_R(\xi) = \eta(o) + \frac{1}{\mu^2}\int_o^\xi \frac{dz}{\eta_R + z^2}. \tag{3.10}$$

Now $\eta(\xi) = O(\mu^{-4/3})$ in R_2 (this follows from Eq. (3.1)), and for a specific choice of $\eta(o) = O(\mu^{-4/3})$, it is evident from Eq. (3.9) that $\eta_R(\xi)$ provides an approximation to the solutions of Eq. (3.5), with an error $O\{\frac{1}{\mu^2} \log [\frac{\eta + \xi^2}{\eta(o)}]\} = O[\frac{1}{\mu^2} \log (1 + C_9\mu^{4/3} \xi^2)]$, where C_9 = const. $>$ o. On inserting $\eta_R(\xi)$ into the right hand side of Eq. (3.9), one obtains

$$\eta_c(\xi) = \eta_R(\xi) - \frac{(N_1 - N_2)}{2\mu^2} \log \left[\frac{\eta_R + \xi^2}{\eta_R(o)}\right] \tag{3.11}$$

as a second approximation. In order to estimate the error involved in using $\eta_c(\xi)$ as the comparison solution for R_2, rather than the solution of Eq. (3.9), we subtract Eq.(3.11) from Eq. (3.9), and use Eq. (3.10) to obtain

$$\left|\eta - \eta_c\right| \leq \frac{1}{\mu^2}\int_o^{\xi} \frac{\left|\eta - \eta_R\right| dz}{(\eta+z^2)(\eta_R+z^2)} + \frac{1}{2\mu^2}\left|(N_1 - N_2)\log\left(\frac{\eta+\xi^2}{\eta_R+\xi^2}\right)\right| .$$

Noting that both $\eta = 0(\mu^{-4/3})$ and $\eta_R = 0(\mu^{-4/3})$, and that $\left|\eta - \eta_R\right| = 0[\frac{1}{\mu^2}\log(1 + C_9\mu^{4/3}\xi^2)]$, by our previous estimate, one has that the second term is $0(\mu^{-2})$ and the integral has order of magnitude

$$\frac{1}{\mu^4}\int_o^{\xi} \frac{\log(1 + C_9\mu^{4/3}z^2)\, dz}{(\mu^{-4/3} + z^2)^2} = 0(\mu^{-2}).$$

Thus, $\eta_c(\xi)$ is a solution of Eq. (3.9), and therefore of Eq. (3.5), to $0(\mu^{-2})$, in R_2. This solution, with the appropriate choice of $\eta(o)$, will provide the comparison solution for Eq. (3.2), in R_2, when we transform back to the original (x,y) variables.

3.3 The comparison solution for R_2, $\eta_c(\xi)$, has been given in terms of $\eta_R(\xi)$, a solution of Eq. (3.6). As mentioned above, Eq. (3.6) can be integrated explicitly.

Let $\eta_R = \mu^{-4/3}u$, $\xi = \mu^{-2/3}s$. On inverting Eq. (3.6) and substituting these new variables, one gets

$$\frac{ds}{du} = u + s^2, \tag{3.12}$$

a Riccati equation whose solutions are shown in Fig. 3. The transformation

$$s = -\frac{d}{du}\log w \tag{3.13}$$

converts the Riccati into the linear differential equation

$$\frac{d^2w}{du^2} + uw = o, \qquad (3.14)$$

which is soluble in terms of Bessel functions of order 1/3.

It is important to observe that the solution $\eta_R(\xi)$ will provide the comparison solution for R_1, when the transformation to the phase plane (x,v) variables is made. In order to satisfy the requirements of Lemma I, the phase plane comparison solutions, $v_1(x)$, must satisfy $o \leq v_1(x_1) \leq O(\mu^{-2})$, or equivalently, in the Liénard plane, $o \leq y_1(x_1) - F_1(x_1) \leq O(\mu^{-2})$ at $x = x_1 = a - O(1)$. On transforming to the (u,s) variables this condition becomes $o \leq u + s^2 \leq O(\mu^{-2/3})$, for $s = - O(\mu^{2/3})$. Any solution of the Riccati equation which crosses the curve $u = - s^2$ to the left of $s = - O(\mu^{2/3})$ certainly satisfies this condition, and can serve as the desired comparison solution. A particularly convenient choice, however, is the solution for which $u \sim -s^2$ as $s \to -\infty$; this solution joins $u = -s^2$, at $s = -\infty$, whatever the value of μ.

We thus select the Riccati solution which has $s \sim -|u|^{1/2}$ as $u \to -\infty$; equivalently, $\log w \sim -2|u|^{3/2}/3$ as $u \to -\infty$. The corresponding solution of Eq. (3.14) is

$$w = C \int_0^\infty \cos(t^3 - 3\,ut)\,dt = C\,\mathrm{Ai}(u), \tag{3.15}$$

where C = const. and Ai(u) is the Airy integral.

The function $w = C\,\mathrm{Ai}(u)$ is exponentially small as $u \to -\infty$, increases monotonically to a maximum at $u = u_o \doteq 1.01879$ (where $\xi = o$), then falls to zero at $u = u_\infty \doteq 2.33811$ ($\xi = \infty$). This solution is shown in Fig. 3, in the (ξ, η) plane

It should be observed that Eq. (3.15) and Eq. (3.13) give an implicit representation for $\eta_R(\xi)$. Thus, the comparison solution for

R_2, $\eta_c(\xi)$, is available from Eq. (3.11) Primary interest, however, centers on the value of $y_1(x)$ at x=o; this value will allow us to continue a description of trajectories into the region R_3.

When x=o, $\xi=0(1)$ so that $\log(\eta_R + \xi^2) = 0(1)$. Furthermore, $\eta_R(o) = 1.01879\mu^{-4/3} = 0(\mu^{-4/3})$. Substituting into Eq. (3.11) one obtains

$$\eta_c(\xi) = \eta_R(\xi) - \frac{2}{3}(N_1 - N_2)\frac{\log \mu}{\mu^2} + 0(\mu^{-2}), \text{ at } \xi=0(1).$$

But, from Eq. (3.10),

$$\eta_R(\xi) = \eta_R(\infty) - \frac{1}{\mu^2}\int_{\xi}^{\infty}\frac{dz}{\eta_R+z^2} = \eta_R(\infty) - 0(\mu^{-2}),$$

since the integral is convergent when $\xi = 0(1)$ $(\eta_R > o)$. Note that $\eta_R(\infty) = 2.33811\ \mu^{-4/3}$.

Returning to the original Liénard plane variables, one has that the periodic orbit, $y_p(x)$, as well as every other member of the bundle, crosses the positive y-axis with ordinate given by

$$y_o = F(a) + 2.33811\left[\frac{-2g^2(a)}{f'^2(a)}\right]^{1/3}\mu^{-4/3}$$

$$-\frac{2}{3}\left\{\frac{2}{3}\frac{g(a)f''(a)}{f'^2(a)} + g'(a)\left[\frac{-4g(a)}{f'^2(a)}\right]^{1/3}\right\}\frac{\log \mu}{\mu^2} + 0(\mu^{-2})$$

3.4 Corollary to Lemma II

The time taken by any one of the bundle of trajectories to traverse the region R_2 is given by

$$T_2 = 1.31932 \left[\frac{2}{g(a)f'(a)}\right]^{1/3} \mu^{-1/3} + \frac{4}{9}\frac{f''(a)}{f'^2(a)}\frac{\log \mu}{\mu} + O(\mu^{-1}).$$

Proof:

The time is determined from the expression

$$T = \frac{1}{\mu}\int_a^x \frac{dx}{y-F(x)} \quad .$$

This integral may be evaluated, to $O(\mu^{-1})$, by replacing $y(x)$ by $y_1(x)$ and $F(x)$ by $F_1(x)$. The error involved is bounded by

$$\frac{1}{\mu}\int_a^x \frac{\left|y-y_1\right| + \left|F-F_1\right|}{v\ v_1}\, dx \ ,$$

where $v=y-F$, $v_1=y_1-F_1$.

On using $\left|y-y_1\right| \leq O(\mu^{-2})$ and $\left|F-F_1\right| \leq \ell_3\left|x-a\right|^4$ and the fact that $v, v_1 \geq \delta + L(x-a)^2$, one has, upon integrating, that the error is bounded by $O(\mu^{-1})$. Consequently, we may use the comparison solution to determine T_2. On scaling the variables, one gets

$$T = \left[\frac{2}{g(a)f'(a)}\right]^{1/3} \frac{1}{\mu}\int_o^{\xi} \frac{dz}{\eta_c + z^2 - N_2 z^3} + O(\mu^{-1}).$$

Expanding the denominator, as was done with Eq. (3.5), and integrating, using Eq.'s (3.9)-(3.11), one finds finally that

$$T = \left[\frac{2}{g(a)f'(a)}\right]^{1/3} \{\mu[\eta_R(\xi) - \eta_R(o)] + \frac{N_2}{2\mu}\log\left[\frac{\eta_R + \xi^2}{\eta_R(o)}\right]\} + O(\mu^{-1}).$$

At $x=o$, then $\xi=O(1)$ and the time of traversal of R_2 is thus

$$T_2 = [\frac{2}{g(a)f'(a)}]^{1/3} [\eta_R(\infty) - \eta_R(o)]\mu + \frac{4}{9}\frac{f''(a)}{f'^2(a)}\frac{\log\mu}{\mu} + 0(\mu^{-1}).$$

Substituting $\eta_R(\infty) = 2.33811\ \mu^{-4/3}$ and $\eta_R(o) = 1.01879\ \mu^{-4/3}$ gives the required result.

3.5 In this section we have completed the explicit description of the bundle of trajectories, in both R_1 and R_2. For the region R_1, we use $\eta_R(\xi)$ and transform to the phase plane ordinate, v. This gives

$$v_1(x) = [\frac{-2g^2(a)}{f'(a)}]^{1/3} [\eta_R(\xi) + \xi^2] \tag{3.16}$$

as the comparison solution for R_1. For R_2 we use $\eta_c(\xi)$ and transform to the Liénard plane ordinate, y, to obtain

$$y_1(x) = F(a) + [\frac{-2g^2(a)}{f'(a)}]^{1/3} \eta_c(\xi) \tag{3.17}$$

as the comparison solution. Note that $\eta_c(o) = \eta_R(o)$, from Eq. (3.11), so that Eq.'s (3.16) and (3.17) give $v_1(a) = y_1(a) - F(a)$, for ξ=o. Thus our description of trajectories is continuous across x=a. Note, too, that $v_1(a)$, required in the corollary to Lemma I, is given by Eq. (3.16).

We continue our description of the bundle of trajectories in the next section.

4.1 The Region R_3: $o \leq x \leq B + \epsilon$

Consider a trajectory, $y(x)$, a member of the bundle which starts at $x=o$ with $y(o) = y_o + O(\mu^{-2})$. This trajectory is within $O(\mu^{-2})$ of the periodic orbit at $x=o$, and the value, y_o, is known to $O(\mu^{-2})$ from the previous sections. Let this trajectory intersect $y=F(x)$ at $x=x_o$; x_o is to be determined.

It is shown, in Lemma III, that a comparison solution exists which remains within $O(\mu^{-2})$ of $y(x)$ for $o \leq x \leq x_o$. The comparison solution, obtained as an exact integral of the comparison equation, is used to determine x_o, from y_o, to $O(\mu^{-2})$. It is shown that this representation is valid, to $O(\mu^{-2})$, for every member of the bundle.

Lemma III:

Let F,g be a pair of functions which satisfy the assumptions of Sec. 1, in R_3, and F_1,g_1 another pair of Lipschitz functions for which

(a) $\left|F(x) - F_1(x)\right| \leq \ell_1\left|x - x_o\right|^2$

(b) $\left|g(x) - g_1(x)\right| \leq \ell_2\left|x - x_o\right|$,

in R_3.

Let the solution of Eq. (3.2) which passes through $x=x_o$, $y=F(x_o)$ be denoted by $y(x)$, and the solution of $\mu^2 dy_1/dx = -g_1(x)/[y_1 - F_1(x)]$ for which $\left|y_1(x_o) - y(x_o)\right| \leq O(\mu^{-2})$ be $y_1(x)$.

Then $\left|y(x) - y_1(x)\right| \leq O(\mu^{-2})$ for $o \leq x \leq x_o$.

Proof:

In the equations for y and y_1, we let $x=x_o - \xi$ and subtract, putting $z=y-y_1$. Then

$$\frac{dz}{d\xi} + \frac{g_1}{\mu^2 vv_1} z = \frac{1}{\mu^2} [\frac{g_1(F-F_1)}{vv_1} + \frac{g-g_1}{v}] ,$$

where $v=y-F(x)$ and $v_1=y_1-F_1(x)$.

Since $z=0(\mu^{-2})$ at $\xi=o$, we have

$$\left|z(x)\right| \leq 0(\mu^{-2}) + \int_o^\xi \{\frac{\left|g_1\right|\left|F-F_1\right|}{vv_1} + \frac{\left|g-g_1\right|}{v}\} \exp [- \frac{1}{\mu^2} \int_u^\xi \frac{g_1 du}{vv_1}] d\xi .$$

Now $v \geq o$, $v_1 \geq o$, and $g_1 \geq o$ in $o \leq \xi \leq x_o$, so that the exponential factor is less than unity. Further, we use (a), (b) above, and $\left|g_1\right| \leq C$ to obtain

$$\left|z(x)\right| \leq 0(\mu^{-2}) + \frac{1}{\mu^2} \int_o^\xi \{\frac{C\ell_1 \xi^2}{vv_1} + \frac{\ell_2 \xi}{v}\} d\xi , \text{ in } R_3.$$

In order to obtain lower bounds on v, v_1, note that both $F(x)$ and $F_1(x)$ may be bounded above, in $o \leq x \leq x_o$, by $F(x_o) + D(x-x_o)$, for some D=const. > o. (See Fig. 4). Then

$$v = y-F(x) \geq y-F(x_o) - D(x-x_o) \geq D\xi , \text{ since } y-F(x_o) \geq o,$$

and similarly for v_1. Substituting these lower bounds on v, v_1, gives

$$\left|z(x)\right| \leq 0(\mu^{-2}) + (\frac{C\ell_1}{D^2} + \frac{\ell_2}{D}) \frac{1}{\mu^2} \int_o^\xi d\xi \leq 0(\mu^{-2})$$

for $o \leq x \leq x_o$. This proves Lemma III.

4.2 Consider the comparison equation for $y_1(x)$, with $F_1(x) = F(x_o) + f(x_o)(x-x_o)$ and $g_1(x) = g(x_o)$. This choice of comparison functions clearly satisfies (a), (b). We have

$$\frac{dy_1}{dx} = \frac{-g(x_o)}{\mu^2[y_1-F(x_o)-f(x_o)(x-x_o)]} .$$

On setting $y_1 = F(x_o) + \eta$ and $x = x_o - \xi$, one obtains

$$\frac{d\eta}{d\xi} = \frac{g(x_o)}{\mu^2\ [\eta+f(x_o)\xi\ ]} . \tag{4.1}$$

The solution of Eq. (4.1) which satisfies the condition $y_1(x_o)=y(x_o)$, or $\eta=o$ at $\xi = o$, is

$$[\eta+f(x_o)\,\xi + g(x_o)/\mu^2 f(x_o)]\ \exp\ [-\mu^2 f(x_o)\eta/g(x_o)] = g(x_o)/\mu^2 f(x_o);$$

after taking logarithms and some rearrangement, this becomes

$$\eta(\xi) = \frac{2g(x_o)}{f(x_o)}\frac{\log\mu}{\mu^2} + \frac{g(x_o)}{\mu^2 f(x_o)}\log[\eta+f(x_o)\,\xi + \frac{g(x_o)}{\mu^2 f(x_o)}] + 0(\mu^{-2}) \tag{4.2}$$

where $\log[g(x_o)/f(x_o)] = 0(1)$ has been used.

For any ξ in $o \le \xi \le x_o$, hence any x in $o \le x \le x_o$, Eq. (4.2) allows one to approximate a trajectory y(x) to $0(\mu^{-2})$ since $y(x) = y_1(x) + 0(\mu^{-2}) = F(x_o) + \eta(x_o - x) + 0(\mu^{-2})$. The value of x_o may be found by noting that $y_o = F(x_o) + \eta(x_o) + 0(\mu^{-2})$. This gives, from Eq. (4.2),

$$y_o - F(x_o) = \frac{2g(x_o)}{f(x_o)}\frac{\log\mu}{\mu^2} + 0(\mu^{-2}), \tag{4.3}$$

as $\log[\eta+f(x_o)\,\xi + g(x_o)/\mu^2 f(x_o)] = 0(1)$, at $\xi = x_o$. In Eq. (4.3), put $x_o = B + \epsilon_1$, where $F(B) = F(a)$. Then

$$\epsilon_1 = \frac{1}{f(B)}\,[y_o - F(a) - \frac{2g(B)}{f(B)}\frac{\log\mu}{\mu^2}] + 0(\mu^{-2}) .$$

A trajectory, y(x), which starts at x=o with $y(o)=y_o+0(\mu^{-2})$, will therefore intersect y=F(x) at $x=x_o$, where

$$x_o = B + \frac{y_o - F(a)}{f(B)} - \frac{2g(B)}{f^2(B)} \frac{\log \mu}{\mu^2} + O(\mu^{-2}) \qquad (4.4)$$

It is important to observe that y(x) is <u>any</u> trajectory which starts on the y-axis, with $y(o)=y_o+O(\mu^{-2})$. Consequently, the value of x_o (Eq. (4.4)), and the comparison solution (Eq. (4.2)), apply to every member of the bundle. Even though the trajectories in the bundle diverge from each other, in going from x=o to $x=x_o$, as can be seen from the proof of Lemma III, the bundle remains of maximum width $O(\mu^{-2})$ in R_3.

We now make use of the comparison solution, $y_1(x)$, to estimate the time of traversal of the interval $o \leq x \leq x_o$.

4.3 Corollary to Lemma III

The time taken to describe the arc of the trajectory, y(x), from x=o to $x=x_o$, is

$$T = \frac{2}{f(B)} \frac{\log \mu}{\mu} + O(\mu^{-1}).$$

Proof:

The time to traverse (o,x_o) may be written $T = -\mu \int_o^{x_o} dy/g(x)$. We divide the interval (o,x_o) into two parts: (o,x_4) and (x_4,x_o), where $x_4 = x_o - \Delta\mu^{-2}$ (Δ = const. > o). In (x_4,x_o) we have $o \leq \xi \leq O(\mu^{-2})$; from Eq. (4.2) one finds that $o \leq \eta \leq O(\mu^{-2})$. Consequently, it follows that $y_1(x_4) - y_1(x_o) \leq O(\mu^{-2})$. Since $y_1(x)$ and y(x) agree to $O(\mu^{-2})$, according to Lemma III, we have $y(x_4) - y(x_o) \leq O(\mu^{-2})$. Thus $-\mu \int_{x_4}^{x_o} dy/g(x)$ is bounded by $O(\mu^{-1})$, since $g(x) = O(1)$ and $\int_{x_4}^{x_o} |dy| \leq O(\mu^{-2})$.

We now have

$$T = -\mu \int_{o}^{x_4} dy/g(x) + O(\mu^{-1}),$$

or, using the equivalent expression for the time,

$$T = \frac{1}{\mu} \int_{o}^{x_4} \frac{dx}{y-F(x)} + O(\mu^{-1}).$$

We evaluate this time by using the comparison solution discussed in Lemma III.

Consider $T_1 = \frac{1}{\mu} \int_{o}^{x_4} dx/[y_1-F_1(x)]$. We have

$$\left|T-T_1\right| \leq \frac{1}{\mu} \int_{o}^{x_4} \frac{\left|F-F_1\right| + \left|y-y_1\right|}{vv_1} dx + O(\mu^{-1}).$$

On setting $\left|F-F_1\right| \leq \ell_1(x-x_o)^2$, $\left|y-y_1\right| = \left|z\right| \leq E\,\mu^{-2}$ (E = const. > o), and $v, v_1 \geq D(x_o-x)$, one finds that $\left|T-T_1\right| \leq O(\mu^{-1})$, where $x_o-x_4 = O(\mu^{-2})$ has been used. Therefore T_1 is an estimate for T, to $O(\mu^{-1})$. Accordingly,

$$T = \frac{1}{\mu} \int_{\Delta\mu^{-2}}^{x_o} \frac{d\xi}{\eta+f(x_o)\xi} + O(\mu^{-1}), \qquad (4.5)$$

since $x=x_o-\xi$ and $y_1=F(x_o)+\eta$.

The integrand in Eq. (4.5) may be written as

$$\frac{1}{f(x_o)} \left\{ \frac{\eta'+f(x_o)}{\eta+f(x_o)\xi} - \frac{g(x_o)}{\mu^2[\eta+f(x_o)\xi]^2} \right\},$$

where $\eta' = g(x_o)/\mu^2[\eta+f(x_o)\xi]$. One obtains

$$T = \frac{1}{\mu f(x_o)} \log[\eta+f(x_o)\xi]\Big|_{\Delta\mu^{-2}}^{x_o} - \frac{g(x_o)}{\mu^3 f(x_o)} \int_{\Delta\mu^{-2}}^{x_o} \frac{d\xi}{[\eta+f(x_o)\xi]^2} ,$$

and the integral may be bounded by $O(\mu^{-1})$ on noting that $\eta \geq D\xi$, and completing the integration. The dominant contribution to the first term occurs at $\xi=\Delta\mu^{-2}$, where $\eta+f(x_o)\ \xi= O(\mu^{-2})$. One obtains, finally,

$$T = \frac{2}{f(x_o)} \frac{\log \mu}{\mu} + O(\mu^{-1}), \qquad (4.6)$$

for the time taken, by any member of the bundle, to transverse $o \leq x \leq x_o$.

Recall, from Sec. 3, that $y_o=F(a)+O(\mu^{-4/3})$ hence, on using Eq. (4.4), $x_o = B + O(\mu^{-4/3})$. Consequently, Eq. (4.6) may be written as

$$T = \frac{2}{f(B)} \frac{\log \mu}{\mu} + O(\mu^{-1}),$$

completing the proof of the corollary.

We have seen that a readily integrable comparison equation, involving a linear approximation to F(x) and a constant in place of g(x), supplies a comparison solution which approximates the bundle to $O(\mu^{-2})$, in R_3. Furthermore, the time of transit for this interval has been given. The discussion of the upper half-orbit of the periodic solution is now complete. The results of the previous sections are collected and summarized in Sec. 5.

5.1 The Amplitude and Period of the Periodic Solution

An explicit representation for the periodic orbit has been obtained, to $0(\mu^{-2})$, in the previous sections. Although the discussion has been limited to the upper half-orbit, in the phase and Liénard planes, the description given applies equally well to the lower half-orbit when the obvious changes in notation are made. The maximum positive x-excursion of the periodic orbit is given by Eq. (4.4), in terms of y_o, the positive y-axis intercept in the Liénard plane. If $x_p(t)$ denotes the periodic solution of Eq. (1.1), then $\mathcal{A}_u = \max [x_p(t)]$, the positive amplitude, is obtained from Eq. (4.4) upon substitution of y_o from Sec. 3. We state the result in the form of a theorem.

Theorem 1:

The equation $\ddot{x} + \mu f(x)\dot{x} + g(x) = o$, with $f(x), g(x)$ and $F(x) = \int_o^x f(u)\, du$ satisfying the requirements of Sec. 1, has a periodic solution, $x_p(t)$, with positive amplitude, $\mathcal{A}_u = \max [x_p(t)]$, given by

$$\mathcal{A}_u = B + \frac{2.33811}{f(B)} \left[\frac{-2g^2(a)}{f'(a)}\right]^{1/3} \mu^{-4/3} - \frac{1}{f(B)} \left\{\frac{2g(B)}{f(B)} + \frac{2}{3}\, g'(a) \left[\frac{-4g(a)}{f'^2(a)}\right]^{1/3} + \frac{4}{9}\, \frac{g(a)f''(a)}{f'^2(a)}\right\} \frac{\log \mu}{\mu^2} + 0(\mu^{-2}). \qquad (5.1)$$

Observe that Eq. (5.1) also gives the negative amplitude of the periodic solution, $\mathcal{A}_L = \min [x_p(t)]$, upon replacing B by A, and a by b.

We may now determine the time taken to describe the upper half of the periodic orbit, from $x = \mathcal{A}_L$ to $x = \mathcal{A}_u$. Summing the various contributions to the half-orbit, from the corollaries to Lemmas I, II, and III, we obtain the time to traverse the upper half-arc of a trajectory which starts at $x=x_1$, $\dot{x}=o$, where, as mentioned in Sec. 1, $A - \epsilon \leq x_1 \leq \alpha$. In order that the time obtained shall refer to the periodic orbit, we choose $x_1 = \mathcal{a}_L$. The result is stated in the following corollary.

Corollary to Theorem 1:

The time taken to describe the arc of the periodic solution, $x_p(t)$, from $x = \mathcal{a}_L$ to $x = \mathcal{a}_u$ is given by

$$T = - \mu \int_{\mathcal{a}_L}^{a} \frac{f(x)dx}{g(x)} + 2.33811 \left[\frac{2}{g(a)f'(a)}\right]^{1/3} \mu^{-1/3}$$

$$+ \left\{ \frac{2}{f(B)} - \frac{2}{3} \frac{g'(a)}{g(a)f'(a)} + \frac{4}{9} \frac{f''(a)}{f'^2(a)}\right\} \frac{\log \mu}{\mu} + 0(\mu^{-1}). \quad (5.2)$$

The contributions of the integral in Eq. (5.2), of $0(\mu^{-1/3})$ and $0(\log\mu/\mu)$, can be evaluated. We write

$$\int_{\mathcal{a}_L}^{a} fdx/g = \int_{\mathcal{a}_L}^{A} fdx/g + \int_{A}^{a} fdx/g,$$

and expand the integrand, in the first integral, about x=A. The first integral becomes $f(A)(A-\mathcal{a}_L)/g(A) + 0(\mu^{-8/3})$, since $\mathcal{a}_L = A + 0(\mu^{-4/3})$ from Eq. (5.1). One thus obtains

$$T = -\mu \int_{A}^{a} \frac{f(x)}{g(x)} dx + 2.33811 \left\{\left[\frac{2}{g(a)f'(a)}\right]^{1/3} + \frac{1}{g(A)} \left[\frac{-2g^2(b)}{f'(b)}\right]^{1/3}\right\} \mu^{-1/3}$$

$$+ \left\{\frac{2}{f(B)} - \frac{2}{f(A)} - \frac{2}{3} \frac{g'(a)}{g(a)f'(a)} + \frac{4}{9} \frac{f''(a)}{f'^2(a)} - \frac{1}{g(A)} \left\{\frac{2}{3} g'(b)\left[\frac{-4g(b)}{f'^2(b)}\right]^{1/3}\right.\right.$$

$$\left.\left. + \frac{4}{9} \frac{g(b)f''(b)}{f'^2(b)}\right\}\right\} \frac{\log \mu}{\mu} + 0(\mu^{-1}). \quad (5.3)$$

As an example of the use of Eq.'s (5.1) and (5.3), we compute the amplitude and period of the unique, nonzero periodic solution of the van der Pol equation: $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$. We have $f(x) = x^2-1$, $F(x) = x^3/3 - x$, $g(x) = x$ so that $b = -a = 1$ and $B = -A = 2$. Since the periodic orbit is symmetrical about the origin, in both planes, then $\mathcal{A}_u = -\mathcal{A}_L$, and the period is twice the value given by Eq. (5.2) or Eq. (5.3). Substitution into Eq.'s (5.1) and (5.3) gives

$$\text{Amplitude} = 2 + \frac{1}{3}\,(2.33811)\,\mu^{-4/3} - \frac{16}{27}\,\frac{\log\mu}{\mu^2} + 0(\mu^{-2}), \quad (5.4)$$

and

$$\text{Period} = (3-\log 4)\,\mu + 3(2.33811)\,\mu^{-1/3} - \frac{2}{3}\,\frac{\log\mu}{\mu} + 0(\mu^{-1}). \quad (5.5)$$

Eq. (5.4) agrees with the expression given by Dorodnicyn [4] and confirmed by Urabe [6]. However, Eq. (5.5) for the period disagrees with both writers. Dorodnicyn obtains a coefficient -22/9 for the $\log\mu/\mu$ term in Eq. (5.5), and Urabe, repeating Dorodnicyn's calculations, obtains -1/3 for this coefficient. The present authors have attempted to verify Eq. (5.5) by numerical integration of the phase and Liénard equations using a digital computer. The period was determined, for several values of μ from 8 to 200, and an estimate made of the terms succeeding the first two, in the asymptotic expansion for the period. The results appear to justify the coefficient -2/3 in Eq. (5.5) [7].

6.1 Nested Periodic Orbits

It will be observed that the conditions on F(x), given in Sec. 1, allow for minor maxima and minima in the interval $a < x < b$. It is quite possible that such extrema might support periodic orbits which lie in this interval. Fig. 5 illustrates this possibility.

The results of previous sections may be applied to the interval $A_1 - \epsilon \leq x \leq B_1 + \epsilon$, and, in particular, Eq.'s (5.1) and (5.3) for the amplitude and period apply, with the appropriate changes in notation, to the periodic orbit depicted.

It might be pointed out that the periodic solutions described by the analysis given here, are orbitally stable [2]. That is, trajectories which begin sufficiently near these periodic orbits will converge to them as $t \to \infty$. Nevertheless, the description may be made to apply, by the following artifice, to orbitally unstable periodic solutions.

In Eq. (1.3) we set $t = -\tau$, in order to consider the motion of trajectories in reversed time, and put $y = -Y$, so as to maintain clockwise motion in the Liénard plane. One then obtains

$$dx/d\tau = \mu[Y+F(x)], \quad dY/d\tau = -g(x)/\mu \tag{6.1}$$

which governs the reverse motions. Note that the characteristic curve is now $Y = -F(x)$, as shown in Fig. 6.

The periodic orbit shown is located, to $O(\mu^{-2})$, by the methods described in the previous sections, and its amplitude and period may be found from Eq.'s (5.1) and (5.3) (with the obvious notational change). Furthermore, this periodic solution is orbitally stable, as $\tau \to \infty$, hence orbitally unstable in the original equations of direct motion, as $t \to \infty$.

REFERENCES

1. LaSalle, J., "Relaxation Oscillations," Quart. Journ. of Applied Math., Vol. 7, pp. 1-19; 1959.

2. Ponzo, P. T. and Wax, N., "On Certain Relaxation Oscillations: Confining Regions" (to appear)

3. Haag, J., "Etudes Asymptotique des Oscillations de Relaxation," Annales de l'Ecole Normale Superieure, vol. 60, pp. 65-111; 1943.

4. Dorodnicyn, A. A., "Asymptotic Solutions of van der Pol's Equation," Prikl Mat. i Mech., vol. 11, pp. 313-328; 1947. (Amer. Math. Soc. Transl. No. 88).

5. Cartwright, Mary L., "Van der Pol's Equation for Relaxation Oscillations," Contrib. to Nonlinear Osc., vol. 2, Princeton, pp. 3-18; 1952.

6. Urabe, M., "Periodic Solutions of van der Pol's Equation with Damping Coefficient $\lambda = 0 \sim 10$," IRE Trans. on Circuit Theory, vol. CT-7, No. 4, pp. 382-386; December 1960.

7. Ponzo, P. J. and Wax, N., "On the Periodic Solution of the van der Pol Equation." (to appear).

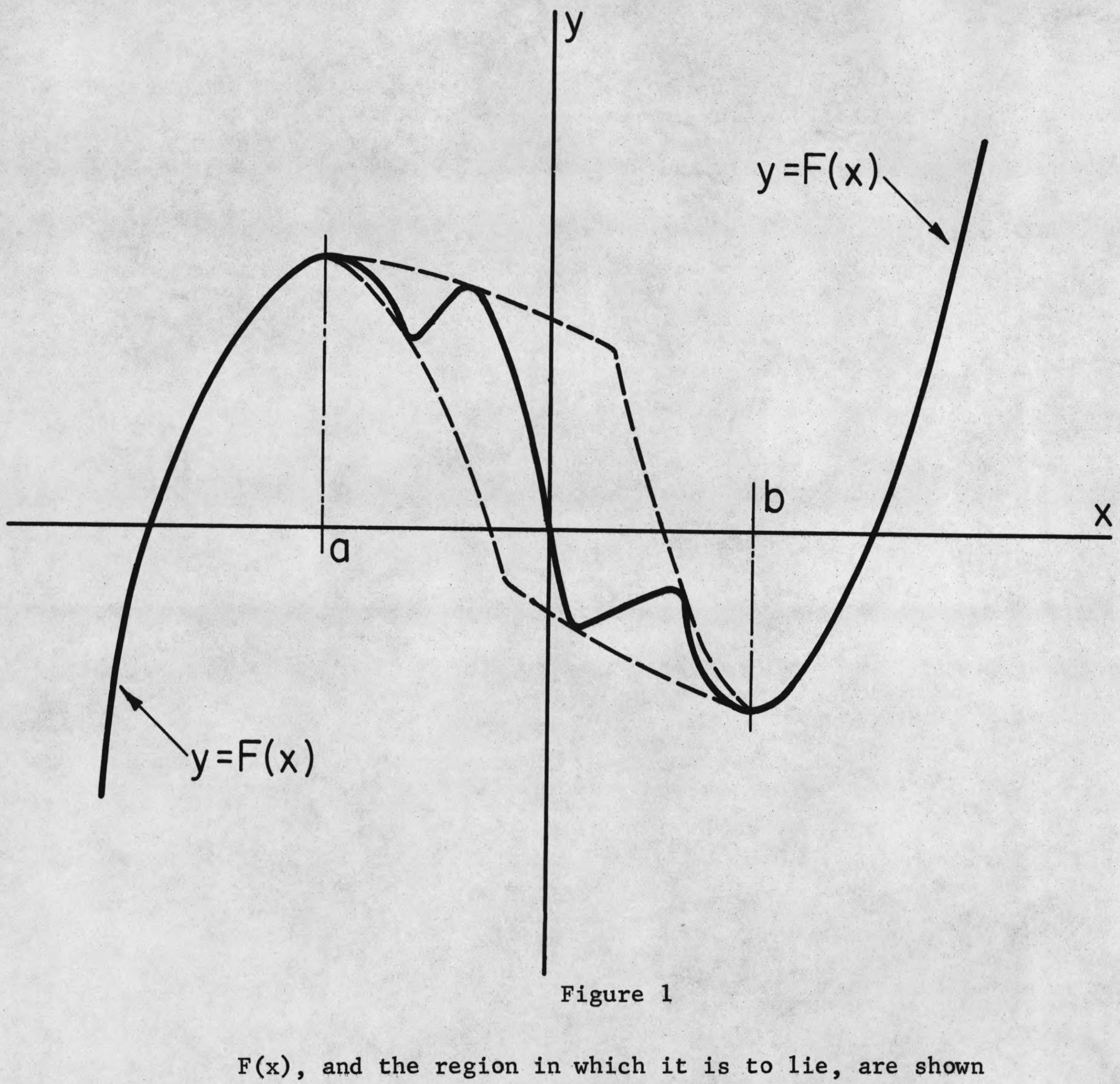

Figure 1

F(x), and the region in which it is to lie, are shown in the Lienard plane.

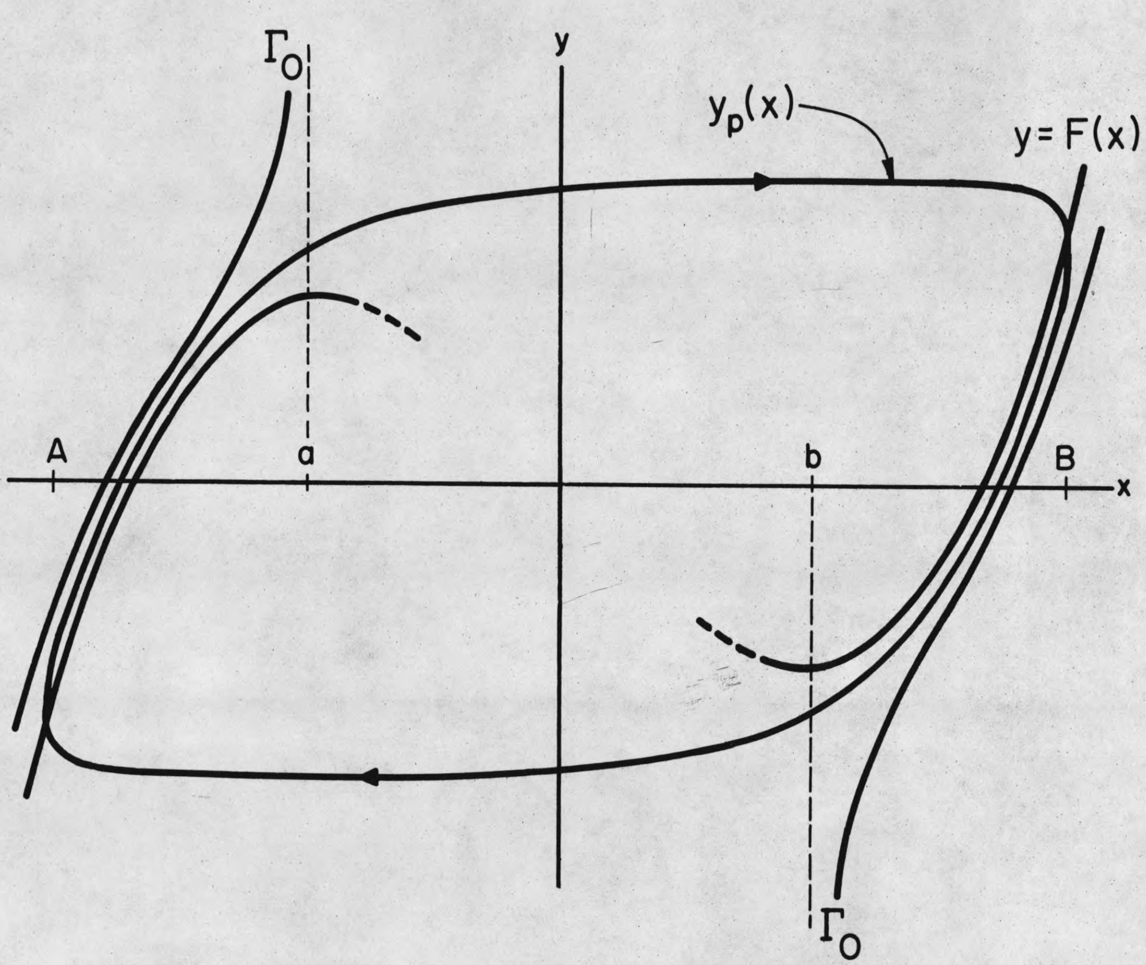

Figure 2

The periodic orbit, $y_p(x)$, together with Γ_o, the zero-slope isocline of the phase plane, is shown in the Lienard plane.

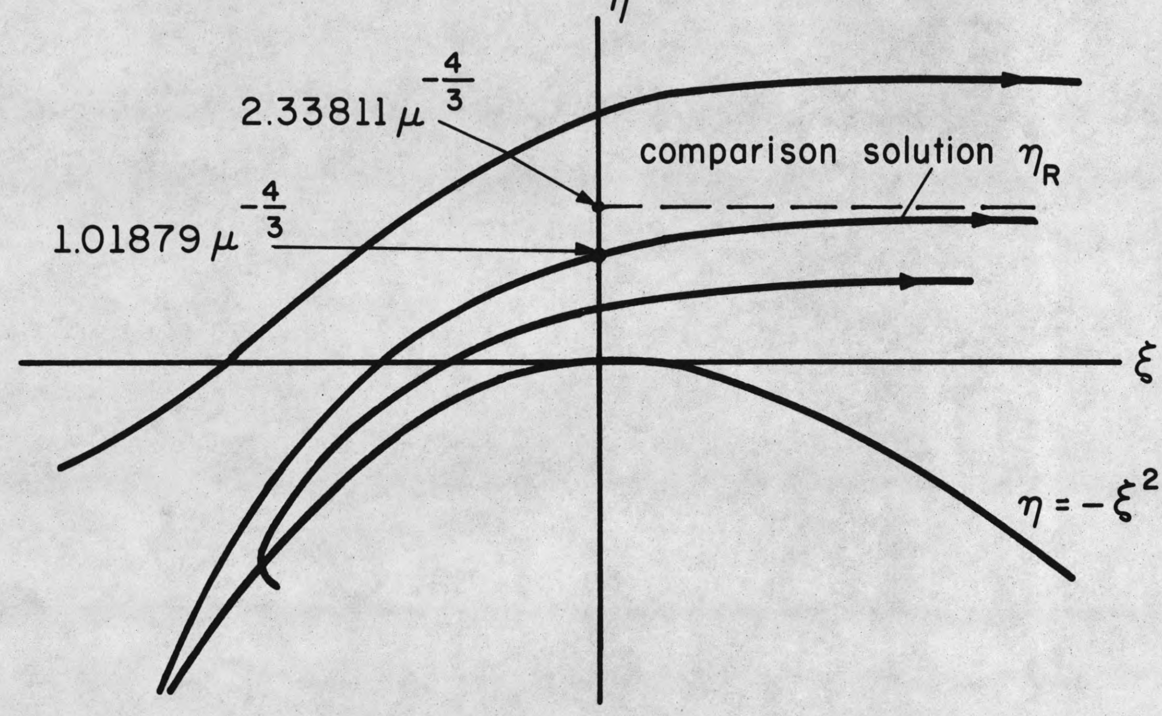

Figure 3

Solutions of the Riccati equation; the comparison solution is shown

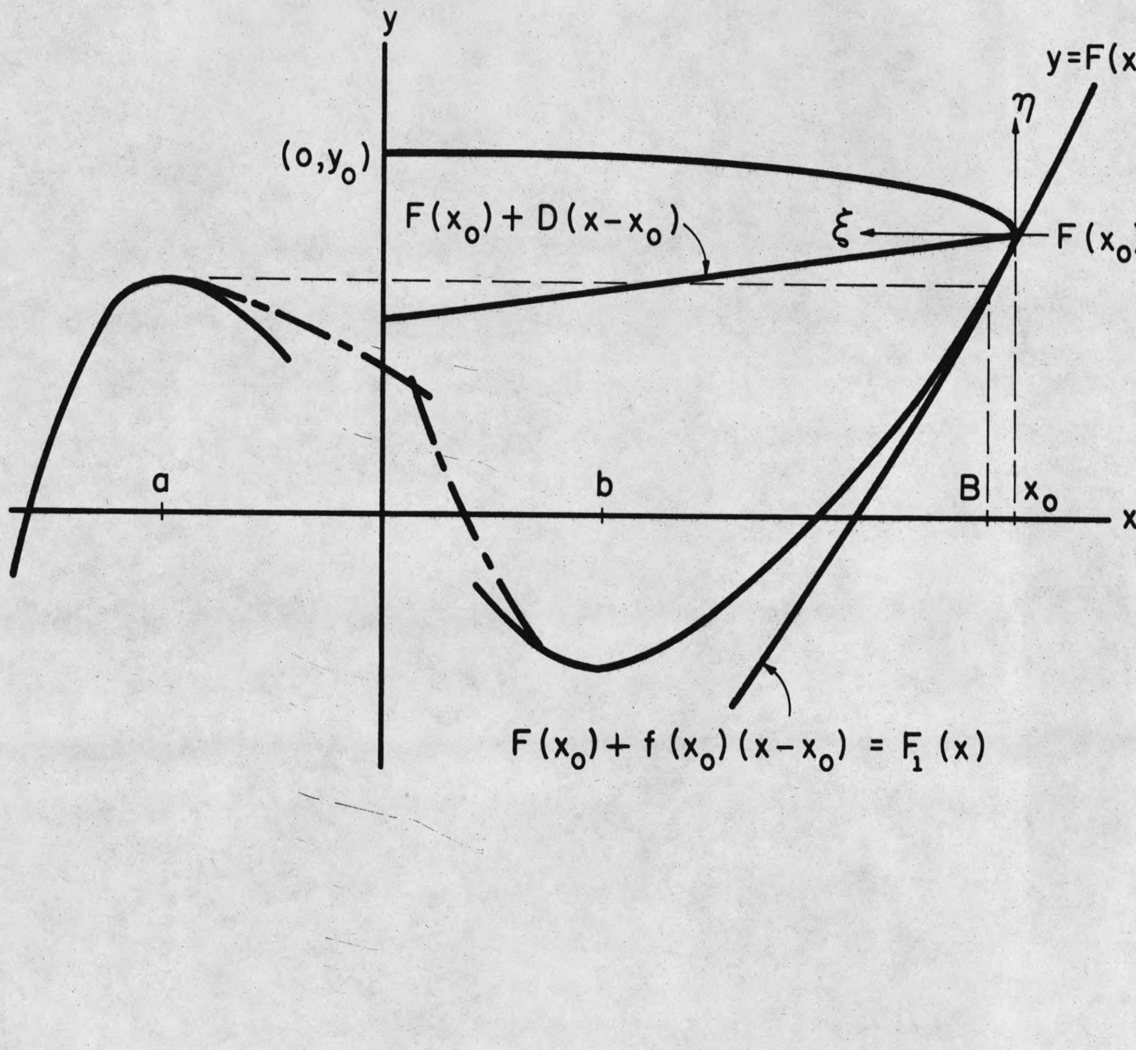

Figure 4

Coordinates and approximations used in R_3.

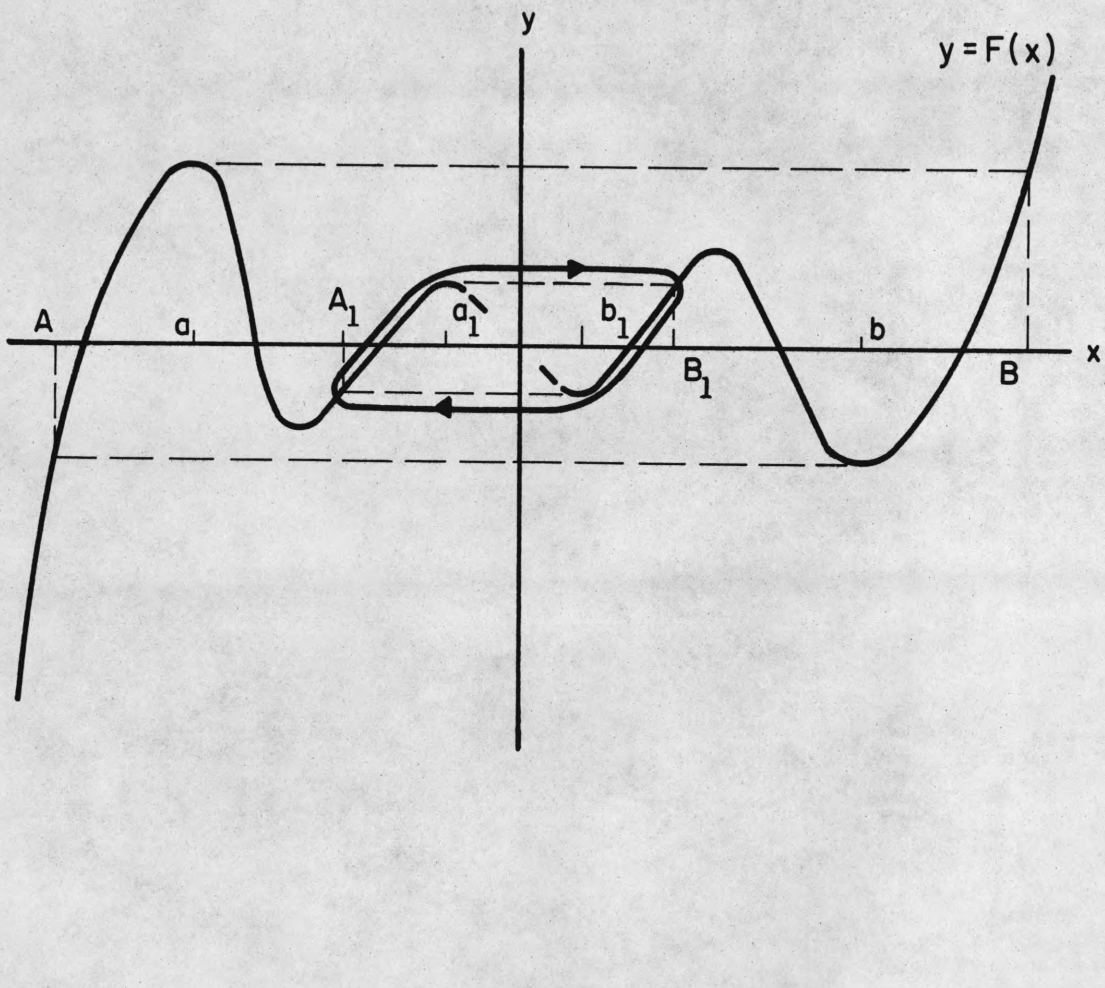

Figure 5

A possible configuration and limit cycle are shown.

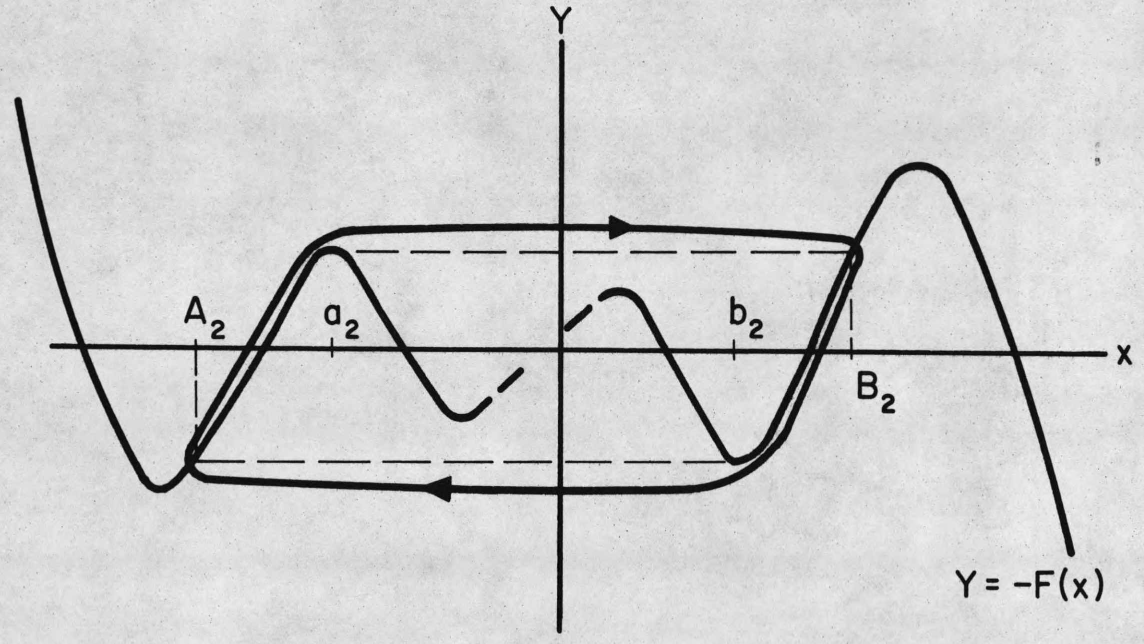

Figure 6

A stable limit cycle for reverse time, hence unstable in direct time, is depicted.

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