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SENSITIVITY CONSIDERATIONS IN OPTIMAL SYSTEM DESIGN

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ABSTRACT

The sensitivity of a control system is usually taken to be the normalized variation of some desired characteristic with the variation of plant or controller parameters. Rather than the usual absolute sensitivity described above, a new definition of relative sensitivity is introduced for the optimal control problem, wherein the system performance is always compared with its optimum under the given circumstances. The implications of the relative sensitivity and its relevance to optimal system design are discussed in detail. Moreover, a theoretical approach to the problem of system optimization when plant parameters are subject to change is presented.

II. Relative Sensitivity

The optimal control problem is taken to be that of minimizing a given performance index,[1]

$$J\left(\underset{\sim}{v},\ \underset{\sim}{u}(t)\right) = \int_{t_o}^{t_f} F\left[\underset{\sim}{v},\ \underset{\sim}{x}(t),\ \underset{\sim}{u}(t),\ t\right] dt, \tag{2}$$

under the restrictions imposed by plant operation,

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}\left[\underset{\sim}{v},\ \underset{\sim}{x}(t),\ \underset{\sim}{u}(t),\ t\right]. \tag{3}$$

The quantities which appear in the performance index (2) and the state equations (3) are the n-dimensional state vector, $\underset{\sim}{x}(t)$, the r-dimensional control vector, $\underset{\sim}{u}(t)$, and the vector $\underset{\sim}{v}$ which represents the variable plant parameters (the dimension of this vector is virtually unlimited in theory). It is assumed, moreover, that the control and plant parameters are members of given closed sets,

$$\underset{\sim}{u}(t) \in U \tag{4a}$$

and

$$\underset{\sim}{v} \in V. \tag{4b}$$

A grossly simplified version of the problem is presented in Figure 1. Although the performance index J is a functional, it is presented here as a function of a single control variable u for various values of the single variable plant parameter v. This picture is adequate for the sake of argument. The controls u_1, u_2, and u_3 are optimal for the plant parameters

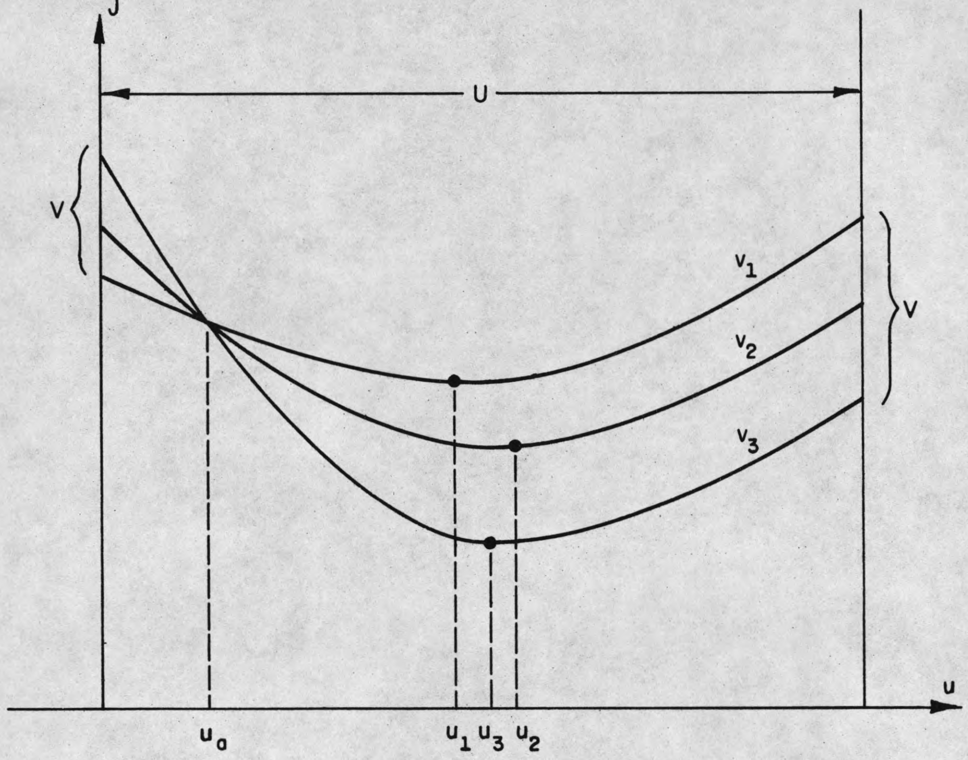

Figure 1. Simplified representation of optimal control problem.

v_1, v_2, and v_3, respectively. Moreover, the picture has been drawn purposefully so that the value of the performance index is "insensitive" to plant parameter variations for the fourth control u_a. But no right-thinking system designer would choose the control u_a merely on the basis of this insensitivity; a control somewhere in the cluster u_1, u_2, u_3 would be better in every sense. It should be noted that although the change in performance index when the plant changes from v_3 to v_1 for control u_3 is large absolutely, there is little that can be done to overcome the situation (the small gain obtained by changing to control u_1 may not even be worth the effort involved). It is considerations such as these which motivate the introduction of relative sensitivity given below.

At a set of plant parameters $\underset{\sim}{v}$ the <u>relative sensitivity</u> for the control $\underset{\sim}{u}$ (t) is defined to be the difference between the actual value of the performance index and that which would be obtained if the control were the optimal <u>for the plant parameters</u> $\underset{\sim}{v}$ (divided by the optimal performance index for normalization):

$$S^{R}\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) = \frac{J\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) - J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right)}{\left|J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right)\right|}, \tag{5}$$

where $\underset{\sim}{u}^{o}$ (t) is the optimal control for plant parameters $\underset{\sim}{v}$,

$$J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right) = \min_{\underset{\sim}{u}(t) \in U} \left[J\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right)\right]. \tag{6}$$

It should be re-emphasized that the relative sensitivity, $S^{R}\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right)$, is that for the given control $\underset{\sim}{u}$ (t) at the plant "operating point" $\underset{\sim}{v}$. For "small" control differences from plant optimal,

$$\delta\, \underset{\sim}{u}(t) = \underset{\sim}{u}(t) - \underset{\sim}{u}^{o}(t), \tag{7a}$$

$$\| \delta\, \underset{\sim}{u}(t) \| \ll 1, \tag{7b}$$

the relative sensitivity (5) takes on an especially simple form in terms of the calculus of variations (cf., Appendix A):

$$S^R\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) \approx \frac{\delta^2 J\left(\underset{\sim}{v}, \underset{\sim}{u}^o(t), \delta \underset{\sim}{u}(t)\right)}{\left|J\left(\underset{\sim}{v}, \underset{\sim}{u}^o(t)\right)\right|}, \tag{8a}$$

when the optimal control $u^o(t)$ is interior to U;

$$S^R\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) \approx \frac{\delta J\left(\underset{\sim}{v}, \underset{\sim}{u}^o(t), \delta \underset{\sim}{u}(t)\right)}{\left|J\left(\underset{\sim}{v}, \underset{\sim}{u}^o(t)\right)\right|}, \tag{8b}$$

when the optimal control $\underset{\sim}{u}^o(t)$ is on the boundary of U. It might be argued that the relative sensitivity should be further normalized by the difference in the control from the optimal, but then it would not be an accurate indication of the folly of very poor control choices.

Among the obvious advantages of the relative sensitivity is that it is always a positive number. Moreover, the relative senitivity reduces to zero at the value of plant parameters $\underset{\sim}{v}$ for which the control $\underset{\sim}{u}(t)$ is the optimal. System performance is always compared with an attainable value; consequently, one is not traumatized by matters over which one has no control. In Figure 2 is a simple illustration illuminating the above discussion; the value of the relative sensitivity for the control u_1 is zero at the plant parameter v_1, $S^R\left(v_1, u_1\right) = 0$, which is also true for the control u_2 at the plant parameter v_2, $S^R\left(v_2, u_2\right) = 0$.

III. Optimal System Design

The relative sensitivity is a normalized measured of optimality. The optimal value of relative sensitivity is zero for every plant regardless of the absolute value of its optimal performance index. Hence, before any

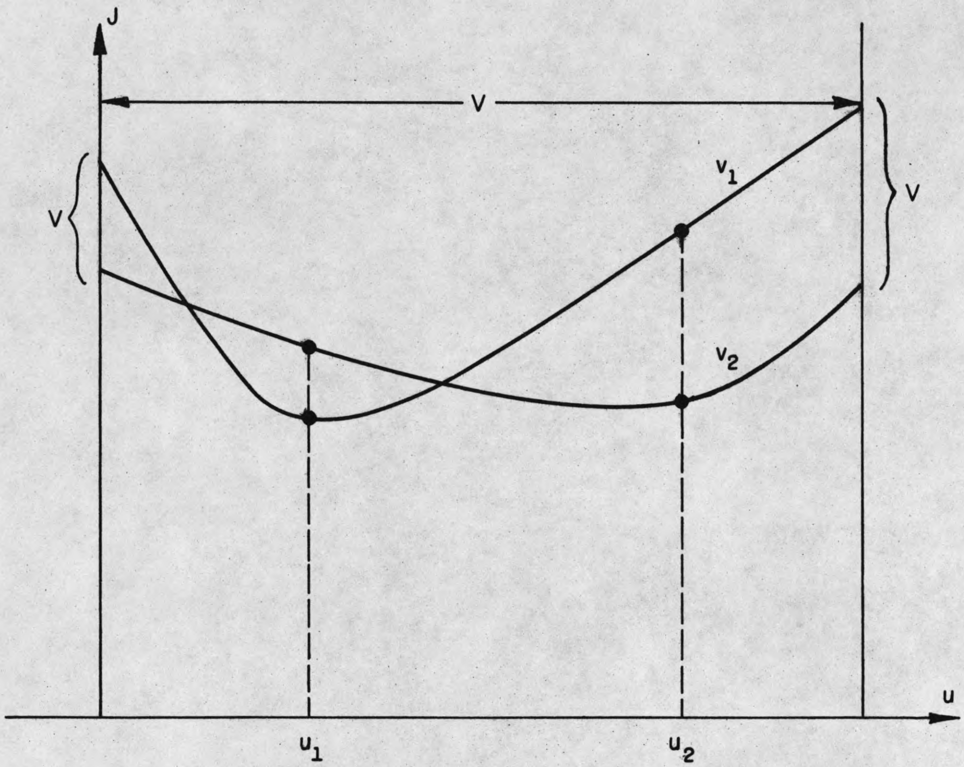

Figure 2. Relative sensitivity illustration.

statistic is introduced, the relative sensitivity provides a basis for comparing "the fact of optimality" among many plants. Those plant parameters which provide values near their optimal over a wide range of controls remain near zero in relative sensitivity -- they are relatively insensitive to the choice of control. When a control is to be found which in some sense provides optimality to a variable plant (or a number of plants), it is good procedure to allow those plant parameters with small relative sensitivity to have a lesser affect on the design decision. Consequently, the relative sensitivity is the quantity which is utilized in the design-oriented plant sensitivities evolved below.

The relative sensitivity, $S^{R}_{-} \left(v, u\ (t)\right)$, is a functional of both the plant parameters and the choice of control. Small relative sensitivity assures a design close to the optimal. One cannot, however, in general choose a control $\underset{\sim}{u}$ (t) for minimum relative sensitivity for any plant. Consequently, the concept of plant sensitivity which relates the design criterion to the relative sensitivity is evolved.

The optimization of a system depends a great deal upon the designer having complete knowledge of the system's behavior. How one deals with a system in partial ignorance of its behavior is in great measure a function of the assumed goals of the design. Two reasonable design criteria can be immediately conjectured:

1) Minimize the maximum deviation from optimal behavior; and

2) Minimize the average deviation from optimal behavior.

Either of these criteria might be applied when the designer is attempting to find a single controller for a number of similar plants or when he is attempting to find a controller for a single, changeable plant. The first criterion is, of course, the more meaningful when critical tolerances are present; the second, however, would probably find more production line use. The word "deviation" employed in the two criteria above can be the victim of various interpretations; here, the plant sensitivity is chosen as a simple quantitative measure of "deviation". Actually, as has been explained above, the sensitivity is a normalized deviation from some unknown optimal behavior; consequently, it must not be counted upon as an absolute measure.

With the first assumed design criterion in mind, one can define the <u>plant sensitivity</u>

$$S^{\underline{M}} \left(\underset{\sim}{u}\, (t) \right) \equiv \max_{\underset{\sim}{v} \in V} \left[S^{\underline{R}} \left(\underset{\sim}{v}, \underset{\sim}{u}\, (t) \right) \right], \tag{9}$$

which is indeed a measure of the maximum deviation from optimal behavior for a given control implementation $\underset{\sim}{u}\,(t)$. With the plant sensitivity (9) the optimal design criterion becomes choose

$$\underset{\sim}{u}\, (t) = \underset{\sim}{u}^{*}\, (t), \tag{10a}$$

where

$$S^{\underline{M}} \left(\underset{\sim}{u}^{*}\, (t) \right) = \min_{\underset{\sim}{u} \in U} \left[S^{\underline{M}} \left(\underset{\sim}{u}\, (t) \right) \right]. \tag{10b}$$

In Figure 3, the above plant sensitivity is illustrated by means of a simplified example; for control u_3 it is

$$S^{\underline{M}} (u_3) = S^{\underline{R}} (v_2, u_3), \tag{11a}$$

and for control u_4,

$$S^{\underline{M}} (u_4) = S^{\underline{R}} (v_1, u_4). \tag{11b}$$

The analytical details of such a minimum plant sensitivity design are presented in Appendix B, while a simplified example is given in Section IV.

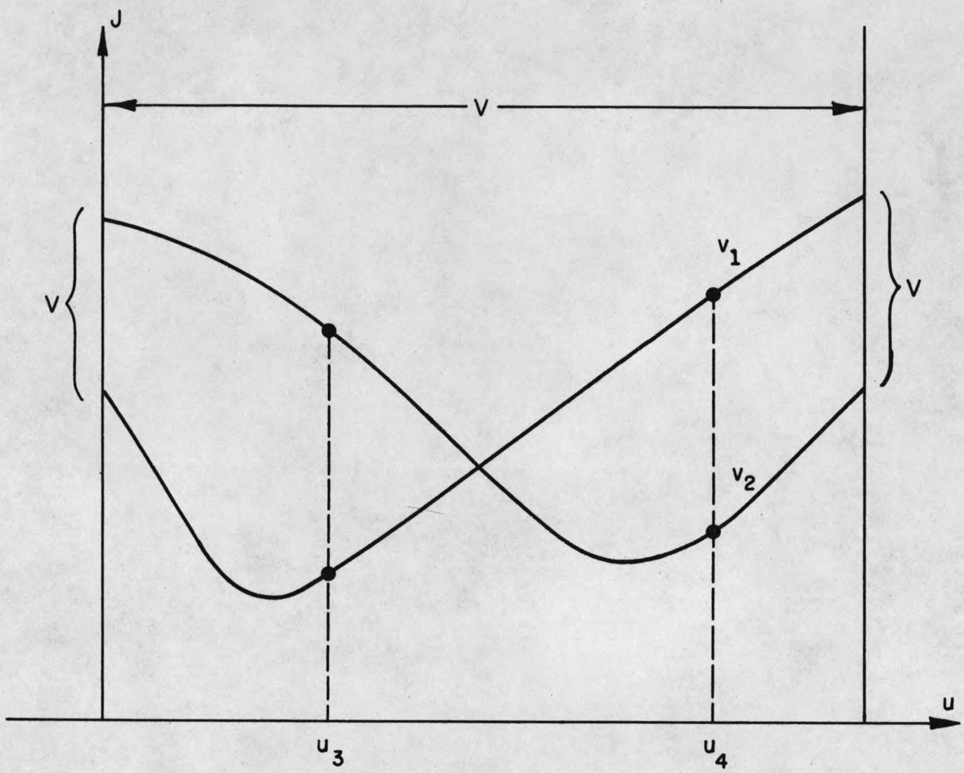

Figure 3. Plant sensitivity illustration.

When the second design criterion listed above is to be employed, an entirely different definition of plant sensitivity can be employed:

$$S^{\underline{E}}\left(\underset{\sim}{u}\,(t)\right) = \underset{\underset{\sim}{v} \in V}{E}\left[S^{\underline{R}}\left(\underset{\sim}{v},\, \underset{\sim}{u}\,(t)\right)\right], \tag{12}$$

where "E" indicates expected value. Plant sensitivity (12) should be particularly effective in situations where plant parameters are given as random variables. Such a situation might arise when a single (universal) controller is to be designed to control a large number of similar plants. The design criterion, of course, becomes

$$\underset{\sim}{u}\,(t) = \underset{\sim}{u}^{*}\,(t), \tag{13a}$$

where

$$S^{\underline{E}}\left(u^{*}\,(t)\right) = \min_{u \in U}\left[S^{\underline{E}}\left(u\,(t)\right)\right]. \tag{13b}$$

Since such considerations must enter the maximization (9) or the expected value (12), the plant sensitivity is affected by the means of implementation of the control $\underset{\sim}{u}\,(t)$; e.g., closed-loop control renders $\underset{\sim}{u}\,(t)$ and $\underset{\sim}{u}^{o}\,(t)$ -- see (5) and (6) -- dependent upon $\underset{\sim}{v}$, whereas open-loop control does not.[2] Since the plant sensitivity varies from one control implementation to another, as well as from one design criterion to another, it can be used as a basis for comparison among them in a manner analogous to that introduced by Cruz and Perkins [4,5].

IV. Example: System Optimization

The simple second order example which follows illustrates some of the principles outlined above. From the amount of manipulation involved in this contrived example, it should be obvious that a computer is in general mandatory for a reasonable design.

Example Given the second-order system characterized by

$$\ddot{x} + v\dot{x} = u, \tag{14}$$

where v is the only variable plant parameter and is assumed to lie in the interval [0,2]; consider the problem of choosing the control u (t) in such a way that the performance index

$$J\left(v, u(t)\right) = \int_0^\infty (x^2 + \dot{x}^2 + u^2)\, dt \tag{15}$$

is minimized. The optimal trajectories satisfy the second-order equation

$$\ddot{x} + \sqrt{3 + v^2}\, \dot{x} + x = 0, \tag{16}$$

as can be shown by any one of a number of well-known optimization techniques [7,8,9]. The optimal control can be realized by linear feedback of the state variables:[3]

$$u^o = c_1^o x + c_2^o \dot{x}, \tag{17}$$

where, from (14) and (16),

$$c_1^o = -1 \tag{18a}$$

and

$$c_2^o = v - \sqrt{3 + v^2} \,. \tag{18b}$$

Since c_1^o is independent of v, the (not necessarily optimal) control is taken as

$$u = -x + c_2 \dot{x}. \tag{19}$$

The performance index (15) now becomes a function of v and c_2; (it coulc be a function of c_1 as well, but such a choice would unnecessarily obscure the discussion) a simple calculation yields

$$J\,(v,\, c_2) = \frac{(c_2 - v)^2 + v^2 + 3}{2\,(v - c_2)}\, x_o^2 + 2\, x_o \dot{x}_o + \frac{c_2^2 + 3}{2(v - c_2)}\, \dot{x}_0^2 \,, \tag{20}$$

where

$$x_o = x\,(0) \tag{21a}$$

and

$$\dot{x}_o = \dot{x}\,(0). \tag{21b}$$

For the specific set of initial conditions $x_o = 1$ and $\dot{x}_o = 0$, J (v, c_2) becomes

$$J\,(v,\, c_2) = \frac{c_2^2 + 3}{2(v - c_2)} + v, \tag{22a}$$

while, from (18b), the optimum is

$$J\ (v,\ c_2^o\ )\ =\ \sqrt{3 + v^2}\ . \tag{22b}$$

Under the above circumstances, then, the relative sensitivity is

$$S^R\ (v,\ c_2)\ =\ \frac{\left(v - c_2 + \sqrt{3 + v^2}\right)^2}{2\left(v - c_2\right)\sqrt{3 + v^2}}. \tag{23}$$

Figure 4 graphs S^R (v, c_2) versus v for several values of c_2. The design procedure being discussed is that of choosing the value of c_2 which minimizes the plant sensitivity (9), $S^M (c_2) = \max_v\ S^R (v, c_2)$. Inspection of Figure 4 shows that

$$S^M\ (c_2 = -1.3) \approx 0.04, \tag{24a}$$

whereas other values of plant sensitivity are

$$S^M\ (c_2 = -1.73) \approx 0.06 \tag{24b}$$

and

$$S^M\ (c_2 = -1.0) \approx 0.15; \tag{24c}$$

for other values of c_2 it is even higher. Consequently, the <u>design optimal</u> c_2 is approximately

$$c_2^* = -1.3. \tag{25}$$

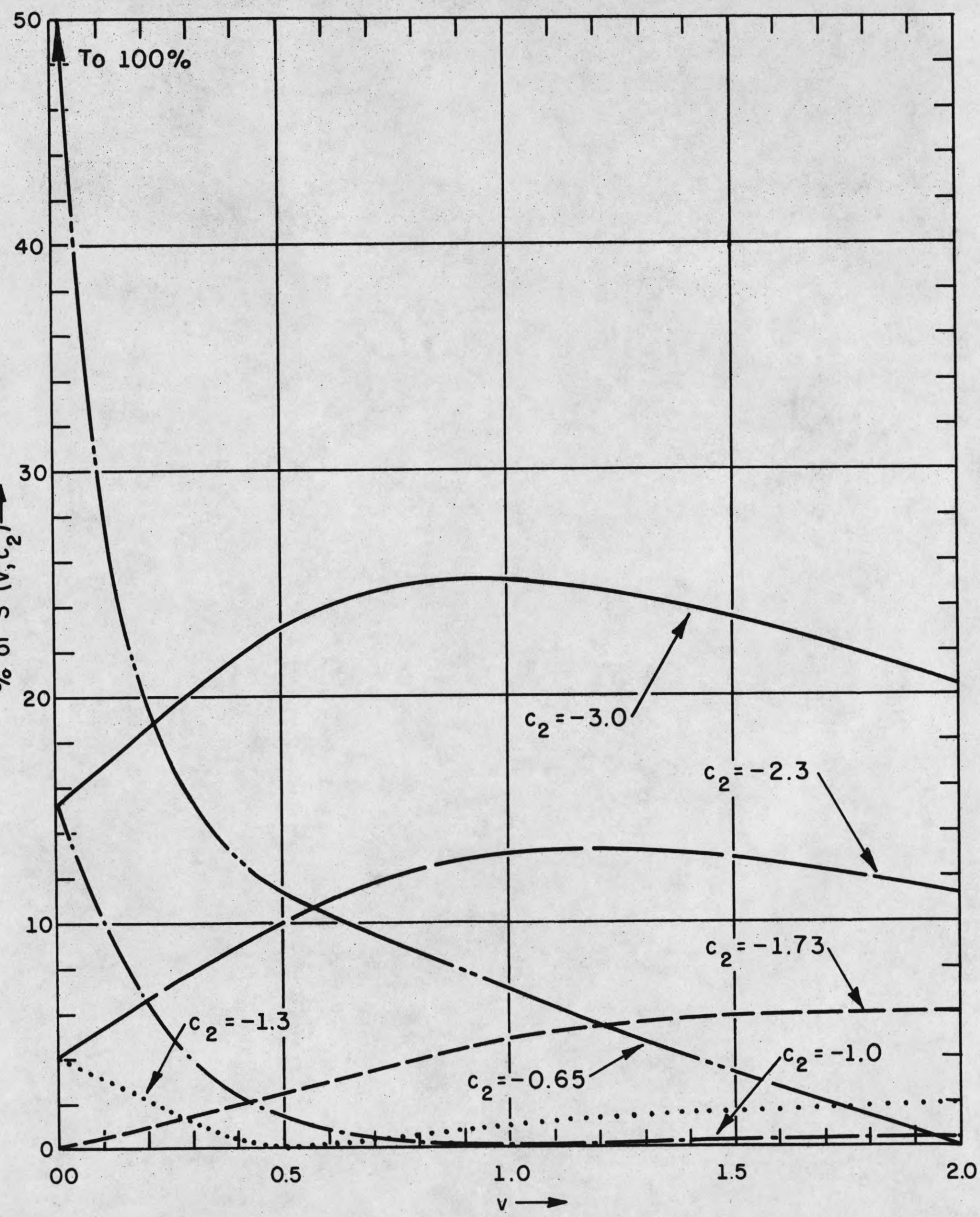

Figure 4. S^R (v, c_2) versus v for the example.

From Figure 5 it can be seen that J (v,-1.3) varies from 1.77 to 2.72, a gross variation of 35% over the interval [0,2] in v; however, these graphs also reveal that J (v,-1.3) is extremely close to the optimal throughout the prescribed range, [0,2].

V. Conclusions

The relative sensitivity (5) has been shown to be a meaningful measure of the performance of an optimal system. Moreover, two different measures of plant sensitivity, (9) and (12), have been indicated as reasonable design criteria for the optimization of incompletely specified (or variable) plants. With the introduction of relative sensitivity to the system optimization problem, system designs can be compared in new and more meaningful ways. Moreover, computer techniques for optimizing whole classes of systems can be developed.

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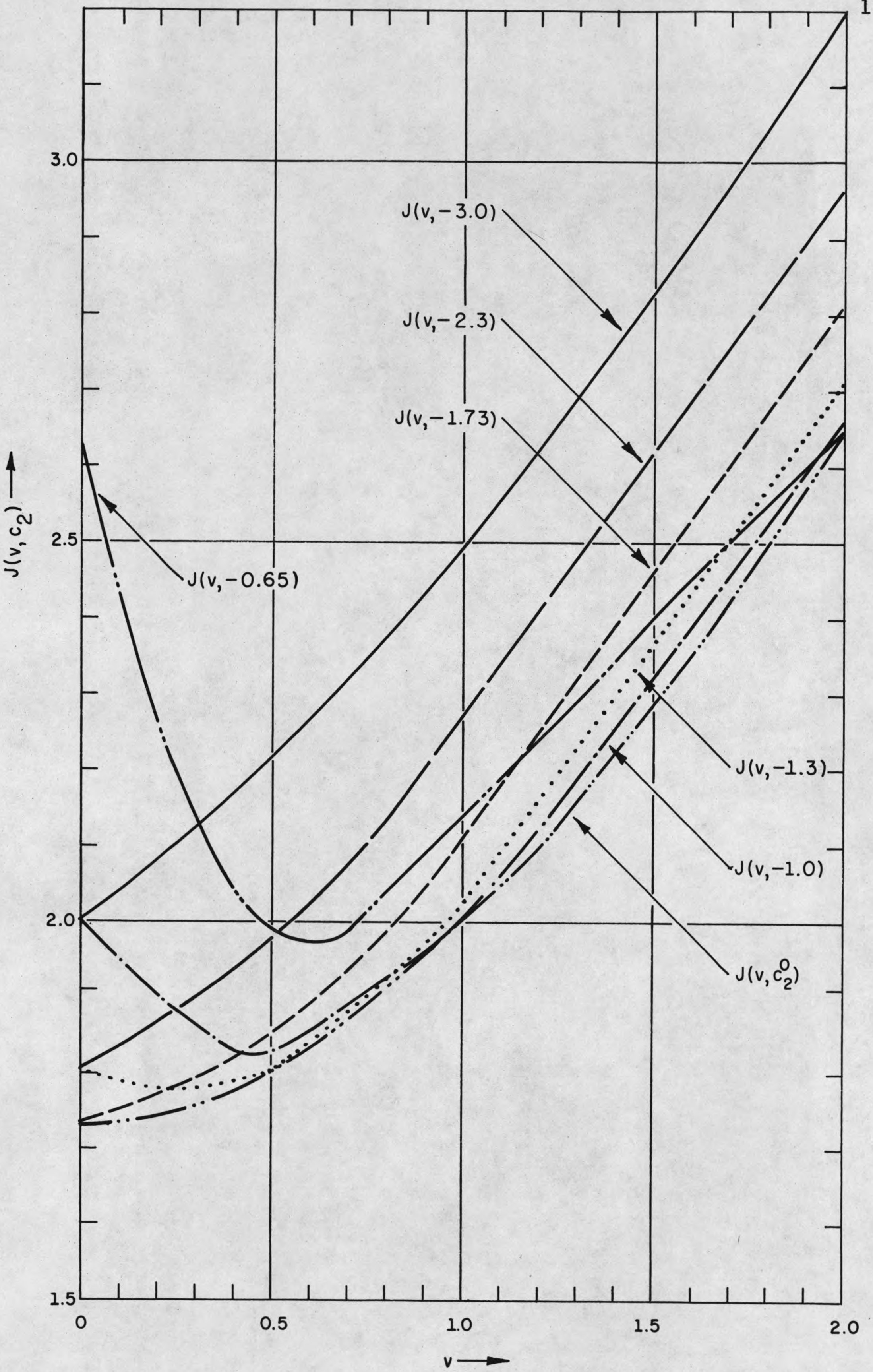

Figure 5. J $(v,\ c_2)$ versus v for the example.

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APPENDIX A

The purpose of this appendix is to demonstrate the reasonability of expressions (8) for the relative sensitivity. The first term in the numerator of (5) is expanded about the minimum in the standard manner of the calculus of variations [6]: if

$$\delta \underset{\sim}{u}(t) = \underset{\sim}{u}(t) - \underset{\sim}{u}^{o}(t), \qquad \text{(A.1)}$$

then

$$\begin{aligned} J\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) &= J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right) \\ &+ \delta J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right) \\ &+ \delta^{2} J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right) \\ &+ \left(\text{terms of order } \left\| \delta \underset{\sim}{u}(t) \right\|^{3}\right). \end{aligned} \qquad \text{(A.2)}$$

Thus, the difference expressed by the numerator of (5) is given approximately by

$$\begin{aligned} J\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) - J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right) &\approx \delta J\left(\underset{\sim}{v}, \underset{\sim}{u}(t), \delta \underset{\sim}{u}(t)\right) \\ &+ \delta^{2} J\left(\underset{\sim}{v}, \underset{\sim}{u}(t), \delta \underset{\sim}{u}(t)\right), \end{aligned} \qquad \text{(A.3)}$$

for "small" variations in $\underset{\sim}{u}(t)$,

$$\left\| \delta \underset{\sim}{u}(t) \right\| \ll 1. \qquad \text{(A.4)}$$

When the optimal control $\underset{\sim}{u}^{o}(t)$ is interior to the region U, the first variation of the performance index vanishes,

$$\delta J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right) = 0, \tag{A.5a}$$

and the second variation is nonnegative,

$$\delta^{2} J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right) \geq 0; \tag{A.5b}$$

consequently, expression (5) becomes approximately (8a)

$$S^{R}\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) \approx \frac{\delta^{2} J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right)}{\left|J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right)\right|} \tag{A.6}$$

to order $\left(\|\delta \underset{\sim}{u}(t)\|\right)^{3}$. On the other hand, when the control $\underset{\sim}{u}^{o}(t)$ is on the boundary of the region of U, the first variation of the performance index does not vanish; the condition for a local minimum becomes rather

$$\delta J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right) \geq 0. \tag{A.7}$$

In this situation expression (5) becomes approximately (8b)

$$S^{R}\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right) \approx \frac{\delta J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t), \delta \underset{\sim}{u}(t)\right)}{\left|J\left(\underset{\sim}{v}, \underset{\sim}{u}^{o}(t)\right)\right|} \tag{A.8}$$

to terms of order $\left(\|\delta \underset{\sim}{u}(t)\|\right)^{2}$. The intuitive notion that interior (stationary) control is qualitatively less sensitive than boundary control <u>most of</u>

<u>the time</u> is thus reinforced (here the difference is of order $\|\delta \underset{\sim}{u}\|$). In the general control problem, of course, both types of control occur and both expressions (A.6) and (A.8) must be used, each where appropriate.

APPENDIX B

To indicate what steps may be involved in a minimum plant sensitivity design, an analytical procedure is sketched below. It is assumed for simplicity that all extremals [6] occur in the interior of admissible regions; the extension of the procedure to more general cases is readily inferred. If the system equations (3),

$$\dot{x}_i = f_i\ (\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{u}, t),\ i = 1, \ldots, n, \tag{B.1}$$

are appended to the performance index (2) by means of lagrange multipliers [6], the integrand becomes

$$\mathcal{L}\ (\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{\dot{x}}, \underset{\sim}{u}, t, \underset{\sim}{\lambda}) = F + \sum_{i=1}^{n} \lambda_i\ (\dot{x}_i - f_i). \tag{B.2}$$

The resulting Euler equations for the optimal control, $\underset{\sim}{u}^o$ (t), are

$$\dot{x}_i = f_i\ (\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{u}^o, t),\ i = 1, \ldots, n, \tag{B.3a}$$

and

$$\dot{\lambda}_i = -\lambda_i \frac{\partial}{\partial x_i} \left[f_i\ (\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{u}^o, t) \right] + \frac{\partial}{\partial x_i} \left[F\ (\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{u}^o, t) \right]$$

$$i = 1, \ldots, n, \tag{B.3b}$$

and

$$\frac{\partial}{\partial u_j^o}\left[F\left(\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{u}^o, t\right) - \sum_{i=1}^{n} \lambda_i f_i \left(\underset{\sim}{v}, \underset{\sim}{x}, \underset{\sim}{u}^o, t\right)\right] = 0,$$

$$j = 1, \ldots, m. \qquad \text{(B.3c)}$$

In the second round one must obtain the plant sensitivity (9) by means of a maximization of the relative sensitivity (5):

$$S^{\underline{M}}\left(\underset{\sim}{u}(t)\right) = \max_{\underset{\sim}{v} \in V}\left[S^{\underline{R}}\left(\underset{\sim}{v}, \underset{\sim}{u}(t)\right)\right] \qquad \text{(B.4a)}$$

$$= \max_{\underset{\sim}{v} \in V}\left\{\frac{\int_{t_o}^{t_f} F\left[\underset{\sim}{v}, \underset{\sim}{x}(t), \underset{\sim}{u}(t), t\right] dt}{\int_{t_o}^{t_f} F\left[\underset{\sim}{v}, \underset{\sim}{x}(t), \underset{\sim}{u}^o(t), t\right] dt} - 1\right\}. \qquad \text{(B.4b)}$$

The constraint equations which must be appended for this maximization are the original system equations (B.1) plus the three sets of equations for the optimal control (B.3); thus, the variation is taken for the augmented quantity

$$\mathscr{L}_1 = S^{\underline{R}} + \sum_{i=1}^{n} \rho_{1i}\left(\dot{x}_i - f_i\right) + \sum_{i=1}^{n} \rho_{2i}\left(\dot{x}_i - f_i^{\,o}\right)$$

$$+ \sum_{i=1}^{n} \rho_{3i}\left(\dot{\lambda}_i + \lambda_i \frac{\partial f_i^o}{\partial x_i} - \frac{\partial F^o}{\partial x_i}\right)$$

$$+ \sum_{j=1}^{m} \rho_{4j} \frac{\partial}{\partial u_j^{\,o}}\left(F^o - \sum_{i=1}^{n} \lambda_i f_i^{\,o}\right), \qquad \text{(B.5)}$$

where superscript zero indicates that a quantity is to be evaluated on the optimal control. It must be kept in mind that the control quantities $\underset{\sim}{u}$ (t) only enter this variation through their dependence on other quantities (i.e., because of the means of control implementation). The maximization leads to a set of equations for $\underset{\sim}{\hat{v}}$, the plant parameters which yield the plant sensitivity, plus 6n + 2m auxilliary equations, 2n + m from (B.3), n from (B.1), and 3n + m for the Lagrange multipliers, ρ_{ij}. All of these constraining relations must be carried in like manner into the final minimization -- that which yields the optimal design $\underset{\sim}{u}$ (t).

It is clear that although such an optimization procedure can be outlined in detail as above, it is quite untractable in practice. The essential features of the design procedure are incorporated in the example of Section IV; in many practical cases similar simplifying assumptions may be employed.

FOOTNOTES

1. This formulation of the problem is merely for notational convenience; relative sensitivity can be obtained for any optimization problem.

2. Strictly speaking, once a control implementation has been chosen, the control must be represented as

$$\underset{\sim}{u}(t) = \underset{\sim}{u}\left[t, \underset{\sim}{x}(t), \underset{\sim}{v}\right].$$

   Such an interdependence between $\underset{\sim}{u}(t)$ and $\underset{\sim}{x}(t)$ and $\underset{\sim}{v}$ obviously affects the maximization (9).

3. Already, the implementation of the optimal control law has been assumed; the reader can convince himself of the futility of attempting to complete the solution without such an assumption.

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