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A CHARACTERIZATION OF THE SMALLEST EIGENVALUE OF A GRAPH

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$$\psi = \min_{S \subseteq V} \frac{e_{\min}(S) + |cut(S)|}{|S|}$$

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A CHARACTERIZATION OF THE SMALLEST EIGENVALUE OF A GRAPH †

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ABSTRACT

It is well known that the smallest eigenvalue of the adjacency matrix of a connected d-regular graph is at least $-d$ and is strictly greater than $-d$ if the graph is not bipartite. More generally, for any connected graph $G = (V, E)$, consider the matrix $Q = D + A$ where D is the diagonal matrix of degrees in the graph G, and A is the adjacency matrix of G. Then Q is positive semi-definite, and the smallest eigenvalue of Q is 0 if and only if G is bipartite. We will study the separation of this eigenvalue from 0 in terms of the following measure of non-bipartiteness of G. For any $S \subseteq V$, we denote by $e_{\min}(S)$ the minimum number of edges that need to be removed from the induced subgraph on S to make it bipartite. Also, we denote by $cut(S)$ the set of edges with one end in S and the other in $V-S$. We define the parameter ψ as

$$\psi = \min_{S \subseteq V} \frac{e_{\min}(S) + |cut(S)|}{|S|}$$

The parameter ψ is a measure of the non-bipartiteness of the graph G. We will show that the smallest eigenvalue of Q is bounded above and below by functions of ψ. For d-regular graphs, this characterizes the separation of the smallest eigenvalue of the adjacency matrix from $-d$. These results can be easily extended to weighted graphs.

Key words: graphs, eigenvalues, bipartite graphs.

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1. INTRODUCTION

Given the graph $G = (V,E)$ with $V = \{1,2, \cdots ,n\}$, define the $n \times n$ *adjacency matrix* $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases} \tag{1.1}$$

Denote the degree of vertex i by $d(i)$. Let $D = diag\,[d(i)]$ be the diagonal matrix whose i^{th} diagonal entry is $d(i)$. We will work with the matrix $Q = D + A$. If G is d-regular, then $d(i) = d$ for all $i \in V$, and thus $Q = dI + A$. For a nonempty $S \subseteq V$, we denote by $E[S]$ the set of edges with both ends in S. The subgraph induced on S is $G_S = (S, E[S])$. Define $e_{\min}(S)$ to be the minimum number of edges needed to be removed from $E[S]$ so that the induced subgraph on S is bipartite. Let $cut(S)$ be the set of edges with one end in S and the other end in it's complement $\overline{S} = V - S$. Thus, $|cut(S)| + e_{\min}(S)$ is the minimum number of edges that need to be removed from E so that S is disconnected from $V-S$ and the induced component on S is bipartite.

Define

$$\psi = \min_{S \neq \phi; S \subseteq V} \frac{e_{\min}(S) + |cut(S)|}{|S|} \tag{1.2}$$

The parameter ψ is a measure of the distance of the graph G from "bipartiteness" in the following sense: to obtain a bipartite component on a subset S of V, we need to remove at least $\psi |S|$ edges from G. We will show that Q is positive semi-definite, and it is singular if and only if $\psi = 0$ (Proposition 2.1). It seems intuitively reasonable that the smallest eigenvalue $\mu_1(Q)$ should be related to ψ. In this paper, we will derive upper and lower bounds for $\mu_1(Q)$ as functions of ψ, and show that this intuition is well founded. For a d-regular graph, this translates to bounds on the smallest eigenvalue $\mu_1(A)$ of the adjacency matrix A.

A more commonly studied matrix associated with a graph is the Laplacian matrix $L = D - A$, which is singular and positive semi-definite. The separation of the second smallest eigenvalue $\mu_2(L)$ of L from 0 is often of interest [1,7]. In [2], $\mu_2(L)$ is characterized by the expansion property (which is a measure of connectedness) of the graph. Similar results have been proved in the more general setting of weighted graphs, and for the second smallest eigenvalues of matrices of the form $I - P$, where P is the transition matrix of a reversible

Markov chain [1,5,7]. In our paper, we will provide a similar characterization of $\mu_1(Q)$ in terms of ψ which is a measure of the non-bipartiteness of the graph. Note that if the graph G is d-regular, then the largest eigenvalue $\mu_n(L)$ of L is equal to $2d - \mu_1(Q)$. In such cases, $\mu_1(Q)$ establishes the distance of the spectrum of L from $2d$. In [8], it is shown that if $\mu_n(L)$ is the largest eigenvalue of L, then $\mu_n(L) \geq 4(|E| - e_{\min}(V))/n$. The results in this paper may be used to obtain a more precise characterization of $\mu_n(L)$ for regular graphs. A lower bound for $\mu_1(Q)$ has been derived by using a path counting idea in [5]. We will show that the same path counting approach yields a lower bound for the parameter ψ defined in (1.2). In other related work, a lower bound for the smallest eigenvalue of a doubly stochastic matrix has been derived in [6].

In this paper, we show that $\mu_1(Q)$ is well characterized by ψ. We will discuss the relation of the lower bound derived here to those derived in [5,6]. The results in this paper may be easily extended to weighted graphs and to reversible Markov chains.

2. PRELIMINARIES

Let $G = (V,E)$ be a graph with vertices $V = \{1,2,\cdots,n\}$ and edges $E = \{b_1,\cdots,b_m\}$ where $b_i = (u_i,v_i)$, for some $u_i, v_i \in V$. For any two disjoint subsets $S, T \subseteq V$, we define the *representation vector* $r_{S,T} \in \mathbb{R}^n$ of the ordered pair $<S,T>$ as

$$r_{S,T}(i) = \begin{cases} 1 \text{ if } i \in S \\ -1 \text{ if } i \in T \\ 0 \quad \textit{else} \end{cases}$$

For $S \subseteq V$, $T \subseteq V$, we define $E[S,T]$ to be the set of edges (u,v) with $u \in S$ and $v \in T$. Let Q be the matrix defined in the introduction.

Proposition 2.1: The matrix Q is positive semidefinite. It is singular if and only if $\psi = 0$.

Proof: Define the $m \times n$ vertex-edge incidence matrix $C = [c_{ij}]$ by $c_{ij} = 1$ if b_i is incident on vertex j and 0 otherwise. It is easily verified that $Q = C^t C$. So, for any $x, y \in \mathbb{R}^n$, we have

$$y^t Q x = (Cy)^t\ (Cx) = \sum_{(i,j) \in E} (x(i) + x(j))(y(i) + y(j)). \tag{2.1}$$

Clearly, Q is positive semidefinite. Next, note that $\psi = 0$ if and only if there exists $S \subseteq V$ with $e_{\min}(S)$=0 and $|cut(S)| = 0$. That is, ψ=0 if and only if G has a bipartite component. Assume that G has a bipartite component. Then, there exist non-empty and disjoint sets $S, T \subseteq V$ with $E[S \bigcup T] = E[S,T]$, and $|cut(S \bigcup T)| = 0$. Let $r_{S,T}$ be the representation vector of $<S,T>$. If $i \in S$ and $j \in T$, we have $r_{S,T}(i) + r_{S,T}(j) = 0$, and thus, from (2.1)

$$r_{S,T}^t\ Q\ r_{S,T} = \sum_{(i,j) \in E[S,T]} (r_{S,T}(i) + r_{S,T}(j))^2 = 0. \tag{2.2}$$

Therefore, if G has a bipartite component, then Q is singular. Conversely, suppose Q is singular, which implies that there exists $x \in \mathbb{R}^n$, $x \neq 0$ with $x^t Q x = 0$. Let $S = \{i \in V\colon x(i) > 0\}$, and $T = \{i \in V\colon x(i) < 0\}$. Since Q has only nonnegative entries, S and T must both be non-empty. Then, from (2.1), we must have

$E[S\cup T, \overline{S\cup T}] = \phi$. Similarly, we must have $E[S] = E[T] = \phi$. Thus, $E[S\cup T] = E[S,T]$, and $cut(S\cup T) = \phi$. Therefore $G_{S\cup T}$ is a bipartite component of G. □

The next two technical lemmas will be useful in obtaining the main results of this paper.

Lemma 2.2 : Let $S, T \subseteq V$ be two disjoint subsets and let $r_{S,T}$ be the representation vector of $<S,T>$. Then,

$$r_{S,T}^t\, Q\, r_{S,T} = 4(\,|E[S]| + |E[T]|\,) + |cut(S\cup T)| \geq 4e_{\min}(S\cup T) + |\, cut(S\cup T)| \tag{2.3}$$

Proof : The first equality follows immediately from (2.1) by noting that

$$|\, r_{S,T}(i) + r_{S,T}(j)\,| = \begin{cases} 2 \text{ if } (i,j)\in E[S] \cup E[T] \\ 1 \text{ if } (i,j)\in E[S\cup T, \overline{S\cup T}] \\ 0 \;\; else \end{cases}$$

The inequality in (2.3) is obtained by using the definition of $e_{\min}(S\cup T)$. □

The next lemma has been proved in a different setting in [5,7]. We will give a proof which is close to that given in [5]. Before stating it, we need a function $h(S)$ defined on a non-empty subset S of V as follows.

$$h(S) = \min_{T\subseteq S} \frac{|cut(T)|}{|T|}. \tag{2.4}$$

where the minimization is carried out over all non-empty subsets T of S.

Lemma 2.3: Let $G=(V,E)$. Suppose S is an arbitrary proper subset of V and x is a non-negative vector with $x(i)>0$ if and only if $i\in S$. Then,

$$\sum_{(i,j)\in E} (x(i) - x(j))^2 \geq \frac{h^2(S)}{2d^*} \sum_{i\in V} x^2(i), \tag{2.4}$$

where d^* is the maximum degree in G.

Proof: Write

$$\sum_{(i,j)\in E} (x(i)-x(j))^2 = \sum_{(i,j)\in E} (x(i)-x(j))^2 \frac{\sum_{(i,j)\in E} (x(i)+x(j))^2}{\sum_{(i,j)\in E} (x(i)+x(j))^2} \geq \frac{\left(\sum_{(i,j)\in E} |x^2(i)-x^2(j)| \right)^2}{\sum_{(i,j)\in E} (x(i)+x(j))^2}, \tag{2.5}$$

using the Cauchy-Schwarz inequality. Now, it is easy to see that

$$\sum_{(i,j)\in E} (x(i)+x(j))^2 \leq \sum_{(i,j)\in E} 2\,x^2(i) + 2\,x^2(j) \leq 2\,d^* \sum_{i\in V} x^2(i). \tag{2.6}$$

We may write

$$\sum_{(i,j)\in E} |x^2(i)-x^2(j)| = \sum_{(i,j)\in E\,:\,x(i)>x(j)} x^2(i)-x^2(j). \tag{2.7}$$

We replace each term inside the summation above by an integral as follows;

$$\sum_{(i,j)\in E\,:\,x(i)>x(j)} x^2(i)-x^2(j) = \sum_{(i,j)\in E\,:\,x(i)>x(j)} 2\int_{x(j)}^{x(i)} t\;dt \tag{2.8}$$

$$= 2\int_0^{\infty} t \Big(\sum_{(i,j)\in E,\, x(i)>t\geq x(j)} 1 \Big) dt. \tag{2.9}$$

Note that for $t>0$, if we let $S_t = \{i\in S,\, x(i)>t\}$, then the summation inside the integral in (2.9) equals $|cut(S_t)|$. But $|cut(S_t)| \geq h(S)\;|S_t|$, by (2.4). Consequently, from (2.7), (2.8) and (2.9), we have

$$\sum_{(i,j)\in E} |x^2(i)-x^2(j)| \geq 2\,h(S)\int_0^{\infty} t\;|S_t|\;dt. \tag{2.10}$$

Consider $f(t) = |S_t|$ to be a function of t and decompose it as $f(t) = \sum_{i=1}^{n} f_i(t)$ where, for each i,

$$f_i(t) = \begin{cases} 1 \text{ if } 0\leq t< x(i) \\ 0 \text{ if } \quad t\geq x(i) \end{cases}$$

That is, f_i is the indicator function of the interval $[0,x(i))$. The integral in (2.10) can then be evaluated as

$$\int_0^{\infty} tf(t)dt = \sum_{i=1}^{n} \int_0^{x(i)} tdt = \frac{1}{2}\sum_{i=1}^{n} x^2(i). \tag{2.11}$$

Combining (2.10), (2.11) and recalling (2.5), (2.6) the proof is complete. □

Finally we derive a simple relationship between the smallest eigenvalue $\mu_1(A)$ of A and the smallest eigenvalue $\mu_1(Q)$ of Q.

Proposition 2.4: Let $G = (V, E)$ be any graph with minimum degree d_* and maximum degree d^*. Let A be the adjacency matrix and let $Q = D + A$ where D is the diagonal matrix of degrees in the graph. Then,

$$\mu_1(Q) - d^* \leq \mu_1(A) \leq \mu_1(Q) - d_* \tag{2.12}$$

Proof: Let x be an eigenvector of Q corresponding to $\mu_1(Q)$ normalized so that $x'x = 1$. Then,

$$\mu_1(Q) = x'Qx = x'Ax + x'Dx. \tag{2.13}$$

But, by the min-max characterization of the eigenvalues of a symmetric matrix, $x'Ax \geq \mu_1(A)$. Also, $x'Dx \geq d_*$. This proves the second inequality in (2.12). For the other inequality, start with an eigenvector of A corresponding to $\mu_1(A)$ and proceed in a similar fashion. □

If G is $d-$ regular, then the inequalities in (2.12) become equalities.

3. MAIN RESULTS

We are in a position to state the main results in the paper. We will first obtain an upper bound (Theorem 3.1) for $\mu_1(Q)$, which is relatively trivial to establish. The lower bound (Theorem 3.2) is more difficult. In addition, we will also establish an alternate form of the lower bound (Theorem 3.3) which may be better than Theorem 3.2 in some instances. To begin with, we can state the upper bound as follows.

Theorem 3.1: Let $G=(V,E)$ be the given graph, and suppose ψ is the parameter defined as in (1.2). Then the smallest eigenvalue of Q is bounded above as

$$\mu_1(Q) \leq 4\psi. \tag{3.1}$$

If G is d-regular, the smallest eigenvalue of the adjacency matrix A is bounded above as $\mu_1(A) \leq -d+4\psi$.

Proof: By the min-max characterization of the eigenvalues of a symmetric matrix, we have

$$\mu_1(Q) = \min_{x \neq 0} \frac{x^t Q x}{x^t x}. \tag{3.2}$$

There exist disjoint subsets S, T of V such that $|E[S]| + |E[T]| + |cut(S \cup T)| = \psi |S \cup T|$. Clearly, we must have $e_{\min}(S \cup T) = |E[S]| + |E[T]|$. Let $r_{S,T}$ be the representation vector of $<S,T>$. From Lemma 2.2, we have

$$\frac{r_{S,T}{}^t\, Q\, r_{S,T}}{r_{S,T}{}^t r_{S,T}} = \frac{4e_{\min}(S \cup T) + |cut(S \cup T)|}{|S \cup T|} \leq 4\psi. \tag{3.3}$$

Combining (3.2) and (3.3), the upper bound (3.1) follows. Finally, if G is d-regular, then $\mu_1(A) = \mu_1(Q) - d$ (Proposition 2.4), which completes the proof. □

Remark: Instead of ψ, we could define the parameter ψ' as

$$\psi' = \min_{S \neq \phi; S \subseteq V} \frac{4\, e_{\min}(S) + |cut(S)|}{|S|} \tag{3.4}$$

The parameters ψ and ψ' are related as $\psi' \geq \psi \geq \psi'/4$. In the case of Theorem 3.1, we could have stated the result as $\mu_1(Q) \le \psi'$. The choice of which parameter to use is a matter of convenience. In the discussion that follows, the lower bound of Theorem 3.2 is more naturally expressed in terms of ψ, whereas the lower bound of Theorem 3.3 can be naturally expressed in terms of ψ'.

The upper bound of Theorem 3.1 says that if there is a subset S of vertices in the graph such that the subset is weakly connected to the rest of the graph (as measured by $|cut(S)|/|S|$), and the induced subgraph on this subset is almost bipartite (as measured by $e_{\min}(S)/|S|$), then $\mu_1(Q)$ is small. The converse notion is more remarkable. We will show that if, for every subset of vertices, either the induced subgraph on that subset is far from bipartite or the subset is well connected to the rest of the graph, then $\mu_1(Q)$ is large.

Theorem 3.2: Let G, Q, ψ be as before. Assume that G is connected. Then the smallest eigenvalue $\mu_1(Q)$ is bounded below as,

$$\mu_1(Q) \ \ge \ \frac{\psi^2}{4\,d^*}, \tag{3.5}$$

where d^* is the largest degree of a vertex in the graph. If G is d-regular, then the smallest eigenvalue $\mu_1(A)$ of the adjacency matrix A is bounded as $\mu_1(A) \ge -d + \psi^2/4d$.

Proof: Let x be a normalized ($x'x = 1$) eigenvector of Q corresponding to eigenvalue $\mu_1(Q)$. Then, from (2.1), we have

$$\mu_1(Q) = x'\,Q\,x = \sum_{(i,j)\in E} (x(i) + x(j))^2. \tag{3.6}$$

Let $S = \{i \in V : x(i) > 0\}$, and $T = \{i \in V : x(i) < 0\}$. Since G is connected, Q is a non-negative irreducible matrix, and it follows from Perron-Frobenius theory [9] that S and T are both non-empty. We will construct a new graph as follows. Create a set S' which consists of copies of vertices in S, i.e., $S' = \{i' : i \in S\}$. Similarly, create a set T' of copies of vertices in T. We will define the graph $G' = (V', E')$ with a vertex set $V' = V \cup S' \cup T'$ and edge set E' defined as follows. If $(i,j) \in E[S]$, then we introduce two edges (i',j) and (i,j') in E'. Similarly, an edge $(i,j) \in E[T]$ gives a pair of edges in E'. For any other edge $(i,j) \in E$, we just introduce the single edge (i,j) in E'. Clearly, G' has the same maximum degree d^* as G. In Figure 1, we illustrate the construction of G' from G for a specified S and T.

Define the function $g: V' \rightarrow \mathbb{R}$ by $g(i) = |x(i)|$ for $i \in S \cup T$, and $g(i) = 0$ otherwise. It can be easily checked that

$$\sum_{(i,j) \in E'} (g(i) - g(j))^2 \leq 2 \sum_{(i,j) \in E} (x(i) + x(j))^2 \tag{3.7a}$$

and

$$\sum_{i \in V'} g^2(i) = \sum_{i \in V} x^2(i) = 1. \tag{3.7b}$$

We will now show that the parameter $h(S \cup T)$ defined for the graph G' as in (2.4) is at least ψ. Let W be an arbitrary subset of $S \cup T$. Suppose $S_1 = W \cap S$ and $T_1 = W \cap T$. Then, in G, we have

$$|E[S_1]| + |E[T_1]| + |cut(S_1 \cup T_1)| \geq \psi \; |S_1 \cup T_1|. \tag{3.8}$$

But for each edge (i,j) in $E[S_1]$ there corresponds a unique edge (i,j') in E', and also, $(i,j') \in cut(S_1 \cup T_1)$ in G'. Similarly, for each edge in $E[T_1]$, there corresponds a uniquely defined edge in $cut(S_1 \cup T_1)$ in G'. It is also easy to see that every edge in $cut(S_1 \cup T_1)$ in G corresponds to a unique edge in $cut(S_1 \cup T_1)$ in G'. Thus, from (3.8), we conclude that in G',

$$|cut(W)| \geq \psi \; |W| \quad \text{for} \quad W \subseteq S \cup T \tag{3.9}$$

From (3.9), $h(S \cup T) \geq \psi$ in G'. Now, we may apply Lemma 2.3 and write

$$\sum_{(i,j) \in E'} (g(i) - g(j))^2 \geq \frac{\psi^2}{2d^*} \sum_{i \in V'} g^2(i). \tag{3.10}$$

From (3.6), (3.7a), (3.7b) and (3.10), we obtain the desired lower bound (3.5). The last part of the Theorem follows from Proposition 2.4. □

Examples :

(i) Let G be the simple cycle on an odd number of vertices $n \geq 3$. It is easy to see that $\psi = 1/n$ and $\psi' = 2/n-1$. From Theorem 3.1, we have $\mu_1(Q) \leq 2/n-1$, and from Theorem 3.2, the lower bound is $\mu_1(Q) \geq 1/8n^2$. The actual value of $\mu_1(Q)$ is $2(1 - \cos 2\pi/n) \sim 4\pi^2/n^2$ [3]. In this case, the lower bound is off by a constant factor.

(ii) Let G be the complete graph on n vertices. The parameter ψ' can be evaluated to be $(n-2)$. As noted in the Remark after Theorem 3.1, $\psi \geq \psi'/4 = (n-2)/4$. Using Theorem 3.1 for the upper bound (in terms of ψ'), and Theorem 3.2 for the lower bound (in terms of ψ), we get

$$(n-2) \geq \mu_1(Q) \geq \frac{(n-2)^2}{64(n-1)}$$

The upper and lower bounds are of the same order in n, and this compares favorably with the actual value of $\mu_1(Q)$, which is $n-2$.

Remark (an application to Markov chains): Given a connected d-regular graph $G=(V,E)$, we may define a random walk on the graph. If the graph is not bipartite, the random walk converges to a uniform equilibrium probability distribution on the vertices of the graph[9]. The speed of convergence of the random walk to its equilibrium is governed by the second largest and smallest eigenvalues of the transition matrix defining the random walk[9]. In this context, Theorems 3.1 and 3.2 provide a precise characterization of the smallest eigenvalue. A characterization of $\mu_2(L)$ provides information about the second smallest eigenvalue[1,5,7].

Remark: A lower bound different from Theorem 3.2 was proved in [5]. We describe this result briefly. Assume G is d-regular. Let Γ be a set of cycles of odd length, such that $|\Gamma| = n$, and each vertex i appears on at least one odd cycle in Γ. Suppose the maximum length of a cycle in Γ is σ and that any edge in a cycle in Γ is used by at most b cycles in Γ. From [5],

$$\mu_1(Q) \geq \frac{2}{\sigma b}$$

Given the set Γ, it is easy to get a lower bound for the parameter ψ in terms of d and b. Let $S \subseteq V$. Then, each vertex in S may be assigned a unique odd cycle in Γ. It is easy to check that this odd cycle must contribute at least one edge to $e_{\min}(S) + |cut(S)|$. Thus, we get a net contribution of $|S|$. However, each edge may be counted up to b times. Therefore, the net contribution is at least $|S|/b$. Hence, we must have $\psi \geq \frac{1}{b}$. The lower bound of Theorem 3.2 is then

$$\mu_1(Q) \geq \frac{1}{4d\ b^2}$$

Depending on the relative values of b, d, and σ, either bound could be better. In any case, choosing Γ so that b is small will yield good lower bounds for ψ.

Fiedler and Ptak [6] considered the problem of obtaining a lower bound for the smallest eigenvalue of the adjacency matrix of a d-regular graph $G = (V,E)$. To summarize, they showed that for a d-regular graph of even order,

$$\mu_1(Q) \geq \frac{h(V)}{d(n-1 + h(V)/e_{\min}(V))},$$

which is worse than Theorem 3.2. This is expected because Theorem 3.1 and Theorem 3.2 show that it is more natural to consider bounds based on ψ.

A different approach can be used to obtain a lower bound for $\mu_1(Q)$ in terms of ψ' instead of ψ. For any vector $x \in \mathbb{R}^n$, we can define the set $Val(x) = \{\ |x(i)| : x(i) \neq 0,\ i=1, \cdots, n\}$. If $x \neq 0$, then $1 \leq |Val(x)| \leq n$.

Theorem 3.3: Suppose $G = (V,E)$ is the given graph, and let ψ' be as defined in (3.4). Then, if x is an eigenvector corresponding to eigenvalue $\mu_1(Q)$, we have

$$\mu_1(Q) \geq \frac{\psi'}{|Val(x)|} \geq \frac{\psi'}{n}$$

Proof: Let $S = \{i \in V : x(i) > 0\}$ and $T = \{i \in V : x(i) < 0\}$. Suppose

$$Val(x) = \{\ |x(i_1)|, |x(i_2)|, \cdots, |x(i_p)|\},$$

with $|x(i_k)| < |x(i_{k+1})|$. Then, for $k=2,3,\cdots,p$, define

$$W_k = \{i \in V:\ |x(i)| > |x(i_{k-1})|\}.$$

Also, define $W_1 = \{i \in V: |x(i)| > 0\}$. We have $W_{k+1} \subset W_k$ for $k=1,\cdots,p-1$. Then, for $k=1,2,\cdots,p$, we define $S_k = W_k \cap S$ and $T_k = W_k \cap T$. Let $\alpha_1 = |x(i_1)|$ and for $k=2,\cdots,p$ define $\alpha_k = |x(i_k)| - |x(i_{k-1})|$. Now, we can write x as

$$x = \sum_{k=1}^{p} \alpha_k \, r_{S_k,T_k} \tag{3.11}$$

Also, note that $\sum_{k=1}^{p} \alpha_k \ |W_k| = \|x\|_1$, where $\|.\|_1$ denotes the l_1–norm. Clearly, there is some $u \in \{1,\cdots,p\}$ such that $\alpha_u \ |W_u| \geq \|x\|_1/p$.

We claim that

$$r^t_{S_i,T_i} \ Q \ r_{S_j,T_j} \ \geq \ 0 \tag{3.12}$$

for all $i, j \in \{1,\cdots,p\}$. If $i=j$, this is obvious since Q is positive semidefinite (Proposition 2.1). Without loss of generality, suppose $i<j$. Then, we have $S_j \subset S_i$ and $T_j \subset T_i$. Consequently, for all $(k,l) \in E$, we can check that

$$(r_{S_i,T_i}(k) + r_{S_i,T_i}(l))(r_{S_j,T_j}(k) + r_{S_j,T_j}(l)) \ \geq \ 0. \tag{3.13}$$

From Proposition 2.1, our claim (3.12) is true.

Using (3.11), (3.12) and Lemma 2.2, we can write

$$x^t \ Q \ r_{S_u,T_u} \geq \alpha_u \ r^t_{S_u,T_u} \ Q \ r_{S_u,T_u} \geq \alpha_u \ |W_u| \ \psi' \geq \psi' \ \frac{\|x\|_1}{p} \tag{3.14}$$

But the left hand side of (3.14) can be written as

$$x^t Q r_{S_u,T_u} = \ \mu_1(Q) x^t \ r_{S_u,T_u} \leq \ \mu_1(Q) \ \|x\|_1 \tag{3.15}$$

Combining (3.14), and (3.15), the Theorem is proven. □

To apply Theorem 3.3 profitably, we need to compute an eigenvector x with $Val(x)$ being as small as possible. In some situations, $Val(x)$ can be related to the symmetry properties of the graph. Let $Aut(G)$ be the automorphism group of G, and suppose that the vertex set V is partitioned into p orbits by the action of $Aut(G)$. Then, if $\mu_1(Q)$ is a *simple* eigenvalue of Q and x is a corresponding eigenvector, we must have $|Val(x)| \leq p$ [3]. Using this in Theorem 3.3, we obtain a lower bound for $\mu_1(Q)$ in terms of the symmetry properties of the graph.

Remark: The bounds of Theorems 3.1, 3.2, and 3.3 may be viewed in two ways. They provide a characterization of $\mu_1(Q)$ in terms of ψ and ψ'. On the other hand, they provide lower and upper bounds for ψ and ψ'. In general, computation of ψ may be difficult and knowledge of $\mu_1(Q)$ provides an upper bound (from Theorem 3.2). If, in addition, we know an eigenvector x corresponding to $\mu_1(Q)$, then Theorem 3.3 provides an alternate upper bound for ψ (or ψ'), which may sometimes be better than Theorem 3.2. For example, consider the complete graph on n vertices. In this case, we are able to choose an eigenvector x corresponding to $\mu_1(Q) = n-2$ which has $|Val(x)| = 1$, and the lower bound of Theorem 3.3 and the upper bound of Theorem 3.1 coincide. From Theorem 3.2, we have an upper bound for ψ as $\psi \leq (4(n-1)(n-2))^{1/2}$, and from Theorem 3.3, we have $\psi \leq \psi' \leq (n-2)$. Thus, Theorem 3.3 provides a better upper bound for ψ in this case. On the other hand, consider the simple cycle on an odd number n of vertices. In this case, it is possible to choose an eigenvector x for $\mu_1(Q)$ with $|Val(x)| = (n-1)/2$[3]. Therefore, from Theorem 3.2, we obtain $\psi \leq 4\pi\sqrt{2}/n$, and from Theorem 3.3, we obtain $\psi \leq 2\pi^2/n$ which is a little worse.

Extensions of Theorems 3.1, 3.2 and 3.3 to weighted graphs are straightforward. A weighted graph is specified as $G = (V,E)$, where V is the set of vertices, E is the set of edges, and each edge has a positive weight associated with it. We may allow self loops to be present in E. The adjacency matrix of the weighted graph has entries $a_{ij} = w_{ij}$, where w_{ij} is the weight of edge (i,j). We define the degree of a vertex i to be $d(i) = \sum_{(i,j)\in E} w_{ij}$. Then, $Q = D+A$ as before. To define ψ and ψ', we sum the weights of edges instead of counting the number of edges. If a self loop is present at some node i, then it's weight must be included when computing $e_{\min}(S)$ for any $S \subseteq V$ which contains i. The results of Theorems 3.1, 3.2, and 3.3 hold under the new definitions. With a little more work, it is possible to extend the ideas developed here to obtain bounds for the smallest eigenvalue of a reversible Markov chain based on the structure of the underlying graph[4].

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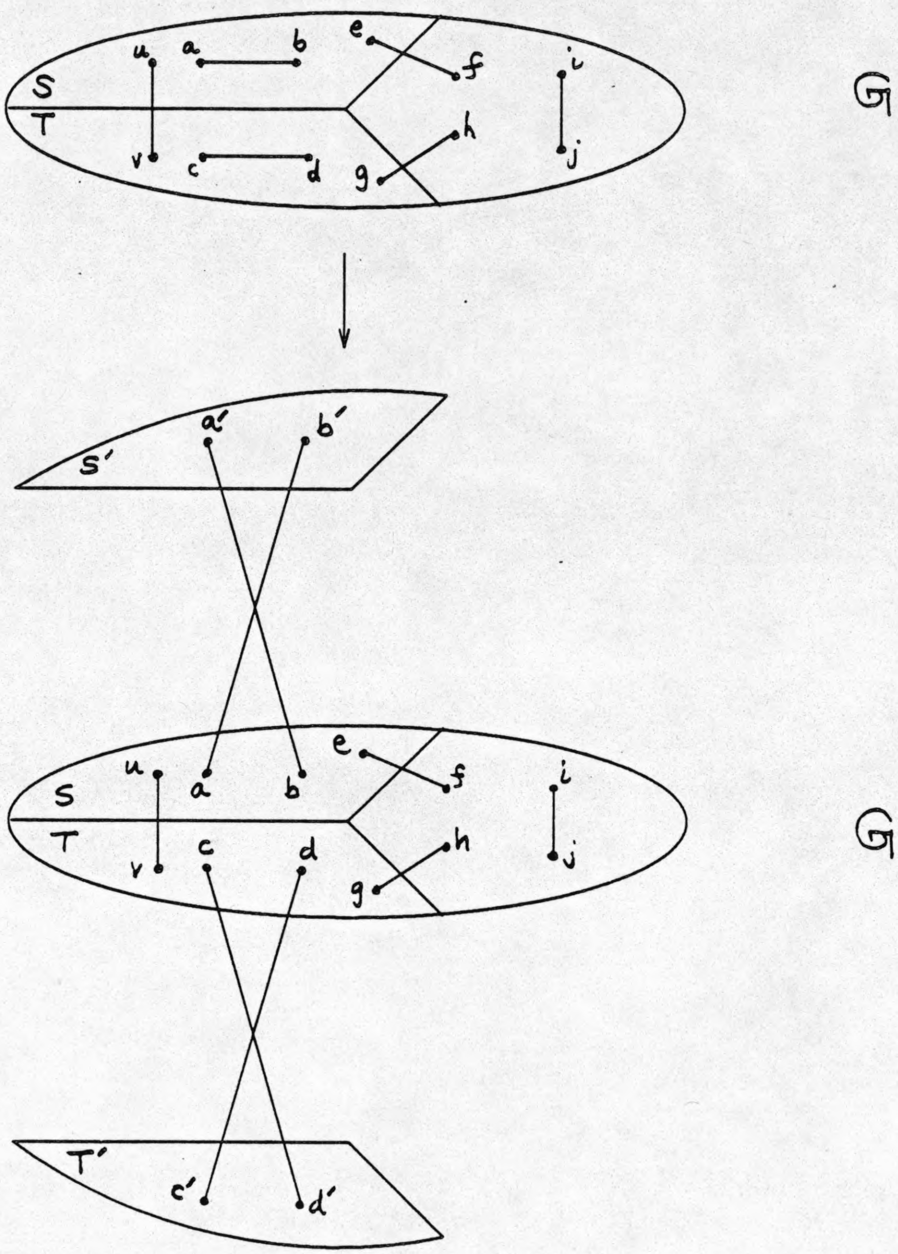

Figure 1. Construction of G' from G.